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On the domain of the assembly map in algebraic K–theory

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Abstract We compare the domain of the assembly map in algebraic K–theory with respect to the family of finite subgroups with the domain of the assembly map with respect to the family of virtually cyclic subgroups and prove that the former is a direct summand of the later.

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1 Introduction

In algebraic K–theory assembly maps relate the algebraic K–theory of a group ring $R\Gamma$ to the algebraic K–theory of R and the group homology of Γ. In the formulation of Davis and Lück [DL98] there is for every family of subgroups $\mathcal{F}$ of Γ an assembly map

$$H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F}); \mathbf{K}R^{-\infty}) \to K_*(R\Gamma) \tag{1.1}$$

and these maps are natural with respect to inclusions of families of subgroups. The notation is reviewed in more detail in Section 2. The Isomorphism Conjecture of Farrell–Jones [FJ93] for algebraic K–theory (and $R = \mathbb{Z}$) states that (1.1) is an isomorphism, provided that $\mathcal{F} = \mathcal{VC}$ is the family of virtually cyclic subgroups. This conjecture has been proven for different classes of groups, cf. [FJ93] [FJ98]. Arbitrary coefficient rings are considered in [BFJR]. The assembly map is also studied with $\mathcal{F} = \mathcal{FIN}$ the family of finite subgroups or $\mathcal{F}$ the family consisting of the trivial subgroup. For the trivial family there are injectivity results for different classes of groups, cf. [BHM93], [CP95]. Both results have been extended to injectivity results for $\mathcal{F} = \mathcal{FIN}$, see [Ros03] and recent work of Lück–Reich–Rognes–Varisco.

In this paper we study the map

$$H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{FIN}); \mathbf{K}R^{-\infty}) \to H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{VC}); \mathbf{K}R^{-\infty}). \tag{1.2}$$

It has been conjectured in [FJ93, p.260] (for $R = \mathbb{Z}$) that this map is split injective. In various cases this follows from the above mentioned results. The purpose of this paper is to verify this conjecture in general.

Theorem 1.3 *The map (1.2) is split injective for arbitrary groups and rings.*

In general the left hand side of (1.2) is much better understood than the right hand side, cf. [Lü02]. Thus modulo the isomorphism conjecture Theorem 1.3 may be viewed as splitting a well understood factor from the K–theory of the group ring.

For virtually cyclic groups Theorem 1.3 asserts that the assembly map for the family $\mathcal{FIN}$ is split injective. This is a special case of [Ros03]. The language of OrΓ–spectra from [DL98] allows us to extend this splitting to the more general setting in (1.2).

There is a corresponding splitting result for L–theory: If we use $L^{-\infty}$–theory and R and Γ are such that $K_{-i}(RV) = 0$ for all virtually cyclic subgroups V of Γ and sufficiently large i, then (1.2) remains split injective. This assumption is satisfied if $R = \mathbb{Z}$ by [FJ95]. We will not give the details of the proof of this L–theory statement. The proof is however completely analogous to the K–theory case. The extra assumption is needed to obtain a suitable compatibility with infinite products, see 4.4. The L–theory statements needed for this transition are provided in [CP95, Section 4].

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2 Equivariant homology theories

First let us briefly fix conventions on spectra. A spectrum $\mathbb{E}$ is given by a sequence $(E_n)_{n\in\mathbb{N}}$ of pointed spaces and structure maps $\Sigma E_n \to E_{n+1}$. A map of spectra is a sequence of maps $E_n \to F_n$ (for $n \in \mathbb{N}$) that commutes with the structure maps. A map of spectra is said to be a weak equivalence if it induces an isomorphism of (stable) homotopy groups. Two spectra $\mathbb{E}$ and $\mathbb{F}$ are said to be weakly equivalent if there is a zig-zag of weak equivalence

$$\mathbb{E} \xrightarrow{\simeq} \mathbb{A} \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \mathbb{F}$$

connecting $\mathbb{E}$ to $\mathbb{F}$.

Let Γ be a group. The *Orbit Category* $\mathrm{Or}\Gamma$ has as objects the homogeneous spaces Γ/H and as morphisms Γ–equivariant maps $\Gamma/H \to \Gamma/K$ [Bre67]. An $\mathrm{Or}\Gamma$–spectrum is a functor from $\mathrm{Or}\Gamma$ to the category of spectra. A map of $\mathrm{Or}\Gamma$–spectra is a natural transformation. A map of $\mathrm{Or}\Gamma$–spectra is called a weak equivalence if it is a weak equivalence evaluated at every Γ/H. Two $\mathrm{Or}\Gamma$–spectra are said to be weakly equivalent if they are connected by a zig-zag of weak equivalences. Our main example of an $\mathrm{Or}\Gamma$–spectrum is given by algebraic K–theory: for a ring R there is an $\mathrm{Or}\Gamma$–spectrum $\mathbf{K}R^{-\infty}$ whose value on Γ/H is the K–theory spectrum of the group ring RH. This functor has been constructed in [DL98, Section 2]. In this paper we will denote spectra by blackboard bold letters (like $\mathbb{E}$) and $\mathrm{Or}\Gamma$–spectra by boldface letters (like $\mathbf{E}$).

