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Realising formal groups

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Abstract We show that a large class of formal groups can be realised functorially by even periodic ring spectra. The main advance is in the construction of morphisms, not of objects.

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1 Introduction

Let FG be the category of formal groups (of the sort usually considered in algebraic topology) over affine schemes. Thus, an object of FG consists of a pair (G, S), where S is an affine scheme, G is a formal group scheme over S, and a coordinate x can be chosen such that $\mathcal{O}_G \simeq \mathcal{O}_S[\![x]\!]$ as $\mathcal{O}_S$-algebras. A morphism from (G_0, S_0) to (G_1, S_1) is a commutative square

$$\begin{tikzcd} G_0 \arrow[r, "\tilde{p}"] \arrow[d] & G_1 \arrow[d] \\ S_0 \arrow[r, "p"'] & S_1 \end{tikzcd}$$

such that the induced map $G_0 \to p^*G_1$ is an isomorphism of formal group schemes over S_0.

Next, recall that an *even periodic ring spectrum* is a commutative and associative ring spectrum E such that $E^1 = 0$ and E^2 contains a unit (which implies that $E \simeq \Sigma^2 E$ as spectra). Here we are using the usual notation $E^k = E^k(\text{point}) = \pi_{-k}E$. We write EPR for the category of even periodic ring spectra. (Everything here is interpreted in Boardman's homotopy category of spectra; there are no E_∞ or A_∞ structures.)

Given an even periodic ring spectrum E, we can form the scheme $S_E := \text{spec}(E^0)$ and the formal group scheme $G_E = \text{spf}(E^0\mathbb{C}P^\infty)$ over S_E. This construction gives rise to a functor $\Gamma\colon \text{EPR}^{\text{op}} \to \text{FG}$.

It is a natural problem to try to define a realisation functor $R\colon \mathrm{FG} \to \mathrm{EPR}^{\mathrm{op}}$ with $\Gamma R(G,S) \simeq (G,S)$, or at least to do this for suitable subcategories of FG. For example, if we let LFG denote the category of Landweber exact formal groups, and put $\mathrm{LEPR} = \{E \in \mathrm{EPR} \mid \Gamma(E) \in \mathrm{LFG}\}$, one can show that the functor $\Gamma\colon \mathrm{LEPR}^{\mathrm{op}} \to \mathrm{LFG}$ is an equivalence; this is essentially due to Landweber, but details of this formulation are given in [5, Proposition 8.43]. Inverting this gives a realisation functor for LFG, and many well-known spectra are constructed using this. In particular, this gives various different versions of elliptic cohomology, based on various universal families of elliptic curves over rings such as $\mathbb{Z}[\frac{1}{6}, c_4, c_6][\Delta^{-1}]$.

It is hard to say more than this unless we invert the prime 2. We therefore make a blanket assumption:

Assumption 1.1 From now on, all rings are assumed to be $\mathbb{Z}[\frac{1}{2}]$-algebras. In particular, we only consider schemes S for which 2 is invertible in $\mathcal{O}_S$. We use the symbol MU for the spectrum that would normally be called $MU[\frac{1}{2}]$.

The other main technique for constructing realisations is the modernised version of Baas-Sullivan theory [2, 4]. This starts with a strictly commutative ring spectrum R, and an algebra A_* over π_*R, and it constructs a homotopically commutative R-algebra spectrum A with $\pi_*A = A_*$, provided that A_* has good structural properties. Firstly, we assume as always that 2 is invertible in A_*. Given this, the construction will work if A_* is a *localised regular quotient (LRQ)* of R_*, in other words it has the form $A_* = (S^{-1}\pi_*R)/I$, where S is a multiplicative set and I is an ideal generated by a regular sequence. The construction can also be extended to cover the case where A_* is a free module over an LRQ of π_*R.

We can apply this taking R to be the periodic bordism spectrum

$$MP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU[\tfrac{1}{2}]$$

(we will verify in the appendix that this can be constructed as a strictly commutative ring). Given a formal group (G,S) we can choose a coordinate x, which gives a formal group law F defined over $\mathcal{O}_S$, and thus a ring map $\pi_0 MP \to \mathcal{O}_S$, making $\mathcal{O}_S$ into a $\pi_0 MP$-algebra. If this algebra has the right properties, then we can use the Baas-Sullivan approach to construct E with $\Gamma(E) \simeq (G,S)$. It is convenient to make the following *ad hoc* definition:

Definition 1.2 A ring R is *standard* if 2 is invertible in R and R is either a field or a ring of the form $T^{-1}\mathbb{Z}$ (for some set T of primes).

An easy argument given below shows that the above method can construct realizations for all formal groups over standard rings. Unfortunately, this construction is not obviously functorial: it depends on a choice of coordinate, and morphisms of formal groups do not generally preserve coordinates. The main result of this paper is to show that with suitable hypotheses we can nonetheless define a functor.

The basic point is to consider the situation where we have several different coordinates, say $x_0, \ldots, x_r$ on a fixed formal group G. In a well-known way, this makes $\mathcal{O}_S$ into an algebra over the ring $\pi_0(MP^{(r+1)})$, and we can ask whether this can be realized topologically by an $MP^{(r+1)}$-algebra; the question will be made more precise in Section 3. We say that G is *very good* if the question has an affirmative answer for all $r \geq 0$ and all $x_0, \ldots, x_r$.

Theorem 1.3 *All formal groups over standard rings are very good.*

This will be proved as Corollary 3.15.

For our sharpest results, we need a slightly more complicated notion. We say that a coordinate x_0 is *multirealisable* if for any list $x_1, \ldots, x_r$ of additional coordinates, the question mentioned above has an affirmative answer. We say that G is *good* if it admits a multirealisable coordinate. Of course, G is very good iff *every* coordinate is multirealisable. We write GFG for the category of good formal groups (considered as a full subcategory of FG). The details are given in Definition 3.12.

Theorem 1.4 *Let x be a coordinate on a formal group (G, S), and suppose that the classifying map $\pi_0 MP \to \mathcal{O}_S$ makes $\mathcal{O}_S$ into a localised regular quotient of $\pi_0 MP$. Then x is multirealisable, and so G is good.*

This will be proved as Proposition 3.14.

Corollary 1.5 *At odd primes, the formal groups associated to 2-periodic versions of BP, $P(n)$, $B(n)$, $E(n)$, $K(n)$, $k(n)$ and so on are all good.* □

This shows that there is a considerable overlap with the Landweber exact case. However, there are many good formal groups that are not Landweber exact. Conversely, there is no reason to expect that Landweber exact formal groups will be good, although we have no counterexamples.

