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Automorphisms of free groups with boundaries

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Abstract The automorphisms of free groups with boundaries form a family of groups $\mathrm{A}_{n,k}$ closely related to mapping class groups, with the standard automorphisms of free groups as $\mathrm{A}_{n,0}$ and (essentially) the symmetric automorphisms of free groups as $\mathrm{A}_{0,k}$. We construct a contractible space $L_{n,k}$ on which $\mathrm{A}_{n,k}$ acts with finite stabilizers and finite quotient space and deduce a range for the virtual cohomological dimension of $\mathrm{A}_{n,k}$. We also give a presentation of the groups and calculate their first homology group.

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1 Introduction

The group $A_{n,k}$ of automorphisms of free groups with k boundaries was introduced in [29] to show that the natural map from the stable mapping class group of surfaces to the stable automorphism group of free groups gives an infinite loop map on the classifying spaces of the groups after plus construction. The "boundaries" make it possible to define appropriate gluing operations.

There are several ways to define these groups.

Geometrically, $\mathrm{A}_{n,k}$ is the group of components of the homotopy equivalences of a certain genus $n+k$ graph $G_{n,k}$ (see Figure 1) fixing k circles in the graph. The mapping class group $\Gamma_{g,k+1}$ of a surface of genus g with $k+1$ boundary components is in fact the subgroup of $\mathrm{A}_{2g,k}$ fixing one more circle (see Section 2). An alternative geometric description is given in [11] in terms of the mapping class groups of a 3-dimensional manifold.

Algebraically, we first define a related group A_n^k, which corresponds to mapping class groups with punctures rather than boundaries. Let F_{n+k} denote the free group on the $n+k$ generators $x_1,\dots,x_n,y_1,\dots,y_k$, and let $\mathrm{Aut}(F_{n+k})$ be its automorphism group. Then $\mathrm{A}_n^k := \{\phi \in \mathrm{Aut}(F_{n+k}) \,|\, \phi(y_i) \sim y_i \text{ for } 1 \le i \le k\}$, where $\sim$ means 'conjugate to'. The groups A_n^k are thus natural intermediates between $\mathrm{A}_n^0 = \mathrm{Aut}(F_n)$ and A_0^k which is known as the pure symmetric automorphism group of F_k. The group $\mathrm{A}_{n,k}$ is a central extension

$$1 \to \mathbb{Z}^k \to \mathrm{A}_{n,k} \to \mathrm{A}_n^k \to 1$$

(see Section 2 for details).

The symmetric automorphisms of free groups have been studied extensively, in particular for their relation to automorphism groups of free products (see for example [1, 4, 14, 25, 26]) and for their relation to motion groups of strings [3, 6, 7, 26]; they give a generalization of the braid groups, which is the mapping class group of a genus 0 punctured surface. Our groups are the higher genus analogues, coming from surfaces with punctures for A_n^k or with boundaries for $\mathrm{A}_{n,k}$.

In [18, Section 4], Levitt defines a class of groups, which he calls *relative automorphism groups*, in order to study automorphism groups of hyperbolic groups and of graphs of groups. The groups $A_{n,k}$ are examples of relative automorphism groups.

In this paper, we construct a contractible space $L_{n,k}$ of dimension $2n+3k-2$ (provided $n \geq 1$; if $n=0$ it is of dimension $2k-1$) on which the group $\mathrm{A}_{n,k}$ acts with finite stabilizers and such that the quotient space is a finite complex. The space $L_{n,k}$ is an analogue of the spine of Auter space X_n. Recall that X_n is a space of graphs of genus n marked by a rose $\vee_n S^1$. In $L_{n,k}$, we consider graphs of genus $n+k$ marked by the graph $G_{n,k}$ of Figure 1. We also construct an analogous space L_n^k for the group A_n^k so that there is a sequence of contractible spaces

$$F_{n,k} \to L_{n,k} \to L_n^k$$

corresponding to the group extension $\mathbb{Z}^k \to A_{n,k} \to A_n^k$. The fiber $F_{n,k}$ is homeomorphic to $\mathbb{R}^k$ with the standard $\mathbb{Z}^k$-action.

We use $L_{n,k}$ to give the following range for the virtual cohomological dimension of $\mathrm{A}_{n,k}$: $2n+2k-1 \leq vcd(\mathrm{A}_{n,k}) \leq 2n+3k-2$ when $k \geq 1$ (Theorem 3.5).

