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Gerbes and homotopy quantum field theories

Ulrich Bunke
Paul Turner
Simon Willerton

Abstract For smooth finite dimensional manifolds, we characterise gerbes with connection as functors on a certain surface cobordism category. This allows us to relate gerbes with connection to Turaev's 1+1-dimensional homotopy quantum field theories, and we show that flat gerbes are related to a specific class of rank one homotopy quantum field theories.

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Introduction

The original motivation for this paper was to reconcile the two "higher" versions of a line bundle with connection mentioned in the title. In the process we came up with a characterization of gerbes-with-connection over a fixed space as functors from a certain cobordism category. Before getting onto that we will give a quick description of the two objects in the title, but first it is pertinent to give a reminder of what a line bundle with connection is.

Line bundles

A line bundle with connection can be viewed in many ways, especially if one wants to generalise to "higher" versions. Here we will mention the idea of it being determined by its holonomy, of it being a functor on the path category of the base space and of it being a functor on the $0+1$ dimensional cobordism category on the base.

The holonomy of a line bundle with connection is a $\mathbb{C}^\times$-valued function on the free loop space of the base manifold X. Barrett [1] and others showed that functions on free loop space which occur as the holonomy of a line bundle with connection are characterised by being invariant under thin-homotopy (see below), being invariant under diffeomorphism of the circle, and satisfying a smoothness condition. Further, such a function on the free loop space uniquely determines a line bundle with connection up to equivalence.

A different characterisation of holonomy is given by thinking of it as a function $\widetilde{S}$ on smooth maps of closed one-dimensional manifolds into the space which is multiplicative under disjoint union, together with a closed two-form $\widetilde{c}$, the curvature of the bundle, such that if $\widetilde{v}\colon \widetilde{V} \to X$ is a map of a surface with boundary into the base space then $\widetilde{S}(\partial\widetilde{v}) = \exp(\int_{\widetilde{V}} \widetilde{v}^*\widetilde{c})$. Such pairs $\widetilde{S}$ and $\widetilde{c}$ satisfying this condition form the Cheeger-Simons group (see [7]) of differential characters $\widehat{H}^2(X)$. This group parametrises line bundles with connection up to equivalence.

One could take a groupoid point of view of a line bundle with connection in the following fashion. The path category PX of a space X is the category whose objects are the points in the space and whose morphisms are, roughly speaking, smooth paths between them, while the category Vect_1 is the category of one-dimensional complex vector spaces with invertible linear maps as morphisms. Any line bundle with connection gives a functor $PX \to \mathrm{Vect}_1$, which to a point in X associates the fibre over that point, and to a path between two points associates the parallel transport along that path. This functor will satisfy some smoothness condition and will in fact descend to a functor on the thin-homotopy path groupoid. Actually, here the category Vect_1 is rather large and could be replaced by something like the small category of lines in infinite projective space.

A variation on this is obtained by considering the 0+1-dimensional cobordism category of the space X, this has finite ordered collections of points in X as its objects, and cobordisms between them as morphisms. A monoidal functor from this to the category of complex lines, with the usual tensor product, should be an appropriate notion of bundle with connection.

Things become a lot simpler when *flat* bundles are considered. In this case geometric notions descend to topological ones. A flat line bundle is a line bundle with connection whose curvature vanishes identically. This means that the holonomy can be considered as an element of the first cohomology group $H^1(X,\mathbb{C}^\times)$. The categorical descriptions become a lot simpler because the morphism sets can be quotiented by homotopy relations, thus becoming discrete sets.

Gerbes

A gerbe is essentially a realization of a degree three cohomology class. The idea of a gerbe was brought to many people's attention by the book of Brylinski [4]. There are several different but equivalent ways to realize gerbes, these include ways involving sheaves of categories [4], classifying space bundles [10], bundle-gerbes [14], and bundle realizations of Cech cocycles [6, 11]. The degree three cohomology class corresponding to a gerbe is called its Dixmier-Douady class and it is the analogue of the first Chern class for a line bundle.

It is possible to introduce an appropriate notion of connection (by which we mean a curving and a connective structure in the language of [4]) on a gerbe over a smooth manifold. Associated to a gerbe with connection, in any of the above mentioned descriptions, are a curvature three-form and two notions of holonomy. The first notion of holonomy is a $\mathbb{C}^\times$-valued function S on the space of maps of closed surfaces into the manifold; and the second is loop holonomy which is a line bundle with connection on the free loop space of the manifold. The curvature and surface holonomy are related by the fact that if $v\colon V \to X$ is a map of a three manifold with boundary into the base manifold then $S(\partial v) = \exp(2\pi i \int_V v^* c)$. Again, pairs S and c which satisfy this condition form a group $\widehat{H}^3(X)$ which parametrises gerbes with connection up to equivalence. (This group is isomorphic to the so called Cheeger-Simons group of differential characters, see Appendix A for more details.) In this paper we will work with this holonomy description of gerbes.

A *flat gerbe* is a gerbe with connection whose curvature vanishes identically. This implies that the Dixmier-Douady class of the underlying gerbe is a torsion class. It also follows that the holonomy around a surface only depends on the homology class of the surface, and so the holonomy can be considered as an element of $\mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{C}^\times) \cong H^2(X,\mathbb{C}^\times)$. This establishes a bijection between flat gerbes on X and $H^2(X,\mathbb{C}^\times)$.

Homotopy quantum field theories

The second generalisation is the notion of a 1+1-dimensional homotopy quantum field theory, which strictly speaking generalises the idea of a *flat* bundle. This notion was introduced by Turaev in [19] and independently by Brightwell and Turner [3], but the idea goes back to the work of Segal [16]. Turaev considered the case of Eilenberg-MacLane spaces and, orthogonally, Brightwell and Turner considered simply-connected spaces. A homotopy quantum field theory is like a topological quantum field theory taking place in a "background space", and it can be given a functorial description as follows.[1] For a space, the homotopy surface category of a space generalizes the $0+1$-dimensional cobordism category by having as objects collection of loops in the space and having as morphisms cobordisms between these considered up to boundary preserving homotopy. A 1+1-dimensional homotopy quantum field theory on a space is a symmetric monoidal functor from the homotopy surface category of a space to the category of finite dimensional vector spaces. As we are comparing these to gerbes we will only consider the rank one theories, ie. those functors taking values in the subcategory of one dimensional vector spaces.

Outline of this paper

One motivation for this paper was to figure out how gerbes are related to homotopy quantum field theories, another was to understand what conditions are necessary and sufficient for a line bundle on loop space to come from a gerbe. These questions are addressed by considering an object that we have dubbed a thin-invariant field theory. The main novelty is that it uses the idea of thin-cobordism: two manifolds in a space are thin cobordant if there is a cobordism between them which has "zero volume" in the ambient space. In 1-dimensional manifolds this is the same as thin-homotopy as defined by Barrett [1] and further developed in [5] and [12], but this is not the case for surfaces — the 1+1-dimensional thin-cobordism category is a groupoid whereas the 1+1-dimensional thin-homotopy category is not. A thin-invariant field theory is essentially a smooth symmetric monoidal functor from the thin-cobordism category to the category of one-dimensional vector spaces. The idea is that this gives an alternative description of a gerbe. The view that a gerbe should be a functor on a cobordism category has been advocated by Segal in [17]. The collection of thin-invariant field theories on X form a group in a natural way. Our main theorem is the following.

Theorem 6.3 *On a smooth finite dimensional manifold, there is an isomorphism from the group of thin-invariant field theories (up to equivalence) to the group of gerbes with connection (up to equivalence).*

[1]It should be noted that we alter Turaev's definition by removing Axiom 2.27 which is not appropriate for non-Eilenberg-MacLane spaces, this does not alter any of the theorems in his paper provided they are all stated for Eilenberg-MacLane spaces. This is the position adopted by Rodrigues in [15], where a connection with gerbes and thinness is also suggested.

To make the connection with homotopy quantum field theories we show that a certain natural subset, the rank one, normalised ones, correspond to flat thin-invariant field theories, this gives the following.

Theorem 6.4 *On a finite dimensional manifold, the group of normalised rank one homotopy quantum field theories (up to equivalence) is isomorphic to the group of flat gerbes (up to equivalence).*

In this context it makes sense to consider homotopy quantum field theories defined over an arbitrary commutative ring with unity. We classify these in the following manner, which generalizes a theorem of Turaev.

Theorem 7.1 *Let K be a commutative ring, and X be a path connected topological space. Then Turaev's construction gives an isomorphism between the group $H^2(X, K^\times)$ and the group of normalised, rank one homotopy quantum field theories defined over K.*

We then show that a thin-invariant field theory is an extension of the usual line bundle of a gerbe over the free loop space.

Theorem 8.1 *A thin-invariant field theory can be restricted to the path category of the free loop space giving a line bundle with connection on the free loop space. This is isomorphic to the transgression of the associated gerbe.*

We include two appendices. In the first we compare our definition of the Cheeger-Simons group with the more familiar one, and in the second we gather together, for ease of reference, a number of categorical definitions used throughout the paper.

It is worth noting here that homotopy quantum field theories are bordism-like in their nature, whereas gerbes are homological creatures. It seems to us that the techniques used and results obtained in this paper rely on the coincidence of bordism and homology in low degree, and will not necessarily generalise to higher degrees.

1 Basic definitions

1.1 Bordism

Here we give, for those unfamiliar with the notion, a very brief introduction to (co)bordism groups and then we present the low-dimensional co-incidence result which is central to the paper.

