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Seifert fibered contact three–manifolds via surgery

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Abstract Using contact surgery we define families of contact structures on certain Seifert fibered three–manifolds. We prove that all these contact structures are tight using contact Ozsváth–Szabó invariants. We use these examples to show that, given a natural number n, there exists a Seifert fibered three–manifold carrying at least n pairwise non–isomorphic tight, not fillable contact structures.

AMS Classification 57R17; 57R57

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1 Introduction and statement of results

The classification problem for tight contact structures on closed oriented three–manifolds is one of the driving forces in present day contact topology. Contact surgery along Legendrian links provides a powerful tool for constructing contact three–manifolds. Tightness of these structures is, however, hard to prove, unless the structures can be shown to be *fillable*, i.e., can be viewed as living on the boundary of a symplectic four–manifold satisfying appropriate compatibility conditions. The question whether any tight contact structure is fillable was open for some time, until the first tight, nonfillable contact three–manifolds were found by Etnyre and Honda [6], followed by infinitely many such examples [12, 13]. The tightness of those examples was proved using a delicate topological method called *state traversal* (see [9]). In this paper we prove tightness by applying the Heegaard Floer theory recently developed by Ozsváth and Szabó [17, 18, 21]. According to our main result, tight, not fillable contact structures are more common than one would expect:

Theorem 1.1 *For any $n \in \mathbb{N}$ there is a Seifert fibered 3–manifold M_n carrying at least n pairwise non–isomorphic tight, not fillable contact structures.*

The construction of the contact structures in Theorem 1.1 relies on contact surgery. We verify nonfillability via the Seiberg–Witten equations, following the approach of [12, 13]. In order to state precisely our results we need a little preparation.

Contact surgery

In a given contact three–manifold (Y,ξ) a knot $K\subset(Y,\xi)$ is *Legendrian* if K is everywhere tangent to ξ. The framing of K naturally induced by ξ is called the *contact framing.* Given a Legendrian knot K in a contact three–manifold (Y,ξ) and a rational number $r\in\mathbb{Q}$ $(r\neq 0)$, one can perform *contact r–surgery* along K to obtain a new contact three–manifold (Y',ξ') [1, 2]. Here Y' is the three–manifold obtained by smooth r–surgery along K, where the surgery coefficient is measured *with respect to the contact framing* defined above, not with respect to the framing induced by a Seifert surface (which, in general, does not exist). The contact structure ξ' is constructed by extending ξ from the complement of a standard neighborhood of K to a tight contact structure on the glued–up solid torus. If $r\neq 0$ such an extension always exists, and for $r=\frac{1}{k}$ $(k\in\mathbb{Z})$ it is unique [9]. When $r=-1$ the corresponding contact surgery coincides with Legendrian surgery along K [5, 8, 22].

Below we outline an algorithm for replacing a contact r–surgery on a Legendrian knot K with a sequence of contact (±1)–surgeries on a suitable Legendrian link. By [2, Proposition 3], contact r–surgery along $K\subset(Y,\xi)$ with $r<0$ is equivalent to Legendrian surgery along a Legendrian link $\mathbb{L}=\cup_{i=0}^{m}L_i$ which is determined via the following simple algorithm by the Legendrian knot K and the contact surgery coefficient r. The algorithm to obtain $\mathbb{L}$ is the following. Let

$$[a_0+1,\dots,a_m],\quad a_0,\dots a_m\leq -2$$

be the continued fraction expansion of r. To obtain the first component L_0, push off K using the contact framing and stabilize it $-a_0-2$ times. Then, push off L_0 and stabilize it $-a_1-2$ times. Repeat the above scheme for each of the remaining pivots of the continued fraction expansion. Since there are $-a_i-1$ inequivalent ways to stabilize a Legendrian knot $-a_i-2$ times, this construction yields $\Pi_{i=0}^{m}(-a_i-1)$ potentially different contact structures. According to [2, Proposition 7], a contact $r=\frac{p}{q}$–surgery $(p,q\in\mathbb{N})$ on a Legendrian knot K is equivalent to a contact $\frac{1}{k}$–surgery on K followed by a contact $\frac{p}{q-kp}$–surgery on a Legendrian pushoff of K for any integer $k\in\mathbb{N}$ such that $q-kp<0$. Therefore, the latter surgery can be turned into a sequence of Legendrian surgeries, as described above. By [1, Proposition 9], a contact $\frac{1}{k}$–surgery ($k \in \mathbb{N}$) on a Legendrian knot K can be replaced by k contact $(+1)$–surgeries on k Legendrian pushoffs of K.

In conclusion, any contact rational r–surgery ($r \neq 0$) can be replaced by contact (± 1)–surgery along a Legendrian link (which is not necessarily uniquely specified); for a related discussion see also [3].

Statement of results

In the following, we shall denote by

$$M(g, n; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

the Seifert fibered 3–manifold obtained by performing $(-\frac{\alpha_1}{\beta_1})$–, $\dots, (-\frac{\alpha_k}{\beta_k})$–surgeries along k fibers of the circle bundle $Y_{g,n} \to \Sigma_g$ over the genus–g surface Σ_g with Euler number $e(Y_{g,n}) = n$. The Seifert invariants

$$(g, n; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

are said to be in *normal form* if

$$\alpha_i > \beta_i \geq 1, \quad i = 1, \dots, k.$$

Using Rolfsen twists (hence changing n if necessary), any tuple

$$(g, n; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

can be transformed into normal form.

Consider the family of contact 3–manifolds defined by the contact surgery diagrams of Figure 1 (the box is repeated $(g-1)$–times, $g \geq 1$).