By the work of McCool [24], we know that A_n^k is finitely presented. We use his work to produce a presentation for A_n^k and $\mathrm{A}_{n,k}$ (Theorems 7.1 and 7.2). We recover the presentation he gave for A_0^k in [25]. Theorem 7.4 says that $H_1(A_{n,k}) = \mathbb{Z}_2$ when $n > 2$. This agrees with the recently proved conjecture that the homology $H_i(\mathrm{A}_{n,k})$ is independent of n and k for $n >> i$ [11].

To show that $L_{n,k}$ is contractible, we show that it is isomorphic to a subcomplex of X_{n+3k}^{Θ}, the set of fixed points of X_{n+3k} under the action of a certain finite subgroup Θ of $\mathrm{Aut}(F_{n+3k})$. Although $\mathrm{A}_{n,k}$ is not a subgroup of $\mathrm{Aut}(F_{n+k})$, it is a subgroup of the normalizer of Θ in $\mathrm{Aut}(F_{n+3k})$. Hence $\mathrm{A}_{n,k}$ acts on X^{Θ}_{n+3k}, which is known to be contractible [13]. The space L_{n+k} can be identified with the essential realization of an initial segment of the essential realization of X^{Θ}_{n+3k} under a certain norm, in the language of [17]. Our Theorem 4.1 says that the initial segments of the essential realization of X^G_n are contractible for any finite group $G \leqslant \mathrm{Aut}(F_n)$.

Our motivation is the study of the stable automorphism group Aut_∞, the colimit of the inclusions $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_{n+1})$. The homology and homotopy type of the classifying space of the stable mapping class group Γ_∞ has been determined recently [19, 20, 28], making use of the mapping class groups $\Gamma_{g,n}$ of surfaces with any number of boundary components, and the fact that the homology groups $H_*(\Gamma_{g,n})$ are independent of g and n when $g >> *$ [8]. Little is known about $B\mathrm{Aut}_\infty$. It was shown recently that the homology of $\mathrm{A}_{n,k}$ is also independent of the number of boundaries in a stable range [11]. This means in particular that we can work with $\mathrm{A}_{n,k}$ instead of $\mathrm{Aut}(F_n)$ if we are only interested in stable results. The boundaries of $\mathrm{A}_{n,k}$ allow us to define gluing operations analogous to the very useful gluing operations for the mapping class groups. These give a wealth of new operations on the automorphism of free groups and are used in [29] to show that the map on classifying spaces $B\Gamma^+_\infty \to B\mathrm{Aut}^+_\infty$ is an infinite loop map after Quillen's plus-construction.

The paper is organized as follows. Section 2 recalls the definition of the groups $\mathrm{A}_{n,k}$ and describes their relationship to mapping class groups of surfaces. Section 3 defines the space $L_{n,k}$ and calculates a range for the virtual cohomological dimension of $\mathrm{A}_{n,k}$. In Section 4, we prove the contractibility of initial segments in a general setting. Section 5 shows that $L_{n,k}$ is an initial segment of X^{Θ}_{n+3k}. In Section 6, we exhibit the sequence $F_{n,k} \to L_{n,k} \to L^k_n$. Finally, Section 7 gives the presentations and calculates H_1.

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2 The groups $\mathrm{A}_{n,k}$

Let $G_{n,k}$ be the graph given in Figure 1, obtained from a wedge of n circles by attaching k circles $C_1,\dots,C_k$ on stems. We think of the circles (together with the basepoint) as the boundary of the graph. The basepoint $*$ of $G_{n,k}$ is the central vertex. Define

$$\mathrm{A}_{n,k} := \pi_0\,\mathrm{Htpy}_*(G_{n,k};\partial),$$

where $\mathrm{Htpy}_*(G_{n,k};\partial)$ is the space of (basepointed) homotopy equivalences of $G_{n,k}$ fixing the circles C_j point-wise. By [9, Prop. 0.19], homotopy equivalences fixing the circles are actually homotopy equivalences relative to the circles. This shows in particular that $A_{n,k}$ is a group.

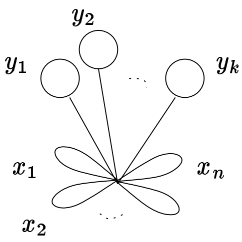

Figure 1: $G_{n,k}$

Labeling the first n petals of $G_{n,k}$ with $x_1,\dots,x_n$ and the k circles with their stems $y_1,\dots,y_k$ induces an identification of $\pi_1(G_{n+k})$ with the free group $F_{n+k} := \langle x_1,\dots,x_n,y_1,\dots,y_k\rangle$. This leads to the following algebraic description of $\mathrm{A}_{n,k}$: Define a map $\alpha : (F_{n+k})^{n+k} \to \mathrm{Hom}(F_{n+k},F_{n+k})$ by

$$\begin{cases} \alpha\langle\nu,w\rangle[x_i] = \nu_i & 1\le i\le n \\ \alpha\langle\nu,w\rangle[y_j] = w_j^{-1}y_jw_j & 1\le j\le k \end{cases}$$

where $\langle\nu,w\rangle := (\nu_1,\dots,\nu_n,w_1,\dots,w_k)\in(F_{n+k})^{n+k}$.

