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A lantern lemma

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Abstract We show that in the mapping class group of a surface any relation between Dehn twists of the form $T_x^j T_y^k = M$ (M a multitwist) is the lantern relation, and any relation of the form $(T_x T_y)^k = M$ (where T_x commutes with M) is the 2-chain relation.

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1 Introduction

There is interesting interplay between the algebraic and topological aspects of the mapping class group of a surface. One instance is the algebraic characterization of certain topological relations between Dehn twists. For example, consider the following well-known relations (see, e.g. [6, Chapter 4]):

	geometric relation	algebraic relation
reflexiveness	$a = b$	$T_a = T_b$
disjointness relation	$\mathrm{i}(a,b) = 0$	$T_a T_b = T_b T_a$
braid relation	$\mathrm{i}(a,b) = 1$	$T_a T_b T_a = T_b T_a T_b$

Here a and b are isotopy classes of simple closed curves on a surface, T_a and T_b are the corresponding Dehn twists, and $\mathrm{i}(a,b)$ is the geometric intersection number between a and b (see below).

One can check directly that the given topological relations imply the corresponding algebraic relations. The algebraic relations *characterize* the topological relations in the sense that the algebraic relations imply the geometric ones. In other words, the algebraic relations *only* come from specific configurations of curves on the surface (see Section 2.4).

Ivanov-McCarthy give even more general statements [7]:

- $T_a^j = T_b^k$ if and only if $a = b$ and $j = k$.

- $T_a^j T_b^k = T_b^k T_a^j$ if and only if $i(a,b) = 0$.
- $T_a^j T_b^k T_a^j = T_b^k T_a^j T_b^k$ if and only if $j = k = \pm 1$ and $i(a,b) = 1$.

McCarthy recently asked whether there was a similar characterization of the *lantern relation*, which is a relation between Dehn twists about curves which lie on a sphere with four punctures:

Lantern relation $T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}$ where the curves are as in Figure 1.

Theorem 1 answers the question in the affirmative.

The lantern relation was discovered by Dehn [3, Section 7g], and later by Johnson [8, Section IV]. Its significance arises in part from the fact that it is one of very few relations needed to give a finite presentation of the mapping class group with the finite generating set of Humphries.

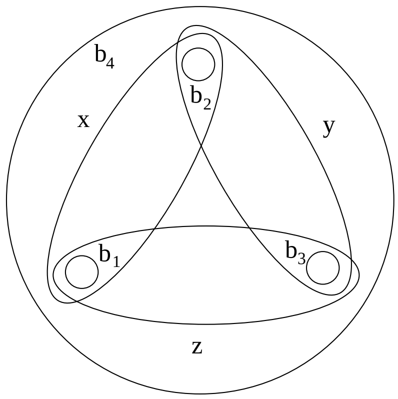

Figure 1: The lantern configuration on a sphere with four punctures

For the statement of Theorem 1, recall that the lantern relation can be written as $T_x T_y = M$, where M is a *multitwist* (see Section 2).

Theorem 1 (Lantern characterization) *Suppose $T_x^j T_y^k = M$, where M is a multitwist word and $j, k \in \mathbb{Z}$, is a nontrivial relation between Dehn twists in* $\mathrm{Mod}(S)$. *Then the given relation (or its inverse) is the lantern relation; that is, $j = k = 1$, a regular neighborhood of $x \cup y$ is a sphere with four boundary components, and $M = T_{b_1} T_{b_2} T_{b_3} T_{b_4} T_z^{-1}$, where the b_i are the boundary components of the sphere, and z is a curve on the sphere which has geometric intersection number 2 with both x and y (the sequence of curves x, y, z should move clockwise around the punctured sphere as in Figure 1).*

In the theorem, the *inverse* of a relation $w_1 = w_2$ between two words w_1 and w_2 is the equivalent relation $w_1^{-1} = w_2^{-1}$.

We also prove a similar theorem for the relation $(T_aT_b)^6 = T_c$, where $\mathrm{i}(a,b) = 1$ and c is the class of the boundary of a regular neighborhood of $a \cup b$.

Theorem 2 (2-chain characterization) *Suppose $M = (T_xT_y)^k$, where M is a multitwist word and $k \in \mathbb{Z}$, is a nontrivial relation between powers of Dehn twists in* $\mathrm{Mod}(S)$*, and $[M, T_x] = 1$. Then the given relation is the 2-chain relation—that is, $M = T_c^j$, where c is the boundary of a neighborhood of $x \cup y$, and $k = 6j$*

In Section 2 we prove the characterizations of the disjointness relation and the braid relation, and introduce ideas required for the proofs of our theorems. Section 3 is a proof of Theorem 1 in the case $j = k = 1$, Section 4 generalizes to arbitrary j and k, Section 5 contains the proof of Theorem 2, Section 6 contains supporting lemmas, and Section 7 contains further questions related to this work.