Throughout the paper we shall assume:

$$g \geq 1, \quad \frac{1}{2} \leq r_1 < 1, \quad r_i < 0, \quad i = 2, \dots, k \quad (r_i \in \mathbb{Q}). \tag{1.1}$$

Under the assumptions (1.1) one can write the coefficients as:

$$r_1 = \frac{(n - 2g + 1)\alpha_1 + \beta_1}{(n - 2g + 2)\alpha_1 + \beta_1}, \quad r_i = \frac{\beta_i - \alpha_i}{\beta_i}, \tag{1.2}$$

where

$$n \geq 2g, \quad \alpha_1 > \beta_1 \geq 0, \quad \alpha_i > \beta_i \geq 1, \; i = 2, \dots, k.$$

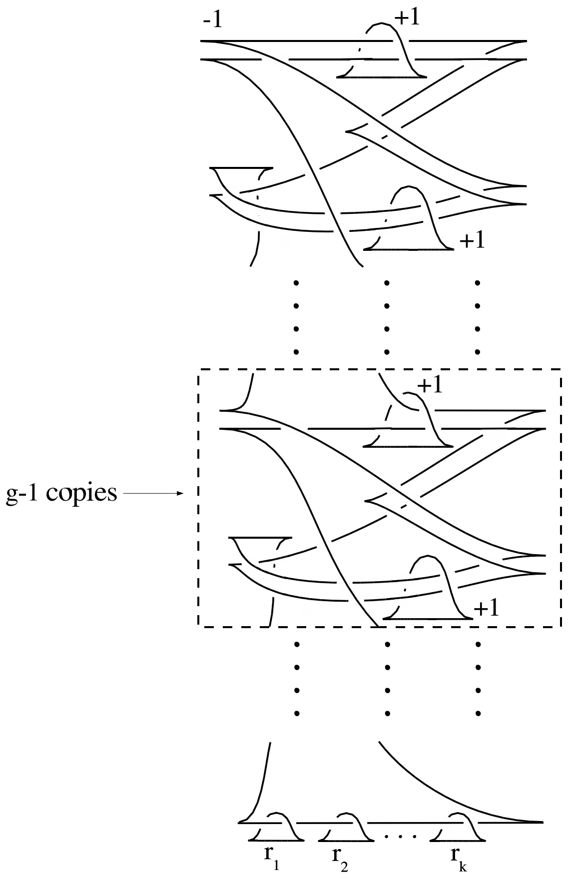

Figure 1: Contact structures on Seifert fibered 3–manifolds

Converting the contact surgery coefficients into smooth coefficients, after $(n - 2g + 1)$ Rolfsen twists on the r_1–framed unknot we conclude that the 3–manifolds underlying the contact structures given by Figure 1 are of the form:

$$M(g, n; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)), \quad n \geq 2g. \tag{1.3}$$

Moreover, if $\beta_1 > 0$ the Seifert invariants are in normal form. Observe that for $\beta_1 = 0$ the $(-\frac{\alpha_1}{\beta_1})$–surgery is trivial.

Conversely, given a Seifert fibered 3–manifold M as in (1.3), Figure 1 provides a contact structure on M as long as the coefficients r_i defined by (1.2) satisfy the conditions (1.1).

Let $\xi_1, \dots, \xi_t$ denote the contact structures obtained by turning the diagrams of Figure 1 into contact (± 1)–surgeries in all possible ways according to the algorithm described in the previous subsection. This paper is devoted to the study of $\xi_1, \dots, \xi_t$. Using the contact Ozsváth–Szabó invariants [21] we prove:

Theorem 1.2 *Fix $k \geq 1$, $g \geq 1$, $\frac{1}{2} \leq r_1 < 1$ and $r_i < 0$ for $i = 2, \dots, k$. Then, all the contact structures defined by Figure 1 are tight.*

It is unclear from the construction whether the contact structures $\xi_1, \dots, \xi_t$ are all distinct up to isotopy. Observe that for $k = 1$ and $r_1 = \frac{\alpha+1}{2\alpha+1}$ the 3–manifold underlying Figure 1 is $M(g, 2g; (\alpha, 1))$.

Theorem 1.3 *Given $g \geq 1$ and $n \in \mathbb{N}$, there is an $\alpha \in \mathbb{N}$ such that at least n of the contact structures defined by Figure 1 for $k = 1$ and $r_1 = \frac{\alpha+1}{2\alpha+1}$ are pairwise non–isomorphic.*

In fact, a more detailed analysis shows that the contact structures defined by Figure 1 on $M(g, 2g; (\alpha, 1))$ are all distinct up to isotopy (see Section 4). This leads us to:

Conjecture 1 All the tight contact structures defined by Figure 1 and satisfying the assumptions (1.1) are distinct up to isotopy.

Recall that a contact 3–manifold (Y, ξ) is *symplectically fillable*, or simply *fillable*, if there exists a compact symplectic four–manifold (W, ω) such that (i) $\partial W = Y$ as oriented manifolds (here W is oriented by $\omega \wedge \omega$) and (ii) $\omega|_\xi \neq 0$ at every point of Y. Our next result concerns fillability properties of some of the contact structures under examination.

Theorem 1.4 *Fix $\alpha \in \mathbb{N}$ and $g \geq 1$ such that $d(d+1) \leq 2g \leq d(d+2) - 1$ for some positive integer d. Then, the tight contact structures defined by Figure 1 for $k = 1$ and $r_1 = \frac{\alpha+1}{2\alpha+1}$ are not symplectically fillable.*

As we show in Section 4, there is some evidence supporting the following:

Conjecture 2 No contact structure defined by Figure 1 and satisfying conditions (1.1) is fillable.

The above results immediately imply Theorem 1.1:

Proof of Theorem 1.1 Fix $n \in \mathbb{N}$ and $g = 1$. Choose $\alpha \in \mathbb{N}$ such that the statement of Theorem 1.3 holds. The contact structures $\xi_1, \dots, \xi_t$ defined by Figure 1 on $M(1, 2; (\alpha, 1))$ are tight by Theorem 1.2 and there are at least n pairwise non–isomorphic among them by Theorem 1.3. By Theorem 1.4 applied with $d = 1$ they are also not fillable. This concludes the proof. □

Our results seem to suggest (see Section 4) that a Seifert fibered 3–manifold

$$M(g, n; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

with Seifert invariants in normal form should support a tight, not fillable contact structure if $n \geq 2g > 0$. This should be contrasted with the result of Gompf [8], who showed that a Seifert fibered 3–manifold with base genus $g \geq 1$ always carries a Stein fillable contact structure.