Associated to an $\mathrm{Or}\Gamma$–spectrum $\mathbf{E}$ is a functor from Γ–CW–complexes to spectra. Its value on a Γ–space X is given by the *balanced smash product*

$$\begin{aligned} \mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{E}) &= X^H_+ \wedge_{\mathrm{Or}\Gamma} \mathbf{E}(\Gamma/H) \\ &= \coprod_{\Gamma/H} X^H_+ \wedge \mathbf{E}(\Gamma/H)/\sim, \end{aligned} \tag{2.1}$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for $x \in X^K_+, y \in \mathbf{E}(\Gamma/H)$ and $\phi\colon \Gamma/H \to \Gamma/K$ (cf. [DL98, Section 5]). The homotopy groups of $\mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{E})$ will be denoted by $H^{\mathrm{Or}\Gamma}_*(X;\mathbf{E})$ and give an equivariant homology theory [DL98, 4.2].

A family of subgroups of Γ is a collection of subgroups of Γ that is closed under conjugation and taking subgroups. For such a family $\mathcal{F}$ there is a classifying space $E\Gamma(\mathcal{F})$, namely a Γ–CW–complex characterized (up to Γ–homotopy equivalence) by the property that $E\Gamma(\mathcal{F})^H$ is contractible if $H \in \mathcal{F}$ and empty otherwise. Given an $\mathrm{Or}\Gamma$–spectrum $\mathbf{E}$ there is for any such family of subgroups $\mathcal{F}$ the assembly map $\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F});\mathbf{E}) \to \mathbb{H}^{\mathrm{Or}\Gamma}(pt;\mathbf{E}) = \mathbf{E}(\Gamma/\Gamma)$, cf. [DL98, Section 5]. This construction is natural in the family $\mathcal{F}$ and in this paper we will compare different families.

We will need the following recognition principle, cf. [DL98, 6.3 2.]. A Γ–$\mathcal{F}$–CW–complex, is a Γ–CW–complex with isotropy groups contained in $\mathcal{F}$.

Lemma 2.2 *Let $\mathbf{E} \to \mathbf{F}$ be a map of $\mathrm{Or}\Gamma$–spectra. Let $\mathcal{F}$ be a family of subgroups of Γ such that $\mathbf{E}(\Gamma/F) \to \mathbf{F}(\Gamma/F)$ is a weak equivalence for all $F \in \mathcal{F}$. Then*

$$\mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{E}) \to \mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{F})$$

is a weak equivalence for any Γ–$\mathcal{F}$–CW–complex. □

It will be useful for us to iterate the construction of $\mathrm{Or}\Gamma$–spectra, i.e. define an $\mathrm{Or}\Gamma$–spectrum using the homology with respect to a different $\mathrm{Or}\Gamma$–spectrum.

Lemma 2.3 *Let X, Y be Γ–CW–complexes and $\mathbf{K}$ be an $\mathrm{Or}\Gamma$–spectrum. Define an $\mathrm{Or}\Gamma$–spectrum $\mathbf{E}$ by*

$$\mathbf{E}(\Gamma/H) = \mathbb{H}^{\mathrm{Or}\Gamma}(\Gamma/H \times Y; \mathbf{K}).$$

Then

$$\mathbb{H}^{\mathrm{Or}\Gamma}(X; \mathbf{E}) \cong \mathbb{H}^{\mathrm{Or}\Gamma}(X \times Y; \mathbf{K}).$$

Proof In the following formula Γ/H will always correspond to the first $\wedge_{\mathrm{Or}\Gamma}$ and Γ/K to the second.

$$\begin{aligned}
\mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{E}) &= X_+^H \wedge_{\mathrm{Or}\Gamma} \left((\Gamma/H \times Y)_+^K \wedge_{\mathrm{Or}\Gamma} \mathbf{K}(\Gamma/K)\right) \\
&= \left(X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H \times Y)_+^K\right) \wedge_{\mathrm{Or}\Gamma} \mathbf{K}(\Gamma/K) \\
&= \left(\left(X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+^K\right) \wedge Y_+^K\right) \wedge_{\mathrm{Or}\Gamma} \mathbf{K}(\Gamma/K) \\
&\cong (X_+^K \wedge Y_+^K) \wedge_{\mathrm{Or}\Gamma} \mathbf{K}(\Gamma/K) \\
&= \mathbb{H}^{\mathrm{Or}\Gamma}(X \times Y; \mathbf{K}).
\end{aligned}$$

In the second, third and fourth line the first $\wedge_{\mathrm{Or}\Gamma}$ is a balanced smash product with a space, that is similarly defined as (2.1). The homeomorphism from the third to the fourth line comes about as follows. There is a natural G–action on $X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+$ (where G acts by multiplication on Γ/H, see [DL98, 7.1]) and by [DL98, 7.4.1] a natural G–homeomorphism

$$X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+ \cong X_+.$$

Moreover, it is not hard to check that,

$$X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+^K = (X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+)^K.$$

Therefore,

$$X_+^H \wedge_{\mathrm{Or}\Gamma} (\Gamma/H)_+^K \cong X_+^K. \qquad \square$$

We finish this section with a formal splitting criterion.