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2 Preliminaries

2.1 Notation

Let S be an orientable surface. We denote by $\mathrm{Mod}(S)$ the mapping class group of S (the group of orientation preserving self-homeomorphisms of S, modulo isotopy). When convenient, we use the same notation for a curve on S, its isotopy class, and its homology class. Brackets around a curve will be used to denote the homology class of the curve.

For two isotopy classes of simple closed curves a and b on S, the *geometric intersection number* of a and b, denoted $\mathrm{i}(a,b)$, is the minimum number of intersection points between representatives of the two classes. By definition, $\mathrm{i}(a,b)=\mathrm{i}(b,a)$. The *algebraic intersection number* of a and b, denoted $\hat{\mathrm{i}}(a,b)$, is the sum of the indices of the intersection points between any representatives of a and b, where an intersection point is of index 1 when the orientation of the intersection agrees with some given orientation of the surface, and -1 otherwise. Note that $\hat{\mathrm{i}}(a,b)=-\hat{\mathrm{i}}(b,a)$.

If a is an isotopy class of simple closed curves on S, we denote by T_a the mapping class of a Dehn twist about a representative of a. As a matter of convention, Dehn twists will be twists to the *left*. Explicitly, if a neighborhood of a representative of a is an annulus A parameterized (with orientation) by $\{(r,\theta)\in\mathbb{R}^2 : 1\le r\le 2, 0\le\theta\le 2\pi\}$, then a representative of T_a is the diffeomorphism which is given by $(r,\theta)\mapsto(r,\theta+(r-1)2\pi)$ on A (in the given coordinates) and the identity elsewhere.

A *multitwist* in $\mathrm{Mod}(S)$ is a product of Dehn twists $\prod_{j=1}^{n} T_{a_j}^{e_j}$, where $\mathrm{i}(a_j,a_k)=0$ for any j and k and $e_j\in\mathbb{Z}$.

The term *multitwist word* is used to describe a word in $\mathrm{Mod}(S)$ consisting of Dehn twists about disjoint curves. If $M=\prod_{j=1}^{n} T_{a_j}^{e_j}$ is a multitwist word in $\mathrm{Mod}(S)$, we can say that (for any j) the curve a_j is *in M*.

2.2 Formulas

Ishida and Poénaru proved Formulas 1 and 2, respectively, using elementary counting arguments [5, Lemma 2.1][1, Appendice Exposé 4]. These inequalities are very useful in computations below.

Formula 1 *Let a, b, and c be any simple closed curves on S, and let $n\in\mathbb{Z}$. Then:*

$$|n|\,\mathrm{i}(a,b)\,\mathrm{i}(a,c)-\mathrm{i}(T_a^n(c),b)\le \mathrm{i}(b,c)$$

Formula 2 *Let $M=\prod_{j=1}^{n} T_{a_j}^{e_j}$ be a multitwist word with $e_j>0$ for all j (or $e_j<0$ for all j), and let b and c be arbitrary simple closed curves on S. Then:*

$$|\,\mathrm{i}(M(c),b)-\Sigma_{j=1}^{n} e_j\,\mathrm{i}(a_j,c)\,\mathrm{i}(a_j,b)|\le \mathrm{i}(b,c)$$

As a special case of Formula 2, where $M=T_a^n$ and $b=c$, we have:

Formula 3 *Let a and b be any simple closed curves on S. Then:*

$$\mathrm{i}(T_a^n(b),b)=|n|\,\mathrm{i}(a,b)^2$$

It will be essential in the proof of the Theorem 1 to be able to compute the action of a product of Dehn twists on the homology of a subsurface of S. The well-known formula below is the pertinent tool.

Formula 4 *Let a and b be simple closed curves on S, and k an integer. Then:*

$$[T_a^k(b)] = [b] + k\,\hat{\mathrm{i}}(b,a)[a]$$

where brackets denote equivalence classes in $\mathrm{H}_1(S)$.

2.3 Basic facts

The following two facts are well-known and elementary [6, Corollary 4.1B, Lemma 4.1C].

Fact 1 *Let a and b be simple closed curves on S. If $T_a = T_b$, then a is isotopic to b.*

Fact 2 *For $f \in \mathrm{Mod}(S)$ and a any simple closed curve on S, $fT_af^{-1} = T_{f(a)}$.*

We now show that a Dehn twist about a given curve has a nontrivial effect on every curve intersecting it. This will be used, for example, in the proof of Proposition 1.