The nth oriented bordism[2] group $\mathrm{MSO}_n(X)$ of a space X for a non-negative integer n is similar but subtly different to the nth ordinary homology group $H_n(X)$. Whereas homology groups are defined using chains of simplices, bordism groups are defined using maps of manifolds. The main ingredient in the definition is the set of pairs (V,v) where V is an oriented smooth n–manifold and $v\colon V \to X$ is a map. Two such pairs (V,v) and (V',v') are said to be *cobordant* if there is an $(n+1)$–manifold W with $\partial W \cong \overline{V} \sqcup V'$ and a map $w\colon W \to X$ such that $\partial w = \overline{v} \sqcup v'$. The group $MSO_n(X)$ is defined to be the set of equivalence classes under this cobordism relation, with the group structure being induced from the disjoint union of manifolds.

These groups share many properties with ordinary homology groups $H_n(X)$, forming an example of what is called an extraordinary homology theory. In fact the only difference between homology and bordism lies in the torsion part, as rationally they are the same: $H_n(X)\otimes\mathbb{Q} \cong \mathrm{MSO}_n(X)\otimes\mathbb{Q}$. The general theory is a well developed topic in the algebraic topology literature, one source for a comprehensive treatment would be [18].

The following lemma on the low-dimensional co-incidence of bordism and homology is key to the ideas of this paper.

Lemma 1.1 *The first homology and bordism groups of a space are isomorphic, as are the second groups: if X is a space then $\mathrm{MSO}_1(X) \cong H_1(X;\mathbb{Z})$ and $\mathrm{MSO}_2(X) \cong H_2(X;\mathbb{Z})$.*

Proof Apply the Atiyah-Hirzebruch spectral sequence for bordism groups (see for example [18]) and use the fact that in low dimensions the coefficients for bordism are given by $\mathrm{MSO}_0(\mathrm{pt}) \cong \mathbb{Z}$ and $\mathrm{MSO}_1(\mathrm{pt}) \cong \mathrm{MSO}_2(\mathrm{pt}) \cong \{1\}$. □

1.2 X-surfaces and thin cobordisms

In this section we introduce the key notions of X-surfaces, thin-cobordism and thin homotopy.

[2]There is a standard problem with terminology here. Initially bordism groups were called cobordism groups, because two things are cobordant if they cobound something else. Unfortunately in this context the prefix "co" usually refers to the contravariant theory, so cobordism was taken to mean the contravariant version (analogous to cohomology) and the word bordism was used for the covariant theory (analogous to homology). In this paper we will always be interested in the covariant theory.

If X is a smooth manifold then an X*-surface* is essentially a smooth map of a surface into X, but with certain technical collaring requirements to ensure that X-surfaces can be glued together. It is perhaps possible to avoid these technical conditions by working with piece-wise smooth maps, but we have not done that.

Boundaries of surfaces will need to be parametrised, so for concreteness, let S^1 be the set of unit complex numbers and fix an orientation for this. Let S_n be the union of n ordered copies of S^1. Fix an orientation on the unit interval $[0,1]$ and define $C_n := S_n \times [0,1]$, so that C_n is n ordered parametrised cylinders. For any oriented manifold Y let $\overline{Y}$ denote the same manifold with the opposite orientation.

A surface will mean a smooth oriented two-manifold Σ together with a *collar*, which will mean a certain type of parametrisation of a neighbourhood N_Σ of the boundary:

$$\iota_\Sigma \colon \overline{C_m} \sqcup C_n \xrightarrow{\cong} N_\Sigma.$$

The m boundary components corresponding to $\overline{C_m}$ will be called *inputs* and the n corresponding to C_n will be called *outputs.* Note that inputs and outputs inherit an order from ι_Σ.

Define an X-surface to be a surface Σ as above, and a smooth map $g\colon \Sigma \to X$ such that $g|_{N_\Sigma} \circ \iota_\Sigma$ factors through the projection $\overline{C_m} \sqcup C_n \to \overline{S_m} \sqcup S_n$, ie. the map g is constant in transverse directions near the boundary. The inputs and outputs of g are the restrictions of g to the inputs and outputs of the underlying surface. If the inputs of $g_1\colon \Sigma_1 \to X$ agree with the outputs of $g_2\colon \Sigma_2 \to X$ then we can glue Σ_1 and Σ_2 together using the given collars to form another surface $\Sigma_1 \cup \Sigma_2$ and using the induced maps form the X-surface $g_1 \circ g_2 \colon \Sigma_1 \cup \Sigma_2 \to X$. If the inputs of g are the same as the outputs, then use the notation $\langle g \rangle$ to denote the closed X-surface obtained by gluing the inputs to the outputs. An X-three-manifold is defined similarly.

Informally two X-surfaces are thin cobordant if there exists a cobounding manifold which has no volume in X. More formally, two X-surfaces $g\colon \Sigma \to X$ and $g'\colon \Sigma' \to X$ are *thin cobordant* if there exists a collared three-manifold W such that $\partial W \cong \Sigma \cup \overline{\Sigma'}$ and a smooth map $w\colon W \to X$ satisfying $w|_{\partial W} = g \cup \overline{g'}$ and dw everywhere having rank at most two.

Thin homotopy is a particular kind of thin cobordism. Let $g\colon \Sigma \to X$ and $g'\colon \Sigma' \to X$ be X-surfaces with the same inputs and the same outputs. The maps g and g' are *thin homotopic* if there exists a thin cobordism homotopic to $\Sigma \times [0,1]$.

One fundamental difference between thin-homotopy and thin-cobordism is that cobordisms are invertible modulo thin-cobordism, but not modulo thin-homotopy. The next proposition shows that if g is a cobordism then its reversal $\overline{g}$ is an inverse modulo thin-cobordism.

Proposition 1.2 *If $g\colon \Sigma \to X$ is an X-surface which is not necessarily closed, then the closed X-surface $\langle g \circ \overline{g} \rangle$ is thin-cobordant to the empty X-surface.*

Proof Consider the manifold with corners $\Sigma \times I$. Smooth this by just removing an arbitrarily small neighbourhood of the corners and call the resulting smooth manifold W. The collaring implies that the boundary of W can be identified with $\Sigma \cup \overline{\Sigma}$. Define the map $w\colon W \to X$ to be the projection to Σ composed with g. The differential dw automatically has rank at most two and thus w provides the requisite thin cobordism. □

2 Gerbe holonomy

In this section we collect together the facts we need about gerbe holonomy.

For a gerbe with connection on a manifold X there is the associated gerbe holonomy which associates a complex number to each closed X-surface. The gerbe holonomy is invariant under diffeomorphism of X-surfaces and it is multiplicative under disjoint union.

The holonomy is related to the curvature of the gerbe connection in the following fashion. Suppose that S is the gerbe holonomy and c, a closed three-form, is the gerbe curvature. If $v\colon V \to X$ is an X-three-manifold then the following holonomy-curvature relation holds:

$$S(\partial v) = \exp\left(2\pi i \int_V v^* c\right).$$

We can take all of the diffeomorphism invariant, multiplicative functions on the set of closed X-surfaces for which there exists a three-form so that the holonomy-curvature relation is satisfied. These form a group $\widehat{H}^3(X)$. This is not exactly the third Cheeger-Simons group, which is defined using smooth two-cycles rather than closed X-surfaces. However these two groups are isomorphic in this degree, this is proved in Appendix A and is due to the fact the bordism and homology agree at low degree. We will therefore refer to $\widehat{H}^3(X)$ as the Cheeger-Simons group. Thus each gerbe with connection gives rise to an element in this Cheeger-Simons group by means of its surface holonomy. It turns out that this sets up a bijection between gerbes with connection and this Cheeger-Simons group (see eg. [10]). Thus specifying a gerbe with connection is the same as specifying its surface holonomy. We will think of $\widehat{H}^3(X)$ as the group of gerbes with connection.

There are two useful exact sequences involving gerbes wich we will now mention (see [4, Section 1.5]). Let $\Omega^*(X)$ denote the smooth complex differential forms on X. By $\Omega^2(X)_{d=0,\mathbb{Z}}$ we denote the subspace of closed forms which have periods in $\mathbb{Z}$. There are the following exact sequences.

$$0 \to H^2(X, \mathbb{C}^\times) \xrightarrow{\phi} \widehat{H}^3(X) \xrightarrow{c} \Omega^3(X);$$
$$0 \to \Omega^2(X)/\Omega^2(X)_{d=0,\mathbb{Z}} \xrightarrow{h} \widehat{H}^3(X) \xrightarrow{D} H^3(X,\mathbb{Z}) \to 0.$$

Here c associates to each gerbe its curvature, D maps a gerbe to its Dixmier-Douady class, and h maps a class $[\omega]$, $\omega \in \Omega^2(X)$, to the gerbe with curvature $d\omega$ and holonomy $h([\omega])\langle g\rangle = \exp(2\pi i \int_\Sigma g^*\omega)$ for all $g : \Sigma \to X$ with Σ an oriented closed surface. The map ϕ can be interpreted as the inclusion of flat gerbes.

Mackaay and Picken [12] observed that gerbe holonomy is invariant under thin-homotopy: we make the stronger, key observation that it is invariant under thin-cobordism.

Proposition 2.1 *Suppose that S is the holonomy of a gerbe with connection on a manifold X. If $g\colon \Sigma \to X$ and $g'\colon \Sigma' \to X$ are closed X-surfaces which are thin cobordant then $S(g) = S(g')$.*

Proof Let the three-form c be the curvature of the gerbe and suppose that $w\colon W \to X$ is a thin cobordism between g and g'. The holonomy-curvature relation implies that $S(\partial w) = \exp(2\pi i \int_W w^*c)$. However, the right-hand side is equal to one as dw has rank at most two, and the left-hand side is equal to $S(g \cup \overline{g'}) = S(g)S(g')^{-1}$, from which the result follows. □

3 Thin-invariant field theories

In order to define thin-invariant field theories we adopt a similar philosophy to [3] (see also [15]) and define a category of cobordisms in a background X and then define a thin-invariant field theory to be a complex representation of this category.