Our main result is as follows:

Theorem 1.6 *There is a realisation functor* $R\colon \mathrm{GFG} \to \mathrm{EPR}$, *with* $\Gamma R \simeq 1\colon \mathrm{GFG} \to \mathrm{GFG}$.

Note that good formal groups are realisable by definition; the content of the theorem is that the realisation is well-defined and functorial.

We next explain the formal part of the construction; in Section 4 we will give additional details and prove that we have the required properties. The functor R actually arises as UV^{-1} for a pair of functors $\mathrm{GFG} \xleftarrow{V} \mathcal{E} \xrightarrow{U} \mathrm{EPR}$ in which V is an equivalence. To explain $\mathcal{E}$, recall that we have a topological category Mod_0 of MP-modules. We write DMod_0 for the derived category, and EPA_0 for the category of even periodic commutative ring objects in DMod_0. The unit map $\eta\colon S \to MP$ gives a functor $\eta^*\colon \mathrm{EPA}_0 \to \mathrm{EPR}$, and the objects of the category $\mathcal{E}$ are the objects $E \in \mathrm{EPA}_0$ for which the associated coordinate on $\Gamma(\eta^* E)$ is multirealisable. The morphism set $\mathcal{E}(E_0, E_1)$ is a subset of $\mathrm{EPR}(\eta^* E_0, \eta^* E_1)$, the functor $V\colon \mathcal{E} \to \mathrm{GFG}$ is given by Γ, and the functor $U\colon \mathcal{E} \to \mathrm{EPR}$ is given by η^*. We say that a map $f\colon \eta^* E_0 \to \eta^* E_1$ in EPR is *good* if there is a commutative ring object A in the derived category of $MP \wedge MP$-modules together with maps $f'\colon E_0 \to (1 \wedge \eta)^* A$ and $f''\colon (\eta \wedge 1)^* A \to E_1$ in EPA_0 such that f'' is an equivalence and f is equal to the composite

$$\eta^* E_0 \xrightarrow{\eta^* f'} (\eta \wedge \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1.$$

The morphisms in the category $\mathcal{E}$ are just the good maps. To prove Theorem 1.6, we need to show that

(3) The composite of two good maps is good, so $\mathcal{E}$ really is a category.

(2) For any map $\Gamma(\eta^* E_0) \to \Gamma(\eta^* E_1)$ of good formal groups, there is a unique good map $\eta^* E_0 \to \eta^* E_1$ inducing it, so that V is full and faithful.

(1) For any good formal group (G, S) there is an object $E \in \mathrm{EPA}_0$ such that $\Gamma(\eta^* E) \simeq (G, S)$, so V is essentially surjective.

To prove statement (k), we need to construct modules over the k-fold smash power of MP. It will be most efficient to do this for all k simultaneously.

2 Preliminaries

2.1 Differential forms

Let (G, S) be a formal group, and let $I \leq \mathcal{O}_G$ be the augmentation ideal. Recall that the cotangent space of G at zero is the module $\omega_G = I/I^2$. If x

is a coordinate on G that vanishes at zero, then we write dx for the image of x in I/I^2, and note that ω_G is freely generated over $\mathcal{O}_S$ by dx. We define a graded ring $D(G,S)^*$ by

$$D(G,S)^k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \omega_G^{\otimes(-k/2)} & \text{if } k \text{ is even.} \end{cases}$$

Here the tensor products are taken over $\mathcal{O}_S$, and $\omega_G^{\otimes n}$ means the dual of $\omega_G^{\otimes|n|}$ when $n<0$. Where convenient, we will convert to homological gradings by the usual rule: $D(G,S)_k = D(G,S)^{-k}$.

Now let E be an even periodic ring spectrum with $\Gamma(E)=(G,S)$. We then have $\mathcal{O}_G = E^0\mathbb{C}P^\infty$ and $I=\widetilde{E}^0\mathbb{C}P^\infty$ and one checks easily that the inclusion $S^2=\mathbb{C}P^1\to\mathbb{C}P^\infty$ gives an isomorphism $\omega_G = I/I^2 = \widetilde{E}^0S^2 = E^{-2}$. Using the periodicity of E, we see that this extends to a canonical isomorphism $D(\Gamma(E))^*\simeq E^*$.

It also follows from this analysis (or from more direct arguments) that a map $f\colon E_0\to E_1$ in EPR is a weak equivalence if and only if $\pi_0 f$ is an isomorphism.

2.2 Periodic bordism

Consider the homology theory $MP_*(X) = MU_*(X)\otimes\mathbb{Z}[u,u^{-1}]$, where u has homological degree 2 (and thus cohomological degree -2). This is represented by the spectrum $MP=\bigvee_{n\in\mathbb{Z}}\Sigma^{2n}MU$, with an evident ring structure. It is well-known that MU is an E_∞ ring spectrum; see for example [3, Section IX]. It is also shown there that MU is an H_∞^2 ring spectrum, which means (as explained in [3, Remark VII.2.9]) that MP is an H_∞ ring spectrum; this is weaker than E_∞ in theory, but usually equivalent in practise. As one would expect, MP is actually an E_∞ ring spectrum; a proof is given in the appendix. It follows from [2, Proposition II.4.3] that one can construct a model for MP that is a strictly commutative ring spectrum (or "commutative S-algebra"). We may also assume that it is a cofibrant object in the category of all strictly commutative ring spectra.

For typographical convenience, we write $MP(r)$ for the $(r+1)$-fold smash power $MP\wedge\ldots\wedge MP$, which is again a strictly commutative ring. The spectra $MP(r)$ fit together into a cosimplicial object in the usual way; for example, we have three maps

$$\eta\wedge 1\wedge 1, 1\wedge\eta\wedge 1, 1\wedge 1\wedge\eta\colon MP(0)\to MP(2).$$

In the category of strictly commutative ring spectra, the coproduct is the smash product. It follows formally that the smash product of cofibrant objects is cofibrant, so in particular the objects $MP(r)$ are all cofibrant.

For $r > 0$, it is well-known that $\pi_*MU^{(r+1)}$ is a polynomial algebra over π_*MU on countably many generators, and it follows that there is a noncanonical isomorphism

$$\pi_0MP(r) \simeq \pi_0MP[x_1, x_2, \ldots][x_1^{-1}, \ldots, x_r^{-1}].$$

There are $r+1$ obvious inclusions $MP \to MP(r)$. We can use these to push forward the standard generator of $MP^0\mathbb{C}P^\infty$, giving $r+1$ different coordinates on the formal group $\Gamma(MP(r))$. We denote these by $\widetilde{x}_0, \ldots, \widetilde{x}_r$.