Section 2 is devoted to the proof of Theorem 1.2, while Theorems 1.3 and 1.4 will be proved in Section 3. In Section 4 we give further evidence supporting Conjectures 1 and 2.

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2 Proof of Theorem 1.2

In a remarkable series of papers [17, 18, 19, 21] Ozsváth and Szabó defined new invariants of many low–dimensional objects — including contact structures on closed 3–manifolds. In this section we apply these invariants to prove Theorem 1.2.

Heegaard Floer theory associates abelian groups $HF^+(Y, \mathbf{t})$ and $\widehat{HF}(Y, \mathbf{t})$ to a closed, oriented Spin^c 3–manifold $(Y, \mathbf{t})$, and homomorphisms

$$F^+_{W,\mathbf{s}}\colon HF^+(Y_1, \mathbf{t}_1) \to HF^+(Y_2, \mathbf{t}_2), \quad \widehat{F}_{W,\mathbf{s}}\colon \widehat{HF}(Y_1, \mathbf{t}_1) \to \widehat{HF}(Y_2, \mathbf{t}_2)$$

to a Spin^c cobordism $(W, \mathbf{s})$ between two Spin^c 3–manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$.

Throughout this paper we shall assume that $\mathbb{Z}/2\mathbb{Z}$ coefficients are being used in the complexes defining the HF^+– and $\widehat{HF}$–groups.

Let $Y_{g,-2g}$ be a circle bundle over the genus–g surface Σ_g with Euler number $-2g$ ($g \geq 1$), and let $D_{g,-2g}$ denote the corresponding disk bundle. Since $H^2(D_{g,-2g};\mathbb{Z})$ has no 2–torsion, each Spin^c structure on $D_{g,-2g}$ is uniquely determined by its first Chern class. Let $\mathbf{s}$ be the unique Spin^c structure on $D_{g,-2g}$ with $c_1(\mathbf{s}) = 0$, and denote by $\mathbf{t}$ the restriction of $\mathbf{s}$ to $Y_{g,-2g}$.

Let W denote the cobordism from $\#_{2g}(S^1 \times S^2)$ to $Y_{g,-2g}$ given by the attachment of a 4–dimensional 2–handle along the $(-2g)$–framed knot $K \subset \#_{2g}(S^1 \times S^2)$ of Figure 2. Let $\mathbf{t}_0 \in \mathrm{Spin}^c(\#_{2g}(S^1 \times S^2))$ be the unique Spin^c

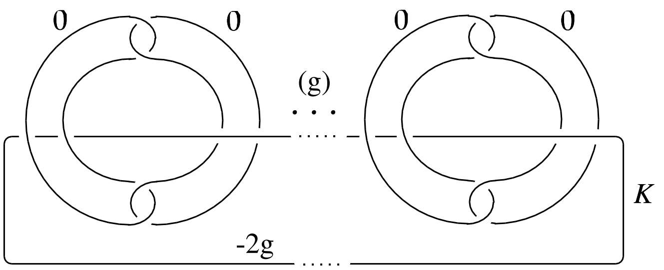

Figure 2: The framed knot K

structure on $\#_{2g}(S^1 \times S^2)$ with vanishing first Chern class. In [20, Lemma 9.17] it is proved that there is an isomorphism

$$HF^+(\#_{2g}(S^1 \times S^2), \mathbf{t}_0) \longrightarrow HF^+(Y_{g,-2g}, \mathbf{t})$$

which can be written as a sum of maps $\sum_{\mathbf{s}} F^+_{W,\mathbf{s}}$ over the set of Spin^c structures on W which restrict to $\mathbf{t}_0$ and $\mathbf{t}$. Application of the 5–lemma to the long exact sequence connecting $HF^+(Y_{g,-2g}, \mathbf{t})$ and $\widehat{HF}(Y_{g,-2g}, \mathbf{t})$ immediately yields the following:

Lemma 2.1 *The homomorphism*

$$\sum_{\{\mathbf{s}\in \mathrm{Spin}^c(W)\ \mid\ \mathbf{s}|_{\partial W} = (\mathbf{t}_0,\mathbf{t})\}} \widehat{F}_{W,\mathbf{s}}\colon\ \widehat{HF}(\#_{2g}(S^1 \times S^2), \mathbf{t}_0) \to \widehat{HF}(Y_{g,-2g}, \mathbf{t}),$$

is an isomorphism. □

Contact Ozsváth–Szabó invariants

Let (Y,ξ) be a closed contact 3–manifold oriented by ξ, and let $\mathbf{t}_\xi \in \mathrm{Spin}^c(Y)$ be the Spin^c structure induced by ξ. In [21], Ozsváth and Szabó define an invariant

$$c(Y,\xi) \in \widehat{HF}(-Y,\mathbf{t}_\xi)$$

whose main properties are summarized in the following two theorems.

Theorem 2.2 [21] *If (Y,ξ) is overtwisted, then $c(Y,\xi) = 0$. If (Y,ξ) is Stein fillable then $c(Y,\xi) \neq 0$. In particular, for the standard contact structure (S^3,ξ_{st}) we have $c(S^3,\xi_{st}) \neq 0$.* □

Theorem 2.3 *Suppose that (Y_2,ξ_2) is obtained from (Y_1,ξ_1) by a contact $(+1)$–surgery. Then we have*

$$F_{-W}(c(Y_1,\xi_1)) = c(Y_2,\xi_2),$$

where $-W$ is the cobordism induced by the surgery with reversed orientation and F_{-W} is the sum of $\sum_{\mathbf{s}} \widehat{F}_{-W,\mathbf{s}}$ over all Spin^c structures $\mathbf{s}$ extending the Spin^c structures induced on $-Y_i$ by ξ_i, $i=1,2$. In particular, if $c(Y_2,\xi_2) \neq 0$ then (Y_1,ξ_1) is tight.