Definition 3.1 The *thin-homotopy surface category*, $\mathcal{T}_X$, of a smooth manifold X, is the category whose objects are smooth maps from n copies of a standard circle, S_n, to X for some integer n; and for which a morphism is an X-surface $g\colon \Sigma \to X$ considered as a morphism from the map defined by its inputs to the map defined by its outputs, with $g'\colon \Sigma' \to X$ being identified with g if there exists a diffeomorphism $T\colon \Sigma \to \Sigma'$ which identifies the collars ($\iota_{\Sigma'} \circ T|_{N_\Sigma} = \iota_{\Sigma'}$) and such that the maps $g' \circ T$ and g are thin homotopic.

Composition of two morphisms g and g' is defined by gluing the outputs of g to the inputs of g' and is denoted by $g' \circ g$. This composition is associative because of the identification of diffeomorphic X-surfaces. If $\gamma\colon S_n \to X$ is an object then the identity morphism is the X-surface $\mathrm{Id}_\gamma\colon C_n \to X$, recalling that C_n is n cylinders, given by composing the projection $C_n \to S_n$ with γ, because gluing Id_γ to an X-surface is thin-homotopic to the original X-surface.

Disjoint union $\sqcup$ of X-surfaces makes $\mathcal{T}_X$ into a strict symmetric monoidal category (see Appendix B for the definition of a symmetric monoidal category). The unit for this monoidal structure is the empty X-surface. For objects $\gamma\colon S_n \to X$ and $\gamma'\colon S_{n'} \to X$ the symmetry structure isomorphism $\kappa\colon \gamma \sqcup \gamma' \to \gamma' \sqcup \gamma$ is given by the cylinder $C_{n+n'} \to S_{n+n'} \to X$ where the first map is the projection and the boundary identification applies the appropriate permutation of boundary circles.

Note that if we did not include thin-homotopy in the definition then we would not have a category as there would not be any identity morphisms.

Now we introduce the main definition of this section.

Definition 3.2 A *rank one, smooth, thin-invariant field theory* for a smooth manifold X is a symmetric monoidal functor $E\colon \mathcal{T}_X \to \mathrm{Vect}_1$ (see Appendix B) from the thin homotopy surface category of X to the category of one-dimension complex vector spaces with tensor product, satisfying the following smoothness condition. If g is a closed surface, then write it as $\langle g \rangle$ to emphasise the fact that it is closed. Such a closed X-surface is an endomorphism of the empty object so $E\langle g \rangle$ is a linear map on $\mathbb{C}$ so can be identified with a complex number, this number is the *holonomy* of $\langle g \rangle$ and will also be written $E\langle g \rangle$. The smoothness condition is then that there exists a closed 3-form c on X such that if $v\colon V \to X$ is an X-three-manifold then

$$E\langle \partial v \rangle = \exp(2\pi i {\textstyle\int_V} v^* c).$$

Two thin-invariant field theories are *isomorphic* if there is a monoidal natural isomorphism between them. If the three-form c is zero we say that the thin-invariant field theory is *flat*.

It is possible to define higher rank thin-invariant field theories, and these should be related to non-Abelian gerbes, but we will not discuss them here. For the rest of this paper "thin-invariant field theory" will mean "rank one, smooth thin-invariant field theory".

Note that according the definition of a symmetric monoidal functor (see Appendix B) a thin-invariant field theory $E\colon \mathcal{T}_X \to \mathrm{Vect}_1$ comes equipped with natural isomorphisms $\Phi_{\gamma,\gamma'}\colon E(\gamma)\otimes E(\gamma') \to E(\gamma \sqcup \gamma')$ for each pair of objects γ and γ'. These are symmetric, that is,

$$\Phi_{\gamma',\gamma} \circ T = E(\kappa) \circ \Phi_{\gamma,\gamma'}$$

where T is the flip in Vect_1 and $\kappa\colon \gamma \sqcup \gamma' \to \gamma' \sqcup \gamma$ is the symmetry structure isomophism in the thin-homotopy surface category.

The definition of isomorphism of thin-invariant field theories requires a natural transformation $\Psi\colon E \to E'$ such that for each object γ, the map $\Psi_\gamma\colon E(\gamma) \to E'(\gamma)$ is an isomorphism and for each pair of objects γ and γ'

$$\Psi_{\gamma\sqcup\gamma'} \circ \Phi^{E}_{\gamma,\gamma'} = \Phi^{E'}_{\gamma,\gamma'} \circ (\Psi_\gamma \otimes \Psi_{\gamma'}).$$

There is a *trivial* thin-invariant field theory defined by setting $E(\gamma) = \mathbb{C}$ for all objects γ, setting $E(g) := \mathrm{Id}$ for all morphisms g, and taking $\Phi_{\gamma,\gamma'}\colon \mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$ to be the canonical identification.

Proposition 3.3 *The set of isomorphism classes of thin-invariant field theories on a manifold X form a group which will be denoted by $\mathrm{TIFT}(X)$. Furthermore the flat thin-invariant field theories on X form a subgroup.*

Proof Given thin-invariant field theories E and F there is thin-invariant field theory $E \otimes F$ formed by defining $(E \otimes F)(\gamma) := E(\gamma) \otimes F(\gamma)$ for objects, $(E\otimes F)(g) = E(g)\otimes F(g)$ for a morphism g and $\Phi_{E\otimes F} = (\Phi_E \otimes \Phi_F)\circ T$ where T is the flip. The three form $c_{E\otimes F}$ is equal to $c_E + c_F$. The identity of this group is the trivial thin-invariant field theory. The inverse E^{-1} of E is defined by setting $E^{-1}(\gamma) = (E(\gamma))^* = \mathrm{Hom}(E(\gamma), \mathbb{C})$ for objects, $E^{-1}(g) = E(\overline{g})^*$ for a morphism g and $\Phi_{E^{-1}} = (\Phi_E^{-1})^*$. □

The next lemma is a useful property coming from the fact that we are only considering the rank one case.

Lemma 3.4 *Suppose that E is a thin-invariant field theory on the smooth manifold X. If $g\colon \Sigma \to X$ is an endomorphism of the object γ of $\mathcal{T}_X$ and $\langle g\rangle$ is the closed X-surface obtained by identifying the inputs and outputs of g then*

$$E(g) = E\langle g\rangle\, \mathrm{Id}_\gamma\,.$$

Proof This is a standard argument in topological field theory. The cylinder, thought as a cobordism from $\gamma \sqcup \gamma$ to the empty map, gives rise to a non-degenerate inner-product on $E(\gamma)$. Evaluating $E\langle g\rangle$ is the same as calculating the trace of $E(g)$ using this inner product. The result follows from this because $E(g)$ is an endomorphism of a one-dimensional space. □

The following theorem gives a fundamental property of thin-invariant field theories.

Theorem 3.5 *A thin-invariant field theory is invariant under thin cobordism of morphisms.*

Proof Suppose that E is a thin-invariant field theory. It suffices to show that if g and g' are thin-cobordant then $E(g') = E(\overline{g})^{-1}$, because, as g is thin-cobordant to itself, we also get $E(g) = E(\overline{g})^{-1}$ and hence $E(g') = E(g)$.

So suppose that w is a thin-cobordism with $\partial w \cong \langle g' \circ \overline{g}\rangle$. Then if c is the three-form of E we get that $\int_W w^*c = 0$ as dw everywhere has rank two, so $E\langle g' \circ \overline{g}\rangle = 1$. By using the previous lemma we find $E(g' \circ \overline{g}) = \mathrm{Id}$ from whence $E(g') \circ E(\overline{g}) = \mathrm{Id}$, and $E(g') = E(\overline{g})^{-1}$ as required. □

This means that a thin-invariant field theory descends to a symmetric monoidal functor on the thin-cobordism surface category of X, the category obtained by replacing "thin-homotopic" by "thin-cobordant" in the above definition. One fundamental property of this category is that it is a groupoid, unlike the thin-homotopy category. This is proved by the proposition in Section 2 and we get the important relation $E(\overline{g}) = E(g)^{-1}$.

4 On flat thin-invariant field theories and homotopy quantum field theories

We will elucidate the connection between thin-invariant field theories and homotopy quantum field theories by showing that a certain subset of homotopy quantum field theories, the rank one, normalised ones, are the same as flat thin-invariant field theories.

Recall that the homotopy surface category is defined by replacing the term "thin-homotopic" by the term "homotopic" in the definition of the thin-homotopy surface category. A 1+1-dimensional homotopy quantum field theory on a space is a symmetric monoidal functor from the homotopy surface category of the space to the category of vector spaces. This is a slight variation on Turaev's original definition better suited to spaces with a non-trivial second homotopy group. We will be interested in rank one homotopy quantum field theories, that is those which are functors to the subcategory of one dimensional vector spaces. In what follows, HQFT means rank one, 1+1-dimensional homotopy quantum field theory.

A flat thin-invariant field theory is one whose three-form is zero, so it descends to a functor on the homotopy surface category and gives rise to an HQFT. However not all HQFTs arise in this way, as is illustrated by the case of a point. An HQFT on a point is the same thing as a topological quantum field theory and a rank one topological quantum field theories is determined by the invariant $\omega \in \mathbb{C}$ of the two-sphere, the genus l surface having invariant ω^{2-l}. On the other hand, all thin-invariant field theories on a point are trivial, as all surfaces are cobordant. We wish to compensate for this, so we make the following definition.

Definition 4.1 An HQFT on a space X together with a point in X gives rise to a topological quantum field theory by considering the constant X-surfaces to the point. Note that if two points are in the same connected component then the topological quantum field theories induced in this way are isomorphic. An HQFT is *normalised* if for every point in X the induced topological quantum field theory is trivial.

The key property of a normalised HQFT is that holonomy of a closed X-surface only depends on the homology class of the X-surface. This is the content of the following proposition.

Proposition 4.2 *For a rank one, normalised, 1+1-dimensional homotopy quantum field theory, the holonomy of a closed X-surface $g\colon \Sigma \to X$ depends only on the homology class $g_*[\Sigma] \in H_2(X;\mathbb{Z})$.*

We delay the proof until after the next theorem.