2.3 Groups and laws

We now define a category FG_r as follows. The objects are systems

$$(G, S, x_0, \ldots, x_r),$$

where (G, S) is a formal group and the x_i are coordinates on G. The morphisms from $(G, S, x_0, \ldots, x_r)$ to $(H, T, y_0, \ldots, y_r)$ are the maps $(\tilde{p}, p)\colon (G, S) \to (H, T)$ in FG for which $\tilde{p}^*y_i = x_i$ for all i. Note that given p, the map $\tilde{p}$ is determined by the fact that $\tilde{p}^*y_0 = x_0$. Thus, the forgetful functor $(G, S, x_0, \ldots, x_r) \mapsto S$ (from FG_r to the category of affine schemes) is faithful.

We also write Alg_r for the category of commutative algebras over the ring $\pi_0MP(r)$.

Proposition 2.1 *There is an equivalence* $\mathrm{FG}_r \simeq \mathrm{Alg}_r^{\mathrm{op}}$.

Proof Recall that we have coordinates $\widetilde{x}_0, \ldots, \widetilde{x}_r$ on $\Gamma(MP(r))$. Given an object $A \in \mathrm{Alg}_r$ we have a structure map $\mathrm{spec}(A) \to \mathrm{spec}(\pi_0MP(r))$, and we can pull back $\Gamma(MP(r))$ to get a formal group G_A over $\mathrm{spec}(A)$. We can also pull back the coordinates $\widetilde{x}_i$ to make G_A an object of FG_r. It is easy to see that this construction defines a functor $U\colon \mathrm{Alg}_r^{\mathrm{op}} \to \mathrm{FG}_r$. By forgetting down to the category of affine schemes, we see that U is faithful.

We now claim that U is an equivalence. We will deduce this from a well-known result of Quillen by a sequence of translations. First, Quillen tells us that maps $\pi_*MU^{(r+1)} \to B_*$ of graded rings biject naturally with systems

$$F_0 \xleftarrow{f_0} F_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{r-1}} F_r,$$

where each F_i is a homogeneous formal group law over B_* and each f_i is a strict isomorphism. By a standard translation to the even periodic case, we see that maps $\pi_0 MP(r) \to A$ of ungraded rings biject naturally with systems

$$F_0 \xleftarrow{f_0} F_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{r-1}} F_r,$$

where each F_i is a formal group law over A and each f_i is a (not necessarily strict) isomorphism.

Now suppose we have an object $(G, S, x_0, \dots, x_r)$ in FG_r. For each i there is a unique formal group law F_i over $\mathcal{O}_S$ such that $x_i(a+b) = F_i(x_i(a), x_i(b))$ for sections a, b of G. Moreover, as x_{i+1} is another coordinate, we can write $x_i = f_i(x_{i+1})$ for a unique power series $f_i \in \mathcal{O}_S[\![t]\!]$. It is easy to check that f_i is an isomorphism from F_{i+1} to F_i, so Quillen's theorem gives us a map $\pi_0 MP(r) \to \mathcal{O}_S$, allowing us to regard $\mathcal{O}_S$ as an object of Alg_r. It is easy to see that this construction gives a functor $\mathrm{FG}_r \to \mathrm{Alg}_r^{\mathrm{op}}$. We leave it to the reader to check that this is inverse to U. □

2.4 Module categories

We write Mod_r for the category of $MP(r)$-modules (in the strict sense, not the homotopical one). Note that a map $f\colon A_0 \to A_1$ of strictly commutative ring spectra gives a functor $f^*\colon \mathrm{Mod}_{A_1} \to \mathrm{Mod}_{A_0}$, which is just the identity on the underlying spectra (and thus preserves weak equivalences). It follows easily that for any two maps $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2$, the functor $f^* g^*$ is actually equal (not just naturally isomorphic or naturally homotopy equivalent) to $(gf)^*$. Thus, the categories Mod_r fit together to give a simplicial category Mod_*.

Remark 2.2 For us, a *simplicial category* means a simplicial object in the category of categories. Elsewhere in the literature, the same phrase is sometimes used to refer to categories enriched over the category of simplicial sets, which is a rather different notion.

Next, we write DMod_r the derived category of Mod_r, as in [2, Chapter III]. As usual, there are two different models for a category such as DMod_r:

(a) One can take the objects to be the cofibrant objects in Mod_r, and morphisms to be homotopy classes of maps; or

(b) One can use all objects in Mod_r and take morphisms to be equivalence classes of "formal fractions", in which one is allowed to invert weak equivalences.

We will use model (b). This preserves the strong functorality mentioned previously, and ensures that DMod_* is again a simplicial category.

We also write EPA_r for the category of even periodic commutative ring objects in DMod_r, giving another simplicial category. (Note that periodicity is actually automatic, because $MP(r)$ is itself periodic.) Various fragments of the simplicial structure will be used in Section 4.

3 Basic realisation results

Let R be a strictly commutative ring spectrum that is even and periodic, such that R_0 is an integral domain (and as always, 2 is invertible). The main examples will be $R = MP(r)$ for $r \geq 0$. Let $\mathcal{D}$ be the derived category of R-modules, and let $\mathcal{R}$ be the category of commutative ring objects $A \in \mathcal{D}$ such that $\pi_1 A = 0$. Recall that if f is a morphism in $\mathcal{R}$ such that $\pi_0 f$ is an isomorphism, then $\pi_* f$ is also an isomorphism and so f is an equivalence.

We also write $\mathcal{R}_0$ for the category of commutative algebras over $\pi_0 R$. We say that an object $A \in \mathcal{R}$ is *strong* if for all $B \in \mathcal{R}$, the map

$$\pi_0 \colon \mathcal{R}(A, B) \to \mathcal{R}_0(\pi_0 A, \pi_0 B)$$

is a bijection. A *realisation* of an object $A_0 \in \mathcal{R}_0$ is a pair (A, u), where $A \in \mathcal{R}$ and $u \colon \pi_0 A \to A_0$ is an isomorphism. We say that (A, u) is a *strong realisation* iff the object A is strong; if so, we have a natural isomorphism $\mathcal{R}(A, B) \simeq \mathcal{R}_0(A_0, \pi_0 B)$. We say that A_0 is *strongly realisable* if it admits a strong realisation. If so, it is easy to check that all realisations are strong, and any two realisations are linked by a unique isomorphism.

