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A lower bound to the action dimension of a group

Sung Yil Yoon

Abstract The *action dimension* of a discrete group Γ, $actdim(\Gamma)$, is defined to be the smallest integer m such that Γ admits a properly discontinuous action on a contractible m–manifold. If no such m exists, we define $actdim(\Gamma) \equiv \infty$. Bestvina, Kapovich, and Kleiner used Van Kampen's theory of embedding obstruction to provide a lower bound to the action dimension of a group. In this article, another lower bound to the action dimension of a group is obtained by extending their work, and the action dimensions of the fundamental groups of certain manifolds are found by computing this new lower bound.

AMS Classification 20F65; 57M60

Keywords Fundamental group, contractible manifold, action dimension, embedding obstruction

1 Introduction

Van Kampen constructed an m–complex that cannot be embedded into $\mathbb{R}^{2m}$ [8]. A more modern approach to Van Kampen's theory of embedding obstruction uses co/homology theory. To see the main idea of this co/homology theoretic approach, let K be a simplicial complex and $|K|$ denote its geometric realization. Define the *deleted product*

$$|\tilde{K}| \equiv \{(x, y) \in |K| \times |K| \mid x \neq y\}$$

such that $\mathbb{Z}_2$ acts on $|\tilde{K}|$ by exchanging factors. Observe that there exists a two-fold covering $|\tilde{K}| \to |\tilde{K}|/\mathbb{Z}_2$ with the following classifying map:

$$\begin{array}{ccc} |\tilde{K}| & \xrightarrow{\tilde{\phi}} & S^\infty \\ \downarrow & & \downarrow \\ |\tilde{K}|/\mathbb{Z}_2 & \xrightarrow{\phi} & \mathbb{R}P^\infty \end{array}$$

Now let $\omega^m \in H^m(\mathbb{R}P^\infty; \mathbb{Z}_2)$ be the nonzero class. If $\phi^*(w^m) \neq 0$ then $|K|$ cannot be embedded into $\mathbb{R}^m$. That is, there is $\Sigma^m \in H_m(|K|/\mathbb{Z}_2; \mathbb{Z}_2)$ such that $\langle \phi^*(w^m), \Sigma \rangle \neq 0$.

A similar idea was used to obtain a lower bound to the *action dimension* of a discrete group Γ [2]. Specifically, the *obstructor dimension* of a discrete group Γ, $obdim(\Gamma)$, was defined by considering an m–*obstructor* K and a *proper*, Lipschitz, *expanding* map

$$f\colon\ cone(K)^{(0)} \to \Gamma.$$

And it was shown that

$$obdim(\Gamma) \leq actdim(\Gamma).$$

See [2] for details. An advantage of considering $obdim(\Gamma)$ becomes clear when Γ has well-defined boundary $\partial\Gamma$, for example, when Γ is $CAT(0)$ or torsion free hyperbolic. In these cases, if an m–obstructor K is contained in $\partial\Gamma$ then $m + 2 \leq obdim(\Gamma)$.

If Γ acts on a contractible m–manifold W properly discontinuously and cocompactly, then it is easy to see that $actdim(\Gamma) = m$. For example, let M be a Davis manifold. That is, M is a closed, aspherical, four-dimensional manifold whose universal cover $\tilde{M}$ is not homeomorphic to $\mathbb{R}^4$. We know that $actdim(\pi_1(M)) = 4$. However, it is not easy to see that $obdim(\pi_1(M)) = 4$. The goal of this article is to generalize the definitions of obstructor and obstructor dimension. To do so, we define *proper obstructor* (Definition 2.5) and *proper obstructor dimension* (Definition 5.2.) The main result is the following.

Main Theorem *The proper obstructor dimension of* $\Gamma \leq actdim(\Gamma)$.

As applications we will answer the following problems:

- Suppose W is a closed aspherical manifold and $\tilde{W}$ is its universal cover so that $\pi_1(W)$ acts on $\tilde{W}$ properly discontinuously and cocompactly. We show that $\tilde{W}$ in this case is indeed an m–proper obstructor and $pobdim(\pi_1(W)) = m$.
- Suppose W_i is a compact aspherical m_i–manifold with all boundary components aspherical and incompressible, $i = 1, ..., d$. (Recall that a boundary component N of a manifold W is called *incompressible* if $i_*\colon\ \pi_j(N) \to \pi_j(W)$ is injective for $j \geq 1$.) Also assume that for each i, $1 \leq i \leq d$, there is a component of ∂W_i, call it N_i, so that $|\pi_1(W_i) : \pi_1(N_i)| > 2$. Let $G = \pi_1(W_1) \times ... \times \pi_1(W_d)$. Then

$$actdim(G) = m_1 + ... + m_d.$$

The organization of this article is as follows. In Section 2, we define proper obstructor. The *coarse Alexander duality* theorem by Kapovich and Kleiner [5], is used to construct the first main example of proper obstructor in Section 3. Several examples of proper obstructors are constructed in Section 4. Finally, the main theorem is proved and the above problems are considered in Sections 5.

2 Proper obstructor

To work in the PL–category we define *simplicial deleted product*

$$\tilde{K} \equiv \{\sigma \times \tau \in K \times K \mid \sigma \cap \tau = \emptyset\}$$

such that $\mathbb{Z}_2$ acts on $\tilde{K}$ by exchanging factors. It is known that $|\tilde{K}|/\mathbb{Z}_2$ ($|\tilde{K}|$) is a deformation retract of $\tilde{K}/\mathbb{Z}_2$ ($\tilde{K}$), see [7, Lemma 2.1]. Therefore, WLOG, we can use $H_m(\tilde{K}/\mathbb{Z}_2;\mathbb{Z}_2)$ instead of $H_m(|\tilde{K}|/\mathbb{Z}_2;\mathbb{Z}_2)$.

Throughout the paper, all homology groups are taken with $\mathbb{Z}_2$–coefficients unless specified otherwise.

To define proper obstructor, we need to consider several definitions and preliminary facts.

Definition 2.1 A proper map $h\colon A \to B$ between proper metric spaces is *uniformly proper* if there is a proper function $\phi\colon [0,\infty) \to [0,\infty)$ such that

$$d_B(h(x), h(y)) \geq \phi(d_A(x,y))$$

for all $x, y \in A$. (Recall that a metric space is said to be *proper* if any closed metric ball is compact, and a map is said to be *proper* if the preimages of compact sets are compact.)

Let W be a contractible m–manifold and define

$$W_0 \equiv \{(x,y) \in W \times W \mid x \neq y\}.$$

Consider a uniformly proper map $\beta\colon Y \to W$ where Y is a simplicial complex and W is a contractible manifold. Since β is uniformly proper, we can choose $r > 0$ such that $\beta(a) \neq \beta(b)$ if $d_Y(a,b) > r$. Note that β induces an equivariant map:

$$\bar{\beta}\colon \{(y,y') \in Y \times Y \mid d_Y(y,y') > r\} \to W_0$$

As we work in the PL–category we make the following definition.

Definition 2.2 If $K \subset Y$ is a subcomplex and r is a positive integer then we define the *combinatorial r-tubular neighborhood of K*, denoted by $N_r(K)$, to be r-fold iterated closed star neighborhood of K.

Recall that when Y is a simplicial complex, $|Y| \times |Y|$ can be triangulated so that each cell $\sigma \times \tau$ is a subcomplex. Let $d\colon Y \to Y^2$ be the diagonal map, $d(\sigma) = (\sigma, \sigma)$, where Y^2 is triangulated so that $d(Y)$ is a subcomplex. Define

$$Y_r \equiv Cls(Y^2 - N_r(d(Y))).$$

Note that a uniformly proper map $\beta\colon Y \to W^m$ induces an equivariant map $\bar{\beta}\colon Y_r \to W_0 \simeq S^{m-1}$ for some $r > 0$.

Definition 2.3 *(Essential $\mathbb{Z}_2$–m–cycle)* An *essential $\mathbb{Z}_2$–m–cycle* is a pair $(\tilde{\Sigma}^m, a)$ satisfying the following conditions:

(i) $\tilde{\Sigma}^m$ is a finite simplicial complex such that $|\tilde{\Sigma}^m|$ is a union of m–simplices and every $(m-1)$–simplex is the face of an even number of m–simplices.

(ii) $a\colon \tilde{\Sigma}^m \to \tilde{\Sigma}^m$ is a free involution.