Theorem 4.3 *Every rank one normalised 1+1-dimensional homotopy quantum field theory can be considered as a flat thin-invariant field theory and vice versa.*

Proof The discussion earlier shows that every flat thin-invariant field theory can be thought of as a normalised HQFT and we now prove that the converse also holds.

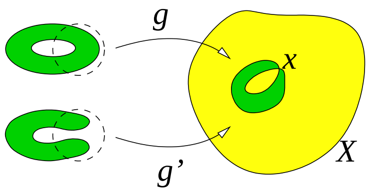

Figure 1: An example of a local surgery. Both parts contained in the dashed circles are mapped to the point x.

To show that every normalised HQFT comes from a thin-invariant field theory we need to show that it satisfies the three-form condition with the three-form equal to zero, ie. if $v\colon V \to X$ is an X-three-manifold then $H(\partial v) = 1$, but that is true because $H(\partial v)$ only depends on the homology class of ∂v, which is zero as it is cobordant to the empty manifold. □

The remainder of this section is dedicated to the proof of Proposition 4.2. First we need some results about local surgery.

Definition 4.4 An X-surface $g'\colon \Sigma' \to X$ is said to be obtained *by a local surgery* from $g\colon \Sigma \to X$ if Σ' with two discs removed is diffeomorphic to Σ with a cylinder removed, the maps g and g' agree on the diffeomorphic parts and they map the discs and cylinder mapped to a single point of X. (See Figure 1.)

Informally this says that while the surfaces may be topologically different, they differ only at the inverse image of a point in X.

Lemma 4.5 *Two closed X-surfaces are cobordant if and only if they can be connected by a sequence of homotopies, local surgeries, and disjoint unions with null-homotopic spheres.*

Proof Suppose that $w\colon W \to X$ is a cobordism from $g\colon \Sigma \to X$ to $g'\colon \Sigma' \to X$. By standard results in Morse theory [13], we can pick a Morse function $f\colon W \to [0,1]$ such that $f^{-1}(0) = \Sigma$ and $f^{-1}(1) = \Sigma'$. Let $0 < f_1 < \cdots < f_c < 1$ be the critical values of f, then the restricted maps $w\colon f^{-1}((f_i, f_{i+1})) \to X$ for $i = 1 \ldots c-1$ give a sequence of homotopies. The critical points of index zero and three correspond to the addition and deletion of null-homotopic two-spheres, while those of index one and two correspond to local surgeries.

Conversely, if we have a sequence of such alterations connecting g and g' then this gives rise to a bordism by reversing the above procedure. □

Lemma 4.6 *The holonomy of a rank one, normalised homotopy quantum field theory around a closed X-surface is unchanged by local surgery and by the disjoint union with a null-homotopic two-sphere.*

Proof Firstly, if $s\colon S^2 \to X$ is a null-homotopic map then it is homotopic to the constant map $S^2 \to \{*\}$ for some point $* \in X$. It follows that $H(S^2 \to \{*\}) = 1$ as H is normalised, ie. induces the trivial topological quantum field theory. Thus if g is any X-surface then

$$H(g \sqcup s) = H(g)H(s) = H(g)H(S^2 \to \{*\}) = H(g).$$

Secondly, consider the union of two discs, $D \sqcup D$ and the cylinder C, as surfaces with two inputs and no output. As the induced TQFT of H is trivial we have $H(D \sqcup D \to \{*\}) = H(C \to \{*\})$ as maps $H(S^1 \sqcup S^1) \to \mathbb{C}$.

If $g'\colon \Sigma' \to X$ is obtained from $g\colon \Sigma \to X$ by a local surgery then let Ω be the surface with two outgoing boundary component such that $\Omega \cup \overline{D} \cup \overline{D} \cong \Sigma$ and $\Omega \cup C \cong \Sigma'$ with $g|_\Omega = g'|_\Omega$ and such that $g|_{D \sqcup D}$ and $g|_C$ are constant maps to the point $\{*\}$. Thus

$$\begin{aligned} H\langle g\rangle &= H\left\langle (D \sqcup D \to \{*\}) \circ g|_\Omega \right\rangle = H\left(D \sqcup D \to \{*\}\right) \circ H\left(g|_\Omega\right) \\ &= H\left(C \to \{*\}\right) \circ H\left(g|_\Omega\right) = H\left\langle (C \to \{*\}) \circ g'|_\Omega \right\rangle \\ &= H\langle g'\rangle. \end{aligned}$$

Which is what was required. □

We can now prove Proposition 4.2, which stated that the holonomy of a normalised HQFT depends only on the homology of X-surfaces.

Proof of Proposition 4.2 Suppose that $g\colon \Sigma \to X$ and $g'\colon \Sigma' \to X$ are homologous closed X-surfaces, in the sense that $g_*[\Sigma] = g'_*[\Sigma'] \in H_2(X, \mathbb{Z})$. We need to show that $H\langle g\rangle = H\langle g'\rangle$.

By Lemma 1.1, g is bordant to g', so Lemma 4.5 implies that there is a sequence of homotopies, local surgeries, and disjoint unions with two-spheres connecting g to g'. The definition of an HQFT ensures that the holonomy does not change under homotopy, and Lemma 4.6 ensures that it is unchanged under the latter two as well. Thus $H(g) = H(g')$ as required. □

5 Examples of thin-invariant field theories

In this section we present a number of examples of thin-invariant field theories.

5.1 Manifolds with trivial first homology

The first example applies to spaces with trivial first integral homology group. For such a space we build a thin-invariant field theory starting from a gerbe with connection. The construction was partially inspired by [8].

Let X be a smooth finite dimensional manifold with $H_1(X,\mathbb{Z})$ trivial, and let S be the holonomy of a gerbe with connection. If γ is an object in the thin-cobordism category $\mathcal{T}_X$ then Lemma 1.1 implies that γ is null cobordant. Define the one dimensional vector space $E(\gamma)$ to be the space of complex linear combinations of null cobordisms of γ modulo a relation involving the gerbe holonomy S:

$$E(\gamma) := \mathbb{C}\mathrm{Hom}_{\mathcal{T}_X}(\emptyset,\gamma) \;/\; \{h_1 = S\langle h_1 \circ \overline{h_2}\rangle h_2\}.$$

This is clearly one-dimensional.

If g is a morphism in $\mathcal{T}_X$ from γ to γ' then define $E(g)\colon E(\gamma) \to E(\gamma')$ by $E(g)h := g \circ h$. This is well defined on $E(\gamma)$ because of the following:

$$\begin{aligned} E(g)(S\langle h_1 \circ \overline{h_2}\rangle h_2) &= S\langle h_1 \circ \overline{h_2}\rangle g \circ h_2 = S\langle g \circ h_1 \circ \overline{h_2} \circ \overline{g}\rangle g \circ h_2 \\ &= g \circ h_1 = E(g)h_1. \end{aligned}$$

Functoriality is immediate: $E(g \circ g') = E(g) \circ E(g')$.

To show thin-invariance we need to show that if g is thin-cobordant to g' then $E(g) = E(g')$. If g is thin-cobordant to g' then by Proposition 2.1, $S\langle \overline{g'} \circ g\rangle = 1$ and we find

$$\begin{aligned} E(g)h = g \circ h &= S\langle g \circ h \circ \overline{h} \circ \overline{g'}\rangle g' \circ h = S\langle h \circ \overline{h} \circ \overline{g'} \circ g\rangle g' \circ h \\ &= S\langle h \circ \overline{h}\rangle S\langle \overline{g'} \circ g\rangle g' \circ h = g' \circ h = E(g')h. \end{aligned}$$

It follows from this invariance that E also respects the identity maps.

The smoothness condition is automatically satisfied by the curvature three-form of the gerbe.

To show that E is symmetric monoidal it is necessary to show that there are symmetric natural isomorphisms:

$$\Phi_{\gamma,\gamma'}\colon E(\gamma) \otimes E(\gamma') \xrightarrow{\cong} E(\gamma \sqcup \gamma').$$

Define $\Phi_{\gamma,\gamma'}(h \otimes h') = h \sqcup h'$ and note this is well defined since

$$\begin{aligned} \Phi_{\gamma,\gamma'}(S\langle h_1 \circ \overline{h_2}\rangle h_2 \otimes S\langle h'_1 \circ \overline{h'_2}\rangle h'_2) S\langle h_1 \circ \overline{h_2}\rangle S\langle h'_1 \circ \overline{h'_2}\rangle h_2 \sqcup h'_2 \\ = S\langle (h_1 \circ \overline{h_2}) \sqcup (h'_1 \circ \overline{h'_2})\rangle h_2 \sqcup h'_2 \\ = S\langle (h_1 \sqcup h'_1) \circ (\overline{h_2 \sqcup h'_2})\rangle h_2 \sqcup h'_2 \\ = h_1 \sqcup h'_1. \end{aligned}$$

Moreover, if $g \in \mathrm{Hom}_{\mathcal{T}_X}(\gamma_1, \gamma_2)$ and $g' \in \mathrm{Hom}_{\mathcal{T}_X}(\gamma_1', \gamma_2')$ then

$$\begin{aligned} E(g \sqcup g') \circ \Phi_{\gamma_1, \gamma_1'}(h \otimes h') E(g \sqcup g')(h \sqcup h') &= (g \sqcup g') \circ (h \sqcup h') \\ &= (g \circ h) \sqcup (g' \circ h') \Phi_{\gamma_2, \gamma_2'}(g \circ h \otimes g' \circ h') \\ &= \Phi_{\gamma_2, \gamma_2'}(E(g)h \otimes E(g')h') \\ &= \Phi_{\gamma_2, \gamma_2'} \circ (E(g) \otimes E(g'))(h \otimes h'), \end{aligned}$$

proving that the $\Phi_{\gamma,\gamma'}$ are natural. Let T be the flip $E(\gamma) \otimes E(\gamma') \to E(\gamma') \otimes E(\gamma)$ and $\kappa \in \mathrm{Hom}_{\mathcal{T}_X}(\gamma \sqcup \gamma', \gamma' \sqcup \gamma)$ be the symmetric structure isomorphism for γ and γ'. Then

$$\begin{aligned} \Phi_{\gamma',\gamma} \circ T(h \otimes h') = \Phi_{\gamma',\gamma}(h' \otimes h) = h' \sqcup h = S\langle (h \sqcup h') \circ \overline{\kappa \circ (h \sqcup h')} \rangle \kappa \circ (h \sqcup h') \\ = E(\kappa)(h \sqcup h') = E(\kappa) \circ \Phi_{\gamma,\gamma'}(h \otimes h'). \end{aligned}$$

This proves that E is a thin-invariant field theory.

