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The universal order one invariant of framed knots in most S^1–bundles over orientable surfaces

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Abstract It is well-known that self-linking is the only $\mathbb{Z}$-valued Vassiliev invariant of framed knots in S^3. However for most 3-manifolds, in particular for the total spaces of S^1-bundles over an orientable surface $F \neq S^2$, the space of $\mathbb{Z}$-valued order one invariants is infinite dimensional.

We give an explicit formula for the order one invariant I of framed knots in orientable total spaces of S^1-bundles over an orientable not necessarily compact surface $F \neq S^2$. We show that if $F \neq S^2, S^1 \times S^1$, then I is the universal order one invariant, i.e. it distinguishes every two framed knots that can be distinguished by order one invariants with values in an Abelian group.

AMS Classification 57M27; 53D99

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1 Main Results

1.1 Introduction

We work in the smooth category.

A *surface* is a not necessarily compact connected 2-dimensional manifold. A *curve* in a manifold is an immersion of S^1 into the manifold. A *framed curve* is a curve equipped with a transverse vector field. A *knot (framed knot)* is an embedded curve. Knots and framed knots are studied up to the corresponding isotopy equivalence relation. An S^1-bundle over a surface is a locally-trivial S^1-fibration.

In [4] we used a new kind of skein relation to construct an order one invariant of unframed knots in an orientable total space M of an S^1-bundle over a surface. The invariant takes value in a quotient of the group ring of $H_1(M)$, and in the case of spherical tangent bundles of surfaces it is a splitting of the Polyak's [9] Arnold-Bennequin type invariant of wave fronts on surfaces.

However, as it is easy to see, the invariant introduced in [4] is not universal i.e. there exist examples of two unframed knots that can not be distinguished by this invariant, but can be distinguished by some other order one Vassiliev invariant.

In this paper we give an explicit geometric construction of the order one invariant I of framed knots in an oriented total space M of an S^1-bundle over an oriented surface F, and we show that for $F \neq S^2, S^1 \times S^1$ the invariant I distinguishes every two knots that can be distinguished by order one invariants with values in an Abelian group.

The invariant I takes values in the group of formal finite integer linear combinations of the free homotopy classes of mappings of the wedge of two circles into the manifold M, factorised by the automorphism of the wedge that interchanges the circles. The geometric ideas we use to construct I are similar to those that allowed us [5] to obtain a formula for the universal order one Arnold's [2] J^+-type invariant of wave fronts on an orientable surface $F \neq S^2$.

In general first order invariants of knots and links in the total spaces of sphere-bundles appear to be very important in the study of wave propagation. In particular, recently the author and Yu. Rudyak [6] applied the order one invariants of links in the sphere-bundles to the study of the causality relation for wave fronts.

1.2 Construction of the invariant

Let M be an oriented total space of an S^1-bundle $p : M \to F$ over an oriented surface $F \neq S^2$. Clearly if two framed knots $K_1, K_2 \subset M$ belong to different components of the space of framed curves in M, then they are not isotopic. For this reason when studying framed knots we restrict ourselves to a connected component $\mathcal{F}$ of the space of framed curves in M. It is easy to verify that each connected component of the space of unframed curves in M corresponds (under forgetting of the framing) to precisely two connected components of the space of framed curves. These two components are distinguished by the values of a spin structure on loops in the principal $SO(3)$-bundle that are naturally associated with the framed curves. In turn, connected components of the space of unframed curves in M are in the natural bijective correspondence with the conjugacy classes of the elements of $\pi_1(M)$.

Let $\mathcal{C}$ be a connected component of the space of unframed curves in M obtained by forgetting the framing on curves from $\mathcal{F}$, and let $\mathcal{L}$ be the connected component of the space of free loops on F that contains the projections of curves from $\mathcal{C}$. (Connected components of the space of free loops on F are naturally identified with the conjugacy classes of the elements of $\pi_1(F)$.)