(iii) There is an equivariant map $\varphi\colon \tilde{\Sigma}^m \to S^m$ with $deg(\varphi) = 1(mod\ 2)$.

Some remarks are in order.

(1) We recall how to find $deg(\varphi)$. Choose a simplex s of S^m and let f be a simplicial approximation to φ. Then $deg(\varphi)$ is the number of m–simplices of $\tilde{\Sigma}^m$ mapped into s by f.

(2) Let $\tilde{\sigma}$ be the sum of all m–simplices of $\tilde{\Sigma}^m$. Condition (i) of Definition 2.3 implies that $[\tilde{\sigma}] \in H_m(\tilde{\Sigma}^m)$. We call $[\tilde{\sigma}]$ the fundamental class of $\tilde{\Sigma}^m$.

(3) Let $\tilde{\Sigma}^m/\mathbb{Z}_2 \equiv \Sigma^m$ and consider a two-fold covering $q\colon \tilde{\Sigma}^m \to \Sigma^m$. As φ is equivariant it induces $\bar{\varphi}\colon \Sigma^m \to \mathbb{R}P^m$. Let $deg_2(\varphi)$ denote $deg(\varphi)(mod\ 2)$.
Note that $deg_2(\varphi) = \langle \bar{\varphi}^*(w^m), q\tilde{\sigma} \rangle$ where $w^m \in H^m(\mathbb{R}P^m; \mathbb{Z}_2)$ is the nonzero element. If $\varphi\colon \tilde{\Sigma}^m \to S^m$ is an equivariant map then $deg_2(\varphi) = 1$. To see this, we prove the following proposition.

Proposition 2.4 *Suppose a map $\varphi\colon \tilde{\Sigma}^m \to S^m$ is equivariant. Then*

$$deg_2(\varphi) = 1.$$

Proof Consider the classifying map and the commutative diagram for a two-fold covering $q\colon \tilde{\Sigma}^m \to \Sigma^m$:

$$\begin{array}{ccc} \tilde{\Sigma}^m & \xrightarrow{\phi} & S^\infty \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ \Sigma^m & \xrightarrow{\bar{\phi}} & \mathbb{R}P^\infty \end{array}$$

We also have:

$$\begin{array}{ccccc} \tilde{\Sigma}^m & \xrightarrow{\varphi} & S^m & \xrightarrow{i} & S^\infty \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \Sigma^m & \xrightarrow{\bar{\varphi}} & \mathbb{R}P^m & \xrightarrow{i} & \mathbb{R}P^\infty \end{array}$$

Because $S^\infty \to \mathbb{R}P^\infty$ is the classifying covering, $i \circ \varphi \simeq \phi$ and $i \circ \bar{\varphi} \simeq \bar{\phi}$. Observe that

$$deg_2(\varphi) = \langle \bar{\varphi}^* w^m,\ q\tilde{\sigma} \rangle = \langle (i \circ \bar{\varphi})^* w^m_\infty,\ q\tilde{\sigma} \rangle$$

where $0 \neq w^m_\infty \in H^m(\mathbb{R}P^\infty)$. But, since $i \circ \bar{\varphi} \simeq \bar{\phi}$,

$$\langle (i \circ \bar{\varphi})^* w^m_\infty,\ q\tilde{\sigma} \rangle = \langle \bar{\phi}^* w^m_\infty,\ q\tilde{\sigma} \rangle = deg_2(\phi). \qquad \square$$

Now we modify the definition of obstructor.

Definition 2.5 *(Proper obstructor)* Let T be a contractible[1] simplicial complex. Recall that $T_r \equiv Cls(T^2 - N_r(d(T)))$ where $N_r(d(T))$ denotes the r–tubular neighborhood of the image of the diagonal map $d\colon T \to T^2$. Let m be the largest integer such that for any $r > 0$, there exists an essential $\mathbb{Z}_2$–m–cycle $(\tilde{\Sigma}^m, a)$ and a $\mathbb{Z}_2$–equivariant map $f\colon \tilde{\Sigma}^m \to T_r$. If such m exists then T is called an *m–proper obstructor.*

The first example of proper obstructor is given by the following proposition.

Proposition 2.6 *Suppose that M is a k–dimensional closed aspherical manifold where $k > 1$ and X is the universal cover of M. Suppose also that X has a triangulation so that X is a metric simplicial complex and a group $G = \pi_1(M)$ acts on X properly discontinuously, cocompactly, simplicially, and freely by isometries. Then X^k is a $(k-1)$–proper obstructor.*

We prove Proposition 2.6 in Section 3. The key ideas are the following:

[1] Contractibility is necessary for Proposition 5.2

(1) Since G acts on X properly discontinuously, cocompactly, simplicially, and freely by isometries, X is *uniformly contractible*. Recall that a metric space Y is *uniformly contractible* if for any $r > 0$, there exists $R > r$ such that $B_r(y)$ is contractible in $B_R(y)$ for any $y \in Y$.

(2) For any $R > 0$, there exists $R' > R$ so that the inclusion induced map

$$i_*\colon \tilde{H}_j(X_{R'}) \to \tilde{H}_j(X_R)$$

is trivial for $j \neq k-1$ and $\mathbb{Z}_2 \cong i_*(\tilde{H}_{k-1}(X_{R'})) \leq \tilde{H}_{k-1}(X_R)$. (See Lemma 3.6.)

(3) We recall the definition of Δ*–complex* and use it to complete the proof as sketched below.

Definition 2.7 A Δ–complex is a quotient space of a collection of disjoint simplices of various dimensions, obtained by identifying some of their faces by the canonical linear homeomorphisms that preserve the ordering of vertices.

Suppose $(\tilde{\Sigma}^m, a)$ is an essential $\mathbb{Z}_2 - m$–cycle with a $\mathbb{Z}_2$–equivariant map $f\colon (\tilde{\Sigma}^m, a) \to T_r$. Let

$$|\tilde{\Sigma}^m| = \cup_{i=1}^n \Delta_i^m$$

(union of n–copies of m–simplices, use subscripts to denote different copies of m–simplices) and

$$f_i \equiv f|_{\Delta_i^m}$$

Then condition (i) of Definition 2.3 implies that $\Sigma_{i=1}^n f_i$ is an m–cycle of T_r (over $\mathbb{Z}_2$). That is, an essential $\mathbb{Z}_2$–m–cycle $(\tilde{\Sigma}^m, a)$ with a $\mathbb{Z}_2$–equivariant map $f\colon (\tilde{\Sigma}^m, a) \to T_r$ can be considered as an m–cycle of T_r (over $\mathbb{Z}_2$). Next suppose that $g = \sum_{i=1}^n g_i$ is an m–chain of T_r (over $\mathbb{Z}_2$) where $g_i\colon \Delta^m \to T_r$ are singular m–simplices. Take an m–simplex for each i and index them as Δ_i^m. Let Δ_i^{m-1} denote a codimension 1 face of Δ_i^m. Construct a Δ–complex Π as follows:

- $|\Pi| = \cup_{i=1}^n \Delta_i^m$
- For each $\ell \neq j$ we identify Δ_ℓ^m with Δ_j^m along Δ_ℓ^{m-1} and Δ_j^{m-1} whenever $g_l|_{\Delta_\ell^{m-1}} = g_j|_{\Delta_j^{m-1}}$.

Subdivide Π if necessary so that Π becomes a simplicial complex. Consider when g is an m–cycle and an m–boundary.

First, suppose g is an m–cycle. Then for any codimension 1 face Δ_i^{m-1} of Δ_i^m there are an even number of j's(including i itself) between 1 and n such that $g|_{\Delta_i^{m-1}} = g|_{\Delta_j^{m-1}}$ So Π satisfies condition (i) of Definition 2.3 and we can consider g as a map

$$g\colon\ \Pi \to T_r$$

by setting $g|_{\Delta_i^m} = g_i$.

Second, suppose g is an m–boundary. Then there is an $(m+1)$–chain $G \equiv \sum_{i=1}^{N} G_i$ where $G_i\colon\ \Delta^{m+1} \to T_r$ are singular $(m+1)$–simplices such that $\partial G = g$. As before one can construct a simplicial complex Ω and consider G as a map

$$G\colon\ \Omega \to T_r$$

Let $\partial\Omega \equiv \cup\{m-$simplices of Ω which are the faces of an odd number of $(m{+}1)-$ simplices$\}$. Note that $\partial\Omega \overset{comb.}{\cong} \Pi$ where $\overset{comb.}{\cong}$ denotes combinatorial equivalence. This observation will be used to construct an essential cycle in the proof of Proposition 2.6.