Fact 3 *Let a and b be simple closed curves on on S. If $\mathrm{i}(a,b) \neq 0$, then $T_a(b) \neq b$.*

Proof Using Formula 3 we have $\mathrm{i}(T_a(b), b) = \mathrm{i}(a,b)^2 \neq 0$. On the other hand, $\mathrm{i}(b,b) = 0$. Therefore $T_a(b) \neq b$. □

2.4 Basic relation characterizations

In the introduction, we stated characterizations of reflexiveness, the disjointness relation, and the braid relation. The characterization of reflexiveness is Fact 1. We present the proofs of the latter two characterizations here for completeness, and as a warmup for our main result.

Proposition 1 *Let a and b be simple closed curves on S. If $T_aT_b = T_bT_a$ then $\mathrm{i}(a,b) = 0$.*

Proof Assume $T_aT_b = T_bT_a$. Then, using Fact 2:

$$\begin{aligned} T_aT_b &= T_bT_a \\ T_aT_bT_a^{-1} &= T_b \\ T_{T_a(b)} &= T_b \end{aligned}$$

So $T_a(b) = b$ by Fact 1. By Fact 3, $\mathrm{i}(a,b) = 0$. □

McCarthy proved the following characterization of the braid relation in $\mathrm{Mod}(S)$ [9, Lemma 4.3]:

Proposition 2 *Let a and b be non-isotopic simple closed curves on S. If $T_aT_bT_a = T_bT_aT_b$, then $\mathrm{i}(a,b) = 1$.*

Proof From the given algebraic relation and Fact 2, we have:

$$\begin{aligned} T_aT_bT_a &= T_bT_aT_b \\ (T_aT_b)T_a(T_aT_b)^{-1} &= T_b \\ T_{T_aT_b(a)} &= T_b \end{aligned}$$

So $T_aT_b(a) = b$ by Fact 1. Applying Formula 3, we have:

$$\mathrm{i}(a,b)^2 = \mathrm{i}(T_b(a),a) = \mathrm{i}(T_aT_b(a),a) = \mathrm{i}(b,a) = \mathrm{i}(a,b)$$

Therefore $\mathrm{i}(a,b) \in \{0,1\}$. If $\mathrm{i}(a,b) = 0$, then we have $T_a^2T_b = T_aT_b^2$, and hence $T_a = T_b$, i.e. a is isotopic to b, which contradicts the assumptions. □

Ivanov-McCarthy showed that actually the following more general phenomena hold [7, Theorems 3.14-3.15]:

Proposition 3 *Let a and b be simple closed curves on S, and let j and k be nonzero integers. If $T_a^j = T_b^k$, then a is isotopic to b and $j = k$.*

Proposition 4 *Let a and b be simple closed curves on S, and let j and k be nonzero integers. If $T_a^jT_b^k = T_b^kT_a^j$ then $\mathrm{i}(a,b) = 0$.*

Proposition 5 *Let a and b be non-isotopic simple closed curves on S, and let j and k be nonzero integers. If $T_a^jT_b^kT_a^j = T_b^kT_a^jT_b^k$, then $\mathrm{i}(a,b) = 1$.*

3 Proof of lantern characterization, $j = k = 1$

The idea is to build up, step by step, the lantern relation using only the given algebraic information. In particular, we show that each of the following must be true for any algebraic relation $T_xT_y = M$, where M is a multitwist: $\mathrm{i}(x,y) > 0$, $[M, T_x] \neq 1$, there is a curve z in the multitwist word M with $\mathrm{i}(x,z) > 0$, $T_xT_y(z) = z$, $\mathrm{i}(x,z) = \mathrm{i}(y,z)$, $\mathrm{i}(x,z) = \mathrm{i}(x,T_y(z))$, $\mathrm{i}(x,y) = 2$, and $\hat{\mathrm{i}}(x,y) = 0$. From this information, it will follow that the given relation is the lantern relation.

Step 1 $\mathrm{i}(x,y) > 0$.

If $\mathrm{i}(x,y) = 0$, then T_xT_y is a multitwist word, and so the multitwist word M must also be T_xT_y by Lemma 1, i.e. the equality between the words M and T_xT_y in $\mathrm{Mod}(S)$ is trivial.

Step 2 $[M, T_x] \neq 1$.

Assuming that $[M, T_x] = 1$, we will arrive at a contradiction:

$$T_xT_yT_x^{-1}T_y^{-1} = MT_x^{-1}T_y^{-1} = T_x^{-1}MT_y^{-1} = T_x^{-1}T_xT_yT_y^{-1} = 1$$

So $T_xT_y = T_yT_x$, which implies $\mathrm{i}(x,y) = 0$ (Proposition 4), contradicting Step 1.

Step 3 There is a curve z in the multitwist word M with $\mathrm{i}(x,z) > 0$.

If $\mathrm{i}(x,z) = 0$ for each curve z in M, then $[M, T_x] = 1$, which contradicts Step 2. Therefore, there is a curve z in M which has nontrivial intersection with the curve x.

Step 4 $T_xT_y(z) = z$.

This is clear since z is one of the curves in the multitwist word M: $T_xT_y(z) = M(z) = z$.