1.2.1 h-principle for curves on F Clearly $\mathcal{L}$ contains many components of the space of curves (immersions of S^1) on F. Put $\mathrm{pr}: STF \to F$ to be the S^1-bundle obtained by the fiberwise spherization of the tangent bundle of F. The h-principle [7] says that the space of curves on F is weak homotopy equivalent to the space of free loops ΩSTF in STF. The equivalence is given by mapping a curve C on F to a loop $\vec{C}$ in STF obtained by mapping $t \in S^1$ to the point in STF that corresponds to the velocity vector of C at $C(t)$. In particular, the connected components of the space of curves on F that are contained in $\mathcal{L}$ are naturally identified with the connected components of the space of free loops in STF that consist of loops projecting to loops from $\mathcal{L}$.

Proposition 1.2.2 *Let $F \neq S^2$ be a (not necessarily compact) oriented surface, and let $\mathcal{C}, \mathcal{L}$ be as above. Then the group $\mathbb{Z}$ acts freely and transitively on the set of connected components of the space of curves on F that realize loops from $\mathcal{L}$. The action is as follows: $i \in \mathbb{Z}$ acts on a connected component $\mathcal{K}$ that contains a curve $C \in \mathcal{L}$ by mapping it to the connected component $\mathcal{K}^i$ that contains the curve obtained from C by the addition to it of i positive kinks, provided that $i \geq 0$; and by the addition of $|i|$ negative kinks provided that $i < 0$, see Figure 1.*

For the proof of Proposition 1.2.2 see Section 2.1.

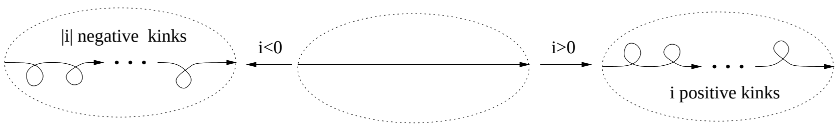

Figure 1

We say that a curve $C \in \mathcal{C}$ is *generic* (with respect to the S^1-bundle $p: M \to F$) if $p(C)$ is a curve and its only singularities are double points of transverse self-intersection.

Fix a point $x_0 \in M$. Let $K_s \subset M$ be a singular knot whose only singularity is a transverse double point d. Let B_2 be the wedge of two oriented circles. Consider a mapping $\phi: S^1 \to B_2$ that maps the preimages of the double point of K_s to the base point of the wedge, respects the orientations of the circles of the wedge, and is injective on the complement of the preimage of the double point of K_s. Then there exists a unique mapping $\psi : B_2 \to M$ such that $\psi \circ \phi = K_s$. Now we pull the base point of $\psi(B_2)$ till it is located at the point x_0 and the two loops of the wedge give an element of $\pi_1(M, x_0) \oplus \pi_1(M, x_0)$.

Clearly the free homotopy class of ψ is well-defined modulo the action of $\mathbb{Z}_2$ that interchanges the two circles. Hence to K_s corresponds a unique element b of the quotient set $\mathcal{B}$ of $\pi_1(M, x_0) \oplus \pi_1(M, x_0)$ modulo the actions of $\pi_1(M)$ via conjugation (this action corresponds to the ambiguity in choice of the path along which we pull the base point of the wedge till it is located at x_0); and by the action of $\mathbb{Z}_2$ via permutation of the two summands (this action corresponds to the ambiguity in the choice of the first of the two loops of K_s). This $b \in \mathcal{B}$ is said to be *the element corresponding to the singular knot* K_s (with one double point).

Let K be a knot in M that is generic with respect to $p : M \to F$. Let d be a double point of $p(K)$. Since F is oriented we can distinguish the two branches of K over d. We call the branch of K over d *the left branch* if the 2-frame that is formed by the projections to F of the velocity vectors of this branch and of the other branch of K over d gives the chosen orientation of the oriented surface F. The other branch is called the *right branch* of K over d.