Proposition 2.1 *There is a group isomorphism*

$$\mathrm{A}_{n,k} \cong \left\{\langle\nu,w\rangle\in(F_{n+k})^{n+k} \;\middle|\; \alpha\langle\nu,w\rangle\in\mathrm{Aut}(F_{n+k})\right\}$$

with the multiplication on the right hand side defined by

$$\langle\nu,w\rangle.\langle\nu',w'\rangle = \langle\ \alpha\langle\nu',w'\rangle[\nu]\ ,\ w'.\,\alpha\langle\nu',w'\rangle[w]\ \rangle,$$

where we apply $\alpha\langle\nu',w'\rangle$ component-wise, and the unit is $(x_1,\dots,x_n,1,\dots,1)$.

Proof We use the labeling of $G_{n,k}$ by x_i's and y_j's as above (with a fixed orientation) to get an identification $\pi_1(G) \cong F_{n+k}$. The "lollies" y_j are made out of a stem, denoted l_j, from the basepoint to the circle, denoted C_j.

For any $f \in \mathrm{Htpy}_*(G;\partial)$, we want to define an element $\langle \nu, w\rangle$ as above which depends only on the homotopy class of f. Set $\nu_i := [f(x_i)]$ and $w_j := [l_j.f(\overline{l_j})]$ in $\pi_1(G)$, where $\overline{l_j}$ means the path l_j taken in the reverse direction. Now $\alpha\langle \nu, w\rangle$ is the automorphism induced by f on $\pi_1(G)$. Indeed, $[f(x_i)] = \nu_i$ and $[f(y_j)] = [f(l_j).f(C_j).f(\overline{l_j})]$ which is equal to $w_j^{-1} y_j w_j$ in $\pi_1(G)$. So $\langle \nu, w\rangle$ is an element of $\alpha^{-1}(\mathrm{Aut}(F_{n+k}))$ as required.

Conversely, we want to show that any such $\langle \nu, w\rangle$ defines an element $[f] \in \pi_0 \mathrm{Htpy}_*(G;\partial)$. Define f piecewise by $f(x_i) = \nu_i$, $f(l_j) = w_j^{-1}.l_j$ and $f(C_j) = C_j$. It is a continuous map, and a homotopy equivalence as it acts on $\pi_1(G)$ via $\alpha\langle \nu, w\rangle$.

The two maps are homomorphisms and inverse of one another. Hence the two groups are isomorphic. □

The map $\alpha : \mathrm{A}_{n,k} \to \mathrm{Aut}(F_{n+k})$ is a homomorphism (where the multiplication of automorphisms $\phi.\psi$ is defined by first applying ϕ and then ψ.) Geometrically, it is the map that forgets that the circles were fixed. It is not injective. There is in fact a short exact sequence

$$1 \to \mathbb{Z}^k \xrightarrow{i} \mathrm{A}_{n,k} \xrightarrow{\alpha} \mathrm{A}_n^k \to 1$$

where $\mathrm{A}_n^k \cong \{\phi \in \mathrm{Aut}(F_{n+k}) \mid \phi(y_j) \sim y_j \text{ for } 1 \leq j \leq k\}$ and where $i(z_1, \ldots, z_k) = (x_1, \ldots, x_n, y_1^{z_1}, \ldots, y_k^{z_k})$. Note that the group A_n^k can be described geometrically as $\pi_0 \mathrm{Htpy}_*(G_{n,k};[\partial])$, where $\mathrm{Htpy}_*(G_{n,k};[\partial])$ is now the space of homotopy equivalences of $G_{n,k}$ fixing the circles up to a rotation.

The group $\mathbb{Z}^k$ maps to the center of $\mathrm{A}_{n,k}$. These k additional generators in $\mathrm{A}_{n,k}$ correspond to "conjugating y_i by itself". Geometrically, these are Dehn twists along the ith boundary component in the corresponding mapping class group $\Gamma_{g,k+1}$.