Proof Let us assume that we are performing contact $(+1)$–surgery along the Legendrian knot $K \subset (Y_1,\xi_1)$. Then, there is an open book decomposition (F,ϕ) on Y_1 compatible with ξ_1 in the sense of Giroux and such that K lies on a page. In fact, the proof of [7, Theorem 3] shows that the 1–skeleton of any contact cellular decomposition of (Y_1,ξ_1) is contained in a page of a compatible open book. Since K can be assumed to lie in the 1–skeleton of a contact cellular decomposition of (Y_1,ξ_1), the conclusion follows. Moreover, up to refining the decomposition, we may assume that K is not homotopic to the boundary of the page. Then, an open book for (Y_2,ξ_2) is given by (F,ϕ'), where $\phi' = \phi \circ R_K^{-1}$ and R_K is the right–handed Dehn twist along K. The first part of the statement now follows applying [21, Theorem 4.2]. The second part of the statement follows immediately from the fact that the invariant of an overtwisted contact structure vanishes. □

Theorem 2.3 immediately yields:

Corollary 2.4 *If $c(Y_2,\xi_2) \neq 0$ and (Y_1,ξ_1) is obtained from (Y_2,ξ_2) by Legendrian surgery along a Legendrian knot, then $c(Y_1,\xi_1) \neq 0$. In particular, (Y_1,ξ_1) is tight.*

Proof Let $K \subset (Y_2,\xi_2)$ be the Legendrian knot along which the Legendrian surgery is performed. A Legendrian pushoff of K gives rise to a Legendrian knot $\widetilde{K}$ in (Y_1,ξ_1). By [1, Proposition 8], contact $(+1)$–surgery on (Y_1,ξ_1) along $\widetilde{K}$ gives back (Y_2,ξ_2). Therefore, by Theorem 2.3 $c(Y_2,\xi_2)\neq 0$ implies $c(Y_1,\xi_1)\neq 0$. □

Let (Z_j,η_j) be the contact 3–manifold obtained by performing contact $(+1)$–surgery on the standard contact three–sphere along the j–component Legendrian unlink depicted in Figure 3.

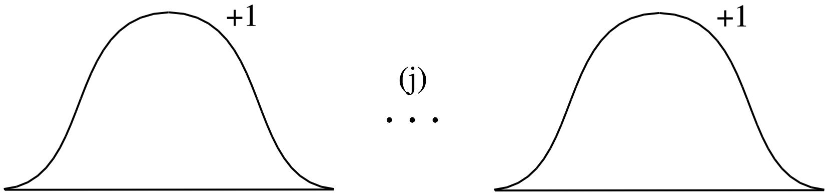

Figure 3: The contact 3–manifold (Z_j,η_j)

Lemma 2.5 *The contact 3–manifold (Z_j,η_j) given by Figure 3 has non–vanishing contact Ozsváth–Szabó invariant for every $j\geq 0$.*

Proof Notice first that Z_j is diffeomorphic to $\#_j(S^1\times S^2)$. We will argue by induction on j. For $j=0$ we have the standard contact 3–sphere, which has non–vanishing contact Ozsváth–Szabó invariant by Theorem 2.2. Now consider η_{j-1} and add the j–th component of the Legendrian unlink to it with contact framing $(+1)$. Let $-W$ be the corresponding cobordism with reversed orientation. By [18, Theorem 9.16] the homomorphism F_{-W} fits into an exact triangle:

$$\begin{tikzcd} \widehat{HF}(\#_{j-1}(S^1\times S^2)) \arrow[rr, "F_{-W}"] & & \widehat{HF}(\#_j(S^1\times S^2)) \arrow[dl] \\ & \widehat{HF}(\#_{j-1}(S^1\times S^2)) \arrow[ul] & \end{tikzcd}$$

In [18, Subsection 3.1 and Proposition 6.1] it is proved that

$$\dim_{\mathbb{Z}/2\mathbb{Z}}\widehat{HF}(\#_j(S^1\times S^2))=2^j.$$

Therefore, the exactness of the triangle implies that the map F_{-W} is injective. Since by Theorem 2.3 we have

$$F_{-W}(c(Z_{j-1},\eta_{j-1}))=c(Z_j,\eta_j)$$

and by the inductive assumption $c(Z_{j-1},\eta_{j-1})) \neq 0$, this concludes the proof. □

Note that when $k=1$ and $r_1=\frac{1}{2}$, Figure 1 specifies a unique contact structure ξ_g for every g because the contact surgery coefficients are of the form $\frac{1}{k}$, $k\in\mathbb{Z}$. Denote the resulting contact 3–manifold by (Y_g,ξ_g). It is a simple exercise to verify that Y_g is an S^1–bundle over a genus–g surface with Euler number $e(Y_g)=2g$.

Proposition 2.6 *The contact Ozsváth–Szabó invariant of (Y_g,ξ_g) is nonzero.*

Proof Let (Y_g',ξ_g') be the contact 3–manifold given by Figure 1 with $k=1$ and $r_1=1$, and perform contact $(+1)$–surgery on a pushoff of the r_1–framed Legendrian knot K. According to the algorithm described in Section 1, the resulting contact structure is (Y_g,ξ_g). Note that Y_g' is diffeomorphic to $\#_{2g}(S^1\times S^2)$. Combining Lemma 2.5 and Corollary 2.4 we conclude $c(Y_g',\xi_g')\neq 0$. In fact, (Y_g',ξ_g') must be the only tight, hence Stein fillable contact structure on $\#_{2g}(S^1\times S^2)$. The cobordism given by the handle attachment induced by the surgery along K can be easily identified (after reversing orientation) with the cobordism appearing in Lemma 2.1, therefore the non–vanishing of $c(Y_g',\xi_g')$ implies, by Theorem 2.3, that $c(Y_g,\xi_g)\neq 0$. □

Remark 2.7 The tightness of the contact structures ξ_g was first proved by Honda [9] (see also [13]).