3 Coarse Alexander duality

We first review the terminology of [5]. Some terminology already defined is modified in the PL category. Let X be (the geometric realization of) a locally finite simplicial complex. We equip the 1-skeleton $X^{(1)}$ with path metric by defining each edge to have unit length. We call such an X with the metric on $X^{(1)}$ a metric simplicial complex. We say that X has *bounded geometry* if all links have a uniformly bounded number of simplices. Recall that $X_r \equiv Cls(X^2 - N_r(d(X)))$, see Definition 2.2. Also denote:

$$\begin{cases} B_r(c) \equiv \{x \in X | d(c,x) \le r\} \\ \partial B_r(c) \equiv \{x \in X | d(c,x) = r\} \end{cases}$$

If $C_*(X)$ is the simplicial chain complex and $A \subset C_*(X)$ then the *support of* A, denoted by $Support(A)$, is the smallest subcomplex of $K \subset X$ such that $A \subset C_*(K)$. We say that a homomorphism

$$h\colon\ C_*(X) \to C_*(X)$$

is *coarse Lipschitz* if for each simplex $\sigma \subset X$, $Support(h(C_*(\sigma)))$ has uniformly bounded diameter. We call a coarse Lipschitz map with

$$D \equiv max_\sigma diam(\ Support(h(C_*(\sigma)))\ )$$

D-Lipschitz. We call a homomorphism h *uniformly proper*, if it is coarse Lipschitz and there exists a proper function $\phi\colon\ \mathbb{R}_+ \to \mathbb{R}$ so that for each subcomplex $K \subset X$ of diameter $\ge r$, $Support(h(C_*(\sigma)))$ has diameter $\ge \phi(r)$.

We say that a homomorphism h has *displacement* $\leq D$ if for every simplex $\sigma \subset X$, *Support*($h(C_*(\sigma))$) $\subset N_D(\sigma)$. A metric simplicial complex is *uniformly acyclic* if for every R_1 there is an R_2 such that for each subcomplex $K \subset X$ of diameter $\leq R_1$ the inclusion $K \to N_{R_2}(K)$ induces zero on reduced homology groups.

Definition 3.1 *(PD group)* A group Γ is called an n-dimensional *Poincaré duality group* ($PD(n)$ group in short) if the following conditions are satisfied:

(i) Γ is of type FP and $n = dim(\Gamma)$.

(ii) $H^j(\Gamma;\mathbb{Z}\Gamma) = \begin{cases} 0 & j \neq n \\ \mathbb{Z} & j = n \end{cases}$

Example 3.2 The fundamental group of a closed aspherical k–manifold is a $PD(k)$ group. See [3] for details.

Definition 3.3 *(Coarse Poincaré duality space [5])* A *Coarse Poincaré duality space of formal dimension k*, $PD(k)$ space in short, is a bounded geometry metric simplicial complex X so that $C_*(X)$ is uniformly acyclic, and there is a constant D_0 and chain mappings

$$C_*(X) \xrightarrow{\bar{P}} C_c^{k-*}(X) \xrightarrow{P} C_*(X)$$

so that

(i) P and $\bar{P}$ have displacement $\leq D_0$,

(ii) $P \circ \bar{P}$ and $\bar{P} \circ P$ are chain homotopic to the identity by D_0-Lipschitz chain homotopies $\Phi\colon C_*(X) \to C_{*+1}(X)$, $\bar{\Phi}\colon C_c^*(X) \to C_c^{*-1}(X)$. We call coarse Poincare duality spaces of formal dimension k a *coarse PD(k) spaces.*

Example 3.4 An acyclic metric simplicial complex that admits a free, simplicial cocompact action by a $PD(k)$ group is a coarse $PD(k)$ space.

For the rest of the paper, let X denote the universal cover of a k-dimensional closed aspherical manifold where $k > 1$.

Assume also that X has a triangulation so that X is a metric simplicial complex with bounded geometry, and $G = \pi_1(M)$ acts on X properly discontinuously, cocompactly, simplicially, and freely by isometries. In particular, G is a $PD(k)$ group and X is a coarse $PD(k)$ space. The following theorem was proved in [5]. Pro-Category theory is reviewed in Appendix A.

Theorem 3.5 (Coarse Alexander duality [5]) *Suppose Y is a coarse $PD(n)$ space, Y' is a bounded geometry, uniformly acyclic metric simplicial complex, and $f\colon C_*(Y') \to C_*(Y)$ is a uniformly proper chain map. Let $K \equiv Support(f(C_*(Y')))$, $Y_R \equiv Cls(Y - N_R(K))$. Then we can choose $0 < r_1 < r_2 < r_3 < \dots$ and define the inverse system $pro\tilde{H}_j(Y_r) \equiv \{\tilde{H}_j(Y_{r_i}), i_*, \mathbb{N}\}$ so that*

$$pro\tilde{H}_{n-j-1}(Y_r) \cong H^j_c(Y').$$

We rephrase the coarse Alexander duality theorem.

Lemma 3.6 Recall that X is a metric simplicial complex with bounded geometry and a group G acts on X properly discontinuously, cocompactly, simplicially, and freely by isometries. Also recall that $X_r \equiv Cls(X^2 - N_r(d(X)))$. One can choose $0 < r_1 < r_2 < r_3 < \dots$ and define the inverse system $pro\tilde{H}_j(X_r)$ $\equiv \{\tilde{H}_j(X_{r_i}), i_*, \mathbb{N}\}$ so that:

$$pro\tilde{H}_j(X_r) = \begin{cases} \mathbf{0}, & j \neq k-1 \\ \mathbb{Z}_2, & j = k-1 \end{cases}$$

Proof Consider the diagonal map

$$d\colon X \to X^2,\ x \mapsto (x, x)$$

and note that d is uniformly proper and X^2 is a $PD(2k)$ space. Theorem 3.5 implies that

$$pro\tilde{H}_{2k-*-1}(X_r) = H^*_c(X).$$

Finally observe that $H^*_c(X) \cong H_{k-*}(\mathbb{R}^k) \cong H^*_c(\mathbb{R}^k)$. □

Now we prove Proposition 2.6.

Proof of Proposition 2.6 Let $r > 0$ be given. First use Lemma 3.6 to choose $r = r_1 < r_2 < \dots < r_{k-1} < r_k$ so that

$$i_*\colon \tilde{H}_j(X_{r_{m+1}}) \to \tilde{H}_j(X_{r_m})$$

is trivial for $j \neq k-1$. In particular, $i\colon X_{r_k} \to X_{r_{k-1}}$ is trivial in π_0. Let $S^0 \equiv \{e, w\}$ and define an involution a_0 on S^0 by $a_0(e) = w$ and $a_0(w) = e$. Let $\theta\colon (S^0, a_0) \to (X_{r_k}, s)$ be an equivariant map where s is the obvious involution on X_{r_i}. Now let

$$\sigma\colon I \to X_{r_{k-1}}$$

be a path so that $\sigma(0) = \theta(e)$ and $\sigma(1) = \theta(w)$. Define

$$\sigma'\colon I \to X_{r_{k-1}}$$

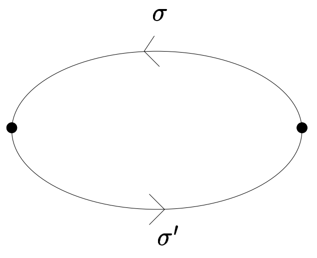

Figure 1: σ_1

by $\sigma'(t) = s\sigma(t)$. Observe that $\sigma_1 \equiv \sigma + \sigma'$ is an 1–cycle in $X_{r_{k-1}}$. See Figure 1.

Let a_1 be the obvious involution on S^1 and consider σ_1 as an equivariant map

$$\sigma_1 \colon (S^1, a_1) \to (X_{r_{k-1}}, s).$$

Since $i_* \colon \tilde{H}_1(X_{r_{k-1}}) \to \tilde{H}_1(X_{r_{k-2}})$ is trivial, σ_1 is the boundary of a 2–chain in $X_{r_{k-2}}$. Call this 2 –chain $\sigma_2^+ = \sum_{i=1}^m g_i$ where g_i are singular 2–simplices. Following Remark (3) after Proposition 2.6, construct a simplicial complex $\tilde{\Sigma}^2_+$ such that

$$\sigma_2^+ \colon \tilde{\Sigma}^2_+ \to X_{r_{k-2}} \quad \text{and} \quad \partial\sigma_2^+ = \sigma_1.$$

See Figure 2. Define the *boundary* of $\tilde{\Sigma}^2_+$, $\partial\tilde{\Sigma}^2_+$, to be the union of 1–simplices, which are the faces of an odd number of 2–simplices. Recall also from Remark (3) that $\partial\tilde{\Sigma}^2_+ \overset{comb.}{\cong} S^1$ where $\overset{comb.}{\cong}$ denotes combinatorial equivalence.