Since both M and F are oriented, the S^1-fibers of $p : M \to F$ are naturally oriented. To a double point d of $p(K)$ we associate two singular knots (with a double point) K^r_d and K^l_d. The singular knots K^r_d and K^l_d are obtained from K by taking respectively the right and the left branch of K over d and pulling it along the oriented S^1-fiber over d in the direction coherent with the orientation of the fiber till it intersects the other branch, see Figure 1.2. Put $[K^r_d], [K^l_d] \in \mathcal{B}$ to be the elements that correspond to K^r_d and K^l_d.

Let $f \in \pi_1(M, x_0)$ be the class of the oriented S^1-fiber of $p : M \to F$. As one can show, see 2.1.1, f is in the center of $\pi_1(M)$. Thus if $b \in \mathcal{B}$ is realizable as $\alpha \oplus \beta \in \pi_1(M, x_0) \oplus \pi_1(M, x_0)$, then for $i, j \in \mathbb{Z}$ the element of $\mathcal{B}$ realized by $\alpha f^i \oplus \beta f^j$ depends only on $b \in \mathcal{B}$ and $i, j \in \mathbb{Z}$ and does not depend on the choice of realization of b as $\alpha \oplus \beta$. We denote this element of $\mathcal{B}$ by $[\alpha f^i, \beta f^j]$. Put $[K, 1]$ to be the element of $\mathcal{B}$ that corresponds to the free homotopy class of the wedge with the base point at $K(1)$, the first loop being K, and the second loop being trivial. Put $\mathbb{Z}[\mathcal{B}]$ to be the group of formal finite linear combinations of the elements of $\mathcal{B}$ with integer coefficients.

Definition 1.2.3 *of the invariant I* Fix a connected component $\mathcal{K}$ of the

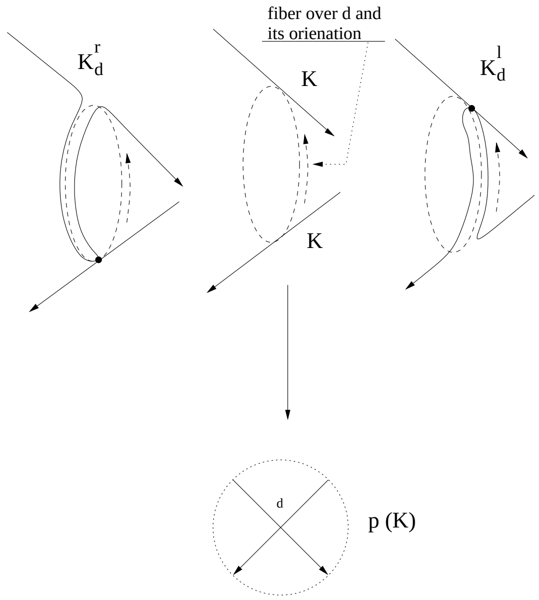

Figure 2

space of curves on F that is contained in the component $\mathcal{L}$ of the space of free loops.

Let $K \in \mathcal{F}$ be a framed knot that is generic with respect to $p: M \to F$ and has all the framing vectors non-tangent to the fibers of p. For simplicity we assume that the projections of the framing vectors point to the right of $p(K)$. (Clearly every framed knot can be deformed to be the one with such properties by a C^0-small deformation.) Put $i(K)$ to be the unique integer such that $p(K) \in \mathcal{K}^{i(K)}$, see Proposition 1.2.2.

Define $I(K) \in \mathbb{Z}[\mathcal{B}]$ by

$$I(K) = i(K)([Kf, f^{-1}] - [Kf^{-1}, f]) + \sum_d 2([K_d^l] - [K_d^r]). \tag{1}$$

Definition 1.2.4 *Vassiliev invariants* A transverse double point t of a singular knot can be resolved in two essentially different ways. We say that a resolution of a double point is *positive (resp. negative)* if the tangent vector to the first strand, the tangent vector to the second strand, and the vector from the second strand to the first form the positive 3-frame (this does not depend on the order of the strands).