The results of [4] provide a good supply of strongly realisable algebras, except that we need a little translation between the even periodic framework and the usual graded framework. Suppose that $A_0 \in \mathcal{R}_0$, and put $T = \mathrm{spec}(A_0)$. We have a unit map $\eta \colon \pi_0 R \to A_0$ and thus a map $\mathrm{spec}(\eta) \colon T \to S_R$; we can pull back the formal group G_R along this to get a formal group $H := \mathrm{spec}(\eta)^* G_R$ over T. From this we get a map $\eta_* \colon R_* = D(G_R, S_R)_* \to D(H, T)_*$, which agrees with η in degree zero. Indeed, if we choose a generator u of R_2 over R_0, then η_* is just the map $R_0[u, u^{-1}] \to A_0[u, u^{-1}]$ obtained in the obvious way from η. It is easy to check that A_0 is strongly realisable (as defined in the previous paragraph) iff $D(H, T)_*$ is strongly realisable over R_* (as defined in [4]).

Definition 3.1 A *short ordinal* is an ordinal λ of the form $n.\omega + m$ for some $n, m \in \mathbb{N}$. A *regular sequence* in a ring R_0 is a system of elements $(x_\alpha)_{\alpha<\lambda}$ for some short ordinal λ such that x_α is not a zero-divisor in the ring $(S^{-1}R_0)/(x_\beta \mid \beta < \alpha)$. An object $A_0 \in \mathcal{R}_0$ is a *localised regular quotient* (or LRQ) of R_0 if $A_0 = (S^{-1}R_0)/I$ for some subset $S \subset R_0$ and some ideal $I \leq S^{-1}R_0$ that can be generated by a regular sequence.

Remark 3.2 We have made a small extension of the usual notion of a regular sequence, to ensure that any LRQ of an LRQ of R_0 is itself an LRQ of R_0; see Lemma 3.8.

Proposition 3.3 *If A_0 is an LRQ of R_0, then it is strongly realisable.*

Proof This is essentially [4, Theorem 2.6], translated into a periodic setting as explained above. Here we are using a slightly more general notion of a regular sequence, but all the arguments can be adapted in a straightforward way. The main point is that any countable limit ordinal has a cofinal sequence, so homotopy colimits can be constructed using telescopes in the usual way. Andrey Lazarev has pointed out a lacuna in [4]: it is necessary to assume that the elements x_α are all regular in $S^{-1}R_0$ itself, which is not generally automatic. However, we are assuming that R_0 is an integral domain so this issue does not arise. □

Proposition 3.4 *Suppose that*

- *A and B are strong realisations of A_0 and B_0*
- *The natural map $A_0 \otimes_{R_0} B_0 \to (A \wedge_R B)_0$ is an isomorphism.*

Then $A \wedge_R B$ is a strong realisation of $A_0 \otimes_{R_0} B_0$.

Proof This follows from [4, Corollary 4.5]. □

Proposition 3.5 *If $A_0 \in \mathcal{R}_0$ is strongly realisable, and B_0 is an algebra over A_0 that is free as a module over A_0, then B_0 is also strongly realisable.*

Proof This follows from [4, Proposition 4.13]. □

Proposition 3.6 *Suppose that R_0 is a polynomial ring in countably many variables over $\mathbb{Z}[\frac{1}{2}]$, that $A_0 \in \mathcal{R}_0$, and that $A_0 = \mathbb{Z}[1/2n]$ as a ring (for some n). Then A_0 is an LRQ of R_0, and thus is strongly realisable.*

Proof Choose a system of polynomial generators $\{x_k \mid k \geq 0\}$ for R_0 over $\mathbb{Z}[\frac{1}{2}]$. Put $a_k = \eta(x_k) \in A_0 = \mathbb{Z}[1/n]$ and $y_k = x_k - a_k \in R_0[1/n]$. It is clear that $R_0[1/2n] = \mathbb{Z}[1/2n][y_k \mid k \geq 0]$, that the elements y_k form a regular sequence generating an ideal I say, and that $A_0 = R_0[1/2n]/I$. □

Proposition 3.7 *Suppose that R_0 is a polynomial ring in countably many variables over $\mathbb{Z}[\frac{1}{2}]$, that $A_0 \in \mathcal{R}_0$, and that A_0 is a field (necessarily of characteristic different from 2). Then A_0 is a free module over an LRQ of R_0, and thus is strongly realisable.*

Proof For notational simplicity, we assume that A_0 has characteristic $p > 2$; the case of characteristic 0 is essentially the same.

Choose a set X of polynomial generators for R_0 over $\mathbb{Z}[\frac{1}{2}]$. Let K be the subfield of A_0 generated by the image of η, or equivalently by $\eta(X)$. We can choose a subset $Y \subseteq X$ such that $\eta(Y)$ is a transcendence basis for K over $\mathbb{F}_p$. This means that the subfield L_0 of K generated by $\eta(Y)$ is isomorphic to the rational function field $\mathbb{F}_p(Y)$, and that K is algebraic over L_0. Put $S = \mathbb{Z}[\frac{1}{2}, Y] \setminus (p\mathbb{Z}[\frac{1}{2}, Y])$, so $L_0 = (S^{-1}\mathbb{Z}[\frac{1}{2}, Y])/p$. Next, list the elements of $X \setminus Y$ as $\{x_1, x_2, \ldots\}$, and let L_k be the subfield of K generated by $\{x_i \mid i \leq k\}$. (We will assume that $X \setminus Y$ is infinite; if not, the notation changes slightly.) As x_k is algebraic over L_{k-1}, there is a monic polynomial $f_k(t) \in L_{k-1}[t]$ with $L_k = L_{k-1}[x_k]/f_k(x_k)$. As L_{k-1} is a quotient of the ring $P_{k-1} := S^{-1}\mathbb{Z}[Y, x_1, \ldots, x_{k-1}]$, we can choose a monic polynomial $g_k(t) \in P_{k-1}[t]$ lifting f_k, and put $z_k := g_k(x_k) \in P_k \subseteq S^{-1}R_0$. It is not hard to check that the sequence $(p, z_1, z_2, \ldots)$ is regular in $S^{-1}R_0$, and that $(S^{-1}R_0)/(z_i \mid i > 0) = K$, so K is an LRQ of R_0. It is clear that A_0 is free over the subfield K. □

Lemma 3.8 *An LRQ of an LRQ is an LRQ.*

Proof Suppose that $B = (S^{-1}A)/(x_\alpha \mid \alpha < \lambda)$ and $C = (T^{-1}B)/(y_\beta \mid \beta < \mu)$, where λ and μ are short ordinals, and the x and y sequences are regular in $S^{-1}A$ and $T^{-1}B$ respectively. Let T' be the set of elements of A that become invertible in $T^{-1}B$; clearly $S \subseteq T'$ and $T^{-1}B = ((T')^{-1}A)/(x_\alpha \mid \alpha < \lambda)$. As $(T')^{-1}A$ is a localisation of $S^{-1}A$ and localisation is exact, we see that x is a regular sequence in $(T')^{-1}A$ as well. After multiplying by suitable elements of T' if necessary, we may assume that y_β lies in the image of A (this does not affect regularity, as the elements of T' are invertible). We then put $z_\alpha = x_\alpha$ for $\alpha < \lambda$, and let $z_{\lambda+\beta}$ be any preimage of y_β in A for $0 \leq \beta < \mu$.