Step 5 $\mathrm{i}(x,z) = \mathrm{i}(y,z)$.

Using Formula 3: $\mathrm{i}(T_y(z), z) = \mathrm{i}(y,z)^2$ and $\mathrm{i}(T_x^{-1}(z), z) = \mathrm{i}(x,z)^2$. But since $T_x^{-1}(z) = T_y(z)$ (Step 4), all four expressions are the same, and so $\mathrm{i}(y,z)^2 = \mathrm{i}(x,z)^2$. Since geometric intersection number is a non-negative integer, we have $\mathrm{i}(x,z) = \mathrm{i}(y,z)$.

Step 6 $\mathrm{i}(y,z) = \mathrm{i}(x, T_y(z))$.

Using $\mathrm{i}(T_y(z), z) = \mathrm{i}(y,z)^2$ (Formula 3) and $z = T_xT_y(z)$ (Step 4), we have:

$$\mathrm{i}(y,z)^2 = \mathrm{i}(T_y(z), z) = \mathrm{i}(z, T_y(z)) = \mathrm{i}(T_x(T_y(z)), T_y(z)) = \mathrm{i}(x, T_y(z))^2$$

So $\mathrm{i}(y,z) = \mathrm{i}(x, T_y(z))$.

Step 7 $\mathrm{i}(x,y) = 2$.

Using Formula 1:

$$\mathrm{i}(y,z)\,\mathrm{i}(y,x) - \mathrm{i}(x, T_y(z)) \leq \mathrm{i}(z,x)$$

But by Steps 5 and 6, $\mathrm{i}(y,z) = \mathrm{i}(x,z) = \mathrm{i}(x,T_y(z))$, so we can rewrite this as:

$$\mathrm{i}(x,z)(\mathrm{i}(x,y) - 2) \leq 0$$

Since $\mathrm{i}(x,z) > 0$ (Step 3), this gives $\mathrm{i}(x,y) \in \{0,1,2\}$. The case $\mathrm{i}(x,y) = 0$ is ruled out by Step 1.

We will now rule out $\mathrm{i}(x,y) = 1$. In this case, a neighborhood of $x \cup y$ on S is a punctured torus S'. We will show that the induced action of T_xT_y on $\mathrm{H}_1(S')$, denoted $(T_xT_y)_\star$, fails to fix any of the nontrivial elements of $\mathrm{H}_1(S')$; this contradicts the assumption that T_xT_y is equal to a multitwist in $\mathrm{Mod}(S)$ (Lemma 2).

Using $\{[x],[y]\}$ as an ordered basis for $\mathrm{H}_1(S')$, and $\hat{\mathrm{i}}(x,y) = 1$, Formula 4 yields:

$$(T_xT_y)_\star = (T_x)_\star(T_y)_\star = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

This matrix does not have an eigenvalue of 1, so $(T_xT_y)_\star$ fixes no nontrivial element of $\mathrm{H}_1(S')$.

Thus, $\mathrm{i}(x,y) = 2$.

Step 8 $\hat{\mathrm{i}}(x,y) = 0$.

Since $\mathrm{i}(x,y) = 2$, either $\hat{\mathrm{i}}(x,y) = 0$ or $\hat{\mathrm{i}}(x,y) = \pm 2$. We assume the latter and arrive at a contradiction.

Assuming $\hat{\mathrm{i}}(x,y) = \pm 2$ and $\mathrm{i}(x,y) = 2$, a neighborhood of $x \cup y$ (call it S') is a genus one surface with two boundary components (Figure 2).

As in Step 7, we will show that $(T_xT_y)_\star$ (the induced action of T_xT_y on $\mathrm{H}_1(S')$) does not fix any nontrivial, *nonperipheral* (see Lemma 2) class in $\mathrm{H}_1(S')$. This

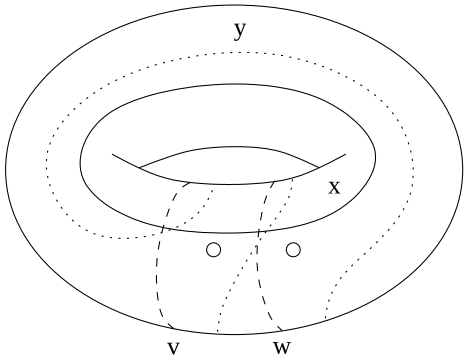

Figure 2: The picture for two simple closed curves with algebraic intersection number 2

will again contradict the assumption that $T_x T_y$ is equal to a multitwist in $\mathrm{Mod}(S)$ (Lemma 2).

Let x, v, and w be generators of $\mathrm{H}_1(S')$ with $\hat{\imath}(x,v) = \hat{\imath}(x,w) = 1$, such that the two boundary components of S' are in the homology classes $v - w$ and $w - v$ (Figure 2).