We will also consider the semidirect product $A_{n,k}^{\Sigma} = \Sigma_k \ltimes \mathrm{A}_{n,k}$. Geometrically, $\mathrm{A}_{n,k}^{\Sigma} := \pi_0 \mathrm{Htpy}_*^{\Sigma}(G_{n,k};\partial)$ with $\mathrm{Htpy}_*^{\Sigma}(G_{n,k};\partial)$ the space of homotopy equivalences fixing the cycles C_j up to a permutation. Algebraically, this means that the y_i's can be permuted.

The groups $\mathrm{A}_{n,k}$ and A_n^k are closely related to mapping class groups when n is even. Indeed, note that the graph $G_{2g,k}$ is homotopy equivalent to a surface $S_{g,k+1}$ of genus g with $k+1$ boundary circles and $\mathrm{A}_{n,k}$ can also be defined as the group of components of the homotopy equivalences of $S_{g,k+1}$ fixing the first k boundary components point-wise and a point on the last one.

Let $\Gamma_{g,k} := \pi_0 \operatorname{Diff}^+(S_{g,k+1};\partial)$ denote the mapping class group of $S_{g,k+1}$ with the boundaries point-wise fixed and let Γ_g^{k+1} be the mapping class group of $S_{g,k+1}$ where the boundaries are fixed, but no longer point-wise. Fricke (1897) and Magnus (1934) for $g = 0, 1$ and Nielsen (1927) for $g \geq 2$ proved that Γ_g^{k+1} is the subgroup of $\operatorname{Out}(F_{2g+k})$ fixing the cyclic words defined by the boundaries [21, p175].

Note that the inner automorphisms of F_{n+k} form a normal subgroup of A_n^k. Let $\mathrm{O}_n^k := \mathrm{A}_n^k \,/\, \operatorname{Inn}(F_{n+k})$. So $\mathrm{O}_n^k = \{\phi \in \operatorname{Out}(F_{n+k}) \mid \phi(y_j) \sim y_j \text{ for } 1 \leq j \leq k\}$. It follows that Γ_g^{k+1} is the subgroup of O_n^k fixing the conjugacy class of the last boundary component, that is the word $c := [x_1, x_2] \ldots [x_{2g-1}, x_{2g}]\, y_1 \ldots y_k$. One actually has the following isomorphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
\mathbb{Z}^{k+1} \arrow[r, "\cong"] \arrow[d] & \big\{(z,\phi) \in \mathbb{Z}^k \times \operatorname{Inn}(F_{2g+k}) \;\big|\; \phi(c) = c\big\} \arrow[d] \\
\Gamma_{g,k+1} \arrow[r, "\cong"] \arrow[d] & \big\{\langle \nu, w\rangle \in \mathrm{A}_{2g,k} \;\big|\; \alpha\langle \nu, w\rangle[c] = c\big\} \arrow[d] \\
\Gamma_g^{k+1} \arrow[r, "\cong"] \arrow[d] & \big\{\phi \in \mathrm{O}_{2g}^k \;\big|\; \phi(c) \sim c\big\} \arrow[d] \\
1 & 1
\end{tikzcd}$$

3 (n, k)-graphs

By a *graph*, we mean a pointed, connected, one-dimensional CW-complex. Let S^1 be the circle with basepoint denoted by $\circ$.

Definition 3.1 An (n, k)*-graph* $(\Gamma, \underline{l})$ is a graph Γ of genus $(n + k)$ with no separating edges, equipped with embeddings $l_j : S^1 \to \Gamma$ for $j = 1, \ldots, k$ such that

(1) For every $j = 1, \ldots, k$, the point $\circ_j := l_j(\circ)$ is a vertex of Γ.

(2) The vertices of Γ have valence at least 2 and if a vertex v has valence 2, then $v = \circ_j$ for some j or v is the basepoint $*$.

(3) Any two $\mathbb{S}_j := l_j(S^1)$ meet in at most a point and the dual graph of the $\mathbb{S}_j$'s is a forest.

We call the embedded circle $(\mathbb{S}_j, \circ_j)$ a *cycle*. In this way, an (n,k)-graph is a graph of genus $n+k$ with k cycles. (See Figure 2 for an example.)

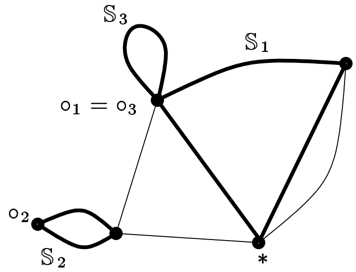

Figure 2: Example of a $(2,3)$-graph

Let $(G_{n,k}, \underline{\kappa})$ be the generalized (n,k)-graph, where $G_{n,k}$ was defined in Section 2 (see Figure 1) and $\kappa_j : S^1 \to G_{n,k}$ identifies S^1 with the jth circle C_j. Note that $G_{n,k}$ has k separating edges, and hence this is not an actual (n,k)-graph.