This gives a regular sequence in $(T')^{-1}A$ indexed by $\lambda+\mu$, such that $C = ((T')^{-1}A)/(z_\gamma \mid \gamma < \lambda+\mu)$ as required. □

We now specialize to the case $R = MP(r)$, so $\mathcal{R}_0 = \mathrm{EPA}_r$. We write Γ_r for the evident composite functor

$$\mathrm{EPA}_r^{\mathrm{op}} \xrightarrow{\pi_0} \mathrm{Alg}_r^{\mathrm{op}} \simeq \mathrm{FG}_r\,.$$

Translating our previous definitions via the equivalence $\mathrm{Alg}_r^{\mathrm{op}} \simeq \mathrm{FG}_r$, we obtain the following.

Definition 3.9 An object $A \in \mathrm{EPA}_r$ is *strong* if for all $B \in \mathrm{EPA}_r$, the map

$$\Gamma_r\colon\ \mathrm{EPA}_r(A,B) \to \mathrm{FG}_r(\Gamma_r(B),\Gamma_r(A))$$

is a bijection.

Definition 3.10 A *realisation* of an object $(G,S,\underline{x}) \in \mathrm{FG}_r$ is a triple $(A,\tilde{p},p)$, where $A \in \mathrm{EPA}_r$ and $(\tilde{p},p)\colon \Gamma_r A \to (G,S,\underline{x})$ is an isomorphism. This is a *strong realisation* if the object A is strong.

We now give more precise versions of the definitions in the introduction.

Definition 3.11 A formal group (G,S) is *very good* if for every nonempty list $\underline{x}$ of coordinates, the object $(G,S,\underline{x}) \in \mathrm{FG}_r$ is strongly realisable.

Definition 3.12 A coordinate x_0 on G is *multirealisable* if for every list $x_1,\dots,x_r$ of coordinates, the object $(G,S,x_0,\dots,x_r) \in \mathrm{FG}_r$ is strongly realisable. A formal group (G,S) is *good* if it admits a multirealisable coordinate. We write GFG for the category of good formal groups.

Remark 3.13 Let $x_0,\dots,x_r$ be coordinates, and suppose that x_0 is multirealisable. Let σ be a permutation of $\{0,\dots,r\}$. Using the evident action of permutations on $MP(r)$, we see that the object $(G,S,x_{\sigma(0)},\dots,x_{\sigma(r)})$ is strongly realisable.

Proposition 3.14 *Suppose that x_0 is such that the classifying map $\pi_0 MP \to \mathcal{O}_S$ makes $\mathcal{O}_S$ an LRQ of $\pi_0 MP$. Then x_0 is multirealisable, so (G,S) is good.*

Proof The coordinate x_0 gives a map $f_0\colon \pi_0MP \to \mathcal{O}_S$. By assumption, there is a multiplicative set $T \subseteq \pi_0MP$ and a regular ideal I such that f_0 induces an isomorphism $(T^{-1}\pi_0MP)/I \to \mathcal{O}_S$.

Now consider a list of additional coordinates $x_1, \ldots, x_r$ say. These give a map $f\colon \pi_0MP(r) \to \mathcal{O}_S$ extending f_0. We know from Section 2.2 that $\pi_0MP(r)$ is a polynomial ring in countably many variables over π_0MP, in which r of the variables have been inverted, so we can write

$$\pi_0MP(r) = \pi_0MP[u_1, u_2, \ldots][u_1^{-1}, \ldots, u_r^{-1}].$$

Put

$$A_0 = \mathcal{O}_S[u_1, u_2, \ldots][u_1^{-1}, \ldots, u_r^{-1}],$$

which is evidently an LRQ of $\pi_0MP(r)$. It is easy to see that f induces a map $f'\colon A_0 \to \mathcal{O}_S$ of $\mathcal{O}_S$-algebras. Put $a_k = f'(u_k) \in \mathcal{O}_S$, and $v_k = u_k - a_k \in A_0$. Clearly A_0 is a localisation of $\mathcal{O}_S[v_k \mid k > 0]$, the sequence of v's is regular in A_0, and $A_0/(v_k \mid k > 0) = \mathcal{O}_S$ as $\pi_0MP(r)$-algebras. It follows that $\mathcal{O}_S$ is an LRQ of an LRQ, and thus an LRQ, over $\pi_0MP(r)$. It is thus strongly realisable as required. □

Corollary 3.15 *If $\mathcal{O}_S$ is a standard ring, then every coordinate is multirealisable, and so (G,S) is very good.*

Proof This now follows from Propositions 3.6 and 3.7. □

4 Proof of the main theorem

Let $\mathcal{E}$ denote the class of objects $E \in \mathrm{EPA}_0$ for which the resulting coordinate on $\Gamma(\eta^*E)$ is multirealisable. Note that this means that Γ_1E is strongly realisable, so every realisation is strong, so in particular E is a strong object.

Proposition 4.1 *For any good formal group (G,S), there exists $E \in \mathcal{E}$ with $\Gamma(\eta^*E) \simeq (G,S)$.*

Proof By the definition of goodness we can choose a multirealisable coordinate x_0 on G. This means in particular that the object $(G,S,x_0) \in \mathrm{FG}_0$ is isomorphic to $\Gamma_0(E)$ for some $E \in \mathrm{EPA}_0$. It follows that $(G,S) \simeq \Gamma(\eta^*E)$, as required. □

Proposition 4.2 *Suppose we have objects $E_0, E_1 \in \mathcal{E}$, together with a map*

$$(\tilde{p}, p) \colon \Gamma(\eta^* E_1) \to \Gamma(\eta^* E_0)$$

in GFG. *Then there is a unique good map $f \colon \eta^* E_0 \to \eta^* E_1$ such that $\Gamma(f) = (\tilde{p}, p)$.*

Proof We first put $(G_i, S_i, x_i) = \Gamma_0 E_i$ for $i = 0, 1$.