Applying Formula 4 and using $y = x + v + w$ (the case $\hat{\imath}(x,y) = +2$), the action of $(T_x T_y)_\star = (T_x)_\star (T_y)_\star$ on $\mathrm{H}_1(S')$ (with ordered basis $\{x, v, w\}$) is found to be:

$$(T_x T_y)_\star = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

In the case $\hat{\imath}(x,y) = -2$, $y = x - v - w$ and the action is:

$$(T_x T_y)_\star = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & -4 & -4 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

A basis for the fixed set of each of these linear operations is $\{v - w\}$, which is the homology class of a boundary component of S', i.e. the set of peripheral classes.

We have a contradiction, so $\hat{\imath}(x,y) = 0$.

Step 9 The relation $T_x T_y = M$ is the lantern relation.

Since x and y have geometric intersection number 2 (Step 7) and algebraic intersection number 0 (Step 8), a neighborhood of $x \cup y$ is a sphere with four boundary components S'. Let M be the word

$$T_{b_1} T_{b_2} T_{b_3} T_{b_4} T_z^{-1}$$

where b_i are the four boundary components of S', and z is one of the two simple closed curves on S' that hits each of x and y twice (the one pictured in Figure 1), then it is well-known that $T_x T_y = M$ (To check this, draw any three arcs which cut S' into a disk, and see that $T_x T_y$ and M have the same effect on each of these arcs. Then apply the Alexander lemma, which says that the mapping class group of a disk is trivial). By Lemma 1, M is uniquely written as a product of twists about disjoint curves, and we are done.

4 Proof of general lantern characterization

To show that any relation of the form $T_x^j T_y^k = M$ (where M is a multitwist word, $j, k \in \mathbb{Z}$) is the lantern relation, we use the same program as in the proof for the case $j = k = 1$ for the first 7 steps. Then, instead of homing in on $\mathrm{i}(x, y)$, and $\hat{\mathrm{i}}(x, y)$, we show that $j = k = 1$, which leaves us in the case of Section 3.

Step 0 Assumptions on j and k.

We only consider ordered pairs of exponents (j, k) in the set $\{(j,k) : j > 0, k > 0\} \cup \{(j,k) : j > 0 > k\}$ because $T_x^j T_y^k$ is equal to a multitwist word if and only if its inverse $T_y^{-k} T_x^{-j}$ is equal to a multitwist word. Also, we can assume that both j and k are nonzero, because if at least one of them is zero, then $T_x^j T_y^k$ is a multitwist about one or no curves, and the relation $T_x^j T_y^k = M$ is trivial by Lemma 1.

Steps 1 through 4 are exactly the same as for the case $j = k = 1$, so we omit the proofs.

Step 1 $\mathrm{i}(x, y) > 0$.

Step 2 $[M, T_x] \neq 1$.

Step 3 There is a curve z in the multitwist word M with $\mathrm{i}(x, z) > 0$.

Step 4 $T_x^j T_y^k(z) = z$.

Step 5 $|k|\,\mathrm{i}(y,z)^2 = |j|\,\mathrm{i}(x,z)^2$.

Using Formula 3: $\mathrm{i}(T_y^k(z),z) = |k|\,\mathrm{i}(y,z)^2$ and $\mathrm{i}(T_x^{-j}(z),z) = |j|\,\mathrm{i}(x,z)^2$. Since $T_x^{-j}(z) = T_y^k(z)$ (Step 4), all four expressions are equal, so we have $|j|\,\mathrm{i}(x,z)^2 = |k|\,\mathrm{i}(y,z)^2$ and $\mathrm{i}(y,z) = \sqrt{|j/k|}\,\mathrm{i}(x,z)$.

Step 6 $\mathrm{i}(x, T_y^k(z)) = \mathrm{i}(x,z)$.

Applying Step 5, Formula 3, Step 4, and again Formula 3, we have:

$$|j|\,\mathrm{i}(x,z)^2 = |k|\,\mathrm{i}(y,z)^2 = \mathrm{i}(z,T_y^k(z)) = \mathrm{i}(T_x^j(T_y^k(z)),T_y^k(z)) = |j|\,\mathrm{i}(x,T_y^k(z))^2$$

So $|j|\,\mathrm{i}(x,z)^2 = |j|\,\mathrm{i}(x,T_y^k(z))^2$, and further $\mathrm{i}(x,T_y^k(z)) = \mathrm{i}(x,z)$.

Step 7 $\mathrm{i}(x,y) \le 2/\sqrt{|jk|}$.