A singular framed knot K_s with $(n+1)$ transverse double points admits $2^{(n+1)}$ possible resolutions of the double points. A sign of the resolution is put to be $+$ if the number of negatively resolved double points is even, and it is put to be $-$ otherwise. Let $\mathcal{A}$ be an Abelian group and let x be an $\mathcal{A}$-valued invariant of framed knots. The invariant x is said to be of *finite order* (or *Goussarov-Vassiliev invariant*) if there exists a positive integer $(n+1)$ such that for any singular knot K_s with $(n+1)$ transverse double points the sum (with appropriate signs) of the values of x on the nonsingular knots obtained by the 2^{n+1} resolutions of the double points is zero. An invariant is said to be of order not greater than n (of order $\leq n$) if n can be chosen as integer in the above definition. The group of $\mathcal{A}$-valued finite order invariants has an increasing filtration by the subgroups of the invariants of order $\leq n$.

Theorem 1.2.5 *Let $p: M \to F$ be an oriented S^1-bundle over a (not necessarily compact) oriented surface $F \neq S^2$. Then*

1 *$I(K)$ is an isotopy invariant of the framed knot K;*

2 *Let K_s be a singular knot with one double point d, let K_s^+ and K_d^- be the nonsingular framed knots obtained by respectively the positive and the negative resolution of d, and let $\alpha, \beta \in \pi_1(M)$ be such that $[\alpha, \beta]$ is the element of $\mathcal{B}$ that corresponds to K_s. Then $I(K_s^+) - I(K_s^-) = 2(-2[\alpha,\beta] + [\alpha f^{-1}, \beta f] + [\alpha f, \beta f^{-1}])$, and thus $I(K)$ is an order one invariant.*

For the Proof of Theorem 1.2.5 see Section 2.2.

The following Theorem says that $I(K)$ distinguishes all pairs of knots that can possibly be distinguished with the order one Vassiliev invariants, provided that $F \neq S^2, S^1 \times S^1$. This means that I is the universal order one invariant of knots in an oriented total space M of an S^1-bundle $p: M \to F$ over a not necessarily compact oriented surface $F \neq S^2, S^1 \times S^1$.

Theorem 1.2.6 *Let M be an oriented 3-manifold, let $F \neq S^2, S^1 \times S^1$ be an oriented (not necessarily compact) surface, and let $p: M \to F$ be an S^1-bundle. Let $\mathcal{F}$ be a connected component of the space of framed curves in M, and let $K_1, K_2 \in \mathcal{F}$ be framed knots. Let $\widetilde{I}$ be an order one Vassiliev invariant (with values in some Abelian group) such that $\widetilde{I}(K_1) \neq \widetilde{I}(K_2)$, then $I(K_1) \neq I(K_2)$.*

For the proof of Theorem 1.2.6 see Section 2.3.

Remark 1.2.7 The statement of Theorem 1.2.6 holds also in the case of $p: S^1 \times S^1 \times S^1 = ST(S^1 \times S^1) \to S^1 \times S^1$. The proof of the Theorem for this case is obtained by a straightforward generalization.

2 Proofs

2.1 Proof of Proposition 1.2.2

We start with the following Propositions.

Proposition 2.1.1 *Let N, L be oriented manifolds and let $q : N \to L$ be an S^1-bundle. Then the class $f \in \pi_1(N)$ of the oriented S^1-fiber of q is in the center of $\pi_1(N)$.*

Take $\alpha \in \pi_1(N)$. Consider $\mu : S^1 \times S^1 \to N$ with $\mu\big|_{t \times S^1}$ being the oriented S^1-fiber of q that contains $\alpha(t)$. Then the restriction of μ to the 2-cell of the torus gives the commutation relation between α and $f \in \pi_1(N)$. □

Proposition 2.1.2 (A. Preissman) *Let $F \neq S^2, S^1 \times S^1$ be an oriented (not necessarily compact) surface and let G be a nontrivial commutative subgroup of $\pi_1(F)$. Then G is infinite cyclic.*

2.1.3 Proof of Proposition 2.1.2

It is well known that any closed oriented F, other than $S^2, S^1 \times S^1$, admits a hyperbolic metric of a constant negative curvature. (It is induced from the universal covering of F by the hyperbolic plane H.) The Theorem by A. Preissman (see [3] pp. 258–265) says that if M is a compact Riemannian manifold with a negative curvature, then any nontrivial Abelian subgroup $G < \pi_1(M)$ is infinite cyclic.

