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L_δ groups are almost convex and have a sub-cubic Dehn function

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Abstract We prove that if the Cayley graph of a finitely generated group enjoys the property L_δ then the group is almost convex and has a sub-cubic isoperimetric function.

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1 Definitions

In this article we show how a new metric property of groups called L_δ is related to some older metric properties of groups. We prove that if a group has a finite generating set that enjoys L_δ then the group is almost convex with respect to this generating set, and has a sub-cubic isoperimetric (or Dehn) function. My thanks to Kim Ruane and Indira Chatterji for introducing me to L_δ, to Andrew Rechnitzer for help with the figures, and to an anonymous reviewer for helpful suggestions.

Let (X, d) be a weakly geodesic metric space (in this paper take the Cayley graph with the word-metric).

Definition 1.1 *δ-path* For any $\delta \geq 0$ and finite sequence of points $x_1, \ldots, x_n$, we say $(x_1, \ldots, x_n)$ is a *δ-path* if

$$d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) \leq d(x_1, x_n) + \delta.$$

Definition 1.2 L_δ For any $\delta \geq 0$ the space X has *property* L_δ if for each three distinct points $x, y, z \in X$ there exists a point $t \in X$ so that the paths $(x, t, y), (y, t, z)$ and (z, t, x) are all δ-paths.

Examples of groups enjoying this property are word-hyperbolic groups, fundamental groups of various cusped hyperbolic 3-manifolds, groups that act on

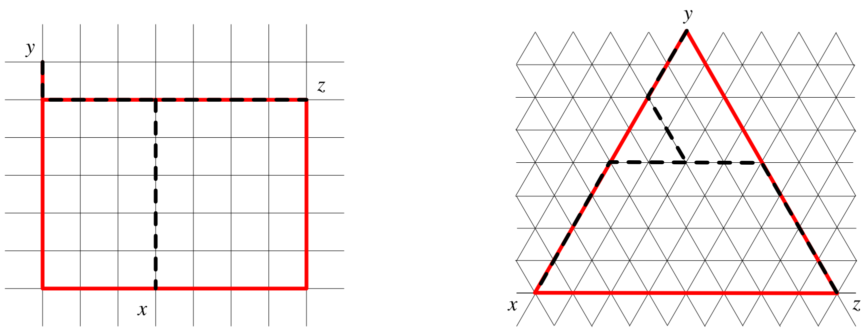

Figure 1: $\mathbb{Z}^2$ is L_δ for one generating set and not another

CAT(0) cube complexes and products of trees, and Coxeter groups [5]. The property is related to the property of *rapid decay* and the Baum-Connes conjecture. The property is not invariant under change of finite generating set. For example, Chatterji and Ruane show that $\mathbb{Z}^2$ with the usual generating set has L_δ, however the generating set $\langle a, b, c \mid ab = ba = c \rangle$ does not enjoy the property, as illustrated in Figure 1.

Define the closed metric ball of radius n to be the set of points that lie within distance n of the identity vertex in the Cayley graph.

Definition 1.3 *Almost convex* A group G with finite generating set $\mathcal{G}$ is *almost convex* if there is a constant $C \geq 0$ so that for any two vertices that lie distance $n \geq 0$ from the identity vertex and at most 2 apart from each other, there is a path connecting them that lies inside the closed ball of radius n and has length at most C.

Cannon [4] showed that if a group has this property then one has an algorithm to construct any finite portion of the Cayley graph. It also implies finite presentability. The property is dependent on choice of finite generating set [10].

Suppose that G is a finitely presented group, with inverse closed finite generating set $\mathcal{G}$ and finite set of relators $\mathcal{R} \subseteq \mathcal{G}^*$. Let $F(\mathcal{G})$ be the free group generated by $\mathcal{G}$. A word in $\mathcal{G}^*$ represents the identity in G if and only if it is freely equal to an expression of the form

$$\Pi_{i=1}^{k} g_i r_i g_i^{-1}$$

where the $g_i \in F(\mathcal{G})$ and $r_i \in \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 1.4 *van Kampen diagram, Area* Let Δ be a labeled, simply connected planar 2-complex, such that each edge is oriented and labeled by and element of $\mathcal{G}$, and reading the labels on the boundary of each 2-cell gives and element of $\mathcal{R} \cup \mathcal{R}^{-1}$. We say Δ is a *van Kampen diagram* for w if reading the labels around the boundary of Δ gives w.