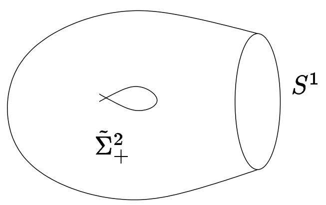

Figure 2: $\tilde{\Sigma}^2_+$

Next, let $\sigma_2^- = s\sigma_2^+ = \sum_{i=1}^m sg_i$. Take a copy of $\tilde{\Sigma}^2_+$, denoted by $\tilde{\Sigma}^2_-$, such that

$$\sigma_2^- \colon \tilde{\Sigma}^2_- \to X_{r_{k-2}} \quad \text{and} \quad \partial\sigma_2^- = \sigma_1.$$

Construct $\tilde{\Sigma}^2$ by attaching $\tilde{\Sigma}^2_+$ and $\tilde{\Sigma}^2_-$ along $S^1 = \partial\tilde{\Sigma}^2_+ = \partial\tilde{\Sigma}^2_-$ by identifying $x \sim a_1(x)$. That is, $\tilde{\Sigma}^2 \equiv \tilde{\Sigma}^2_+ \cup_{S^1} \tilde{\Sigma}^2_-$. See Figure 3. Define an involution a_2

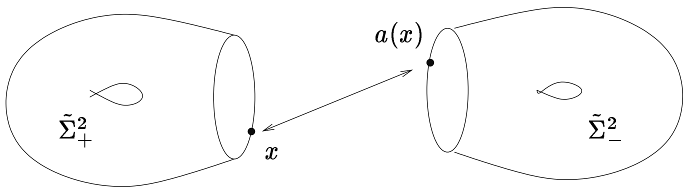

Figure 3: Constructing $\tilde\Sigma^2$

on $\tilde\Sigma^2$ by setting

$$a_2(x) = \begin{cases} x \in \tilde\Sigma^2_+ & \text{if } x \in \tilde\Sigma^2_- - S^1 \\ x \in \tilde\Sigma^2_- & \text{if } x \in \tilde\Sigma^2_+ - S^1 \\ a_1(x) & \text{if } x \in S^1 \end{cases}$$

Observe that $\sigma_2 \equiv \sigma_2^+ + \sigma_2^-$ is a 2–cycle in X_{r_2} and we can consider σ_2 as an equivariant map

$$\sigma_2 \colon (\tilde\Sigma^2, a_2) \to (X_{r_{k-2}}, s).$$

Continue inductively and construct a $(k-1)$–cycle

$$\sigma_{k-1} \colon (\tilde\Sigma^{k-1},\ a_{k-1}) \to (X_{r_1} = X_{r_{k-(k-1)}},\ s)'$$

Simply write a instead of a_{k-1}, and note that $X_{r_1} \subset X_r$. So $(\tilde\Sigma^{k-1},\ a)$ satisfies conditions (i)–(ii) of Definition 2.3 and we only need to show that it satisfies condition (iii).

It was proved in [2] that there exists a $\mathbb{Z}_2$–equivariant homotopy equivalence $\tilde h \colon X_0 \to S^{k-1}$. So $\tilde h$ induces a homotopy equivalence

$$h \colon X_0/\sim \to \mathbb{R}P^{k-1}.$$

Let $g \equiv hi\sigma_{k-1} \colon \tilde\Sigma^{k-1} \xrightarrow{\sigma_{k-1}} X_r \xrightarrow{i} X_0 \xrightarrow{h} S^{k-1}$. Note that g is equivariant. We shall prove that $deg(g) = 1 (mod\ 2)$ by constructing another map

$$f_{k-1} \colon \tilde\Sigma^{k-1} \to S^{k-1}$$

with odd degree and applying Proposition 2.4.

Observe that

$$S^1 \subset \tilde\Sigma^2 \subset \tilde\Sigma^3 \subset \ldots \subset \tilde\Sigma^{k-2} \subset \tilde\Sigma^{k-1}$$

and for each i, $2 \le i \le k-1$:

$$\tilde\Sigma^i = \tilde\Sigma^i_+ \cup_{\tilde\Sigma^{i-1}} \tilde\Sigma^i_-$$

Now construct a map $f_{k-1}\colon \tilde{\Sigma}^{k-1} \to S^{k-1}$ as follows: First let $f_1\colon S^1 \to S^1$ be the identity and extend f_1 to $f_2^+\colon \tilde{\Sigma}^2_+ \to B^2$ by Tietze Extension theorem. Without loss of generality assume that $(f_2^+)^{-1}(S^1) \subset S^1 \overset{comb.}{\cong} \partial\tilde{\Sigma}^2_+$. Then extend equivariantly to $f_2\colon \tilde{\Sigma}^2 \to S^2$. Note that $f_2^{-1}(B^2_+) \subset \tilde{\Sigma}^2_+$, $f_2^{-1}(B^2_-) \subset \tilde{\Sigma}^2_-$, and $f_2^{-1}(S^1) \subset S^1$.

Continue inductively and construct an equivariant map

$$f_{k-1}\colon \tilde{\Sigma}^{k-1} \to S^{k-1}.$$

By construction, we know that

$$f_j^{-1}(B^j_+) \subset \tilde{\Sigma}^j_+,\ f_j^{-1}(B^j_-) \subset \tilde{\Sigma}^j_-,\ \text{ and }\ f_j^{-1}(S^{j-1}) \subset \tilde{\Sigma}^{j-1},\ 2 \le j \le k-1.$$

Observe that $deg(f_{k-1}) = deg(f_{k-2}) = \ldots = deg(f_2) = deg(f_1)$. (Recall that $deg(f_m) \equiv$ the number of m–simplices of $\tilde{\Sigma}^m$ mapped into a simplex s of S^m by f.) But $deg(f_1) = id_{S^1} = 1(mod\ 2)$ so $f_{k-1}\colon \tilde{\Sigma}^{k-1} \to S^{k-1}$ has nonzero degree. Now Proposition 2.4 implies that $deg(g) = 1(mod\ 2)$. Therefore $(\tilde{\Sigma}^{k-1}, a)$ with equivariant map

$$\sigma_{k-1}\colon \tilde{\Sigma}^{k-1} \to X_r.$$

satisfies conditions (i),(ii), and (iii) of Definition 2.3. Now the proof of Proposition 2.6 is complete.

4 New proper obstructors out of old

In this Section, we construct a k–proper obstructor from a $(k-1)$–proper obstructor X.

Definition 4.1 Let (Y, d_Y) be a proper metric space and (α, d_α) be a metric space isometric to $[0,\infty)$. Let $\phi\colon [0,\infty) \to \alpha$ be an isometry and denote $\phi(t)$ by α_t. Define a metric space $(Y \vee \alpha, d)$, called Y *union a ray*, as follows:

(i) As a set $Y \vee \alpha$ is the wedge sum. That is, $Y \vee \alpha = Y \cup \alpha$ with $Y \cap \alpha = \{\alpha_0\}$

(ii) The metric d of $Y \vee \alpha$ is defined by

$$\begin{cases} d(v,w) = d_Y(v,w), & \text{if } v,w \in Y \\ d(v,w) = d_\alpha(v,w), & \text{if } v,w \in \alpha \\ d(v,w) = d_Y(v,\alpha_0) + d_\alpha(\alpha_0,w), & \text{if } v \in Y,\ w \in \alpha \end{cases}$$

Proposition 4.2 *Let X be a k-dimensional contractible manifold without boundary and $k > 1$. Suppose also that X has a triangulation so that X is a metric simplicial complex and a group G acts on X properly discontinuously, cocompactly, simplicially, and freely by isometries. In particular, X is a $(k-1)$–proper obstructor. Then $X \vee \alpha$ is a k–proper obstructor.*