5.2 Gerbes with trivial Dixmier-Douady class

The second example does not require any restrictions on the manifold X, and builds a thin-invariant field theory from a gerbe with connection whose Dixmier-Douady class is zero. By the exact sequence of Section 2 such a gerbe may be represented, non-uniquely, by a 2-form ω, with the holonomy around a closed X-surface $g\colon \Sigma \to X$ given by $\exp(2\pi i \int_\Sigma g^*\omega)$. Now define a thin-invariant field theory by setting $E(\gamma) = \mathbb{C}$ for each object γ in $\mathcal{T}_X$, and for each morphism $g\colon \Sigma \to X$ defining $E(g)\colon \mathbb{C} \to \mathbb{C}$ to be multiplication by $\exp(2\pi i \int_\Sigma g^*\omega)$. The monoidal structure $\Phi_{\gamma,\gamma'}\colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ is the canonical isomorphism. Thin-invariance follows from Stokes' Theorem. It is evident that the holonomy of this thin-invariant field theory is the same as that of the original gerbe.

5.3 Flat thin-invariant field theories from two-cocycles

The third example uses the identification of flat thin-invariant field theories with normalised HQFTs (Theorem 4.3) to get examples of flat thin-invariant field theories from a construction of Turaev [19]. Note that we use a slightly different convention to Turaev to ensure that we get the correct holonomy, and not its inverse. Let $\theta \in C^2(X, \mathbb{C}^\times)$ be a two-cocycle, and define $E^\theta(\gamma)$ for an object $\gamma\colon S_m \to X$ by taking all one-cycles which represent the fundamental class of S_m and quotienting by a certain relation:

$$E^\theta(\gamma) := \mathbb{C}\left\{ a \in C_1(S_m) \;\middle|\; [a] = [S_m] \right\} \Big/ \mathbb{C}\left\{ a - \gamma^*\theta(e) b \;\middle|\; \begin{array}{c} e \in C_2(S_m) \\ \partial e = a - b \end{array} \right\}.$$

Write $|a|$ for the equivalence class in $E^\theta(\gamma)$ of the one-cycle a. To a cobordism $g\colon \Sigma \to X$ from γ_0 to γ_1 we need to associate a linear map. This is done by picking a singular two-cycle representative $f \in C_2(\Sigma)$ of the fundamental class $[\Sigma] \in H_2(\Sigma, \partial\Sigma)$. Then $E^\theta(g)\colon E^\theta(\gamma_0) \to E^\theta(\gamma_1)$ is defined by $E^\theta(g)|a_0| := g^*\theta(f)|a_1|$, where $\partial f = a_0 - a_1$. For objects γ_0 and γ_1 the monoidal structure map $\Phi_{\gamma_0,\gamma_1}\colon E^\theta(\gamma_0) \otimes E^\theta(\gamma_1) \to E^\theta(\gamma_0 \sqcup \gamma_1)$ is defined such that $\Phi_{\gamma_0,\gamma_1}(|a_0| \otimes |a_1|) := |a_0 \sqcup a_1|$.

Turaev shows that this is well-defined and gives a normalised HQFT, hence, by Theorem 4.3, we get a flat thin-invariant field theory. In actual fact this gives rise to a group homomorphism from $H^2(X, \mathbb{C}^\times)$ to the group of flat thin-invariant field theories up to equivalence, as if two two-cocycles differ by a coboundary then the thin-invariant field theories are non-canonically isomorphic in the following manner. If $\theta = \theta' + \delta f$ where $f \in C^1(X)$ then for an object γ in $\mathcal{T}_X$ define $\Psi_\gamma\colon E^\theta(\gamma) \to E^{\theta'}(\gamma)$ by $\Psi_\gamma(|a|) := (f(\gamma_* a))^{-1}|a|$: it transpires that Ψ is a natural transformation giving an isomorphism of thin-invariant field theories. We have a group homomorphism because the theory $E^{\theta_1+\theta_2}$ constructed from $\theta_1 + \theta_2 \in C^2(X)$ is isomorphic to $E^{\theta_1} \otimes E^{\theta_2}$.

6 Thin-invariant field theories and gerbes

The goal of this section is to show that a gerbe with a connection is the same thing as a thin-invariant field theory.

6.1 Ext groups and monoidal functors

To prove the main theorem we are going to need an aside on Ext groups. We will start with a little reminder. If Γ and A are abelian group then $\mathrm{Ext}(\Gamma, A)$ is the set of all abelian extensions of Γ by A, that is to say, all abelian groups $\hat\Gamma$ which fit into an exact sequence $0 \to A \to \hat\Gamma \to \Gamma \to 0$, modulo some suitable notion of equivalence. Similarly the group cohomology group $H^2_{\mathrm{gp}}(\Gamma, A)$ can be identified with the set of central extensions of Γ by A, that is those $\hat\Gamma$ as above which are not necessarily abelian, but in which A is a central subgroup.

We will be interested in the case that A is $K^\times$, the group of units of a commutative ring K. In this case a $K^\times$-extension of Γ is like a K-line bundle over the discrete space Γ. We will need the notion of a $K^\times$-torsor which is just another name for a principal $K^\times$-homogeneous space, ie. a space with a transitive and free $K^\times$-action. Of course such a space is homeomorphic to $K^\times$, but in general there will be no canonical homeomorphism. The collection of $K^\times$-torsors forms a symmetric monoidal category, in which the morphisms are the maps commuting with the action, the monoidal product is given by $A \cdot B := A \times_{K^\times} B$, and the unit object is just $K^\times$.

Lemma 6.1 *If K is a commutative ring then there is an equivalence of symmetric monoidal categories between the category of $K^\times$-torsors and the category of rank one K-modules.*

Proof The functor $\{K^\times\text{–torsors}\} \to \{\text{rank one } K\text{–modules}\}$ is given by $A \mapsto K \times_{K^\times} A$, and the functor $\{\text{rank one } K\text{–modules}\} \to \{K^\times\text{–torsors}\}$ is given by $M \mapsto \{\text{generators of } M \text{ over } K\}$. The reader is left to fill in the uninspiring details. □

Given an abelian group Γ we can form a strict symmetric monoidal category $\underline{\Gamma}$ whose objects are the elements of the group, whose morphisms are just the identity morphisms and whose monoidal structure is just given by the group multiplication. In the next section we will be using $\underline{H_1(X,\mathbb{Z})}$, which is just the category consisting of the "connected components" of the thin-cobordism category. Now we will relate functors from $\underline{\Gamma}$ with extensions of the group Γ.

Lemma 6.2 *Suppose that Γ is an abelian group and K is a commutative ring. There is a naturual bijection between $\operatorname{Ext}(\Gamma, K^\times)$, the abelian extensions of Γ by $K^\times$, and the set of symmetric monoidal functors from the category $\underline{\Gamma}$ defined above to the category of rank one K-modules.*

Proof Firstly, in view of the previous lemma, we can equivalently consider symmetric monoidal functors from $\underline{\Gamma}$ to the category of $K^\times$-torsors.

So suppose that $E\colon \underline{\Gamma} \to \{K^\times\text{–torsors}\}$ is a symmetric monoidal functor. Define $\hat{\Gamma} := \bigcup_{x\in\Gamma} E(x)$. We need to show that this is an abelian extension of Γ by $K^\times$. The multiplication comes from the monoidal structure $E(x)\cdot E(y) \to E(xy)$, this is associative because of the associativity axiom for monoidal structure. Inverses exist for the following reason: if $\alpha \in \hat{\Gamma}$ lives in $E(x)$ then there is a map $\Phi_x\colon E(x)\cdot E(x^{-1}) \to K^\times$, pick any element $\beta \in E(x^{-1})$ and take α^{-1} to be $(\Phi_x(\alpha,\beta))^{-1}\beta$. There is the obvious quotient map $p\colon \hat{\Gamma} \to \Gamma$, which is automatically a group homomorphism, and there is the inclusion homomorphism $K^\times \cong E(1) \hookrightarrow \hat{\Gamma}$ coming from the unit axiom for a symmetric monoidal functor. The symmetric axiom then gives that $\hat{\Gamma}$ is abelian.

Conversely, if $0 \to K^\times \xrightarrow{i} \hat{\Gamma} \xrightarrow{p} \Gamma \to 0$ is an abelian extension of Γ by $K^\times$ then define $E\colon \underline{\Gamma} \to \{K^\times\text{–torsors}\}$ by $E(x) := p^{-1}(x)$. Note that $K^\times$ acts transitively and freely on $p^{-1}(x)$ via i. The structure maps $E(x){\cdot}E(y) \to E(xy)$ come directly from the product in $\hat{\Gamma}$ and these are symmetric because $\hat{\Gamma}$ is abelian. Isomorphic extensions give isomorphic functors. □

Note that if we replace the phrase "abelian extension" by "central extension" and "symmetric monoidal functor" by "monoidal functor" in the above proof then we get a bijection between $H^2_{\mathrm{gp}}(\Gamma, K^\times)$ and monoidal functors from $\underline{\Gamma}$ to the category of rank one K–modules.