Proof of Theorem 1.2 Let K_1,K_2 denote two Legendrian pushoffs of the r_1–framed Legendrian unknot K of Figure 1. According to the algorithm of Section 1 all contact structures of Figure 1 can be given as negative contact surgery on the diagram obtained erasing the r_i–framed circles $(i=2,\dots,k)$ from Figure 1 and performing contact $(+1)$–surgeries on K,K_1 and contact $\frac{r_1}{1-2r_1}$–surgery on K_2. (Here we use the assumption $r_i<0$ for $i=2,\dots,k$.) Since $r_1\geq\frac{1}{2}$, the surgery coefficient of K_2 is also negative (or infinity), therefore all the contact structures defined by Figure 1 (obeying the restrictions on the r_i) can be given as Legendrian surgery on (Y_g,ξ_g) for an appropriate $g\geq 1$. Since negative contact surgery can be replaced by a sequence of Legendrian surgeries, Corollary 2.4 and Proposition 2.6 imply that these contact structures have non–vanishing contact Ozsváth–Szabó invariants, hence by Theorem 2.2 they are tight. This concludes the proof of the theorem. □

3 The proof of non–fillability

Suppose that (Y,ξ) is given by a contact (±1)–surgery diagram and denote the corresponding 4–manifold by X. Then, the Spin^c structure of the 0–handle of X extends to a Spin^c structure $\mathbf{s}\in \mathrm{Spin}^c(X)$ with the property that $\mathbf{s}|_{\partial X}=\mathbf{t}_\xi$ and $c_1(\mathbf{s})$ evaluates on a homology class $[\Sigma_K]$ given by an oriented surgery curve K as $\mathrm{rot}(K)$. This statement was proved for (-1)–surgeries by Gompf [8] — in this case the complex structure of D^4 also extends over the 2–handles — and in [13] for the case of $(+1)$–surgeries; see also [3].

Consider the diagram obtained from Figure 1 for $k=1$ and $r_1=\frac{\alpha+1}{2\alpha+1}$; this diagram represents contact structures on $M(g,2g;(\alpha,1))$. According to the algorithm outlined in Section 1, these contact structures are also representable by replacing the Legendrian knot K with three Legendrian pushoffs K_1,K_2,K_3 having contact surgery coefficients $(+1)$, $(+1)$ and $-(\alpha+1)$, respectively. This last diagram can be turned into a contact (±1)–surgery diagram by stabilizing the Legendrian curve K_3 α times. There are $(\alpha+1)$ different ways to do this. Choose an orientation for K_3 and define ξ_r as the result of the surgery along the diagram with $\mathrm{rot}(K_3)=r$. (Notice that $r\equiv\alpha \pmod 2$ and $-\alpha\le r\le\alpha$.) The above observation regarding Spin^c structures yields:

Lemma 3.1 *Let* $\mathbf{s}\in\mathrm{Spin}^c(X)$ *be the unique* Spin^c *structure such that* $\langle c_1(\mathbf{s}),[\Sigma_{K_3}]\rangle = r$ *and* $\langle c_1(\mathbf{s}),[\Sigma_j]\rangle=0$ *on the 2–homology classes defined by the remaining surgery circles. Then, the restriction of* $\mathbf{s}$ *to* ∂X *is the* Spin^c *structure* $\mathbf{t}_{\xi_r}\in\mathrm{Spin}^c(M(g,2g;(\alpha,1)))$ *induced by the contact structure* ξ_r. □

Recall that, since X is simply connected, the Chern class $c_1(\mathbf{s})$ uniquely specifies the Spin^c structure $\mathbf{s}\in\mathrm{Spin}^c(X)$. For $M=M(g,2g;(\alpha,1))$ let $\mu\in H_1(M;\mathbb{Z})$ denote the homology class of the normal circle to the knot K_3 — or, equivalently, the homology class represented by the singular fiber of the Seifert fibration. Then, Lemma 3.1 implies that

$$c_1(\xi_r)=c_1(\mathbf{t}_{\xi_r})=r\,\mathrm{PD}(\mu).$$

In particular, since the order of μ in $H_1(M;\mathbb{Z})$ is equal to $2g\alpha+1$, $\mathbf{t}_{\xi_r}$ is a torsion Spin^c structure for all r.

Proof of Theorem 1.3 By the classical Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes of the form $2gm+1$ as m varies among the natural numbers. Therefore, we can choose natural numbers $a_1,\dots,a_n$ so that

$$p_1=2ga_1+1,\dots,p_n=2ga_n+1$$

are distinct odd primes. Define a so that

$$2ga + 1 = p_1 \cdots p_n.$$

If a is odd, let $\alpha = a$, otherwise let $\alpha = a(2g+1)+1$. With this choice $2g\alpha+1$ is divisible by $p_1 \cdots p_n$ and α is odd. Therefore,

$$\alpha \equiv p_i \bmod 2, \quad i = 1, \dots, n,$$

and we can choose the stabilizations of K_3 so that $c_1(\xi_i) = p_i\mu$. This implies that the order of $c_1(\xi_i)$ is $\frac{2g\alpha+1}{p_i}$, and since the p_i's are all distinct, the orders of the $c_1(\xi_i)$'s are all different for $i = 1, \dots, n$. This shows that the contact structures ξ_i, $i = 1, \dots, n$, are pairwise non–isomorphic, concluding the proof. □