If F is not closed, then the statement of the Proposition is true, since $\pi_1(F)$ is a free group on a countable or finite set of generators, see Ahlfors and Sario [1], chapter 1, or [8], pp. 143 and 199–200. □

2.1.4 The proof is based on the h-principle, see 1.2.1. Let $f \in \pi_1(STF)$ be the class of the oriented S^1-fiber of $\mathrm{pr} : STF \to F$. Proposition 2.1.1 says that f is in the center of $\pi_1(STF)$.

Let C be a curve from $\mathcal{L}$. Take $i \in \mathbb{Z}$ and put C^i to be a curve obtained from C by the addition of i positive kinks for $i \geq 0$; and by the addition of $|i|$ negative kinks for $i < 0$. It is easy to see that $\vec{C^i} = \vec{C} f^i \in \pi_1(STF)$. Since $\ker(\mathrm{pr}_* : \pi_1(STF) \to \pi_1(F))$ is generated by f, the h-principle implies that the action of $\mathbb{Z}$ (introduced in Proposition 1.2.2) on the set of connected components of the space of curves that are contained in $\mathcal{L}$ is well defined and transitive.

To show that the action is free it suffices to show that for any $\alpha, \beta \in \pi_1(STF)$ if

$$\alpha\beta\alpha^{-1} = \beta f^k, \tag{2}$$

then $k = 0$.

If $F = T^2 = S^1 \times S^1$, then $STT^2 = S^1 \times S^1 \times S^1$ and the fact that $k = 0$ is obvious. For this reason below we assume that $F \neq T^2$. Clearly for such α, β the elements $\mathrm{pr}_*(\alpha)$ and $\mathrm{pr}_*(\beta)$ commute in $\pi_1(F)$. Proposition 2.1.2 implies that there exist $\bar{g} \in \pi_1(F)$ and $i, j \in \mathbb{Z}$ such that $\mathrm{pr}_*(\alpha) = \bar{g}^i$ and $\mathrm{pr}_*(\beta) = \bar{g}^j$. Take $g \in \pi_1(STF)$ such that $\mathrm{pr}_*(g) = \bar{g}$. Since f is in the center of $\pi_1(STF)$ and f generates $\ker \mathrm{pr}_*$, we get that there exist $l, m \in \mathbb{Z}$ such that $\alpha = g^i f^l$ and $\beta = g^j f^m$. Substitute these expressions for α and β into (2) and use the fact that f is in the center of $\pi_1(STF)$ to get that $f^k = 1$. Since $\pi_2(STF) = 0$ for our manifolds STF, we get that f has infinite order in $\pi_1(STF)$. Thus $k = 0$. This finishes the proof of Proposition 1.2.2. □

2.2 Proof of Theorem 1.2.5

Let K_1, K_2 be two isotopic oriented framed knots such that $p(K_1), p(K_2)$ are immersions, the framing vectors of knots are nowhere tangent to the fibers of $p : M \to F$ and project to the nonzero vectors pointing to the right of $p(K_1)$ and $p(K_2)$, respectively.

Then it is clear that there is an isotopy between K_1 and K_2 that can be decomposed into

1 isotopies that project to the ambient isotopies of projections with the framing vectors nowhere tangent to the fibers of p and projections of them pointing to the right from the oriented knot projections; and

2 the sequence of moves such that

- **a** they happen in the charts of M homeomorphic to $\mathbb{R}^3 = (x, y, z)$ with the lines (x_0, y_0, z) for fixed (x_0, y_0) being the arcs of the S^1-fibers of p;
- **b** projections of the moves to the (x, y)-plane correspond to the second and third Reidemeister moves, and the first Reidemeister move for framed knots with blackboard framing shown in Figure 3 and its reflections. (At the start and end of these moves the framing vectors

are assumed to be nowhere tangent to the fibers of p and their projections point to the right from the oriented knot projections.)