6.2 The Main Theorem

Now we can prove the main result of this paper.

Theorem 6.3 *On a smooth finite dimensional manifold, there is an isomorphism from the group of thin-invariant field theories (up to equivalence) to the group of gerbes with connection (up to equivalence).*

Proof We will show that the holonomy $S\colon\ \mathrm{TIFT}(X) \to \widehat{H}^3(X)$ is an isomorphism.

Firstly to show that S is injective we will identify the kernel of S with the set of symmetric monoidal functors $\underline{H_1(X,\mathbb{Z})} \to \mathrm{Vect}_1$. Then we can use Lemma 6.2 to identify this set with $\mathrm{Ext}(H_1(X,\mathbb{Z}), \mathbb{C}^\times)$ which we know to be trivial as $\mathbb{C}^\times$ is a divisible group.

To identify the kernel of S with the collection of symmetric monoidal functors $\underline{H_1(X,\mathbb{Z})} \to \mathrm{Vect}_1$ we proceed as follows. Suppose $E \in \mathrm{Ker}(S)$, then we will construct a symmetric monoidal functor $\mathcal{E}\colon \underline{H_1(X,\mathbb{Z})} \to \mathrm{Vect}_1$. Suppose γ_1 and γ_2 are objects in $\mathcal{T}_X$, and suppose $g, g' \in \mathrm{Hom}_{\mathcal{T}_X}(\gamma_1,\gamma_2)$. Since E has trivial holonomy, we have $E\langle g \circ \overline{g'}\rangle = 1$ and it follows from Lemma 3.4 that $E(g)\circ E(\overline{g'}) = \mathrm{Id}$ and hence from the discussion after Theorem 3.5 that $E(g') = E(g)$. Thus, for each cobordant γ_1 and γ_2 there is a canonical identification of $E(\gamma_1)$ with $E(\gamma_2)$. By Lemma 1.1, γ_1 and γ_2 are cobordant if and only if they belong to the same homology class, and we can therefore associate in a natural way a one dimensional vector space $\mathcal{E}(x)$ to each homology class $x \in H_1(X,\mathbb{Z})$.

There are natural isomorphisms $\Phi_{x,x'}\colon \mathcal{E}(x)\otimes\mathcal{E}(x') \to \mathcal{E}(x+x')$ obtained by choosing γ to represent x and γ' to represent x', so that $\mathcal{E}(x) \cong E(\gamma)$ and

$\mathcal{E}(x') \cong E(\gamma')$, and then setting $\Phi_{x,x'} = \Phi_{\gamma,\gamma'}$. It follows from properties of $\Phi_{\gamma,\gamma'}$ that these are well defined natural isomorphisms. Thus, $\mathcal{E}$ is a symmetric monoidal functor.

If the functor just defined is isomorphic to the trivial one then the field theory giving rise to it must also be trivial. Conversely, any monoidal functor from $\underline{H_1(X,\mathbb{Z})}$ to Vect_1 can be extended to a field theory with trivial holonomy. Thus there is a bijection of $\mathrm{Ker}(S)$ with symmetric monoidal functors $\underline{H_1(X,\mathbb{Z})} \to \mathrm{Vect}_1$ as required. As explained at the beginning of the proof, this shows that $\mathrm{Ker}(S) = 0$.

The second step is to show that S is surjective. Example 5.1 shows that if X is simply connected then every gerbe is the image under S of some thin-invariant field theory. Similarly Example 5.2 shows that for arbitrary X, every gerbe with zero Dixmier-Douady class comes, via S, from a thin-invariant field theory.

For the general case suppose we have a gerbe holonomy $\mathcal{S}$. Let M be a smooth manifold which is $\dim(X)+1$-homotopy equivalent to $K(\mathbb{Z},3)$. By identifying $H^3(X,\mathbb{Z})$ with homotopy classes of maps $X \to M$ we can choose a smooth map $f\colon X \to M$ representing the Dixmier-Douady class of $\mathcal{S}$. Now let $\mathcal{S}_1$ be a gerbe over M whose Dixmier-Douady class is the generator of $H^3(M,\mathbb{Z})$. Since M is simply connected we can apply Example 5.1 to obtain a thin-invariant field theory E_1 over M such that $S_{E_1} = \mathcal{S}_1$. The gerbe-holonomy $\mathcal{S} \otimes f^*\mathcal{S}_1^{-1}$ has Dixmier-Douady class zero and we can apply Example 5.2 to obtain a thin-invariant field theory E_0 over X. Finally, the thin-invariant field theory $E = E_0 \otimes f^*E_1$ satisfies $S_E = \mathcal{S}$. □

Note that the second half of this proof could have been done more neatly if we could classify the gerbe holonomy by a smooth map $X \to BBS^1$, but this does not seem to be in [10].

Combining this with Theorem 4.3, we obtain the identification of normalised HQFTs with flat gerbes:

Theorem 6.4 *On a finite dimensional manifold, the group of normalised rank one homotopy quantum field theories (up to equivalence) is isomorphic to the group of flat gerbes (up to equivalence).*

The definition of thin-invariant field theory requires the functor $E\colon \mathcal{T}_X \to \mathrm{Vect}_1$ to be symmetric. In view of the motivation of this paper, namely to reconcile homotopy quantum field theories and gerbes, this is an entirely natural assumption to make. However, one can drop this assumption, to get *non-symmetric* thin-invariant field theories. In this case there is an analogue of Theorem 6.3.

Theorem 6.5 *There is a split short exact sequence*

$$0 \to H^2_{gp}(H_1(X,\mathbb{Z}),\mathbb{C}^\times) \to \left\{ \begin{array}{c} \textit{non-symmetric} \\ \textit{thin-invariant} \\ \textit{field theories on } X \end{array} \right\} \to \widehat{H}^3(X) \to 0.$$

Proof The proof of Theorem 6.3 goes through almost exactly, by replacing the Ext group with the cohomology group, as in the comment after Lemma 6.2. The splitting comes from the fact that we have already identified the group $\widehat{H}^3(X)$ with (symmetric) thin-invariant field theories. □

7 Normalised homotopy quantum field theories and flat gerbes

Homotopy quantum field theories can be defined over rings other than $\mathbb{C}$, so let K be any commutative ring, and recall that HQFT is used to mean rank one 1+1-dimensional homotopy quantum field theory. Turaev's construction in Example 5.3 can be generalised to give a map

$$\tau\colon H^2(X,K^\times) \to \{\text{normalised } K\text{-HQFTs}\}.$$

In [19], using his classification of homotopy quantum field theories for Eilenberg-MacLane spaces in terms of crossed algebras, Turaev proved that when X is a $K(\pi,1)$ and K is a field of characteristic zero then τ is an isomorphism. We generalize this in the following manner.

Theorem 7.1 *Let K be a commutative ring, and X be a path connected topological space. Then Turaev's construction gives an isomorphism between the group $H^2(X,K^\times)$ and the group of normalised, rank one homotopy quantum field theories defined over K.*

Proof The proof will proceed like so. We will construct the following diagram.

$$\begin{tikzcd} \mathrm{Ext}(H_1(X,\mathbb{Z}),K^\times) \arrow[r, hook, "\iota"] \arrow[d, equal] & H^2(X,K^\times) \arrow[r, two heads] \arrow[d, "\tau"] & \mathrm{Hom}(H_2(X,\mathbb{Z}),K^\times) \arrow[d, equal] \\ \mathrm{Ext}(H_1(X,\mathbb{Z}),K^\times) \arrow[r, hook] & \{\text{normalised HQFTs}\} \arrow[r, "S"] & \mathrm{Hom}(H_2(X,\mathbb{Z}),K^\times) \end{tikzcd}$$

We will show that it is commutative and then invoke the Five Lemma to deduce that the map τ is an isomorphism.

First then we describe the morphisms in the diagram. The top row is the short exact sequence from the Universal Coefficient Theorem. The map τ is Turaev's construction described in Example 5.3. The map S is the holonomy map which is well defined by Proposition 4.2. The kernel of S is $\mathrm{Ext}(H_1(X,\mathbb{Z}),K^\times)$ by the proof of Theorem 6.3.

Now we consider the commutativity of the diagram. The right-hand square is commutative by the definition of τ. The commutativity of the left-hand square will take up the rest of the proof.

We need to describe the inclusion $\iota\colon \mathrm{Ext}(H_1(X,\mathbb{Z}),K^\times) \hookrightarrow H^2(X,K^\times)$, in which we are considering $\mathrm{Ext}(H_1(X,\mathbb{Z}),K^\times)$ as the group of equivalence classes of extensions of abelian groups. Let B_1 and Z_1 be the groups of one-boundaries and one-cycles on X, then $H_1(X,\mathbb{Z})$ has the following free-resolution: $B_1 \hookrightarrow Z_1 \overset{q}{\twoheadrightarrow} H_1(X,\mathbb{Z})$. Now given ϵ an abelian extension $K^\times \hookrightarrow \hat{\Gamma} \overset{p}{\twoheadrightarrow} H_1(X,\mathbb{Z})$, we can lift the morphism q to a morphism $\hat{q}\colon Z_1 \to \hat{\Gamma}$. Now we can define $\theta \in C^2(X,K^\times)$ by

$$\theta_\epsilon(e) := \hat{q}(\partial e); \quad \text{for } e \in C_2(X,K^\times).$$

The map θ is immediately seen to be a cocycle and it is $K^\times$-valued because $\partial e \in B_1$ so $q(\partial e) = 0$. It transpires that the cohomology class $[\theta]$ is precisely $\iota([\epsilon]) \in H^2(X,K^\times)$.