The proof of Theorem 1.4 will follow the approach used in [10] and further exploited in [12]. Fix a Seifert fibration

$$M = M(g, n; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)) \to \Sigma_g$$

over the orbifold Σ_g. The surface Σ_g can be thought of as the underlying space of an orbifold with k marked points of multiplicities $\alpha_1, \dots, \alpha_k$. An orbifold line bundle $L \to \Sigma_g$ can be pulled back to an honest line bundle $\overline{L} \to M$ with torsion first Chern class, and if the invariants α_i are mutually coprime, all line bundles on M with torsion first Chern class arise in this way. An orbifold line bundle $L \to \Sigma_g$ can be described by its *Seifert data* $(c; \gamma_1, \dots, \gamma_k)$, where c is the background degree of L and the numbers γ_i determine the orbifold bundle around the orbifold points of Σ_g (see [14, §2] for further details). For example, the orbifold canonical bundle K_Σ has Seifert data $(2g-2; \alpha_1 - 1, \dots, \alpha_k - 1)$. The *degree* of the orbifold line bundle L is equal by definition to the rational number

$$\deg(L) = b + \sum_{i=1}^{k} \frac{\gamma_i}{\alpha_i}.$$

For more about Seifert fibered three–manifolds and line bundles on them see [14, 16].

Theorem 3.2 [14] *The moduli space of Seiberg–Witten solutions for the Seifert fibered 3–manifold $M = M(g, 2g; (\alpha, 1))$ and Spin^c structure $\mathbf{t}_{\xi_r} \in \mathrm{Spin}^c(M)$ contains only reducible solutions, for all of which the associated Dirac operator has trivial kernel.*

Proof We need to express the Spin^c structure $\mathbf{t}_{\xi_r}$ in the coordinates used in [14] and then appeal to the description of the Seiberg–Witten moduli spaces on Seifert fibered 3–manifolds as given in [14, Theorem 5.19]. In that paper the Spin^c structures are parametrized by their twisting relative to the canonical Spin^c structure $\mathbf{t}_{\mathrm{can}}$ induced by any tangent 2–plane field transverse to the S^1–fibration. As explained in [14, §3], the orbifold disk bundle associated to M can be desingularized to a smooth complex surface X with $\partial X = M$. The group $H_2(X;\mathbb{Z})$ is generated by the classes of a genus–g smooth complex curve C and a smooth rational curve R, satisfying:

$$C\cdot C = 2g, \quad C\cdot R = 1, \quad R\cdot R = -\alpha.$$

The restriction to ∂X of the complex bundle TX is isomorphic to the pull–back of

$$\underline{\mathbb{C}} \oplus K_\Sigma^{-1} \to \Sigma_g,$$

where $\underline{\mathbb{C}}$ is the trivial complex line bundle and K_Σ is the orbifold canonical bundle of Σ_g.

Therefore, denoting by $\mathbf{s}^{\mathbb{C}}$ the Spin^c structure on X induced by the complex structure, we have $\mathbf{s}^{\mathbb{C}}|_{\partial X} = \mathbf{t}_{\mathrm{can}}$ (cf. text following [14, Lemma 5.10]). The adjunction formula gives:

$$\langle c_1(X), C\rangle = 2, \quad \langle c_1(X), R\rangle = 2-\alpha.$$

Thus, if $\Gamma_r \in H^2(X;\mathbb{Z})$ is a cohomology class satisfying

$$\langle \Gamma_r, C\rangle = -1 \quad \langle \Gamma_r, R\rangle = \frac{1}{2}(r+\alpha-2),$$

setting $\mathbf{s}_r = \mathbf{s}^{\mathbb{C}} + \Gamma_r$, we have $\mathbf{s}_r|_{\partial X} = \mathbf{t}_{\xi_r}$. This implies:

$$\mathbf{t}_{\xi_r} = \mathbf{t}_{\mathrm{can}} + \Gamma_r|_{\partial X} = \mathbf{t}_{\mathrm{can}} + \frac{1}{2}(r-\alpha-2)\,\mathrm{PD}(\mu). \tag{3.1}$$

Now [14, Theorem 5.19] can be restated in the following form, more convenient for our present purposes. Fix a torsion Spin^c structure

$$\mathbf{t}_k = \mathbf{t}_{\mathrm{can}} + k\,\mathrm{PD}(\mu) \in \mathrm{Spin}^c(M).$$

Let $L_k \to \Sigma_g$ be an orbifold line bundle which pulls back to a line bundle $\overline{L}_k \to M$ with $c_1(\overline{L}_k) = k\,\mathrm{PD}(\mu)$. Then, the moduli space $\mathfrak{M}_k$ of Seiberg–Witten solutions on M in the Spin^c structure $\mathbf{t}_k$ has a component of reducible solutions (homeomorphic to the Jacobian torus of Σ_g), and by [14, Corollary 5.17] the associated Dirac operators have trivial kernels if and only if either α is even or

$$\deg L_k \not\in \frac{1}{2}\deg K_\Sigma + (2g+\frac{1}{\alpha})\cdot\mathbb{Z} \subset \mathbb{Q}. \tag{3.2}$$

In addition, $\mathfrak{M}_k$ contains irreducible solutions if and only if there exists some orbifold line bundle $L \to \Sigma_g$ satisfying:

$$\deg L \in [0, \deg K_\Sigma] \setminus \{\frac{1}{2} \deg K_\Sigma\}, \quad \deg L \in \deg L_k + (2g + \frac{1}{\alpha}) \cdot \mathbb{Z}. \qquad (3.3)$$

In view of (3.1), in our case we have:

$$k = \frac{1}{2}(r - \alpha - 2) \equiv 2g\alpha + \frac{1}{2}(r - \alpha) \mod (2g\alpha + 1).$$

Therefore, since $r \in [-\alpha, \alpha]$,

$$\deg K_\Sigma = 2g - 1 - \frac{1}{\alpha} < \deg L_k = 2g + \frac{1}{2\alpha}(r - \alpha) < 2g + \frac{1}{\alpha}.$$

It follows that L_k satisfies (3.2) and there is no orbifold line bundle $L \to \Sigma_g$ satisfying (3.3). Hence, $\mathfrak{M}_k$ consists entirely of reducible solutions with associated Dirac operators having trivial kernels. □