We introduce a category $\mathcal{B} = \mathcal{B}(E_0, E_1, \tilde{p}, p)$ as follows. The objects are triples (A, f', f'') where

(a) A is an object of EPA_1.

(b) f' is a morphism $E_0 \to (1 \wedge \eta)^* A$ in EPA_0.

(c) f'' is an isomorphism $(\eta \wedge 1)^* A \to E_1$ in EPA_0.

(d) The composite

$$f = \theta(A, f', f'') := (\eta^* E_0 \xrightarrow{\eta^* f'} (\eta \wedge \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1)$$

satisfies $\Gamma(f) = (\tilde{p}, p)$.

The morphisms from (A, f', f'') to (B, g', g'') in $\mathcal{B}$ are the isomorphisms $u \colon A \to B$ in EPA_1 for which $((1 \wedge \eta)^* u) f' = g'$ and $g''((\eta \wedge 1)^* u) = f''$.

The maps of the form $\theta(A, f', f'')$ are precisely the good maps that induce $(\tilde{p}, p)$, and isomorphic objects of $\mathcal{B}$ have the same image under θ. It will thus suffice to show that $\mathcal{B} \neq \emptyset$ and all objects of $\mathcal{B}$ are isomorphic.

First, as x_1 is multirealisable, we can find an object $A \in \mathrm{EPA}_1$ and an isomorphism $(\tilde{q}, q) \colon \Gamma_1 A \to (G_1, S_1, \tilde{p}^* x_0, x_1)$ displaying A as a strong realisation of $(G_1, S_1, \tilde{p}^* x_0, x_1)$. We write $(H, T, y_0, y_1) = \Gamma_1 A$, so $(\tilde{q}, q) \colon (H, T) \xrightarrow{\simeq} (G_1, S_1)$ and $(\tilde{p}\tilde{q})^* x_0 = y_0$ and $\tilde{q}^* x_1 = y_1$. We can thus regard $(\tilde{p}\tilde{q}, pq)$ as a morphism

$$\Gamma_0((1 \wedge \eta)^* A) = (H, T, y_0) \to (G_0, S_0, x_0) = \Gamma_0 E_0,$$

and E_0 is a strong realisation of (G_0, S_0, x_0), so this must come from a map $f' \colon E_0 \to (1 \wedge \eta)^* A$ in EPA_0. Similarly, we can regard $(\tilde{q}, q)$ as an isomorphism

$$\Gamma_0((\eta \wedge 1)^* A) = (H, T, y_1) \xrightarrow{\simeq} (G_1, S_1, x_1) = \Gamma_0^* E_1.$$

As E_1 is a strong realisation of (G_1, S_1, x_1), this comes from a map $E_1 \to (\eta \wedge 1)^* A$; this is easily seen to be an isomorphism, and we let $f'' \colon (\eta \wedge 1)^* A \to E_1$ be the inverse map. It is then clear that the map

$$f = (\eta^* f'') \circ (\eta^* f') \colon \eta^* E_0 \to \eta^* E_1$$

is good and satisfies $\Gamma(f) = (\tilde{p}, p)$, so $(A, f', f'') \in \mathcal{B}$. Thus $\mathcal{B} \neq \emptyset$.

Now suppose we have another object $(B, g', g'') \in \mathcal{B}$, with $\Gamma_1 B = (K, U, z_0, z_1)$ say. We put

$$\begin{aligned}(\tilde{r}, r) = \Gamma_1 g'' &\colon (G_1, S_1, x_1) \xrightarrow{\simeq} \Gamma_1((\eta \wedge 1)^* B) = (K, U, z_1)\\ (\tilde{s}, s) = \Gamma_1 g' &\colon \Gamma_1((1 \wedge \eta)^* B) = (K, U, z_0) \to (G_0, S_0, x_0).\end{aligned}$$

By hypothesis we have $(\tilde{s}\tilde{r}, sr) = (\tilde{p}, p) \colon (G_1, s_1) \to (G_0, S_0)$. We display all these maps in the following commutative diagram:

$$\begin{tikzcd}
(H,T) \arrow[r, "(\tilde{q},q)", "\simeq"'] \arrow[d, "(\tilde{p}\tilde{q},pq)"'] & (G_1,S_1) \arrow[dl, "(\tilde{p},p)"'] \arrow[d, "(\tilde{r},r)", "\simeq"'] \\
(G_0,S_0) & (K,U). \arrow[l, "(\tilde{s},s)"]
\end{tikzcd}$$

We now claim that $(\tilde{r}\tilde{q}, rq)$ can be regarded as a map

$$(H, T, y_0, y_1) \to (K, U, z_0, z_1).$$

Indeed, it is clear from the data recorded above that it is a map $(H, T, y_1) \to (K, U, z_1)$, so it will suffice to check that $(\tilde{r}\tilde{q})^* z_0 = y_0$. We are given that $z_0 = \tilde{s}^* x_0$ and $\tilde{s}\tilde{r} = \tilde{p}$ and $(\tilde{p}\tilde{q})^* x_0 = y_0$; the claim follows. As r and q are isomorphisms, we have an isomorphism

$$(\tilde{r}\tilde{q}, rq)^{-1} \colon \Gamma_1 B = (K, U, z_0, z_1) \to (H, T, y_0, y_1) = \Gamma_1 A$$

in FG_1. As A is a strong realization, this comes from a unique map $u \colon A \to B$ in EPA_1, which is easily seen to be an isomorphism.

We must show that u is a morphism in our category $\mathcal{B}$, or equivalently that in EPA_0 we have

$$\begin{aligned}((1 \wedge \eta)^* u) f' = g' &\colon E_0 \to (1 \wedge \eta)^* B\\ g''((\eta \wedge 1)^* u) = f'' &\colon (\eta \wedge 1)^* B \to E_1.\end{aligned}$$

Note that E_0 and E_1 are strong, and f'' is an isomorphism, so $(\eta \wedge 1)^* B$ is strong. It is thus enough to check our two equations after applying π_0 (here we have used the original definition rather than the equivalent one in Definition 3.9). By definition or construction, we have

$$\begin{aligned}
\mathrm{spec}(\pi_0 f') &= pq \\
\mathrm{spec}(\pi_0 f'') &= q^{-1} \\
\mathrm{spec}(\pi_0 g') &= s \\
\mathrm{spec}(\pi_0 g'') &= r \\
\mathrm{spec}(\pi_0 u) &= (rq)^{-1} \\
sr &= p.
\end{aligned}$$