Each van Kampen diagram with k 2-cells for w gives a way of expressing w as a product $\Pi_{i=1}^{k} g_i r_i g_i^{-1}$. Conversely, each product $\Pi_{i=1}^{k} g_i r_i g_i^{-1}$ gives a van Kampen diagram for w with at most k 2-cells. Thus van Kampen's Lemma states that w has a van Kampen diagram if and only if w represents the identity element. Define the *area* $A(w)$ of a word $w \in \mathcal{G}^*$ which represents the identity to be the minimum k in any such expression for w, or equivalently the minimum number of 2-cells in a van Kampen diagram for w.

In terms of the Cayley graph, if a word evaluates to the identity element it corresponds to a closed path, and its area is the least number of relators needed to fill in this closed path. See [3] p.155 for more details on van Kampen's Lemma.

Definition 1.5 *Dehn function, Isoperimetric function* The *Dehn function* for $\langle \mathcal{G} | \mathcal{R} \rangle$ is defined to be $\mathcal{D}(n) = \max\{A(w) : w$ has at most n letters, and w evaluates to the identity$\}$. An *isoperimetric function* for $\langle \mathcal{G} | \mathcal{R} \rangle$ is any function which satisfies $f(n) \geq \mathcal{D}(n)$.

Two functions f, g are said to be equivalent if there are constants A, A', B, B', C, C', D, D', E, E' so that $f(n) \leq Ag(Bn+C)+Dn+E$, and $g(n) \leq A'f(B'n+C') + D'n + E'$. Up to this notion of equivalence, a Dehn function for a group is invariant of the finite presentation. If G has a sub-quadratic isoperimetric function then it has a linear isoperimetric function [1, 9]. For more details about isoperimetric functions and van Kampen diagrams, see for example [8, 7].

2 Results

Theorem 2.1 *If the Cayley graph for a group has property L_δ for some $\delta \geq 0$ then it is almost convex with constant $3\delta + 2$.*

Proof Consider two vertices g, g' that lie at distance n from the identity in the Cayley graph, and distance at most 2 apart. We will call the identity vertex z. Let w and w' be geodesic paths of length n from z to g and g' respectively.

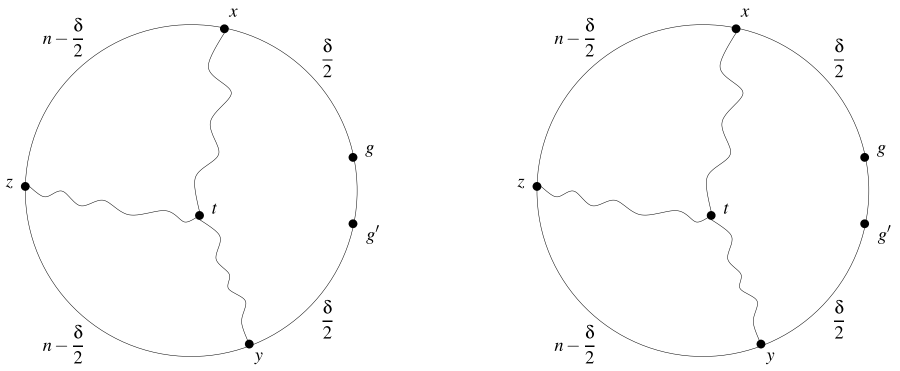

Figure 2: L_δ implies almost convex

If $n \le \delta$ then the path $w^{-1}w'$ lies inside the closed ball of radius n, connects g to g' and has length $2n \le 2\delta$.

If $n > \delta$ then let x be the point that lies distance exactly $n - \frac{\delta}{2}$ from z along w, and y the point that lies distance $n - \frac{\delta}{2}$ from z along w'. Note that x and y need not be vertices (they may lie in the interior of edges). See Figure 2.