Corollary 3.3 *Let (W, ω) be a weak filling of the contact 3–manifold (M, ξ_r). Then, $b_2^+(W) = 0$ and the homomorphism $H^2(W; \mathbb{R}) \to H^2(\partial W; \mathbb{R})$ induced by the inclusion $\partial W \subset W$ is the zero map.*

Proof The statement follows from Theorem 3.2 in exactly the same way as [12, Proposition 4.2] follows from [12, Lemma 4.1]. □

Proof of Theorem 1.4 Let ξ_r be one of the contact structures on $M = M(g, 2g; (\alpha, 1))$ given by Figure 1. We shall argue as in [12, Theorem 1.1], therefore we shall need to find a 4–manifold $Z = Z(g, 2g; (\alpha, 1))$ with $b_2^+(Z) = 0$, $\partial Z = -M$ and such that the intersection form Q_Z does not embed into the diagonal lattice $\mathbb{D}_m = (\mathbb{Z}^m, m(-1))$ for any m.

We shall use a construction similar to the one given in [12, Proposition 4.4]. To this end, fix $g, d \in \mathbb{N}$ with $d(d+1) \le 2g \le (d+1)^2 - 2$, let $C \subset \mathbb{CP}^2$ be a smooth complex curve of degree $d+2$, and let $\widehat{\mathbb{CP}^2}$ be the blow–up of $\mathbb{CP}^2$ at $(d+2)^2 - 2g - 1$ distinct points of C. Denote by $\widehat{C} \subset \widehat{\mathbb{CP}^2}$ the proper transform of C. Let $\widetilde{C} \subset \widehat{\mathbb{CP}^2}$ be a smooth, oriented surface obtained by adding $g - \frac{1}{2}d(d+1)$ fake handles to $\widehat{C}$. Blow up $\widehat{\mathbb{CP}^2}$ at one more point of $\widetilde{C}$, then blow up repeatedly at distinct points of the last exceptional sphere until the corresponding proper transform in the resulting rational surface X is an embedded sphere S with self–intersection $-\alpha$. Define Z as the complement in X of a tubular neighborhood of $\widetilde{C} \cup S$.

The group $H_2(X;\mathbb{Z})$ is generated by classes $h, e_1, e_2, \dots, e_t$, where h corresponds to the standard generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$ and the e_i's are the classes of the exceptional curves. Let q be a positive integer such that $2q \le t$, and define $\Lambda_q = (H_q, Q_q)$ as the intersection lattice given by the subgroup

$$H_q = \langle e_1 - e_2, e_2 - e_3, \dots, e_{2q-1} - e_{2q}, h - e_1 - e_2 - \dots - e_q \rangle \subset H_2(X;\mathbb{Z})$$

together with the restriction Q_q of the intersection form Q_X.

As in the proof of [12, Proposition 4.4], the inequality $2g \le d(d+2) - 1$ guarantees that $2(d+2) \le t$, hence the lattice $\Lambda_{d+2} = (H_{d+2}, Q_{d+2})$ embeds into $(H_2(Z;\mathbb{Z}), Q_Z)$. Since by [12, Lemma 4.3] $\Lambda_{d+2} = (H_{d+2}, Q_{d+2})$ does not embed into any diagonal lattice $\mathbb{D}_m$, the same holds for $(H_2(Z;\mathbb{Z}), Q_Z)$.

By Corollary 3.3, a filling (W, ω) would give rise to a negative definite closed 4–manifold $V = W \cup Z$ with nonstandard intersection form, contradicting Donaldson's famous diagonalizability result [4]. □

4 Concluding remarks

With a little more work, essentially the same proof as the one given in Section 3 yields non–fillability for all structures defined by Figure 1 on $M(g, n; (\alpha, \beta))$ and satisfying

$$d(d+1) \le 2g \le n \le d(d+2) - 1$$

for $g \ge 1$ and some integer d. In fact, a slightly more general argument in the computation of the Spin^c structures allows one to check that the statement of Theorem 3.2 still holds.

In another direction, Theorem 1.4 generalizes to all $M(g, n; (\alpha, 1))$ with $n \ge 2g > 0$. In this case, one needs to consider Figure 1 for $k = 1$ and

$$r_1 = \frac{(n - 2g + 1)\alpha + 1}{(n - 2g + 2)\alpha + 1}.$$

According to the algorithm described in Section 1, the corresponding contact surgery can be expressed as a contact (± 1)–surgery by replacing the r_1–framed unknot K with two pushoffs of K, $n - 2g$ pushoffs of a stabilization $K^{\pm}$ of K, and one pushoff of $K^{\pm}$ stabilized $\alpha - 1$ times. Depending on the choice of stabilization of K, the result looks either like Figure 4 or Figure 5. Denoting by r the rotation number of the last knot (after a choice of orientation), this gives a contact structure ξ_r^+ for every $-\alpha < r \le \alpha$ and a contact structure ξ_r^- for every $-\alpha \le r < \alpha$ (and $r \equiv \alpha \bmod 2$ in both cases).

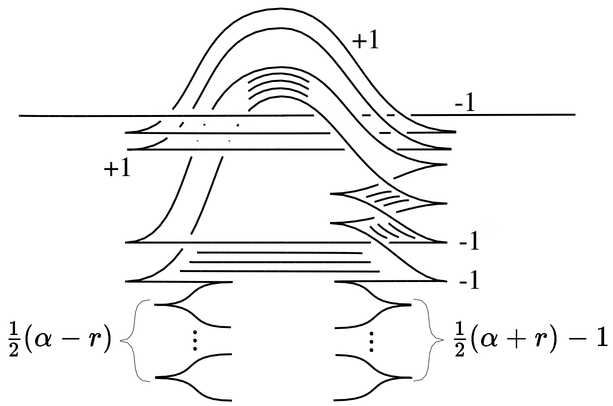

Figure 4: The contact structures ξ_r^+

A computation as in Section 3 gives

$$\mathbf{t}_{\xi_r^\pm} = \mathbf{t}_{\text{can}} + \frac{1}{2}(r - \alpha - 2 \pm \alpha(n-2g) - \alpha(n-2g))\,\mathrm{PD}(\mu).$$

This already shows that the contact structures defined on $M(g,n;(\alpha,1))$ by Figure 1 are all distinct up to homotopy, providing further evidence for Conjecture 1.