It follows easily that $(\pi_0 u)(\pi_0 f') = \pi_0 g'$ and $(\pi_0 g'')(\pi_0 u) = \pi_0 f''$, as required. This shows that u is an isomorphism in $\mathcal{B}$, and thus that f is the unique good map inducing the map $(\tilde{p}, p)$. $\square$

Lemma 4.3 *For any $E \in \mathcal{E}$, the identity map $1\colon \eta^*E \to \eta^*E$ is good.*

Proof Note that the multiplication map $MP(1) = MP \wedge MP \to MP$ is a map of ring spectra (in the strict sense) and so induces a functor $\mu^*\colon \mathrm{EPA}_0 \to \mathrm{EPA}_1$ with $(1 \wedge \eta)^*\mu^*E = (\eta \wedge 1)^*\mu^*E = E$ on the nose. We can thus take $A = \mu^*E$ and $f' = f'' = 1_E$ to show that 1_E is good. $\square$

Proposition 4.4 *Suppose we have objects $E_0, E_1, E_2 \in \mathcal{E}$ and good morphisms $\eta^*E_0 \xrightarrow{f} \eta^*E_1 \xrightarrow{g} \eta^*E_2$. Then the composite gf is also good.*

Proof Write $(G_i, S_i, x_i) = \Gamma_0 E_i$ and $(\tilde{p}, p) = \Gamma(f)\colon (G_1, S_1) \to (G_0, S_0)$ and $(\tilde{q}, q) = \Gamma(g)\colon (G_2, S_2) \to (G_1, S_1)$.

Choose objects $A, B \in \mathrm{EPA}_1$ and maps

$$\begin{aligned}
f'&\colon E_0 \to (1 \wedge \eta)^* A \\
f''&\colon (\eta \wedge 1)^* A \xrightarrow{\simeq} E_1 \\
g'&\colon E_1 \to (1 \wedge \eta)^* B \\
g''&\colon (\eta \wedge 1)^* B \xrightarrow{\simeq} E_2
\end{aligned}$$

exhibiting the goodness of f and g. This gives rise to isomorphisms

$$\begin{aligned}
\Gamma_1 A &= (G_1, S_1, \tilde{p}^* x_0, x_1) \\
\Gamma_1 B &= (G_2, S_2, \tilde{q}^* x_1, x_2).
\end{aligned}$$

Next, observe that we have an object $(G_2, S_2, (\tilde{p}\tilde{q})^* x_0, \tilde{q}^* x_1, x_2) \in \mathrm{FG}_2$, which is strongly realisable because x_2 is a multirealisable coordinate. We can thus choose an object $P \in \mathrm{EPA}_2$ and an isomorphism

$$(\tilde{r}, r)\colon \Gamma_2 P \to (G_2, S_2, (\tilde{p}\tilde{q})^* x_0, \tilde{q}^* x_1, x_2)$$

making P a strong realisation. We can also regard $(\tilde{r}, r)$ as an isomorphism

$$\Gamma_1((\eta \wedge 1 \wedge 1)^* P) \to (G_2, S_2, \tilde{q}^* x_1, x_2) = \Gamma_1 B.$$

As B is strong, this comes from a unique isomorphism $v\colon (\eta \wedge 1 \wedge 1)^* P \to B$ in EPA_1.

Similarly, we can regard $(\tilde{r}, r)$ as an isomorphism

$$\Gamma_1((1 \wedge 1 \wedge \eta)^* P) \to (G_2, S_2, \tilde{q}^* \tilde{p}^* x_0, \tilde{q}^* x_1),$$

and we can regard $(\tilde{q}, q)$ as a morphism

$$(G_2, S_2, \tilde{q}^* \tilde{p}^* x_0, \tilde{q}^* x_1) \to (G_1, S_1, \tilde{p}^* x_0, x_1) \simeq \Gamma_1 A.$$

As A is strong, the composite $(\tilde{q}\tilde{r}, qr)$ must come from a unique map $u\colon A \to (1 \wedge 1 \wedge \eta)^* P$ in EPA_1.

We now put

$$\begin{aligned}
C &= (1 \wedge \eta \wedge 1)^* P \in \mathrm{EPA}_1 \\
h' &= (E_0 \xrightarrow{f'} (1 \wedge \eta)^* A \xrightarrow{(1\wedge\eta)^* u} (1 \wedge \eta \wedge \eta)^* P = (1 \wedge \eta)^* C) \\
h'' &= ((\eta \wedge 1)^* C = (\eta \wedge \eta \wedge 1)^* P \xrightarrow{(\eta\wedge 1)^* v} (\eta \wedge 1)^* B \xrightarrow{g''} E_2).
\end{aligned}$$

As v and g'' are isomorphisms, the same is true of h''. We claim that after forgetting down to EPR, we have $h''h' = gf$; this will prove that gf is good as claimed. We certainly have $h''h' = g''vuf'$ and $gf = g''g'f''f'$ so it will suffice to show that $vu = g'f''\colon A \to B$ in EPR. For this, it will be enough to prove that the following diagram in EPA_0 commutes.

$$\begin{tikzcd}
(\eta \wedge 1)^* A \arrow[r, "(\eta\wedge 1)^* u"] \arrow[d, "f''"', "\simeq"] & (\eta \wedge 1 \wedge \eta)^* P \arrow[d, "\simeq"', "(1\wedge\eta)^* v"] \\
E_1 \arrow[r, "g'"'] & (1 \wedge \eta)^* B.
\end{tikzcd}$$

As this is a diagram in EPA_0 and $(\eta \wedge 1)^* A \simeq E_1$ is strong, it will be enough to check that the diagram commutes after applying π_0. By construction we have $\pi_0(u) = w^{-1} \circ \psi \circ \pi_0(f'')$ and $\psi = \pi_0(g) = \pi_0(g'') \circ \pi_0(g')$ and $\pi_0(v) = \pi_0(g'')^{-1} \circ w$. It follows directly that the above diagram commutes on homotopy, groups, so it commutes in EPA_0, so it commutes in EPR, so $gf = h''h'$ in EPR as explained previously. Thus, the map gf is good, as claimed. □

Proof of Theorem 1.6 We merely need to collect results together and explain the argument in the introduction in more detail. Lemma 4.3 and Proposition 4.4 show that we can make $\mathcal{E}$ into a category by taking the good maps from $\eta^* E_0$ to $\eta^* E_1$ as the morphisms from E_0 to E_1. Tautologically, we can define a faithful functor $U: \mathcal{E} \to \text{EPR}$ by $U(E) = \eta^* E$ and $U(f) = f$. We then define $V = \Gamma U: \mathcal{E} \to \text{FG}$; by the definition of $\mathcal{E}$, this actually lands in GFG. Proposition 4.1 says that V is essentially surjective, and Proposition 4.2 says that V is full and faithful. This means that V is an equivalence, so we can invert it and define $R = UV^{-1}: \text{GFG} \to \text{EPR}$. As $V = \Gamma U$ we have $\Gamma R = 1$, so R is the required realisation functor. □

A Appendix : The product on MP

In this appendix we verify that MP can be constructed as an E_∞ ring spectrum.