Using Formula 1:

$$|k|\,\mathrm{i}(y,z)\,\mathrm{i}(y,x) - \mathrm{i}(x,T_y^k(z)) \le \mathrm{i}(z,x)$$

But $\mathrm{i}(y,z) = \sqrt{|j/k|}\,\mathrm{i}(x,z)$ (Step 5) and $\mathrm{i}(x,T_y^k(z)) = \mathrm{i}(x,z)$ (Step 6), so we can rewrite this as:

$$\mathrm{i}(x,z)(\sqrt{|jk|}\,\mathrm{i}(x,y) - 2) \le 0$$

Since $\mathrm{i}(x,z) > 0$ (Step 3), this gives $\mathrm{i}(x,y) \le 2/\sqrt{|jk|}$.

Step 8 $0 < |jk| \le 4$.

If $|jk| > 4$, then the inequality of Step 7 says $\mathrm{i}(x,y) < 1$, which contradicts Step 1. The inequality $|jk| > 0$ is part of Step 0.

Step 9 $(j,k) = (1,\pm 1)$.

If $(j,k) \ne (1,\pm1)$, then $|jk| > 1$ and Step 7 implies that $\mathrm{i}(x,y) < 2$. This, coupled with $\mathrm{i}(x,y) > 0$ (Step 1), gives $\mathrm{i}(x,y) = 1$. In this case, a neighborhood of $x \cup y$ is a punctured torus S', and $(T_x^j T_y^k)_\star$ acts on $\mathrm{H}_1(S')$ (with basis elements represented by x and y) via the matrix:

$$(T_x^j T_y^k)_\star = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} = \begin{pmatrix} 1-jk & j \\ -k & 1 \end{pmatrix}$$

which has eigenvalues:

$$e(j,k) = \frac{(2-jk) \pm \sqrt{(jk)^2 - 4jk}}{2}$$

By Lemma 2, since $T_x^jT_y^k$ is equal to a multitwist supported on S', $(T_x^jT_y^k)_\star$ must have a fixed point on $\mathrm{H}_1(S')$, so it must have an eigenvalue of 1. We will show, however, that $e(j,k)$ does not equal 1 for $1 < |jk| \leq 4$, and so $|jk| = 1$.

By the standing assumption that either j and k are positive or $j > 0 > k$ (Step 0), and the fact that $e(j,k) = e(k,j) = e(1,jk)$, it suffices to check $e(1,jk)$ for $2 \leq |jk| \leq 4$. Using the formula, we have $e(1,4) = -1$, $e(1,3) = (-1 \pm \sqrt{3}i)/2$, $e(1,2) = \pm i$, $e(1,-2) = 2 \pm \sqrt{3}$, $e(1,-3) = (5 \pm \sqrt{21})/2$, and $e(1,-4) = 3 \pm 2\sqrt{2}$.

Step 10 $(j,k) = (1,1)$.

By Step 9, the only possibilities left for (j,k) are $(1,1)$ and $(1,-1)$. Our goal now is to show that $(j,k) = (1,-1)$ leads to a contradiction. Step 7 implies that $\mathrm{i}(x,y) \leq 2$ in this case.

As in Step 9, we know $\mathrm{i}(x,y) \neq 1$ because $e(1,-1) = (3 \pm \sqrt{5})/2$. In particular $(T_x^1T_y^{-1})_\star$ does not have an eigenvalue of 1, contradicting Lemma 2.

We can also check that $\mathrm{i}(x,y) \neq 2$. There are three subcases: $\hat{\mathrm{i}}(x,y) = 2$, $\hat{\mathrm{i}}(x,y) = -2$, and $\hat{\mathrm{i}}(x,y) = 0$.

For $\hat{\mathrm{i}}(x,y) = 2$, we can compute $(T_xT_y^{-1})_\star = (T_x)_\star(T_y)_\star^{-1}$ as follows (using the ordered basis $\{x,v,w\}$ as in Section 3, Step 8):

$$(T_xT_y^{-1})_\star = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

And for $\hat{\mathrm{i}}(x,y) = -2$, we have:

$$(T_xT_y^{-1})_\star = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -2 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 2 \\ -2 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$$

The only fixed points of the above two matrices are peripheral classes (multiples of $v - w$). By Lemma 2, both of these cases are impossibilities.

The final subcase for $(j,k) = (1,-1)$ and $\mathrm{i}(x,y) = 2$ is $\hat{\mathrm{i}}(x,y) = 0$. In this situation, a regular neighborhood of $x \cup y$ is a sphere with four punctures S'. Since $\mathrm{H}_1(S')$ contains only peripheral elements, Lemma 2 does not apply. We employ a similar idea, with curve classes playing the role of homology classes. In particular, we will show that $T_xT_y^{-1}$ is *irreducible* on S' (i.e. it does not fix any nontrivial isotopy class of simple closed curves on S'). By Lemma 3, this is a contradiction.