Proposition 2.4 *Let $\mathbf{E} \to \mathbf{F} \to \mathbf{G}$ be maps of $\mathrm{Or}\Gamma$–spectra. Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of Γ. Assume that $\mathbf{E}$ is weakly equivalent to $\Gamma/H \mapsto \mathbb{H}^{\mathrm{Or}\Gamma}(\Gamma/H \times E\Gamma(\mathcal{F}); \mathbf{K})$ for some $\mathrm{Or}\Gamma$–spectrum $\mathbf{K}$. Assume moreover that $\mathbf{E}(\Gamma/F) \to \mathbf{F}(\Gamma/F)$ and $\mathbf{E}(\Gamma/G) \to \mathbf{G}(\Gamma/G)$ are weak equivalences for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Then*

$$H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F}); \mathbf{F}) \to H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G}); \mathbf{F})$$

is split injective.

Proof Consider the following commutative diagram.

$$\begin{tikzcd}
\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F});\mathbf{E}) \arrow[r, "\alpha"] \arrow[d, "\beta_0"] & \mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G});\mathbf{E}) \arrow[d] \arrow[dd, bend left, "\beta_1"] \\
\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F});\mathbf{F}) \arrow[r] \arrow[d] & \mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G});\mathbf{F}) \arrow[d] \\
\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F});\mathbf{G}) \arrow[r] & \mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G});\mathbf{G})
\end{tikzcd}$$

By the first assumption and 2.3 we have

$$\begin{aligned}
\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F});\mathbf{E}) &\simeq \mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F})\times E\Gamma(\mathcal{F});\mathbf{K}),\\
\mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G});\mathbf{E}) &\simeq \mathbb{H}^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G})\times E\Gamma(\mathcal{F});\mathbf{K}).
\end{aligned}$$

Now $\mathcal{F}\subset\mathcal{G}$ implies that both $E\Gamma(\mathcal{F})\times E\Gamma(\mathcal{F})$ and $E\Gamma(\mathcal{G})\times E\Gamma(\mathcal{F})$ are Γ–homotopy equivalent to $E\Gamma(\mathcal{F})$. Thus α is a weak equivalence. The second assumption and 2.2 imply that the maps labeled β_i are also weak equivalences. □

3 Homotopy fixed points

A useful tool in proving injectivity results for assembly maps are homotopy fixed points, cf. [CP95]. Given an action of a group Γ on a space X the homotopy fixed points with respect to $\mathcal{F}$ are by definition,

$$X^{h_{\mathcal{F}}\Gamma} = \mathrm{Map}_\Gamma(E\Gamma(\mathcal{F}),X).$$

We will also need actions of Γ on spectra. By definition Γ acts on a spectrum $\mathbb{E}$, by acting (pointed) on each E_n compatible with the structure maps. This allows to take (homotopy) fixed points level wise. We will call a map $X\to Y$ a weak $\mathrm{Or}\Gamma$–equivalence, if it is Γ–equivariant and induces a weak equivalence on all fixed point sets.

Proposition 3.1 *Let* $\mathbf{A},\mathbf{B}$ *be* $\mathrm{Or}\Gamma$*–spectra with a* Γ*–action (i.e. functors from* $\mathrm{Or}\Gamma$ *to spectra with* Γ*–action) and* $\mathcal{F}\subset\mathcal{G}$ *two families of subgroups of* Γ*. Assume that there is a* Γ*–equivariant map of* $\mathrm{Or}\Gamma$*–spectra* $\mathbf{A}\to\mathbf{B}$ *such that the following holds.*

(1) *There is an* $\mathrm{Or}\Gamma$*–spectrum* $\mathbf{K}$ *such that the* $\mathrm{Or}\Gamma$*–spectra* $\mathbf{A}^\Gamma$ *and* $\Gamma/H\mapsto \mathbb{H}^{\mathrm{Or}\Gamma}(\Gamma/H\times E\Gamma(\mathcal{F});\mathbf{K})$ *are weakly equivalent.*

(2) *For all $G \in \mathcal{G}$ there are weak* $\mathrm{Or}\Gamma$*–equivalences*

$$\begin{aligned}\mathbf{A}(\Gamma/G) &\simeq \mathrm{Map}_G(\Gamma, \mathbb{A}_0(G)) \\ \mathbf{B}(\Gamma/G) &\simeq \mathrm{Map}_G(\Gamma, \mathbb{B}_0(G))\end{aligned}$$

for spectra $\mathbb{A}_0(G), \mathbb{B}_0(G)$ with a G–action. Moreover, there is a G–map $\mathbb{A}_0(G) \to \mathbb{B}_0(G)$ compatible with the Γ–map $\mathbf{A}(\Gamma/G) \to \mathbf{B}(\Gamma/G)$.