Proof Recall that by Lemma 3.6, we can choose $0 < r_1 < r_2 < r_3 \dots$ and define $pro\tilde{H}_{k-1}(X_r) \equiv \{\tilde{H}_{k-1}(X_{r_i}), i_*, \mathbb{N}\}$ so that $pro\tilde{H}_{k-1}(X_r) = \mathbb{Z}_2$. This means that for any $r > 0$ we can choose $R > r$ so that

$$r' \geq R \Rightarrow \mathbb{Z}_2 = i_*(H_{k-1}(X_{r'})) \leq H_{k-1}(X_r).$$

Now let $r > 0$ be given and choose $R > r$ as above. Let $(\tilde{\Sigma}^{k-1}, a)$ be an essential $\mathbb{Z}_2$–$(k-1)$–cycle with a $\mathbb{Z}_2$–equivariant map

$$f\colon \tilde{\Sigma}^{k-1} \to X_R.$$

Next consider composition $i \circ f\colon \tilde{\Sigma}^{k-1} \xrightarrow{f} X_R \xrightarrow{i} X_r$. If $i \circ f \in Z_{k-1}(X_r)$ is the boundary of a k–chain then we can construct an essential $\mathbb{Z}_2$–k–cycle with $\mathbb{Z}_2$–equivariant map into X_r using the method used in the proof of Proposition 2.6. But this implies X is a k–proper obstructor. (Recall that X^k is a $(k-1)$–proper obstructor.) So we can assume $i \circ f \in Z_{k-1}(X_r) - B_{k-1}(X_r)$. That is, $0 \neq [i \circ f] = i_*[f] \in H_{k-1}(X_r)$. Let $p_i\colon X_r \to X$ denote the projection to the i-th factor, $i = 1, 2$.

We need the following lemma.

Lemma 4.3 Define $j\colon X - B_R \to X_R$, $x \mapsto (\alpha_0, x)$. Then the composition

$$i_* \circ j_*\colon H_{k-1}(X - B_R(\alpha_0)) \xrightarrow{j_*} H_{k-1}(X_R) \xrightarrow{i_*} H_{k-1}(X_r)$$

is nontrivial.

The proof of Lemma 4.3 Consider a map $\lambda\colon H_{k-1}(X_0) \to \mathbb{Z}_2$ given by

$$[f] \mapsto Lk(f, \Delta)(mod\ 2)$$

where $Lk(f, \Delta)$ denote the linking number of f with the diagonal Δ.[2] Now consider the composition:

$$\zeta\colon H_{k-1}(X - B_R(\alpha_0)) \xrightarrow{j_*} H_{k-1}(X_R) \xrightarrow{i_*} H_{k-1}(X_0) \xrightarrow{\lambda} \mathbb{Z}_2$$

[2]We can compute $Lk(f, \Delta)$ by letting f bound a chain $\tilde{f}$ transverse to Δ and setting $Lk(f, \Delta) = Card(\tilde{f}^{-1}(\Delta))$.

We shall show that ζ is nontrivial. Choose $[f_1] \in H_{k-1}(X - B_R)$ so that $Lk(f_1, \alpha_0) \neq 0$ where $[\alpha_0] \in H_0(X)$. Then $Lk(i_*j_*([f_1]), \Delta) \neq 0$.(We can choose the same chain transverse to Δ.) Hence ζ is nontrivial. In particular, $i_* \circ j_*$ and j_* are nontrivial. □

Since $j_*\colon H_{k-1}(X - B_R) \to H_{k-1}(X_R)$ is nontrivial, we can choose $h \in Z_{k-1}(X - B_R) - B_{k-1}(X - B_R)$ with $g \equiv j \circ h \in Z_{k-1}(X_R) - B_{k-1}(X_R)$. That is, $0 \neq [g] \in H_{k-1}(X_R)$. We can consider g as a map $g\colon \Pi \to X_R$ where Π is a $(k-1)$–dimensional simplicial complex satisfying condition (i) of Definition 2.3 such that

- $0 \neq i_*[g] \in H_{k-1}(X_r)$
- $i \circ g\colon \Pi \xrightarrow{g} X_R \xrightarrow{i} X_r$ with $p_1(\ i \circ g(\Pi)\ ) = \{\alpha_0\} = X \cap \alpha(0)$.

Next define $g' = sg$, that is,

$$g'\colon \Pi \xrightarrow{g} X_R \xrightarrow{s} X_R.$$

Note that $i \circ g'$ is a cycle in X_R and $p_2(\ i \circ g'(\Pi)\ ) = \alpha_0$. Also $[f], [g] \in H_{k-1}(X_R)$ and $i_*[f], i_*[g] \in H_{k-1}(X_r)$ are nonzero. Observe that $i \circ f$ and $i \circ g$ must be homologous in X_r since $\mathbb{Z}_2 = i_*(H_{k-1}(X_R)) \leq H_{k-1}(X_r)$. We simply write f, g, and g' instead of $i \circ f$, $i \circ g$, and $i \circ g'$. There exists a k–chain $G \in C_k(X_r)$ such that

$$\partial G = f + g.$$

Again consider G as a map $G\colon \Omega \to X_r$ where Ω is a simplicial complex so that

$$\partial \Omega = \tilde{\Sigma}^{k-1} \sqcup \Pi.$$

See Figure 4.

Next define $G' = sG$, that is,

$$G'\colon \Omega \xrightarrow{G} X_r \xrightarrow{s} X_r.$$

Note that

$$\partial G' = f + g'.$$

Now take two copies of Ω and index them as Ω_1 and Ω_2. Similarly $\Pi_1 \subset \partial\Omega_1$ and $\Pi_2 \subset \partial\Omega_2$. Hence

$$\partial \Omega_i = \tilde{\Sigma}^{k-1} \cup \Pi_i\ , i = 1, 2.$$

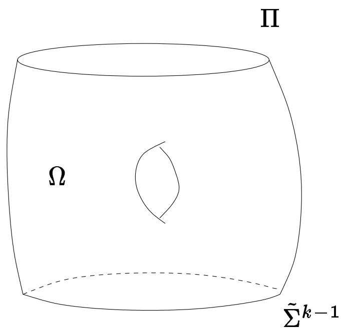

Figure 4: Ω

Denote $id(x) = x'$ for $x \in \Omega_1 - \tilde{\Sigma}^k$ where $id\colon \Omega_1 \to \Omega_2$. Construct a k–dimensional simplicial complex $\tilde{\Omega}$ by attaching Ω_1 and Ω_2 along $\tilde{\Sigma}^{k-1}$ by $a\colon \tilde{\Sigma}^{k-1} \to \tilde{\Sigma}^{k-1}$. That is,

$$\tilde{\Omega} = (\Omega_1 \cup \Omega_2)/x \sim ax,\ x \in \tilde{\Sigma}^{k-1}.$$

See Figure 5.

We can define an involution $\bar{a}$ on $\tilde{\Omega}$ by

$$\begin{cases} \bar{a}(x) = a(x), & x \in \tilde{\Sigma}^{k-1}, \\ \bar{a}(x) = x', & x \in \Omega_1 - \tilde{\Sigma}^{k-1}, \\ \bar{a}(x') = x, & x' \in \Omega_2 - \tilde{\Sigma}^{k-1}. \end{cases}$$

Also we can define a $\mathbb{Z}_2$–equivariant map $\Phi\colon \tilde{\Omega} \to X_r$ by:

$$\begin{cases} \Phi|_{\Omega_1} = G \\ \Phi|_{\Omega_2} = G' \end{cases}$$

We define

$$\tilde{\Sigma}^k = (\ \Pi_1 \times [0,1]/(\Pi_1, 1) \sim *\ ) \cup_{\Pi_1} \tilde{\Omega} \cup_{\Pi_2} (\Pi_2 \times [0,-1]/(\Pi_2,-1) \sim *\ ).$$

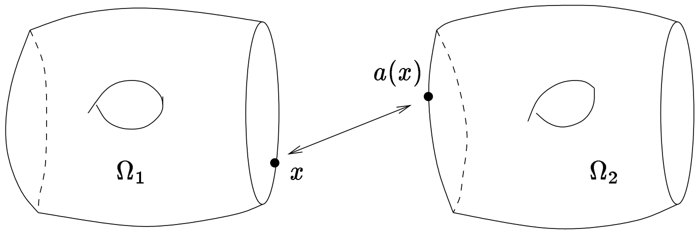

Figure 5: Constructing $\tilde{\Omega}$

See Figure 6. Now extend $\bar{a}$ over $\tilde{\Sigma}^k$, and denote $\tilde{\Sigma}^k/x \sim a(x)$ by Σ^k.