It is well known that the isotopy classes of simple closed curves on S' are in one-to-one correspondence with the set $\{(p,q) : \gcd(p,q) = 1\}/\sim$, where $(p,q) \sim (-p,-q)$, that $\mathrm{PMod}(S')$ (the subgroup which preserves punctures) is isomorphic to a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with a matrix A acting on a (p,q) curve by matrix multiplication, and that a Dehn twist about the $(1,0)$ curve is given by the matrix $((1,2),(0,1))$ [10, Section 3]. Therefore:

$$T_xT_y^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

This matrix does not fix any (p,q) (since it does not have an eigenvalue of ± 1). In other words, the mapping class is irreducible, and by Lemma 3 this contradicts the assumption that $T_xT_y^{-1}$ is equal to a multitwist.

Step 11 The relation $T_x^jT_y^k = M$ is the lantern relation.

We have eliminated all possibilities for the exponents except $j = k = 1$. By Section 3, the given relation (or its inverse) is the lantern relation.

5 Proof of 2-chain characterization

Theorem 2 follows from a result proven by Ishida and Hamidi-Tehrani [5, Theorem 1.2] [4]:

Theorem *If* $\mathrm{i}(x,y) \geq 2$*, then there are no relations between* T_x *and* T_y*.*

If we have the conditions of the theorem: $(T_xT_y)^k = M$, where M is a multitwist word, and $[T_x, M] = 1$. Then:

$$(T_xT_y)^kT_x = MT_x = T_xM = T_x(T_xT_y)^k$$

which is a relation between T_x and T_y, assuming $|k| > 1$ (by Section 3, Step 2 there are no relations with $|k| = 1$ and $[T_x, M] = 1$) . Therefore, $\mathrm{i}(x,y) \in \{0,1\}$. We can rule out $\mathrm{i}(x,y) = 0$, because then the relation $T_xT_y = M$ is trivial by Lemma 1. Thus, $\mathrm{i}(x,y) = 1$, and a neighborhood of $x \cup y$ is a punctured torus S'.

As in Section 3, Step 7, we consider the action of $(T_xT_y)^k$ on $\mathrm{H}_1(S')$ with generators represented by x and y. The first 6 powers of $(T_xT_y)_\star$ are

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The first five of these matrices fix no nontrivial vector. Hence, by Lemma 2, $(T_xT_y)^k$ cannot equal a multitwist in $\mathrm{Mod}(S)$ for k not a multiple of 6. When $k = 6j$ for some integer j, then it is well-known that $(T_xT_y)^k = T_c^j$ where c is the boundary component of S' [6, Lemma 4.1G]. One can check this relation by using the Alexander lemma, as in Section 3, Step 9. By Lemma 1, the multitwist word M is unique, and we are done.

6 Technical lemmas

Lemma 1 uses some new terminology: An *essential reduction class* of $f \in \mathrm{Mod}(S)$ is a class of simple closed curves α such that $f(\alpha) = \alpha$, and if $\mathrm{i}(\alpha, \gamma) \neq 0$ then $f^n(\gamma) \neq \gamma$ for any $n \in \mathbb{N}$. The *canonical reduction system* for $f \in \mathrm{Mod}(S)$ is the set of essential reduction classes of f. Lemma 1 is really a special case of the theorem of Birman-Lubotzky-McCarthy which states that canonical reduction systems are unique.

Lemma 1 *Suppose $M = \prod_{j=1}^m T_{x_j}^{e_j}$ and $N = \prod_{j=1}^n T_{y_j}^{f_j}$ are multitwist words in $\mathrm{Mod}(S)$. If $M = N$ in $\mathrm{Mod}(S)$, then $m = n$ and $\{(x_j, e_j)\} = \{(y_j, f_j)\}$.*

Proof Since M and N are multitwist words, $\mathrm{i}(x_j, x_k) = \mathrm{i}(y_j, y_k) = 0$, and so $M(x_i) = x_i$ and $N(y_j) = y_j$ for all i and j. It then follows from the work of Birman-Lubotzky-McCarthy that both $\{x_j\}$ and $\{y_j\}$ are canonical reduction systems for $M = N$ [2, Lemma 2.5], and hence the sets are the same by uniqueness of such systems [2, Theorem C]. It then follows that the exponents are the same: consider the surface obtained by cutting S along $\{x_k\}_{k \neq j}$; the mapping class induced by M on this surface is $T_{x_j}^{e_j} = T_{x_j}^{f_j}$ (assuming $x_j = y_j$), and no two different powers of a Dehn twist are the same element (Proposition 4), so $e_j = f_j$. □

For Lemma 2, a *peripheral homology class* $\alpha \in \mathrm{H}_1(S')$ on a subsurface $S' \subset S$ is one which is contained in the subgroup of $\mathrm{H}_1(S')$ generated by the classes of components of $\partial S'$. For $f \in \mathrm{Mod}(S)$, $f_\star$ denotes the induced action of f on homology.