(3) *For all $G \in \mathcal{G}$ the induced map $\mathbb{A}_0(G)^G \to \mathbb{B}_0(G)^{h_{\mathcal{F}}G}$ is a weak homotopy equivalence. (Here $\mathcal{F}$ is viewed as the obvious family of subgroups of G it induces.)*

Then the map $H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{F}); \mathbf{B}^\Gamma) \to H_*^{\mathrm{Or}\Gamma}(E\Gamma(\mathcal{G}); \mathbf{B}^\Gamma)$ is split injective.

In our application in Section 5 $\mathcal{F}$ will be the family of finite subgroups and $\mathcal{G}$ will be the family of virtual cyclic subgroups. In order to prove 3.1, we need three lemmata. They will be used to relate fixed points of $\mathbf{B}$ (and $\mathbf{A}$) to homotopy fixed points of $\mathbf{B}$. The proof of the first lemma is straightforward.

Lemma 3.2 *Let H be a subgroup of Γ, X a Γ–space and Y an H–space. Then there is a natural homeomorphism*

$$\mathrm{Map}_\Gamma(X, \mathrm{Map}_H(\Gamma, Y)) \cong \mathrm{Map}_H(X, Y). \qquad \square$$

Lemma 3.3 *Let H be a subgroup of Γ and Y be an H–space. Let $S = \mathrm{Map}_H(\Gamma, Y)$. Then*

$$Y^H \cong S^\Gamma \quad \text{and} \quad Y^{h_{\mathcal{F}}H} \simeq S^{h_{\mathcal{F}}\Gamma}.$$

If moreover $H \in \mathcal{F}$ then

$$S^\Gamma \simeq S^{h_{\mathcal{F}}\Gamma}.$$

Proof Using 3.2 we have

$$\begin{aligned}S^\Gamma &= \mathrm{Map}_\Gamma(pt, S) \\ &= \mathrm{Map}_\Gamma(pt, \mathrm{Map}_H(\Gamma, Y)) \\ &\cong \mathrm{Map}_H(pt, Y) \\ &= Y^H. \\ S^{h_{\mathcal{F}}\Gamma} &= \mathrm{Map}_\Gamma(E\Gamma(\mathcal{F}), S) \\ &= \mathrm{Map}_\Gamma(E\Gamma(\mathcal{F}), \mathrm{Map}_H(\Gamma, Y)) \\ &\cong \mathrm{Map}_H(E\Gamma(\mathcal{F}), Y) \\ &\simeq \mathrm{Map}_H(EH(\mathcal{F}), Y) \\ &= Y^{h_{\mathcal{F}}H}.\end{aligned}$$

To prove the last assertion, observe that if $H \in \mathcal{F}$, then $EH(\mathcal{F})$ is a point and $Y^{h_{\mathcal{F}}H} = Y^H$. Therefore $S^{h_{\mathcal{F}}\Gamma} \simeq S^\Gamma$. □

Lemma 3.4 *For $F \in \mathcal{F}$ and $G \in \mathcal{G}$ the induced maps*

$$\begin{array}{rcl} \mathbf{B}(\Gamma/F)^\Gamma & \to & \mathbf{B}(\Gamma/F)^{h_{\mathcal{F}}\Gamma}, \\ \mathbf{A}(\Gamma/G)^\Gamma & \to & \mathbf{B}(\Gamma/G)^{h_{\mathcal{F}}\Gamma} \end{array}$$

are homotopy equivalences.

Proof The first homotopy equivalence follows easily from 3.1 (2) and the second part of 3.3. The second map is by 3.1 (2) and the first part of 3.3 equivalent to $\mathbb{A}_0(G)^G \to \mathbb{B}_0(G)^{h_{\mathcal{F}}G}$ and a homotopy equivalence by 3.1 (3). □

Proof of Proposition 3.1 Set $\mathbf{E} = \mathbf{A}^\Gamma$, $\mathbf{F} = \mathbf{B}^\Gamma$ and $\mathbf{G} = \mathbf{B}^{h_{\mathcal{F}}\Gamma}$. In order to apply 2.4, we need to check that $\mathbf{A}(\Gamma/G)^\Gamma \to \mathbf{B}(\Gamma/G)^{h_{\mathcal{F}}\Gamma}$ and $\mathbf{A}(\Gamma/F)^\Gamma \to \mathbf{B}(\Gamma/F)^\Gamma$ are weak equivalences for $G \in \mathcal{G}$ and $F \in \mathcal{F}$. This a consequence of 3.4. □