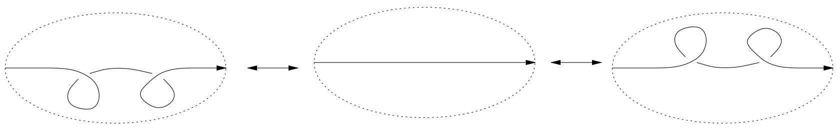

Figure 3

The invariance of I under the isotopies that project to the ambient isotopies of projections is obvious.

The increments into I that correspond to the double points of $p(K)$ that do not participate in the Reidemeister moves are unchanged under the moves.

Clearly the connected component of the space of curves on F that contains $p(K)$ is unchanged under the second and third Reidemeister moves.

A straightforward verification shows that the summands $2([K^l_{d_1}] - [K^r_{d_1}])$ and $2([K^l_{d_2}] - [K^r_{d_2}])$ corresponding to the two extra double points d_1 and d_2 of $p(K)$ that participate in an oriented version of the second Reidemeister move cancel out, and thus I is invariant under the second move.

There is a natural correspondence between the three branches of K that are present on the diagram before and after the third move. This correspondence induces the natural identification between the three double points of $p(K)$ before the move and after the move. (We identify the two points that are the double points of the projection of the corresponding pairs of branches.) Now it is easy to see that for the corresponding double points d and d' of $p(K)$ the summands $2([K^l_d] - [K^r_d])$ and $2([K^l_{d'}] - [K^r_{d'}])$ are equal. Thus I is invariant under the third move.

Clearly the number $i(K)$ is changed by ± 2 under the first move (depending on the version of the move that takes place). Thus the summand $i(K)([Kf, f^{-1}] - [Kf^{-1}, f])$ increases by $\pm 2([Kf, f^{-1}] - [Kf^{-1}, f])$. On the other hand it is easy to see that the increments into $\sum_d 2([K^l_d] - [K^r_d])$ corresponding to the two double points of $p(K)$ that participate in the first move also do not cancel out and the sum $\sum_d 2([K^l_d] - [K^r_d])$ increases by $\mp 2([Kf, f^{-1}] - [Kf^{-1}, f)$. Thus $I(K) = i(K)([Kf, f^{-1}] - [Kf^{-1}, f]) + \sum_d 2([K^l_d] - [K^r_d])$ does not change under the first move. This shows that $I(K)$ is invariant under isotopy.

The proof of the second statement of the Theorem is a straightforward calculation. □

2.3 Proof of Theorem 1.2.6

Let $\mathcal{A}$ be an Abelian group and let $\widetilde{I}$ be an $\mathcal{A}$-valued order one Vassiliev invariant of framed knots from $\mathcal{F}$. Since $\widetilde{I}$ is an order one invariant, it is defined (up to an additive constant) by its derivative $\widetilde{I}'$ i.e. by the values of its increments under the passages through the codimension zero strata of the discriminant subspace of $\mathcal{F}$. (The discriminant is the subspace of $\mathcal{F}$ formed by singular knots, and the codimension zero strata of the discriminant are formed by singular knots whose only singularity is one transverse double point.) Since $\widetilde{I}$ is an order one invariant, we get that the values of the increments depend only on the elements of $\mathcal{B}$ that correspond to singular knots with one double point that we obtain when we cross the discriminant.

Since we have fixed the connected component $\mathcal{C}$, we get that for any $[\alpha_1, \alpha_2] \in \mathcal{B}$ that corresponds to a singular knot from $\mathcal{F}$ the loop $\alpha_1\alpha_2$ is free homotopic to a curve from $\mathcal{C}$. Observe that for all $[\alpha_1, \alpha_2]$ that participate in the definition of $I(K)$ the loop $\alpha_1\alpha_2$ is also free homotopic to a loop from $\mathcal{C}$.

For this reason by abuse of notation below in the proof we denote by $\mathcal{B}$ the subset of $\mathcal{B}$ that consists of the elements realizable by $\alpha_1 \oplus \alpha_2 \in \pi_1(M) \oplus \pi_1(M)$ with the loop $\alpha_1\alpha_2$ free homotopic to curves from $\mathcal{C}$.