One can also compute the 3–dimensional invariant $d_3([\xi_r^\pm])$ of the homotopy class $[\xi_r^\pm]$ of tangent 2–plane fields containing the contact structure $\xi_r^\pm$ (as discussed in [13]), obtaining:

$$d_3([\xi_r^\pm]) = \frac{1}{4(n\alpha+1)}((n-2g)^2\alpha - r^2 n \pm 2(n-2g)r) + \frac{2g-1}{2}.$$

On the other hand, the statement of Theorem 3.2 holds for all contact structures defined on $M = M(g,n;(\alpha,1))$ by Figure 1 for $n \geq 2g$. Therefore, the argument of [11, Theorem 2.1] and [13, Theorem 4.1] applies, showing that there is a unique homotopy class $\Xi(\mathbf{t}_{\xi_r^\pm})$ of 2–plane fields inducing the Spin^c structure $\mathbf{t}_{\xi_r^\pm}$ and which might potentially contain a fillable contact structure. The proof of this observation rests on the fact that, assuming Theorem 3.2 to hold, the 3–dimensional invariant of $\Xi(\mathbf{t}_{\xi_r^\pm})$ is determined by some topological terms plus an η–invariant of $(M, \mathbf{t}_{\xi_r^\pm})$ as follows.

By the formula preceding [15, Section 3] (when $\rho(L) \neq 0$, which always holds in our case), the dimension d_1 of the Seiberg–Witten moduli space with fixed boundary limit can be expressed as

$$d_1 = d_3(\Xi(\mathbf{t}_{\xi_r^\pm})) + \omega_{red}(\mathbf{t}_{\xi_r^\pm}) - (2g-1),$$

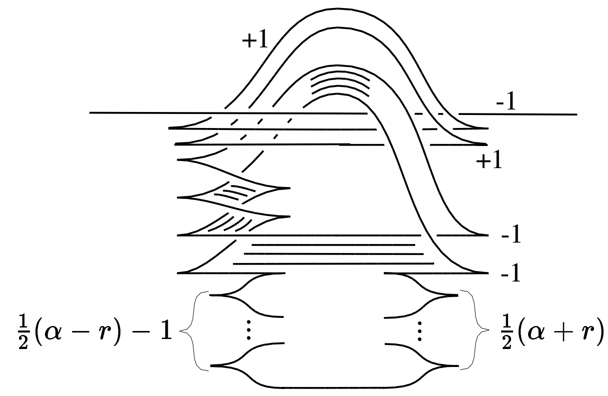

Figure 5: The contact structures ξ_r^-

where $d_3(\Xi(\mathbf{t}_{\xi_r^\pm}))$ is the 3–dimensional invariant of $\Xi(\mathbf{t}_{\xi_r^\pm})$ and $\omega_{red}(\mathbf{t}_{\xi_r^\pm})$ is given, in the notations of [15], by the formula:

$$\frac{2g-1}{2} - \frac{l - sign(l)}{4} + l\rho^\pm(1-\rho^\pm) - \rho^\pm + \frac{1-\alpha}{2\alpha}(1-2\rho^\pm) + S(1,\alpha) + F_{\rho^\pm}(\alpha,1,\gamma) + 2S_{\rho^\pm}(1,\alpha,\gamma).$$

In our situation we have:

$$l = n + \frac{1}{\alpha}, \quad \text{sign}(l) = 1, \quad \rho^\pm = \frac{\alpha(n \mp (n-2g)) - r + 1}{2n\alpha + 2},$$

$$\gamma = \frac{1}{2}(r + \alpha - 2), \quad S(1,\alpha) = \frac{\alpha^2+2}{12\alpha} - \frac{1}{4}, \quad F_{\rho^\pm}(\alpha,1,\gamma) = \frac{\gamma + \rho^\pm}{\alpha},$$

$$S_{\rho^\pm}(1,\alpha,\gamma) = \frac{\alpha^2 - 3\alpha(1+2\gamma) + 2(1+3\gamma+3\gamma^2)}{12\alpha}.$$

This shows that

$$\omega_{red}(\mathbf{t}_{\xi_r^\pm}) = -\frac{1}{4(n\alpha+1)}((n-2g)^2\alpha - r^2 n \pm 2(n-2g)r) + \frac{2g-1}{2}.$$

On the other hand, by the argument of [11, Theorem 2.1] we have

$$d_1 = -1 - b_1(M) = -1 - 2g,$$

therefore

$$d_3(\Xi(\mathbf{t}_{\xi_r^\pm})) = -\omega_{red}(\mathbf{t}_{\xi_r^\pm}) - 2,$$

yielding

$$d_3(\Xi(\mathbf{t}_{\xi_r^\pm})) = \frac{1}{4(n\alpha+1)}((n-2g)^2\alpha - r^2 n \pm 2(n-2g)r) - \frac{2g+3}{2}.$$

Since

$$d_3([\xi_r^{\pm}]) - d_3(\Xi(\mathbf{t}_{\xi_r^{\pm}})) = 2g + 1 \neq 0,$$

none of the contact structures defined by Figure 1 on $M(g,n;(\alpha,1))$ $(n \geq 2g > 0)$ are symplectically fillable.

We believe that the same idea should work for all the tight contact structures given by Figure 1 (with the constraints (1.1)). The verification of non–fillability, however, seems to be much more tedious in the general case. The difficulty is number–theoretic in nature: it is hard to see that $d_3([\xi]) \neq d_3(\Xi(\mathbf{t}_\xi))$, because the formulas involve sums which are hard to write in closed form.

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