Here we will take a slightly different but equivalent and more convenient point of view of HQFTs, which is entirely analogous to thinking of *principal* $\mathbb{C}^\times$*-bundles* rather than *complex line bundles.* We will think of HQFTs as associating to an X-one-manifold a $K^\times$-torsor rather than a rank one K-module. In view of Lemma 6.1 this does not alter anything. From this point of view, the HQFT associated to θ by τ, which will be denoted E^θ, is defined on $\gamma\colon S_m \to X$ by

$$E^\theta(\gamma) := K^\times \cdot \left\{ a \in C_1(S_m) \,\middle|\, [a] = [S_m] \right\} \Big/ \left\{ a = \gamma^*\hat{q}(\partial e) b \,\middle|\, \begin{matrix} e \in C_2(S_m) \\ \partial e = a - b \end{matrix} \right\}.$$

Remember that we use the notation $|a|$ for the equivalence class of a in $E^\theta(\gamma)$.

The homomorphism sending the extension ϵ to E^θ gives the composition of two of the sides of the left-hand square in the diagram. The composition of the other two is got by looking at the proof of Theorem 6.3, by which we see that the HQFT, E^ϵ, obtained from ϵ, is as follows. Recalling that ϵ is the abelian extension $K^\times \hookrightarrow \hat{\Gamma} \overset{p}{\twoheadrightarrow} H_1(X,\mathbb{Z})$, to each object $\gamma\colon S_m \to X$ we associate the $K^{\times}$-torsor $E^{\epsilon}(\gamma) := p^{-1}([\gamma])$, where $[\gamma] \in H_1(X, \mathbb{Z})$ denotes the class represented by γ. The morphisms are mapped to identities, and the symmetric monoidal structure $\Phi^{\epsilon}_{\gamma,\gamma'} \colon E^{\epsilon}(\gamma) \otimes E^{\epsilon}(\gamma') \to E^{\epsilon}(\gamma \sqcup \gamma')$ is given by the group structure of $\hat{\Gamma}$.

It suffices to define an equivalence of symmetric monoidal functors $\Psi \colon E^{\theta} \to E^{\epsilon}$. This is defined by $\Psi_{\gamma}|a| := \gamma^* \hat{q}(a)$. This is well-defined as

$$\begin{aligned}\Psi_{\gamma}|a| &= \gamma^* \hat{q}(a) = \gamma^* \hat{q}(b) \gamma^* \hat{q}(b)^{-1} \gamma^* \hat{q}(a) = \gamma^* \hat{q}(a-b) \gamma^* \hat{q}(b) \\ &= \gamma^* \hat{q}(\partial e) \gamma^* \hat{q}(b) = \Psi_{\gamma}(\gamma^* \hat{q}(\partial e)|b|).\end{aligned}$$

We now verify that Ψ is a natural transformation. Let $g : \Sigma \to X$ be a cobordism from γ_0 to γ_1. Let $f \in C_2(\Sigma)$ represent $[\Sigma] \in H_2(\Sigma, \partial\Sigma)$ and be such that $a_0 - a_1 = \partial f$, where $|a_i| \in E^{\theta}(\gamma_i)$. Then we have

$$\begin{aligned}\Psi_{\gamma_1}\big(E^{\theta}(g)|a_0|\big) &= \Psi_{\gamma_1}\big(g^*\theta_{\epsilon}(f)|a_1|\big) = g^* \hat{q}(\partial f) \Psi_{\gamma_1}|a_1| = g^* \hat{q}(a_0 - a_1) \gamma_1^* \hat{q}(a_1) \\ &= \gamma_0^* \hat{q}(a_0) \gamma_1^* \hat{q}(a_1)^{-1} \gamma_1^* \hat{q}(a_1) = \Psi_{\gamma_0}|a_0| = E^{\epsilon}(g)(\Psi_{\gamma_0}|a_0|).\end{aligned}$$

Finally we verify, that Ψ is compatible with the monoidal structures.

$$\begin{aligned}\Psi_{\gamma_0 \sqcup \gamma_1}\big(\Phi^{\theta}_{\gamma_0,\gamma_1}(|a_0| \otimes |a_1|)\big) &= \Psi_{\gamma_0 \sqcup \gamma_1}(|a_0 \sqcup a_1|) = (\gamma_0 \sqcup \gamma_1)^* \hat{q}(a_0 + a_1) \\ &= \gamma_0^* \hat{q}(a_0) \gamma_1^* \hat{q}(a_1) = \Phi^{\epsilon}_{\gamma_0,\gamma_1}(\gamma_0^* \hat{q}(a_0) \otimes \gamma_1^* \hat{q}(a_1)) \\ &= \Phi^{\epsilon}_{\gamma_0,\gamma_1}\big(\Psi_{\gamma_0}|a_0| \otimes \Psi_{\gamma_1}|a_1|\big).\end{aligned}$$

Thus E^{θ} and E^{ϵ} are isomorphic and the left-hand square of the diagram commutes. As mentioned above, the Five Lemma can now be invoked to prove that τ is an isomorphism. □

8 The line bundle on loop space

Associated to a gerbe with connection on a manifold X is a line bundle with connection on $\mathcal{L}X$, the free loop space on X thought of as an infinite dimensional manifold (see [4, Chapter 6]). Recalling from the introduction that the second Cheeger-Simons group classifies line bundles with connection, this association can be viewed as the transgression map $t \colon \widehat{H}^3(X) \to \widehat{H}^2(\mathcal{L}X)$ which is described below. Alternatively, given a thin-invariant field theory $E \colon \mathcal{T}_X \to \mathrm{Vect}_1$ we can restrict this to a functor on the path category of the loop space, which gives us a line bundle with connection on the free loop space. Not surprisingly, since this is where the origins of the definition of a thin-invariant field theory lie, these two ways of getting a line bundle coincide.

Theorem 8.1 *A thin-invariant field theory can be restricted to the path category of the free loop space giving a line bundle with connection on the free loop space. This line bundle with connection is isomorphic to the transgression of the gerbe associated to the thin-invariant field theory.*

Proof First we need to describe the transgression map $t\colon \widehat{H}^3(X) \to \widehat{H}^2(\mathcal{L}X)$. Given $S \in \widehat{H}^3(X)$ a gerbe holonomy, define $t(S) \in \widehat{H}^2(\mathcal{L}X)$ as follows. If $\gamma\colon S_n \to \mathcal{L}X$ is a smooth map, then we have an induced map $\check{\gamma}\colon S^1 \times S_n \to X$ given by $\check{\gamma}(r,s) := \gamma(s)(r)$: set $t(S)(\gamma) := S(\check{\gamma})$. (The curvature of S is transgressed as $t(c) = \pi_* \mathrm{ev}^* c$ where $\mathrm{ev}\colon S^1 \times \mathcal{L}X \to X$ is the evaluation map, and $\pi_*\colon \Omega^3(S^1 \times \mathcal{L}X) \to \Omega^2(\mathcal{L}X)$ is integration over the fibre.)

Now if E is a thin-invariant field theory with holonomy S_E to verify that the restriction $E'\colon P\mathcal{L}X \to \mathrm{Vect}_1$ is a line bundle coinciding with the transgression $t(S_E)$ it suffices to compare holonomies. Let $\gamma\colon S^1 \to \mathcal{L}X$ be a smooth loop. We view γ as a map $\check{\gamma}\colon S^1 \times S^1 \to X$, $\check{\gamma}(r,s) := \gamma(s)(r)$. On the one hand the holonomy of E' along γ is then given by $E(\check{\gamma}) \in \mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^\times$. On the other hand, the holonomy of $t(S_E)$ along γ is (by the definition of S_E) equal to $S_E(\check{\gamma}) = E(\check{\gamma})$. □

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A Appendix: The Cheeger-Simons group

The definition of the Cheeger-Simons group given in the main body of the text is non-standard. Our definition is based on manifolds and maps to X whereas the original definition of Cheeger and Simons uses chains in X. In this appendix we prove that these two definitions are equivalent.

First recall the usual definition of the Cheeger-Simons group. Let Z_2X be the group of smooth two-cycles in X. A *differential character* is a pair (f,c) where f is a homomorphism $f\colon Z_2X \to \mathbb{C}^\times$ and c is a closed three-form such that if B is a three-chain then

$$f(\partial B) = \exp\left(2\pi i \int_B c\right). \tag{1}$$

The collection of differential characters forms a group which we will denote by $\widehat{\mathbb{H}}^3(X)$. This is the usual definition of the Cheeger-Simons group, though the index "3" is *á la* Brylinski as opposed to the "2" used by Cheeger and Simons.

In this paper we considered the group $\widehat{H}^3(X)$ consisting of pairs (S,c) where S is a $\mathbb{C}^\times$-valued function on the space of maps of closed surfaces to X, and c is a three-form such that if $v\colon V \to X$ is a map of a three-manifold to X then

$$S(\partial v) = \exp\left(2\pi i \int_V v^* c\right). \tag{2}$$

We can now show that these two groups are essentially the same.