Consider the homomorphism $g : \mathbb{Z}[\mathcal{B}] \to \mathbb{Z}[\mathcal{B}]$ that maps

$$[s_1, s_2] \to 2(-2[s_1, s_2] + [s_1 f, s_2 f^{-1}] + [s_1 f^{-1}, s_2 f]). \tag{3}$$

(This homomorphism describes the behavior of I under crossings of the discriminant, see Theorem 1.2.5.2. Recall that by 2.1.1 f is in the center of $\pi_1(M)$.) To prove the Theorem it suffices to show that $\ker g = 0$.

Let $\overline{\mathcal{B}}$ be the quotient set of $\pi_1(F) \oplus \pi_1(F)$ via the actions of $\pi_1(F)$ that acts by conjugation of both summands and by the action of $\mathbb{Z}_2$ that acts by permuting the summands. Once again by abuse of notation below we denote by $\overline{\mathcal{B}}$ the part of $\overline{\mathcal{B}}$ that is formed by the classes of $\alpha_1 \oplus \alpha_2 \in \pi_1(F) \oplus \pi_1(F)$ such that $\alpha_1\alpha_2$ is free homotopic to the loops from $\mathcal{L}$. Let $q : \mathcal{B} \to \overline{\mathcal{B}}$ be the natural mapping induced by $p_* : \pi_1(M) \to \pi_1(F)$. (One verifies that this mapping is really well-defined.)

One verifies that $\mathbb{Z}[\mathcal{B}]$ splits into the direct sum over $\overline{\mathcal{B}}$ of $\mathbb{Z}$-submodules that are finite linear combinations of the elements of $\mathcal{B}$ projecting to the same element of $\overline{\mathcal{B}}$. Clearly g maps every summand to itself. Thus it suffices to show that the restriction of g to every summand has trivial kernel.

Fix $\bar{b} \in \overline{\mathcal{B}}$. Below we construct the ordering on $q^{-1}(\bar{b})$, that makes it isomorphic (as an ordered set) to $\mathbb{N}$ or to $\mathbb{Z}$ (depending on $\bar{b}$). One verifies that the matrix of the restriction of g to $\mathbb{Z}[q^{-1}(\bar{b})]$ written with respect to the basis that is the ordered set $q^{-1}(\bar{b})$ appears to be tridiagonal with all nonzero entries on the diagonal below the main one. Thus the restriction of g to $\mathbb{Z}[q^{-1}(\bar{b})]$ has trivial kernel, and this proves the Theorem.

To construct the ordering on $q^{-1}(\bar{b})$ we need the following proposition.

Proposition 2.3.1 *Let $F \neq S^2, S^1 \times S^1$ be a (not necessarily compact) oriented surface, let $p: M \to F$ be an S^1-bundle with oriented M, let $f \in \pi_1(M)$ be the class of the oriented fiber of p, and let α_1, α_2 be elements of $\pi_1(M)$.*

a *α_1 and α_2 commute in $\pi_1(M)$ if and only if $p_*(\alpha_1)$ and $p_*(\alpha_2)$ commute in $\pi_1(F)$.*

b *If $p_*(\alpha_1)$ and $p_*(\alpha_2)$ are conjugate in $\pi_1(F)$, then there exists a unique $i \in \mathbb{Z}$ such that α_1 and $\alpha_2 f^i$ are conjugate in $\pi_1(M)$.*

c *Let $\beta_1, \beta_2 \in \pi_1(M)$ be such that $(\delta\alpha_1\delta^{-1}, \delta\alpha_2\delta^{-1}) = (\beta_1, \beta_2) \in \pi_1(M) \oplus \pi_1(M)$, for some $\delta \in \pi_1(M)$. If there exists $\xi \in \pi_1(F)$ such that $p_*(\alpha_2) = \xi p_*(\alpha_1)\xi^{-1}$ and $p_*(\alpha_1) = \xi p_*(\alpha_2)\xi^{-1}$, then $p_*(\alpha_1) = p_*(\alpha_2)$, $p_*(\beta_1) = p_*(\beta_2)$; and hence there exist unique $i, j \in \mathbb{Z}$ such that $\alpha_1 = \alpha_2 f^i$, $\beta_1 = \beta_2 f^j$. Moreover $i = j$.*