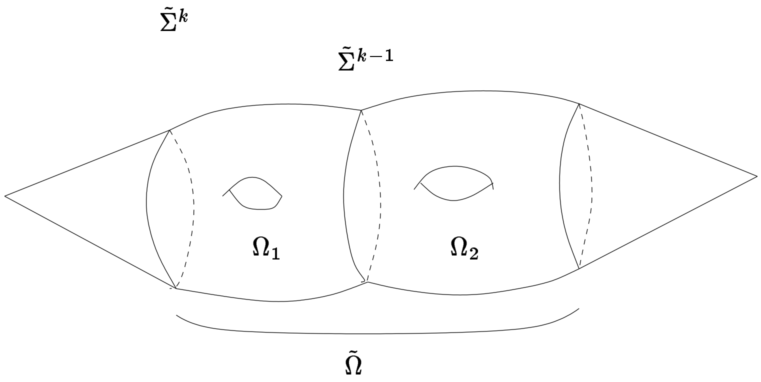

Figure 6: Constructing $\tilde{\Sigma}^k$

Suppose that Σ^k classifies into $\mathbb{R}P^m$ where $m < k$. Let

$$h\colon\ \Sigma^k \to \mathbb{R}P^m$$

be the classifying map and

$$\tilde{h}\colon\ \tilde{\Sigma}^k \to S^m$$

be the equivariant map covering h. Observe that

$$deg\ \tilde{h}|_{\tilde{\Sigma}^{k-1}} = deg\ \tilde{h} = 0\ (mod\ 2)$$

This is a contradiction since there already exists a $\mathbb{Z}_2$–equivariant map

$$\varphi\colon\ \tilde{\Sigma}^{k-1} \to S^{k-1}$$

of odd degree. Hence $(\tilde{\Sigma}^k, \bar{a})$ is an essential $\mathbb{Z}_2$–k–cycle.

Finally, we need to define a $\mathbb{Z}_2$–equivariant map:

$$F\colon\ \tilde{\Sigma}^k \to (X \vee \alpha)_r$$

Recall that $p_1 g(\Pi) = \alpha_0$ and let

$$c\colon\ p_2 g(\Pi_1) \times I \to X$$

be a contraction to α_0. Similarly $p_2 g'(\Pi) = \alpha_0$ and let

$$c'\colon\ p_1 g'(\Pi_1) \times I \to X$$

be a contraction to α_0. Define a $\mathbb{Z}_2$–equivariant map

$$F\colon\ \tilde{\Sigma}^k \to (X \vee \alpha)_r$$

as follows: Recall that $\phi(t) = \alpha_t$ in Definition 4.1, so $d(\alpha_0, \alpha_s) = s$ and $d(\alpha_s, x) \geq s$ for any $x \in X$.

$$\begin{cases} F|_{\tilde{\Omega}} = \Phi \\ F(x,t) = (\ \alpha_{2rt},\ p_2 g(x)\ ), & x \in \Pi_1,\ t \in [0, \frac{1}{2}] \\ F(x,t) = (\ \alpha_r,\ c_{(2t-1)}(p_2 g(x))\ ), & x \in \Pi_1,\ t \in [\frac{1}{2}, 1] \\ F(x,t) = (\ p_1 g'(x),\ \alpha_{-2rt}\ ), & x \in \Pi_2,\ t \in [0, -\frac{1}{2}] \\ F(x,t) = (\ c_{(-2t-1)}(p_1 g'(x)),\ \alpha_r\ ), & x \in \Pi_2,\ t \in [-\frac{1}{2}, -1] \end{cases}$$

The proof of Proposition 4.2 is now complete. □

If Y and Z are metric spaces we use the *sup metric* on $Y \times Z$ where

$$d_{sup}((y_1, z_1), (y_2, z_2)) \equiv max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\}.$$

Proposition 4.4 *Suppose X_1, X_2 are m_1, m_2–proper obstructors, respectively. Then $X_1 \times X_2$ is an (m_1+m_2+1)–proper obstructor.*

Proof Let $r > 0$ be given and let

$$\begin{cases} f_1\colon \tilde{\Sigma}_1^{m_1} \to (X_1)_r \\ f_2\colon \tilde{\Sigma}_2^{m_2} \to (X_2)_r \end{cases}$$

be $\mathbb{Z}_2$–equivariant maps for essential $\mathbb{Z}_2$–cycles. Note that

$$(X_1 \times X_2)_r = ((X_1)_r \times (X_2)^2) \cup_{(X_1)_r \times (X_2)_r} ((X_1)^2 \times (X_2)_r).$$

Let a_1 be the involution on $(X_1)_r$ and a_2 be the involution on $(X_2)_r$. Recall that the join $\tilde{\Sigma}_1^{m_1} * \tilde{\Sigma}_2^{m_2}$ is obtained from $\tilde{\Sigma}_1^{m_1} \times \tilde{\Sigma}_2^{m_2} \times [-1, 1]$ by identifying $\tilde{\Sigma}_1^{m_1} \times \{y\} \times \{1\}$ to a point for every $y \in \tilde{\Sigma}_2^{m_2}$ and identifying $\{x\} \times \tilde{\Sigma}_2^{m_2} \times \{-1\}$ to a point for every $x \in \tilde{\Sigma}_1^{m_1}$. Define an involution a on $\tilde{\Sigma}_1^{m_1} * \tilde{\Sigma}_2^{m_2}$ by

$$a(v, w, t) = (a_1(v), a_2(w), t).$$

Let

$$\begin{cases} g_1\colon \tilde{\Sigma}_1^{m_1} \to S^{m_1} \\ g_2\colon \tilde{\Sigma}_2^{m_2} \to S^{m_2} \end{cases}$$

be equivariant maps of odd degree. Then:

$$\begin{aligned} g_1 * g_2\colon \tilde{\Sigma}_1^{m_1} * \tilde{\Sigma}_2^{m_2} &\to S^{m_1} * S^{m_2} = S^{m_1+m_2+1} \\ (v, w, t) &\mapsto (g_1(v), g_2(w), t) \end{aligned}$$

is also an equivariant map of an odd degree. Hence $(\tilde{\Sigma}_1^{m_1} * \tilde{\Sigma}_2^{m_2}, a)$ is an essential $\mathbb{Z}_2$–(m_1+m_2+1)–cycle.

Now let

$$c\colon\ f_1(\tilde{\Sigma}_1^{m_1})\times[-1,1]\to X_1^2$$

be a $\mathbb{Z}_2$–equivariant contraction to a point such that $c_t = id$ for $t\in[-1,0]$. Similarly let

$$d\colon\ f_2(\tilde{\Sigma}_2^{m_2})\times[-1,1]\to X_2^2$$

be a $\mathbb{Z}_2$–equivariant contraction to a point such that $d_t = id$ for $t\in[0,1]$.

Finally define

$$f\colon\ \tilde{\Sigma}_1^{m_1} * \tilde{\Sigma}_2^{m_2}\to (X_1\times X_2)_r \text{ by } f(v,w,t) = (\,c_t(f_1(v)), d_t(f_2(w))\,).$$

We note that f is $\mathbb{Z}_2$–equivariant. □

5 Proper obstructor dimension

We review one more notion from [2].

Definition 5.1 The *uniformly proper dimension*, *updim*(G), of a discrete group G is the smallest integer m such that there is a contractible m–manifold W equipped with a proper metric d_W, and there is a $g\colon\ \Gamma\to W$ with the following properties:

- g is Lipschitz and uniformly proper.
- There is a function $\rho\colon\ (0,\infty)\to(0,\infty)$ such that any ball of radius r centered at a point of the image of h is contractible in the ball of radius $\rho(r)$ centered at the same point.

If no such n exists, we define *updim*$(G)=\infty$.

It was proved in [2] that

$$updim(G)\le actdim(G).$$

Now we generalize the obstructor dimension of a group.

Definition 5.2 The *proper obstructor dimension* of G, *pobdim*(G), is defined to be 0 for finite groups, 1 for 2–ended groups, and otherwise $m+1$ where m is the largest integer such that for some m–proper obstructor Y, there exists a uniformly proper map

$$\phi\colon\ Y\to T_G$$

where T_G is a proper metric space with a quasi-isometry $q\colon\ T_G\to G$.

Lemma 5.3 Let Y be an m–proper obstructor. If there is a uniformly proper map $\beta\colon Y \to W^d$ where W is a contractible d–manifold then $d > m$.