Theorem A.1 *For a smooth manifold X, the group $\widehat{H}^3(X)$ is canonically isomorphic to the Cheeger-Simons group $\widehat{\mathbb{H}}^3(X)$.*

Proof Define a function $\widehat{H}^3(X) \to \widehat{\mathbb{H}}^3(X)$ via $(S,c) \mapsto (f_S, c)$ where f_S is defined as follows. By the isomorphism from $\mathrm{MSO}_2(X)$ to $H_2(X,\mathbb{Z})$, for a smooth two-cycle y, there is a closed smooth X-surface $g\colon \Sigma \to X$ with fundamental cycle $d \in H_2(\Sigma,\mathbb{Z})$ such that $[g_*(d)] = [y]$ in $H_2(X,\mathbb{Z})$, ie. there is a smooth three-chain B such that $\partial B = -g_*(d) + y$. Now define

$$f_S(y) = S(g)\exp\left(2\pi i \int_B c\right).$$

First we must show that this is well-defined, ie. that it is independent of the choices made and that (f_S, c) satisfies (1). Suppose we are given g', d' and B' such that $\partial B' - g'_*(d') + y$ then

$$g_*(d) - g'_*(d') = (-\partial B + y) - (-\partial B' + y) = \partial(B' - B)$$

showing that $[g_*(d)] = [g'_*(d')] \in H^2(X,\mathbb{Z})$ and so $[g] = [g'] \in MSO_2(X)$, meaning that there is an X-three-manifold $v\colon V \to X$ such that $\partial v g - g'$. Now observe that

$$S(g)/S(g') = S(g - g') = S(\partial v) = \exp\left(2\pi i \int_V v^* c\right).$$

Choosing a fundamental cycle (relative to the boundary) D for V such that $\partial D = d - d'$ we get that $v_*(D) + B - B'$ is a cycle, and so $\exp\left(2\pi i \int_{D+B-B'} c\right) = 1$. Hence

$$\begin{aligned} S(g)\exp\left(2\pi i \int_B c\right) &= S(g')\exp\left(2\pi i \int_D c\right)\exp\left(2\pi i \int_B c\right) \\ &= S(g')\exp\left(2\pi i \int_{B'} c\right), \end{aligned}$$

so that f_S is independent of the choices made.

We must also show that (f_S, c) satisfies (1). If B is a smooth three-chain, then to apply f_S to ∂B, we may choose g and d above to be trivial, so we see immediately that $f_S(\partial B) = \exp\left(2\pi i \int_B c\right)$ as required.

We have a well defined map going one way, so we wish to define a map going the other way, $\widehat{\mathbb{H}}^3(X) \to \widehat{H}^3(X)$, which we do via $(f, c) \mapsto (S_f, c)$ where $S_f(g\colon \Sigma \to X) := f(g_*(d))$, for d a fundamental two-cycle for Σ. We will show that this is an inverse to the above map. We must first show this is a well-defined. Suppose we are given another fundamental two-cycle d', then we can find a three-cycle e in Σ such that $\partial e = d - d'$. Observe that

$$\begin{aligned} f(g_*(d))/f(g_*(d')) &= f(g_*(\partial e)) = f(\partial(g_*(e))) \\ &= \exp\left(2\pi i \int_{g_*(e)} c\right) = \exp\left(2\pi i \int_\Sigma g^* c\right). \end{aligned}$$

However, $\int_\Sigma g^* c = 0$ since c is a three-form and Σ a surface. This shows that $f\left(g_*\left(d\right)\right) = f(g_*(d'))$.

Now we will verify that (S_f, c) satisfies (2). Given $v\colon V \to X$, choose a (relative) fundamental cycle B such that ∂B is a fundamental cycle for ∂V, then

$$S(\partial v) = f\left(\partial v_*(\partial B)\right) = f(\partial(v_* B)) = \exp\left(2\pi i \int_{v_* B} c\right) = \exp\left(2\pi i \int_V v^* c\right).$$

This shows that (S_f, c) is a well defined element of $\widehat{H}^3(X)$.

Finally we show that the two maps are inverses, ie. that $f_{S_f} = f$ and $S_{f_S} = S$. For the first equality, let y be a smooth two-cycle in X and choose (as before) a map $g\colon \Sigma \to X$, a fundamental two-cycle d and a three-chain B such that $\partial B = -g_*(d) + y$. Then

$$\begin{aligned} f_{S_f}(y) = S_f(g) \exp\left(2\pi i \int_B c\right) &= f(g_*(d)) \exp\left(2\pi i \int_B c\right) = f(g_*(d)) f(\partial B) \\ &= f(g_*(d)) f(-g_*(d) + y) = f(g_*(d)) f(g_*(d))^{-1} f(y) = f(y). \end{aligned}$$

For the second equality, the B can be chosen trivially so that

$$S_{f_S}(g) = f_S(g_*(d)) = S(g) \exp\left(2\pi i \int_B c\right) = S(g). \qquad \square$$

The equivalence between the bordism and chain definition of the Cheeger-Simons group presented in this section is a phenomenon of the particular low dimension we are working in. For higher dimensions there is a difference between bordism and homology. It is possible, however, to define a variant of thin invariant field theory based on chains in X for any dimension n and such theories are related to the Cheeger-Simons groups in a similar fashion to that presented in this paper (see [20]) .

B Appendix: Symmetric monoidal categories

In this appendix we reproduce, for convenient ease of access, the categorical definitions pertinent to this paper. For further details see for example [2].

Definition B.1 A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an object $\mathbf{1}$, the unit, together with the following structure isomorphisms:

(i) for every triple A, B, C of objects, an isomorphism

$$a_{A,B,C}\colon (A\otimes B)\otimes C \to A\otimes(B\otimes C)$$

(ii) for every object A, isomorphisms

$$l_A\colon \mathbf{1}\otimes\mathbf{A}\to\mathbf{A} \qquad \text{and} \qquad \mathbf{r_a}\colon \mathbf{A}\otimes\mathbf{1}\to\mathbf{A}.$$

The above are subject to the following axioms:

(1) The structure isomorphisms are natural (in all variables).

(2) For each quadruple of objects A, B, C, D the following diagram commutes.

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes 1"'] & (A\otimes B)\otimes(C\otimes D) \arrow[dd, "a_{A,B,C\otimes D}"] \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "A_{A,B\otimes C,D}"'] & \\
A\otimes((B\otimes C)\otimes D) \arrow[r, "1\otimes a_{B,C,D}"'] & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

(3) For each pair of objects A, B the following diagram commutes.

$$\begin{tikzcd}
(A\otimes\mathbf{1})\otimes\mathbf{B} \arrow[r, "a_{A,\mathbf{1},\mathbf{B}}"] \arrow[dr, "r_A\otimes 1"'] & A\otimes(\mathbf{1}\otimes\mathbf{B}) \arrow[d, "1\otimes l_B"] \\
& A\otimes B
\end{tikzcd}$$

The category is *strict* if the structure isomorphisms are identities.

Definition B.2 A *symmetric* monoidal category is a monoidal category $\mathcal{C}$ equipped with natural isomorphisms

$$s_{A,B} \colon A \otimes B \to B \otimes A$$

satisfying the following.

(1) For every triple A, B, C of objects the following diagram commutes.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "s_{A,B} \otimes 1"] \arrow[d, "a_{A,B,B}"'] & (B \otimes A) \otimes C \arrow[d, "a_{B,A,C}"] \\
A \otimes (B \otimes C) \arrow[d, "s_{A,B \otimes C}"'] & B \otimes (A \otimes C) \arrow[d, "1 \otimes s_{A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "a_{B,C,A}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

(2) For every object A the following diagram commutes.

$$\begin{tikzcd}
A \otimes \mathbf{1} \arrow[r, "s_{A,1}"] \arrow[dr, "r_A"'] & \mathbf{1} \otimes \mathbf{A} \arrow[d, "l_A"] \\
& A
\end{tikzcd}$$

(3) For every pair A, B of objects the following diagram commutes.

$$\begin{tikzcd}
A \otimes B \arrow[r, "s_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "s_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

Definition B.3 Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A *monoidal functor* is a functor $E \colon \mathcal{C} \to \mathcal{D}$ together with the following morphisms in $\mathcal{D}$:

(i) for each pair A, B of objects in $\mathcal{C}$ a morphism

$$\Phi_{A,B} \colon E(A) \otimes E(B) \to E(A \otimes B),$$

(ii) a morphism $\epsilon \colon \mathbf{1}_{\mathcal{D}} \to \mathbf{E}(\mathbf{1}_{\mathcal{C}})$.

These must satisfy the following axioms.

(1) The $\Phi_{A,B}$ are natural in both A and B.

(2) For each triple A, B, C of objects in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd}
(E(A) \otimes E(B)) \otimes E(C) \arrow[r, "a_{E(A),E(B),E(C)}"] \arrow[d, "\Phi_{A,B}\otimes 1"'] & E(A) \otimes (E(B) \otimes E(C)) \arrow[d, "1\otimes\Phi_{B,C}"] \\
E(A \otimes B) \otimes E(C) \arrow[d, "\Phi_{A\otimes B,C}"'] & E(A) \otimes E(B \otimes C) \arrow[d, "\Phi_{A,B\otimes C}"] \\
E((A \otimes B) \otimes C) \arrow[r, "E(a_{A,B,C})"'] & E(A \otimes (B \otimes C))
\end{tikzcd}$$

(3) For each object A of $\mathcal{C}$ the following two diagrams commute.

$$\begin{tikzcd}
E(\mathbf{1}) \otimes \mathbf{E}(\mathbf{A}) \arrow[r, "\Phi_{\mathbf{1},\mathbf{A}}"] & E(\mathbf{1} \otimes \mathbf{A}) \arrow[d, "E(l_A)"] \\
\mathbf{1} \otimes \mathbf{E}(\mathbf{A}) \arrow[u, "\epsilon\otimes 1"] \arrow[r, "l_{E(A)}"'] & E(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
E(A) \otimes E(\mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] & E(A \otimes \mathbf{1}) \arrow[d, "E(r_A)"] \\
E(A) \otimes \mathbf{1} \arrow[u, "1\otimes\epsilon"] \arrow[r, "r_{E(A)}"'] & E(A)
\end{tikzcd}$$

If the categories $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal then a *symmetric monoidal functor* is a monoidal functor as above such that for every pair A, B of objects in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd}
E(A) \otimes E(B) \arrow[r, "s_{E(A),E(B)}"] \arrow[d, "\Phi_{A,B}"'] & E(B) \otimes E(A) \arrow[d, "\Phi_{B,A}"] \\
E(A \otimes B) \arrow[r, "E(s_{A,B})"'] & E(B \otimes A)
\end{tikzcd}$$

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Mathematisches Institut, Universität Göttingen, 37073 Göttingen, Germany

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland

Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, England

Email: `bunke@uni-math.gwdg.de, paul@ma.hw.ac.uk, s.willerton@shef.ac.uk`

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