By property L_δ there exists a point t such that $d(z,t) + d(t,x) \le d(z,x) + \delta = n - \frac{\delta}{2} + \delta = n + \frac{\delta}{2}$ and $d(z,t) + d(t,y) \le d(z,y) + \delta = n - \frac{\delta}{2} + \delta = n + \frac{\delta}{2}$ since x and y lie on geodesics. It follows that the geodesic paths from x to t and t to y lie in the closed ball of radius n.

Also, $d(x,t) + d(t,y) \le d(x,y) + \delta \le (\frac{\delta}{2} + 2 + \frac{\delta}{2}) + \delta = 2\delta + 2$, where $(\frac{\delta}{2} + 2 + \frac{\delta}{2})$ is the length of the path from x to y that goes via g and g'.

So we can find a path of length $\frac{\delta}{2} + (2\delta + 2) + \frac{\delta}{2} = 3\delta + 2$ from g to g' that lies in the closed ball of radius n. □

It can be shown that if a group is almost convex with constant C with respect to some finite generating set, then the set of all words that evaluate to the identity and have length at most $C + 2$ form a finite set of relators for a presentation for the group, so the group is finitely presented. See [8]. In this case, however, we prove finite presentability directly in the following result.

Theorem 2.2 *If the Cayley graph for a group G with respect to some finite generating set $\mathcal{G}$ has property L_δ for some $\delta \ge 0$ then G is finitely presented and has an isoperimetric function equivalent to $n^{\frac{1}{1-\log_3 2}}$.*

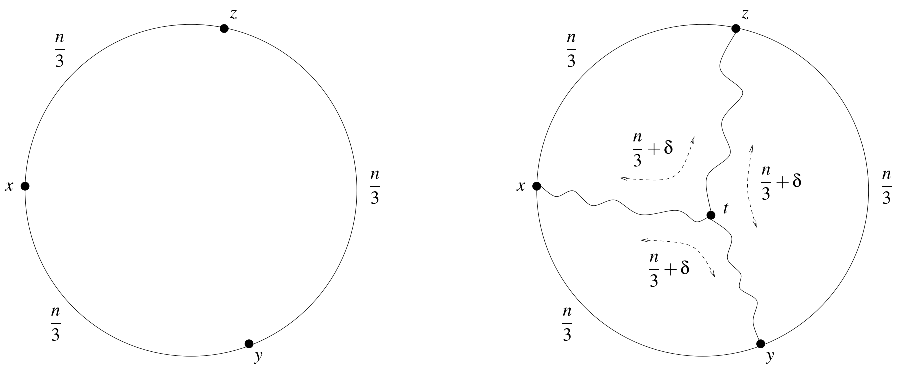

Figure 3: Subdividing a closed path into three shorter ones

Proof Assume that the finite generating set $\mathcal{G}$ is inverse closed. We will show that every word in $\mathcal{G}^*$ that evaluates to the identity has a van Kampen diagram consisting of 2-cells of perimeter (at most) $3\delta + 2$. Then taking as a set of relators the set of all words that represent the identity that have length at most $3\delta + 2$, it follows that G is finitely presented. Moreover, we will show that the area of a diagram with respect to this presentation is at most sub-cubic in the length of the word.

Let $w \in \mathcal{G}^*$ be a word of length n that evaluates to the identity. We wish to construct a van Kampen diagram for w from 2-cells of perimeter at most $3\delta + 2$. So if $n \leq 3\delta + 2$ then it will be a relator, and if $n > 3\delta + 2$ then proceed as follows.

The word w represents a closed path in the Cayley graph, which starts and ends at some vertex z. Choose two points x and y that lie at distance exactly $\frac{n}{3}$ along the paths w and w^{-1} from z. So x, y and z are equally spaced around w, that is, pairwise at most $\frac{n}{3}$ apart. Note that x and y need not be vertices (they may lie in the interior of edges). See the left side of Figure 3.