Proof Assume $d \le m$ $(d-1 \le m-1)$. Observe that if β is uniformly proper then β induces an equivariant map $\bar{\beta}\colon Y_r \to W_0$ for some large $r > 0$. Now let $f\colon \tilde{\Sigma}^m \to Y_r$ be an essential $\mathbb{Z}_2$–m–cycle where f is equivariant. Let $h\colon W_0 \to S^{d-1}$ be an equivariant homotopy equivalence. We have an equivariant map

$$g = ih\bar{\beta}f\colon \tilde{\Sigma}^m \xrightarrow{f} Y_r \xrightarrow{\bar{\beta}} W_0 \xrightarrow{h} S^{d-1} \xrightarrow{i} S^{m-1} \xrightarrow{i} S^m$$

where $i\colon S^{d-1} \to S^{m-1} \to S^m$ is the inclusion. Note that g is equivariant but $deg(g) = 0 (mod\ 2)$. This is a contradiction by Proposition 2.4. □

Suppose that G is finite so that $pobdim(G) = 0$ by definition.

Clearly, $actdim(G) = 0$ if G is finite. Hence $pobdim(G) = actdim(G) = 0$ in this case. Next suppose that G has two ends so that $pobdim(G) = 1$. Note that there exists $\mathbb{Z} \cong H \le G$ with $|G : H| < \infty$. And this implies that

$$actdim(G) = actdim(H) = actdim(\mathbb{Z}) = 1.$$

Therefore, $pobdim(G) = actdim(G) = 1$ when G has two ends. Now we prove the main theorem for the general case.

Main Theorem $pobdim(G) \le updim(G) \le actdim(G)$

Proof We only need to show the first inequality. Let $pobdim(G) = m + 1$ for some $m > 0$. That is, there exists an m–proper obstructor Y, a proper metric space T_G, a uniformly proper map $\psi\colon Y \to T_G$, and a quasi-isometry $q\colon T_G \to G$. Let $updim(G) \equiv d$ such that there exists a uniformly proper map $\phi\colon G \to W^d$ where W is a contractible d–manifold. But the composition

$$\phi \circ q \circ \psi\colon Y \to T_G \to G \to W^d$$

is uniformly proper. Therefore

$$m + 1 = pobdim(G) \le updim(G)$$

by the previous lemma. □

Before we consider some applications, we make the following observation about compact aspherical manifolds with incompressible boundary.

Lemma 5.4 Assume that W is a compact aspherical m–manifold with all boundary components incompressible. Let $\pi\colon \tilde{W} \to W$ denote the universal cover of W. Suppose that there is a component of ∂W, call it N, so that $|\pi_1(W)\colon\pi_1(N)| > 2$. Then $|\pi_1(W)\colon\pi_1(N)|$ is infinite.

Proof Observe that N is aspherical also. First, we show that if

$$1 < |\pi_1(W)\colon\pi_1(N)| < \infty$$

then $\tilde{M} \equiv \tilde{W}/\pi_1(N)$ has two boundary components and W has one boundary component. We claim that $\tilde{M}$ has a boundary component homeomorphic to N which is still denoted by N. To see this consider the long exact sequence:

$$\cdots \to H_1(\partial\tilde{M}) \xrightarrow{i_*} H_1(\tilde{M}) \to H_1(\tilde{M}, \partial\tilde{M}) \to \tilde{H}_0(\partial\tilde{M}) \to \tilde{H}_0(\tilde{M}) = 0$$

Since $\pi_1(N) = \pi_1(\tilde{M})$, $i_*\colon H_1(\partial\tilde{M}) \to H_1(\tilde{M})$ is surjective. So we have:

$$0 \to H_1(\tilde{M}, \partial\tilde{M}) \to \tilde{H}_0(\partial\tilde{M}) \to 0$$

Since $|\pi_1(W)\colon\pi_1(N)|$ is finite $\tilde{M}$ is compact. Now $H_1(\tilde{M}, \partial\tilde{M}) \cong H^{m-1}(\tilde{M})$ by duality. But $H^{m-1}(\tilde{M}) \cong H^{m-1}(N)$ and $H^{m-1}(N) \cong \mathbb{Z}_2$ since N is a closed $(m{-}1)$–manifold. That is, $\tilde{H}_0(\partial\tilde{M}) \cong \mathbb{Z}_2$ so $\tilde{M}$ has two boundary components. Next let N and N' denote two boundary components of $\partial\tilde{M}$ both of which are mapped to $N \subset W$ by $p\colon \tilde{M} \to W$. Hence ∂W has one component.

Now assume that $m \equiv |\pi_1(W)\colon\pi_1(N)| > 2$. Suppose m is finite. Note that $p|_N\colon N(\subset \tilde{M}) \to N(\subset W)$ has index 1, and $p|_{N'}\colon N'(\subset \tilde{M}) \to N(\subset W)$ has index $m-1$. This means that $|\pi_1(\tilde{M})\colon\pi_1(N')| = m{-}1$ since $\pi_1(\tilde{M}) = \pi_1(N)$. There are two alternative arguments:

- If $m > 2$ then $\tilde{M}$ is an aspherical manifold with two boundary components N and N' with $|\pi_1(\tilde{M})\colon\pi_1(N')| = m{-}1 > 1$. Consider $\tilde{W}/\pi_1(N')$. The same argument applied to $\tilde{W}/\pi_1(N')$ shows that $\tilde{M}$ has one boundary component, which is a contradiction. Therefore $|\pi_1(W)\colon\pi_1(N)|$ is infinite.
- Suppose $m > 2$. Choose a point $x \in N \subset \partial W$ and let $\tilde{x} \in N \subset \partial\tilde{M}$ so that $p(\tilde{x}) = x$. Next choose two loops α and β in W based at x so that $\{\pi_1(N), [\alpha]\pi_1(N), [\beta]\pi_1(N)\}$ are distinct cosets. (We are assuming $|\pi_1(W)\colon\pi_1(N)| > 2$.) Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the liftings of α and β respectively so that $\tilde{\alpha}(0) = \tilde{x} = \tilde{\beta}(0)$. Note that $\tilde{y}_1 \equiv \tilde{\alpha}(1), \tilde{y}_2 \equiv \tilde{\beta}(1) \in N'$ and $\tilde{y}_1 \neq \tilde{y}_2$ since $[\alpha]\pi_1(N) \neq [\beta]\pi_1(N)$. Now consider a path $\tilde{\gamma}$ in N' from $\tilde{y}_1$ to $\tilde{y}_2$. Observe that $p\tilde{\gamma} \equiv \gamma$ is a loop based at x, and $[\gamma] \in p_*(\pi_1(N')) \leq \pi_1(N)$. But $[\alpha][\gamma][\beta]^{-1} = 1$ and this implies that $[\alpha]^{-1}[\beta] \in \pi_1(N)$ contary to $[\alpha]\pi_1(N) \neq [\beta]\pi_1(N)$. □

Corollary 5.5 (Application) *Suppose that W is a compact aspherical $m-$manifold with incompressible boundary. Also assume that there is a component of ∂W, call it N, so that $|\pi_1(W):\pi_1(N)|>2$.*
Then $actdim(\pi_1(W))=m$.

Proof Let $p\colon \tilde{W} \to W$ be the universal cover of W. It is obvious that $actdim(\pi_1(W)) \leq m$ as $\pi_1(W)$ acts cocompactly and properly discontinuously on $\tilde{W}$. Denote $G \equiv \pi_1(W)$ and $H \equiv \pi_1(N)$. Let $\tilde{N}$ be a component of $p^{-1}(N)$. Therefore $\tilde{N}$ is the contractible universal cover of $N^{(m-1)}$. Note that $\tilde{N}$ is an $(m-2)$–proper obstructor by Proposition 2.6. Now $\tilde{W}/H$ has a boundary component homeomorphic to N. Call this component N also. $|G:H|$ is infinite by the previous lemma, and this implies that $\tilde{W}/H$ is not compact. In particular, there exists a map $\alpha'\colon [0,\infty) \to \tilde{W}/H$ with the following property: For each $D>0$ there exists $T \in [0,\infty)$ such that for any $x \in N$, $d(\alpha'(t),x)>D$ for $t>T$, and $\alpha'(0) \in N$. Let $\tilde{\alpha}\colon [0,\infty) \to \tilde{W}$ be a lifting of α' such that $\tilde{\alpha}(0) \in \tilde{N}$. Now we define a uniformly proper map:

$$\phi\colon\ \tilde{N} \vee \alpha \to \tilde{W}$$

$$\begin{cases} \phi|_{\tilde{N}} = \mathit{inclusion} \\ \phi(\alpha_t) = \tilde{\alpha}(t) \end{cases}$$

Observe that ϕ is a uniformly proper map. Since $\tilde{N} \vee \alpha$ is an $(m-1)$–proper obstructor and $\tilde{W}$ is quasi-isometric to G, $pobdim(G) \geq m$. But

$$pobdim(G) \leq updim(G) \leq actdim(G) \leq m.$$

The last inequality follows from the fact that G acts on $\tilde{W}$ properly discontinuously. Therefore $pobdim(G)=m$. □

The following corollary answers Question 2 found in [2].