The proof of the proposition is a straightforward calculation (similar to the one we did when proving that the action of $\mathbb{Z}$ introduced in 1.2.2 is free) and is based on Propositions 2.1.1, 2.1.2, and the fact that f generates $\ker p_*$ and has infinite order.

The ordering of the basis $q^{-1}(\bar{b})$ of $\mathbb{Z}[q^{-1}(\bar{b})]$ such that the matrix of $g\big|_{\mathbb{Z}[q^{-1}(\bar{b})]}$ written with respect to this ordered basis is tridiagonal with all the elements on the diagonal below the main one being nonzero is constructed as follows:

a) If $b \in q^{-1}(\bar{b})$ can be realized as (α_1, α_2) such that $\xi p_*(\alpha_1)\xi^{-1} = p_*(\alpha_2)$ and $\xi p_*(\alpha_2)\xi^{-1} = p_*(\alpha_1)$, for some $\xi \in \pi_1(F)$, then any realization of any element of $q^{-1}(\bar{b})$ has this property. From 2.3.1.c we get that every element $b \in q^{-1}(\bar{b})$ determines a unique $i \in \mathbb{N}$ such that b can be realized as (α_1, α_2) with $\alpha_1 f^i = \alpha_2$. One verifies that these natural numbers are different for different elements of $q^{-1}(\bar{b})$. (Recall that as it was said in the beginning of the proof of the Theorem, $\mathcal{B}$ in this proof denotes the subset of the original $\mathcal{B}$ that consists of elements realizable by $\alpha_1 \oplus \alpha_2$ with $\alpha_1\alpha_2$ being a loop free homotopic to curves from the fixed connected component $\mathcal{C}$ of the space of curves in M.) The ordering on $q^{-1}(\bar{b})$ is induced by the magnitude of $i \in \mathbb{N}$ and it makes $q^{-1}(\bar{b})$ isomorphic to $\mathbb{N}$ as the ordered set.

b) If $b \in q^{-1}(\bar{b})$ can not be realized as an element of the type described above, then none of the elements of $q^{-1}(\bar{b})$ can. This allows us to distinguish one loop of $\bar{b}$, and consequently to distinguish one loop of the elements of $q^{-1}(\bar{b})$. We use the $\mathbb{Z}_2$ action on $\pi_1(M) \oplus \pi_1(M)$ (used to introduce $\mathcal{B}$) to interchange the two loops, so that the first loop projects to the distinguished loop of $\bar{b}$. We get that every element of $q^{-1}(\bar{b})$ can be realized in a unique way as an element of the set $\widetilde{\mathcal{B}}$ that is the quotient of $\pi_1(M) \oplus \pi_1(M)$ modulo the action of $\pi_1(M)$ by conjugation of both summands. If (s_1, s_2) and $(s_3, s_4) \in \widetilde{\mathcal{B}}$ realize two elements of $q^{-1}(\bar{b})$, then there exists a unique $i \in \mathbb{Z}$ such that $s_1 f^i$ is conjugate to s_3, see 2.3.1.b. As it was said in the beginning of the proof, $s_1 s_2$ and $s_3 s_4$ are conjugate in $\pi_1(M)$, since they correspond to knots from the same connected component $\mathcal{C}$ of the space of curves in M. One uses this to verify that if $i = 0$, then (s_1, s_2) and (s_3, s_4) realize the same element of $q^{-1}(\bar{b})$. The ordering on $q^{-1}(\bar{b})$ is induced by the magnitude of i, and it makes $q^{-1}(\bar{b})$ isomorphic to $\mathbb{Z}$ as the ordered set.

This finishes the proof of Theorem 1.2.6. □

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