Lemma 2 *Suppose $M \in \mathrm{Mod}(S)$ is a multitwist with support on a subsurface S', and that there is a nontrivial and nonperipheral element of $\mathrm{H}_1(S')$. Then there is a nontrivial and nonperipheral $\alpha \in \mathrm{H}_1(S')$ with $M_\star(\alpha) = \alpha$.*

Proof Since M has its support on S', it must be of the form:

$$M = \prod_{i=1}^{m} T_{a_i}^{e_i} \prod_{j=1}^{n} T_{b_j}^{f_j} \prod_{k=1}^{p} T_{c_k}^{g_k}$$

where the a_i represent the trivial class in $\mathrm{H}_1(S')$, the b_j represent peripheral homology classes in $\mathrm{H}_1(S')$, and the c_k represent nontrivial, nonperipheral classes in $\mathrm{H}_1(S')$. If p is nonzero, i.e. M consists of at least one twist about a representative of a nontrivial, nonperipheral homology class, then $M_\star([c_k]) = [c_k]$ for any k since $M(c_k) = c_k$, and we are done. Otherwise, if $p = 0$, let s be a simple closed curve on S' representing any nontrivial, nonperipheral class in $\mathrm{H}_1(S')$. Then $\hat{\mathrm{i}}([s],[a_i]) = 0$ (the $[a_i]$ can be represented by the trivial curve class) and $\hat{\mathrm{i}}([s],[b_i]) = 0$ (the $[b_i]$ can be represented by boundary curves), so $M_\star([s]) = [s]$ by Formula 4. □

Recall that an irreducible mapping class is one which fixes no isotopy class of curves. Lemma 3 states that a multitwist in $\mathrm{Mod}(S)$ cannot restrict to an irreducible mapping class on a subsurface of S.

Lemma 3 *Suppose $M \in \mathrm{Mod}(S)$ is a multitwist with support on a subsurface S', and that there is a nontrivial (not homotopic to a point or a boundary component) isotopy class of curves on S'. Then there is a nontrivial isotopy class of curves on S' which is fixed by M.*

Proof Since M has support on S', it is of the form:

$$M = \prod_{i=1}^{m} T_{a_i}^{e_i} \prod_{j=1}^{n} T_{b_j}^{f_j}$$

where the a_i are boundary components of S' and the b_j are nontrivial curve classes on S'. If $n \neq 0$, then $M(b_j) = b_j$ for any $1 \leq j \leq n$. If $n = 0$, then $M(\alpha) = \alpha$ for any nontrivial curve class on S'. □

7 Questions

Powers of T_xT_y This paper gives a partial classification of relations of the form $(T_xT_y)^k = M$, where M is a multitwist word. If $k = 1$, then it is the lantern relation. If $k \neq 1$ and $[T_x, M] = 1$, then it is the 2-chain relation. The author is unaware of relations where $k \neq 1$ and $[T_x, M] \neq 1$. One way to generalize this is to consider relations of the form $W(T_x, T_y) = M$, where $W(T_x, T_y)$ is any word in T_x and T_y.[1]

[1]Hamidi-Tehrani has successfully addressed this question [4].

Noncommutativity The results of this paper rely heavily on the assumption that certain mapping classes are multitwists. This is a strong assumption, as multitwists a priori consist of *disjoint* curves. A more general problem is to classify all relations of the form $T_xT_yT_z = T_aT_bT_cT_d$, with no hypotheses of commutativity or disjointness.

Multiple lanterns A natural question to ask is under what assumptions is $XY = M$ (M a multitwist word) the lantern relation for arbitrary mapping classes. This is certainly not true for any X and Y. For example, there are *multiple lanterns*: Let $X = T_{x_1}T_{x_2}$ and $Y = T_{y_1}T_{y_2}$, where $T_{x_1}T_{y_1} = M_1$ and $T_{x_2}T_{y_2} = M_2$ are lantern relations. Then XY is a multitwist if $[M_1, M_2] = 1$. If the two lanterns have the same boundary components, then $M = M_1^2 = M_2^2$.

Chain relations There is a canonical relation for any n-chain of curves on S (a sequence of curves $\{a_1, \ldots, a_n\}$ with $\mathrm{i}(a_j, a_k) = 1$ for $k = j \pm 1$ and $\mathrm{i}(a_j, a_k) = 0$ otherwise). When n is odd, the boundary of a neighborhood of the n-chain consists of two curves d_1 and d_2, and we have the relation:

$$(T_{a_1} \ldots T_{a_n})^{n+1} = T_{d_1}T_{d_2}$$

and when n is even, a neighborhood of the n-chain consists of one curve d_1, and we have:

$$(T_{a_1} \ldots T_{a_n})^{2n+2} = T_{d_1}$$

One can ask how well these relations can be characterized. Note that Theorem 2 is the special case $n = 2$, and that the case of $n = 1$ is Fact 1.

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