4 Controlled algebra

Let Z be a topological space and R be a ring. Controlled algebra is concerned with categories of R–modules over Z ($M = \bigoplus_{z\in Z} M_z$) and R–module maps over Z ($\phi = (\phi_{z,z'}\colon M'_z \to M_z)$). We will need an equivariant version of this theory that has been studied in [BFJR]. Let Γ be a group and X be a Γ–space. The equivariant continuous control condition $\mathcal{E}_{\Gamma cc}(X)$ (consisting of subsets of $(X \times [1,\infty))^{\times 2}$) is defined in [BFJR, 2.5]. Let $p\colon Y \to X$ be a continuous Γ–map. We define a category $\mathcal{C}(Y;p)$ of R–modules over $Y \times \Gamma \times [1,\infty)$: Its objects are locally finite (see [BFJR, Section 2.2]) free R–modules $M = \bigoplus M_{(y,\gamma,t)}$ subject to the condition that there is a compact subset $K \subset Y \times \Gamma$ (depending on M) such that $M_{(y,\gamma,t)} = 0$ unless $(y,\gamma) \in \Gamma K$. Morphisms $\phi = (\phi_{(y,\gamma,t),(y',\gamma',t')})$ are required to satisfy the following condition: there is $E \in \mathcal{E}_{\Gamma cc}(X)$ (depending on ϕ) such that $\phi_{(y,\gamma,t),(y',\gamma',t')} = 0$ unless $((p(y),t),(p(y'),t')) \in E$. Note that this definition depends on the group action we have in mind. The objects of the full subcategory $\mathcal{C}_0(Y;p) \subset \mathcal{C}(Y;p)$ have by definition support in $Y \times \Gamma \times [1,\alpha]$, i.e. for every module M there is $\alpha > 0$ such that $M_{y,\gamma,t} = 0$ unless $t \le \alpha$. This inclusion is a Karoubi filtration ([CP95, 1.27]) and we denote the quotient by $\mathcal{D}(Y;p)$. The group Γ acts on all these categories. The fixed point category $\mathcal{D}^\Gamma(Y;p)$ appeared in [BFJR]. We abbreviate

$$\mathbb{K}(p) = \mathbb{K}^{-\infty}\mathcal{D}(Y;p).$$

If $p = \mathrm{id}_X$ we will write $\mathbb{K}(X)$ for $\mathbb{K}(\mathrm{id}_X)$. An important application of controlled algebra has been the construction of homology theories [PW89]. The following equivariant version of this result is proven in [BFJR, Section 5 and 6.2].

Theorem 4.1 *The functor*

$$X \mapsto \Omega\mathbb{K}(X)^\Gamma$$

from Γ*–CW–complexes to spectra is weakly equivalent to*

$$X \mapsto \mathbb{H}^{\mathrm{Or}\Gamma}(X;\mathbf{K}R^{-\infty}). \qquad \square$$

We will later on need the following simple observation.

Lemma 4.2 $\mathbb{K}(X \times Y \to Y) \to \mathbb{K}(Y)$ *is a weak* $\mathrm{Or}\Gamma$*–homotopy equivalence.*

Proof It is not hard to check that $\mathcal{D}^H(X \times Y; X \times Y \to Y)) \to \mathcal{D}^H(Y;\mathrm{id}_Y)$ is an equivalence of categories for any subgroup H. $\square$

The next lemma will later on be the key ingredient in checking condition 3.1 (2).

Lemma 4.3 *Let* $p\colon X \to Y \times \Gamma/H$ *be a* Γ*–map. Let* $X_0 = p^{-1}(Y \times \{eH\})$ *and denote by* $p_0^H\colon X_0 \to Y$ *the* H*–map induced by* p*. Then there is a weak* $\mathrm{Or}\Gamma$*–equivalence*

$$\mathbb{K}(p) \simeq \mathrm{Map}_H(\Gamma, \mathbb{K}(p_0^H)).$$

Proof For $U \subset \Gamma/H$ let $X[U] = p^{-1}(Y \times U)$ and $p[U] = p|_{X[U]}$. For a subgroup F we abbreviate $\mathcal{C}^F[U] = \mathcal{D}^F(X[U];p[U])$. Clearly $\mathbb{K}(p_0^H) = \mathbb{K}^{-\infty}\mathcal{C}[eH]$. The continuous control condition $\mathcal{E}_{\Gamma cc}(Y \times \Gamma/H)$ separates in particular different path components. Therefore we get

$$\mathcal{D}(X;p) \cong \prod_{\gamma H \in \Gamma/H} \mathcal{C}[\gamma H].$$

Projections induce a map

$$\mathbb{K}^{-\infty}\mathcal{D}(X;p) \to \prod_{\gamma H \in \Gamma/H} \mathbb{K}^{-\infty}\mathcal{C}[\gamma H] \cong \mathrm{Map}_H(\Gamma, \mathbb{K}(p_0^H)).$$

We have to show that this map is a weak $\mathrm{Or}\Gamma$–equivalence. Let F be a subgroup of Γ. Again, the continuous control condition implies

$$\mathcal{D}^F(X;p) \cong \prod_{F\gamma H \in F\backslash\Gamma/H} \mathcal{C}^F[F\gamma H].$$