Let $\mathcal{U}$ be a complex universe. For any finite-dimensional subspace U of $\mathcal{U}$, we write $U_L = U \oplus 0 < \mathcal{U} \oplus \mathcal{U}$ and $U_R = 0 \oplus U < \mathcal{U} \oplus \mathcal{U}$. We let $\text{Grass}(U \oplus U)$ denote the Grassmannian of all subspaces of $U \oplus U$ (of all possible dimensions), and we write γ_U for the tautological bundle over this space, and $\text{Thom}(U \oplus U)$ for the associated Thom space. If $U \leq U' < \mathcal{U}$ then we define $i\colon \text{Grass}(U^2) \to \text{Grass}((U')^2)$ by $i(A) = A \oplus (U' \ominus U)_R$. On passing to Thom spaces we get a map $\sigma\colon \Sigma^{U' \ominus U}\,\text{Thom}(U^2) \to \text{Thom}((U')^2)$. These maps can be used to assemble the spaces $\text{Thom}(U^2)$ into a Σ-inclusion prespectrum indexed by the complex subspaces of $\mathcal{U}$. We write $T_{\mathcal{U}}$ for this prespectrum, and $MP_{\mathcal{U}}$ for its spectrification.

Now let $\mathcal{V}$ be another complex universe, so we have a prespectrum $T_{\mathcal{V}}$ over $\mathcal{V}$, and thus an external smash product $T_{\mathcal{U}} \wedge_{\text{ext}} T_{\mathcal{V}}$ indexed on the complex subspaces of $\mathcal{U} \oplus \mathcal{V}$ of the form $U \oplus V$. The direct sum gives a map $\text{Grass}(U^2) \times \text{Grass}(V^2) \to \text{Grass}((U \oplus V)^2)$ which induces a map $\text{Thom}(U^2) \wedge \text{Thom}(V^2) \to \text{Thom}((U \oplus V)^2)$. These maps fit together to give a map $T_{\mathcal{U}} \wedge_{\text{ext}} T_{\mathcal{V}} \to T_{\mathcal{U} \oplus \mathcal{V}}$, and thus a map $MP_{\mathcal{U}} \wedge_{\text{ext}} MP_{\mathcal{V}} \to MP_{\mathcal{U} \oplus \mathcal{V}}$ of spectra over $\mathcal{U} \oplus \mathcal{V}$. Essentially the same construction gives maps

$$MP_{\mathcal{U}_1} \wedge_{\text{ext}} \ldots \wedge_{\text{ext}} MP_{\mathcal{U}_r} \to MP_{\mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_r}.$$

If $\mathcal{U}_1 = \ldots \mathcal{U}_r = \mathcal{U}$, then this map is Σ_r-equivariant.

Now suppose instead that we have a complex linear isometry $f: \mathcal{U} \to \mathcal{V}$. This gives evident homeomorphisms $\text{Thom}(U^2) \to \text{Thom}((fU)^2)$, which fit together to induce a map $MP_{\mathcal{U}} \to f^* MP_{\mathcal{V}}$, which is adjoint to a map $f_* MP_{\mathcal{U}} \to MP_{\mathcal{V}}$. We next observe that this construction is continuous in all possible variables, including f. (This statement requires some interpretation, but there are no new issues beyond those that are well-understood for MU; the cleanest technical framework is provided by [1].) It follows that they fit together to give a map $\mathcal{L}_{\mathbb{C}}(\mathcal{U},\mathcal{V}) \ltimes MP_{\mathcal{U}} \to MP_{\mathcal{V}}$ of spectra over $\mathcal{V}$.

We now combine this with the product structure mentioned earlier to get a map

$$\mathcal{L}_{\mathbb{C}}(\mathcal{U}^r,\mathcal{U}) \ltimes_{\Sigma_r} (MP_{\mathcal{U}} \wedge_{\text{ext}} \dots \wedge_{\text{ext}} MP_{\mathcal{U}}) \to MP_{\mathcal{U}}.$$

This means that $MP_{\mathcal{U}}$ has an action of the E_∞ operad of complex linear isometries, as required.

All that is left is to check that the spectrum $MP = MP_{\mathbb{C}^\infty}$ constructed above has the right homotopy type. As T is a Σ-inclusion prespectrum, we know that spectrification works in the simplest possible way and that MP is the homotopy colimit of the spectra

$$\Sigma^{-2n}\operatorname{Thom}(\mathbb{C}^n \oplus \mathbb{C}^n) = \bigvee_{k=-n}^{n} \Sigma^{-2n}\operatorname{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma,$$

where $\operatorname{Grass}_d(V)$ is the space of d-dimensional subspaces of V. It is not hard to see that the maps of the colimit system preserve this splitting, so that MP is the wedge over all $k \in \mathbb{Z}$ of the spectra

$$X_k := \underset{\substack{\longrightarrow \\ n}}{\operatorname{holim}}\, \Sigma^{-2n}\operatorname{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma.$$

This can be rewritten as

$$X_k = \Sigma^{2k} \underset{\substack{\longrightarrow \\ n,m}}{\operatorname{holim}}\, \Sigma^{-2(k+n)}\operatorname{Grass}_{k+n}(\mathbb{C}^m \oplus \mathbb{C}^n)^\gamma.$$

We can reindex by putting $n = i - k$ and $m = j + k$, and then pass to the limit in j. We find that

$$X_k = \Sigma^{2k} \underset{\substack{\longrightarrow \\ i}}{\operatorname{holim}}\, \Sigma^{-2i}\operatorname{Grass}_i(\mathbb{C}^\infty \oplus \mathbb{C}^i)^\gamma.$$

It is well-known that $\operatorname{Grass}_i(\mathbb{C}^\infty \oplus \mathbb{C}^i)$ is a model for $BU(i)$, and it follows that $X_k = \Sigma^{2k}MU$, so $MP = \bigvee_k \Sigma^{2k}MU$ as claimed. We leave it to the reader to check that the product structure is the obvious one.

All the above was done without inverting 2. Inverting 2 is an example of Bousfield localisation, and this can always be performed in the category of strictly commutative ring spectra.

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