Corollary 5.6 (Application) *Suppose that W_i is a compact aspherical m_i-manifold with incompressible boundary for $i=1,\ldots,d$. Also assume that for each i, $1 \leq i \leq d$, there is a component of ∂W_i, call it N_i, so that $|\pi_1(W_i):\pi_1(N_i)|>2$. Let $G \equiv \pi_1(W_1)\times \ldots \times\pi_1(W_d)$. Then:*

$$actdim(G) = m_1 + \ldots + m_d$$

Proof It is easy to see that

$$actdim(G) \leq m_1 + \ldots + m_d$$

as G acts cocompactly and properly discontinuously on $\tilde{W}_1 \times \cdots \times \tilde{W}_d$. Denote $\pi_1(W_i) \equiv G_i$ and $\pi_1(N_i) \equiv H_i$. Let

$$p\colon \tilde{W}_i \to W_i$$

be the contractible universal cover and let $\tilde{N}_i$ be a component of $p^{-1}(N_i)$. Since N_i is incompressible, $\tilde{N}_i$ is the contractible universal cover of $N_i^{(m_i-1)}$.

By the previous Corollary, there are uniformly proper maps:

$$\phi_1\colon \tilde{N}_1 \vee \alpha \to \tilde{W}_1$$

$$\phi_2\colon \tilde{N}_2 \vee \beta \to \tilde{W}_2$$

So there exists a uniformly proper map:

$$\phi_1 \times \phi_2\colon (\tilde{N}_1 \vee \alpha) \times (\tilde{N}_2 \vee \beta) \to \tilde{W}_1 \times \tilde{W}_2$$

Recall that $(\tilde{N}_1 \vee \alpha) \times (\tilde{N}_2 \vee \beta)$ is an $(m_1 + m_2 - 1)$–proper obstructor by Proposition 4.4. Since $\tilde{W}_1 \times \tilde{W}_2$ is quasi-isometric to $G_1 \times G_2$:

$$pobdim(G_1 \times G_2) \geq m_1 + m_2$$

But $G_1 \times G_2$ acts on $\tilde{W}_1 \times \tilde{W}_2$ properly discontinuously, and this implies that:

$$pobdim(G_1 \times G_2) \leq actdim(G_1 \times G_2) \leq m_1 + m_2$$

Therefore, $pobdim(G_1 \times G_2) = m_1 + m_2$.

Continue inductively and we conclude that:

$$pobdim(G) = pobdim(G_1 \times \cdots \times G_d) = m_1 + \ldots + m_d$$

Finally we see that

$$pobdim(G) \leq updim(G) \leq actdim(G) \Rightarrow actdim(G) = m_1 + \ldots + m_d \qquad \square$$

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A Pro-Category of Abelian Groups

With every category $\mathcal{K}$ we can associate a new category $pro(\mathcal{K})$. We briefly review the definitions, see [1] or [6] for details. Recall that a partially ordered set $(\Lambda, \leq)$ is *directed* if, for $i, j \in \Lambda$, there exists $k \in \Lambda$ so that $k \geq i, j$.

Definition A.1 *(Inverse system)* Let $(\Lambda, \leq)$ be a directed set. The system $\mathbf{A} = \{A_\lambda, p_\lambda^{\lambda'}, \Lambda\}$ is called an *inverse system* over $(\Lambda, \leq)$ in the category $\mathcal{K}$, if the following conditions are true.

(i) $A_\lambda \in Ob_{\mathcal{K}}$ for every $\lambda \in \Lambda$

(ii) $p_\lambda^{\lambda'} \in Mor_{\mathcal{K}}(A_{\lambda'}, A_\lambda)$ for $\lambda' \geq \lambda$

(iii) $\lambda \leq \lambda' \leq \lambda'' \Rightarrow p_\lambda^{\lambda'} p_{\lambda'}^{\lambda''} = p_\lambda^{\lambda''}$

Definition A.2 *(A map of systems)* Given two inverse systems in $\mathcal{K}$,

$$\mathbf{A} = \{A_\lambda, p_\lambda^{\lambda'}, \Lambda\}, \quad \text{and} \quad \mathbf{B} = \{B_\mu, q_\mu^{\mu'}, M\}$$

the system

$$\bar{f} = (f, f_\lambda)\colon \mathbf{A} \to \mathbf{B}$$

is called a *map of systems* if the following conditions are true.

(i) $f\colon M \to \Lambda$ is an increasing function

(ii) $f(M)$ is cofinal with Λ

(iii) $f_\mu \in Mor_{\mathcal{K}}(A_{f(\mu)}, B_\mu)$

(iv) For $\mu' \geq \mu$ there exists $\lambda \geq f(\mu), f(\mu')$ so that:

$$q_\mu^{\mu'} \circ f_\mu \circ p_{f(\mu')}^\lambda = f_\mu \circ p_\mu^\lambda$$

$$\begin{tikzcd}
A_{f(\mu)} \arrow[d, "f_\mu"'] & A_{f(\mu')} \arrow[l, "p_{f(\mu)}^{f(\mu')}"'] \arrow[d, "f_{\mu'}"] & A_\lambda \arrow[l, "p_\mu^\lambda"'] \\
B_\mu & B_{\mu'} \arrow[l, "q_\mu^{\mu'}"']
\end{tikzcd}$$

$\bar{f}$ is called a *special map of systems* if $\Lambda = M$, $f = id$, and $f_\lambda p_\lambda^{\lambda'} = q_\lambda^{\lambda'} f_{\lambda'}$. Two maps of systems $\bar{f}, \bar{g}\colon \mathbf{A} \to \mathbf{B}$ are considered equivalent, $\bar{f} \simeq \bar{g}$, if for every $\mu \in M$ there is a $\lambda \in \Lambda, \lambda \geq f(\mu), g(\mu)$, such that $f_\mu p_{f(\mu)}^\lambda = g_\mu p_{g(\mu)}^\lambda$. This is an equivalence relation.

Definition A.3 *(Pro-category)* $pro(\mathcal{K})$ is a category whose objects are inverse systems in $\mathcal{K}$ and morphisms are equivalence classes of maps of systems. The class containing $\bar{f}$ will be denoted by $\mathbf{f}$.

Our main interest is the following pro-category.

Example A.4 *Pro-category of abelian groups* Let $\mathcal{A}$ be the category of abelian groups and homomorphisms. Then corresponding $pro(\mathcal{A})$ is called the category of pro-abelian groups.

Example A.5 *Homology pro-groups* Suppose $\{(X, X_0)_i, p_i^{i'}, \mathbb{N}\}$ is an object in the pro-homotopy category of pairs of spaces having the homotopy type of a simplicial pair. Then $\{H_j((X, X_0)_i), (p_i^{i'})_*, \mathbb{N}\}$ is an object of $pro(\mathcal{A})$. Denote $\{H_j((X, X_0)_i), (p_i^{i'})_*, \mathbb{N}\}$ by $proH_j(X, X_0)$.

We list useful properties of $pro(\mathcal{A})$:

(1) A system $\mathbf{0}$ consisting of a single trivial group is a zero object in $pro(\mathcal{A})$.

(2) A pro-abelian group $\{G_i, p_i^{i'}, \mathbb{N}\} \cong \mathbf{0}$ iff every i admits a $i' \geq i$ such that $p_i^{i'} = 0$.

(3) Let $\mathbf{A}$ denote a constant pro-abelian group $\{A, id_A, \mathbb{N}\}$. If a pro-abelian group $\{G_i, p_i^{i'}, \mathbb{N}\} \cong \mathbf{A}$ then

$$\lim_{\leftarrow} G_i = A.$$

See [4, Lemma 4.1].

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110 8th Street RPI, Troy, NY 12180, USA

Email: `yoons@rpi.edu`

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