Using the fact that $\mathbb{K}^{-\infty}$ commutes with fixed points and up to weak equivalence with infinite products [Car95] we obtain

$$\begin{aligned}(\mathbb{K}^{-\infty}\mathcal{D}(X;p))^F &\cong \mathbb{K}^{-\infty}\mathcal{D}^F(X;p) \\ &\simeq \prod_{F\gamma H\in F\backslash\Gamma/H} \mathbb{K}^{-\infty}\mathcal{C}^F[F\gamma H].\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{K}^{-\infty}\mathcal{C}^F[F\gamma H] &\cong \mathbb{K}^{-\infty}\mathcal{C}^{F\cap\gamma H\gamma^{-1}}[\gamma H] \\ &\cong (\mathbb{K}^{-\infty}\mathcal{C}[\gamma H])^{F\cap\gamma H\gamma^{-1}} \\ &\cong \Big(\prod_{f\gamma H\in F(\gamma H)} \mathbb{K}^{-\infty}\mathcal{C}[f\gamma H]\Big)^F.\end{aligned}$$

(Here $F(\gamma H)$ denotes the F–orbit of γH in Γ/H.) We finish the argument by observing that

$$\Big(\prod_{\gamma H\in\Gamma/H} \mathbb{K}^{-\infty}\mathcal{C}[\gamma H]\Big)^F \cong \prod_{F\gamma H\in F\backslash\Gamma/H} \Big(\prod_{f\gamma H\in F(\gamma H)} \mathbb{K}^{-\infty}\mathcal{C}[f\gamma H]\Big)^F. \qquad \square$$

Remark 4.4 In the proof above we used the compatibility of K–theory with infinite products from [Car95]. At this point the L–theory version of our splitting result needs the additional assumption stated in the introduction. It is explained in [CP95, p. 756] that for additive categories with involutions $\mathcal{A}_n$ there is a weak equivalence

$$\mathbb{L}^{-\infty}\Big(\prod \mathcal{A}_n\Big) \cong \prod \mathbb{L}^{-\infty}\mathcal{A}_n,$$

provided there is i_0 independent of n such that $K_{-i}\mathcal{A}_n = 0$ for all $i \geq i_0$. Thus, an L–theory version of 4.3 needs an additional assumption. A sufficient assumption is that $K_{-i}RH = 0$ for all sufficiently large i.

Under sufficient control conditions, there is no difference between fixed points and homotopy fixed points. This is an important ingredient in the proof of injectivity of assembly maps in [CP95] and [Ros03]. We will need the following version of this result.

Lemma 4.5 *Let X be a cocompact Γ–CW–complex with isotropy groups contained in a family of subgroups $\mathcal{F}$. Then the obvious map*

$$\mathbb{K}(X)^\Gamma \to \mathbb{K}(X)^{h_{\mathcal{F}}\Gamma}$$

is a homotopy equivalence.

Proof This is [Ros03, 6.2]. One proceeds by induction on the equivariant cells of X. The induction step uses 4.3 and 3.3. □

The following result is closely related to [Ros03, 7.1]. Using what is sometimes called the descent principle it can be used to show split injectivity of (1.2) in the base case, i.e. for virtually cyclic Γ. (The point of the descent principle is that it requires only knowledge about fixed points of finite subgroups.) The infinite cyclic and the infinite dihedral group act properly on $\mathbb{R}$. Virtually cyclic groups map either onto the integers or the infinite dihedral group ([FJ95, 2.5]), and act therefore also properly on $\mathbb{R}$. The restriction of this action to finite subgroups is either trivial or factors through the action of $\mathbb{Z}/2$ by a reflection.

Proposition 4.6 *Consider $\mathbb{R}$ with the aforementioned proper action of a virtual cyclic group V. If H is a finite subgroup of V, then*

$$\mathbb{K}(\mathbb{R})^H \to \mathbb{K}(\mathbb{R} \to pt)^H$$

is a weak equivalence.

In order to prove this we will need a slightly different construction of $\mathcal{D}(Y;p)$ for a continuous Γ–map $p\colon Y \to X$ where X carries a Γ–equivariant metric d. Define the subcategory $\tilde{\mathcal{C}}(Y;p) \subset \mathcal{C}(Y;p)$ whose morphisms have to satisfy the additional condition, that there is $\alpha > 0$ (depending on ϕ) such that $\phi_{(y,\gamma,t),(y',\gamma',t')} = 0$ unless $d(p(y),p(y')) \leq \alpha$. The corresponding inclusion $\tilde{\mathcal{C}}_0(Y;p) \subset \tilde{\mathcal{C}}(Y;p)$ is again a Karoubi filtration. It is not to hard to check, that its quotient $\tilde{\mathcal{D}}(Y;p)$ is equivalent to $\mathcal{D}(Y;p)$ and that this is compatible with the Γ–actions, cf. [BFJR, 8.8]. However, one has to be a little careful with the definitions to get this even before taking fixed points. In particular, it is at this point important that all $E \in \mathcal{E}_{\Gamma cc}(X)$ are required to be Γ-invariant, [BFJR, 2.5(iii)].