Property L_δ says that there is a point t in the Cayley graph such that $d(x,t) + d(t,y) \leq d(x,y) + \delta = \frac{n}{3} + \delta$ and similarly for x, z and y, z.

Thus we can find three closed paths containing x, y, t, y, z, t and x, z, t each of perimeter at most $\frac{n}{3} + (\frac{n}{3} + \delta) = 2\frac{n}{3} + \delta$. See the right side of Figure 3.

These three closed paths are each strictly shorter than the original closed path, since $n > 3\delta + 2 > 3\delta$ so $\delta < \frac{n}{3}$ so $\frac{2n}{3} + \delta < n$.

Note that if t in fact lies on the path w then these paths do not embed in the Cayley graph. For the purpose of the argument we are not concerned with whether these shorter closed paths embed or immerse in the Cayley graph, we merely want to construct a van Kampen diagram for w with 2-cells bounded by closed paths of length at most $3\delta+2$, and we allow that some of these 2-cells could be non-embedded. So we will iterate this partitioning process until the maximum perimeter of an internal closed path is not more than $3\delta+2$.

Find k so that $(\frac{3}{2})^k \leq n < (\frac{3}{2})^{k+1}$. After one iteration we have three closed paths of length at most $\frac{2}{3}n+\delta$. After a second iteration we get at most nine closed paths of length at most $\frac{2}{3}(\frac{2}{3}n+\delta)+\delta = (\frac{2}{3})^2 n + \frac{2}{3}\delta+\delta$. After a third iteration we get at most 3^3 closed paths of length at most $\frac{2}{3}(\frac{2}{3}(\frac{2}{3}n+\delta)+\delta)+\delta$ $= (\frac{2}{3})^3 n + (\frac{2}{3})^2\delta + \frac{2}{3}\delta+\delta$.

Iterating k times we will get at most 3^k closed paths of perimeter at most

$$\begin{aligned} & (2/3)^k n + (2/3)^{k-1}\delta + \ldots + (2/3)\delta + \delta \\ < \ & (2/3)^k (3/2)^{k+1} + \sum_{i=0}^{k-1} (2/3)^i \delta \\ = \ & 3/2 + \left(\frac{1-(2/3)^k}{1-2/3}\right)\delta \\ < \ & 3/2 + 3\delta. \end{aligned}$$

Thus after k iterations we will have partitioned the original closed path down into closed paths of length less than $3\delta+2$, so we will have succeeded in finding a van Kampen diagram for w using only words that evaluate to 1 and have length less than $3\delta+2$.

Then we have established that G is finitely presented by the presentation described above.

Now $(\frac{3}{2})^k \leq n$ so taking $\log_3$ of both sides gives $k\log_3(\frac{3}{2}) \leq \log_3 n$. Let $c = \frac{1}{\log_3(3/2)}$. Then $k \leq c\log_3 n$. So the number of relators in a van Kampen diagram for w is at most $3^k \leq 3^{c\log_3 n} = (3^{\log_3 n})^c = n^c$. It follows that the isoperimetric function is sub-cubic since $c = \frac{1}{\log_3(1.5)} = \frac{\ln 3}{\ln 1.5}$ which is approximately equal to 2.7. Alternatively, $c = \frac{1}{\log_3(3/2)} = \frac{1}{\log_3 3 - \log_3 2} = \frac{1}{1-\log_3 2}$.

Note that the number of iterations required is independent of δ, so the isoperimetric bound is not improved by smaller δ (such as 0). □

3 Remarks

The 3-dimensional integral Heisenberg group has a cubic isoperimetric function (see [6] p.165), so in the light of Theorem 2.2 cannot enjoy property L_δ for any $\delta \geq 0$.

An open question is whether there is a group with an isoperimetric function greater than quadratic and less than the sub-cubic bound given in Theorem 2.2 that has L_δ. Brady and Bridson have a family of groups having isoperimetric functions with exponent $2\log_2 \frac{2p}{q}$ for all $p \geq q$ [2], however a quick investigation of these suggests that they would not enjoy L_δ.

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