Lemma 4.7 *The K–theory of $\tilde{\mathcal{C}}^H(\mathbb{R};\mathrm{id}_{\mathbb{R}})$ vanishes under the assumption of 4.6. (Here we consider the standard metric on $\mathbb{R}$.)*

Proof Let $x_0 \in \mathbb{R}$ be a fixed point for the action of H. We will need various full subcategories of $\tilde{\mathcal{C}}^H(\mathbb{R};\mathrm{id}_{\mathbb{R}})$. Let $\tilde{\mathcal{S}}$ be the full subcategory whose objects have support in $[x_0-\alpha,x_0+\alpha] \times V \times [1,\infty)$ for some $\alpha > 0$; $\tilde{\mathcal{S}}_+$ be the full subcategory whose objects have support in $[x_0,x_0+\alpha] \times V \times [1,\infty)$ for some $\alpha > 0$; $\tilde{\mathcal{S}}_-$ be the full subcategory whose objects have support in $[x_0-\alpha,x_0] \times V \times [1,\infty)$ for some $\alpha > 0$; $\tilde{\mathcal{C}}_+$ be the full subcategory whose objects have support in $[x_0,\infty)\times V\times[1,\infty)$; $\tilde{\mathcal{C}}_-$ be the full subcategory whose objects have support in $(-\infty,x_0]\times V\times[1,\infty)$. Then $\tilde{\mathcal{S}}\subset\tilde{\mathcal{C}}^H(\mathbb{R};\mathrm{id}_{\mathbb{R}})$, $\tilde{\mathcal{S}}_+\subset\tilde{\mathcal{C}}_+$ and $\tilde{\mathcal{S}}_-\subset\tilde{\mathcal{C}}_-$ are Karoubi filtrations and we denote the quotient categories by $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{Q}}_+$ and $\tilde{\mathcal{Q}}_-$. It is not hard to check that the first of these quotients is equivalent to the direct sum of the two later. The K–theory of $\tilde{\mathcal{Q}}$ is therefore the sum of the K–theories of $\tilde{\mathcal{Q}}_+$ and $\tilde{\mathcal{Q}}_-$. Applying $\mathbb{K}^{-\infty}$ to Karoubi filtrations gives a homotopy fibration by [CP95, 1.28]. Putting all this together, we see that it suffices to show that the K–theory of each of our five full subcategories is trivial. The map $(x,v,t)\mapsto((x-x_0)/2+x_0,v,t+1)$ induces an Eilenberg swindle on $\tilde{\mathcal{S}}$, $\tilde{\mathcal{S}}_+$ and $\tilde{\mathcal{S}}_-$; the maps $(x,v,t)\mapsto(x+1,v,t)$ and $(x,v,t)\mapsto(x-1,v,t)$ induce Eilenberg swindles on $\tilde{\mathcal{C}}_+$ and $\tilde{\mathcal{C}}_-$. □

Note that it is important to use the category $\tilde{\mathcal{C}}$ rather than $\mathcal{C}$ for this argument. For example, the corresponding subcategory $\mathcal{S}$ of $\mathcal{C}^H(\mathbb{R};\mathrm{id}_{\mathbb{R}})$ is not a Karoubi filtration.

Proof of 4.6 Let p denote the projection $\mathbb{R}\to pt$. We will use the following diagram.

$$\begin{tikzcd}
\tilde{\mathcal{C}}_0(\mathbb{R};\mathrm{id}_{\mathbb{R}})^H \arrow[r] \arrow[d, "\mathcal{F}_1"'] & \tilde{\mathcal{C}}(\mathbb{R};\mathrm{id}_{\mathbb{R}})^H \arrow[r] \arrow[d, "\mathcal{F}_2"'] & \tilde{\mathcal{D}}(\mathbb{R};\mathrm{id}_{\mathbb{R}})^H \arrow[d, "\mathcal{F}_3"'] \\
\tilde{\mathcal{C}}_0(\mathbb{R};p)^H \arrow[r] & \tilde{\mathcal{C}}(\mathbb{R};p)^H \arrow[r] & \tilde{\mathcal{D}}(\mathbb{R};p)^H
\end{tikzcd}$$

It is not hard to check that $\mathcal{F}_1$ is an equivalence of categories. The K–theory of $\tilde{\mathcal{C}}(\mathbb{R};\mathrm{id}_{\mathbb{R}})^H$ vanishes by 4.7. The map $(x,v,t)\mapsto(x,v,t+1)$ gives an Eilenberg swindle on $\tilde{\mathcal{C}}(\mathbb{R};p)^H$ and its K–theory also vanishes. As used before, applying $\mathbb{K}^{-\infty}$ to Karoubi filtrations gives a homotopy fibration by [CP95, 1.28]. Thus $\mathcal{F}_3$ induces an isomorphism in K–theory. The result follows, since $\tilde{\mathcal{D}}(\mathbb{R};q)=\mathcal{D}(\mathbb{R};q)$ for any q as noted before 4.7. □

5 The coefficient spectra

This section contains the proof of Theorem 1.3 from the introduction. As before, we fix a ring R and a group Γ. For a subgroup H of Γ let

$$p_{\Gamma/H}\colon \Gamma/H\times E\Gamma(\mathcal{FIN})\to\Gamma/H$$

be the obvious projections. We define two $\mathrm{Or}\Gamma$ spectra $\mathbf{A}$ and $\mathbf{B}$ by

$$\begin{aligned}\mathbf{A}(\Gamma/H) &= \mathbb{K}(\Gamma/H\times E\Gamma(\mathcal{FIN})),\\ \mathbf{B}(\Gamma/H) &= \mathbb{K}(p_{\Gamma/H}).\end{aligned}$$

Both, $\mathbf{A}$ and $\mathbf{B}$ are naturally equipped with a Γ–action. There is an obvious Γ–equivariant map of $\mathrm{Or}\Gamma$–spectra $\mathbf{A} \to \mathbf{B}$.

We will show that these spectra satisfy the hypothesis of 3.1 with respect to the families $\mathcal{FIN} \subset \mathcal{VC}$. For 3.1 (1) this follows from 4.1, where $\mathbf{K}$ is the algebraic K–theory $\mathrm{Or}\Gamma$–spectrum $\mathbf{K}R^{-\infty}$. In 5.1 we will prove that 3.1 (2) is satisfied. The final condition 3.1 (3) will follow from 5.2. Moreover, it is an easy consequence of 4.1 and 4.2 that $\Omega\mathbf{B}^\Gamma$ is weakly equivalent to $\mathbf{K}R^{-\infty}$ and therefore Theorem 1.3 will be a consequence of the splitting result 3.1.

For a subgroup H of Γ let

$$\begin{aligned} \mathbb{A}_0(H) &= \mathbb{K}(\mathrm{res}_\Gamma^H E\Gamma(\mathcal{FIN})), \\ \mathbb{B}_0(H) &= \mathbb{K}(\mathrm{res}_\Gamma^H (E\Gamma(\mathcal{FIN}) \to pt)). \end{aligned}$$

Here res_Γ^H denotes the forgetful functor from Γ–spaces to H–spaces.

The next statement is an immediate consequence of 4.3 and verifies 3.1 (2).

Lemma 5.1 *There are natural weak* $\mathrm{Or}\Gamma$ *equivalences*

$$\begin{aligned} \mathbf{A}(\Gamma/H) &= \mathrm{Map}_H(\Gamma, \mathbb{E}_0(H)), \\ \mathbf{B}(\Gamma/H) &= \mathrm{Map}_H(\Gamma, \mathbb{B}_0(H)). \end{aligned}$$

□

Finally, we verify 3.1 (3).

Proposition 5.2 *For $V \in \mathcal{VC}$ the obvious map*

$$\begin{array}{c} \mathbb{A}_0(V)^V = \mathbb{K}(\mathrm{res}_\Gamma^V E\Gamma(\mathcal{FIN}))^V \\ \downarrow \\ \mathbb{B}_0(V)^{h_{\mathcal{F}}V} = \mathbb{K}(\mathrm{res}_\Gamma^H (E\Gamma(\mathcal{FIN}) \to pt))^{h_{\mathcal{FIN}}V} \end{array}$$

is a homotopy equivalence.

Proof We can choose $EV(\mathcal{FIN}) = \mathbb{R}$ with the proper action used towards the end of the previous section. We will use the following commutative diagram.

$$\begin{tikzcd} \mathbb{K}(\mathbb{R})^V \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"] & \mathbb{A}_0(V)^V \arrow[dd] \\ \mathbb{K}(\mathbb{R})^{h_{\mathcal{FIN}}V} \arrow[d, "\beta_0"] & \\ \mathbb{K}(\mathbb{R} \to pt)^{h_{\mathcal{FIN}}V} \arrow[r, "\beta_1"] & \mathbb{B}_0(V)^{h_{\mathcal{FIN}}V} \end{tikzcd}$$

The maps labeled α_i and β_i are all homotopy equivalences: α_0 by the fact that $\mathrm{res}_\Gamma^V E\Gamma(\mathcal{FIN}))$ is also an $EV(\mathcal{FIN})$ and 4.1 and α_1 by 4.5. To study the maps labeled β_i we need a fact about homotopy fixed points: if an equivariant map induces a homotopy equivalence on fixed points for finite subgroups, then it induces a homotopy equivalence on homotopy fixed points with respect to $\mathcal{FIN}$, see [Ros03, 4.1]. Thus β_1 is a homotopy equivalence by 4.2. The map $\mathbb{K}(\mathbb{R}) \to \mathbb{K}(\mathbb{R} \to pt)$ induces a homotopy equivalence on fixed points under all finite subgroups of V by 4.6 and therefore β_0 is also a homotopy equivalence. □

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