Definition 3.2 A *marked* (n,k)*-graph* (Γ, ϕ) is a graph Γ equipped with a pointed homotopy equivalence $\phi : G_{n,k} \to \Gamma$ such that $(\Gamma, \phi \circ \underline{\kappa})$ is a (n,k)-graph.

Two marked (n,k)-graphs (Γ^1, ϕ^1) and (Γ^2, ϕ^2) are *equivalent* if there is a graph isomorphism $\psi : \Gamma^1 \to \Gamma^2$ such that $\psi \circ \phi^1(C_j) = \phi^2(C_j)$ and $\psi \circ \phi^1 \simeq \phi^2$ (rel. to C_j for all j.)

The intuition (which is made precise in the proof of Theorem 5.3) is that condition (3) in Definition 3.1 prevents the existence of Whitehead moves (blowing an edge and then collapsing it) which would take a 'good marking' to a 'bad marking'.

For an (n,k)-graph $(\Gamma, \underline{l})$, let $\hat{\Gamma}$ be the graph of genus n obtained from Γ by collapsing all the edges in the cycles $\mathbb{S}_1, \ldots, \mathbb{S}_k$. Let e be an edge of Γ which does not define a loop in Γ or in $\hat{\Gamma}$. Then composition with the edge collapse $\Gamma \to \Gamma/e$ induces embeddings $l_j/e : S^1 \to \Gamma/e$ such that $(\Gamma/e, \underline{l}/e)$ is again a (n,k)-graph. By an edge collapse in an (n,k)-graph, we will always mean the collapse of an edge satisfying the above hypothesis. For a marked (n,k)-graph, the marking of the collapsed graph is also obtained by composition with the collapse.

A forest collapse in a (n,k)-graph $(\Gamma,\underline{l})$ is a sequence of edge collapses. As the order of the collapses does not matter, it is defined by the union of the edges to be collapsed, which is a forest F in Γ. We denote the collapsed graph by $(\Gamma/F,\underline{l}/F)$.

Let $L_{n,k}$ denote the set of equivalence classes of marked (n,k)-graphs. Define a poset structure on $L_{n,k}$ by saying that $(\Gamma^1,\phi^1)\leq(\Gamma^2,\phi^2)$ if there is a forest F in Γ^2 such that $(\Gamma^2/F,\phi^2/F)$ is equivalent to (Γ^1,ϕ^1). By abuse of notation, we will also denote by $L_{n,k}$ the simplicial complex which is the geometric realization of this poset.

Using the geometric description of $A^{\Sigma}_{n,k}$, we obtain a natural action of $A^{\Sigma}_{n,k}$ (and thus of $\mathrm{A}_{n,k}$) on $L_{n,k}$: For $\lambda\in A^{\Sigma}_{n,k}$ and $(\Gamma,\phi)\in L_{n,k}$, define

$$\lambda\cdot(\Gamma,\phi):=(\Gamma,\phi\circ\lambda^{-1}).$$

We consider two equivalence relations on (n,k)-graphs: Two (n,k)-graphs $(\Gamma^1,\underline{l}^1)$ and $(\Gamma^2,\underline{l}^2)$ are Σ*-equivalent* if there is a graph isomorphism $\phi:\Gamma^1\to\Gamma^2$ and a permutation $\sigma\in\Sigma_k$ such that $\phi\circ l^1_j(\circ)=l^2_{\sigma(j)}(\circ)$ and $\phi\circ l^1_j\simeq l^2_{\sigma(j)}$ (rel. to $\circ$) for each $j=1,\dots,k$. The (n,k)-graphs are *equivalent* if the permutation σ is trivial.

Let $QL^{\Sigma}_{n,k}$ and $QL_{n,k}$ denote the set of all equivalence classes of (n,k)-graphs under each equivalence relation. Define spaces $QL^{\Sigma}_{n,k}$ and $QL_{n,k}$ by taking the elements of the sets as 0-simplices and attaching a p-simplex for each equivalence class of chain of forests $\emptyset\neq F_1\subsetneq F_2\subsetneq\cdots\subsetneq F_p$, where two chains F^1_i and F^2_i are equivalent if there is a $(\Sigma$-$)$equivalence $\phi:(\Gamma,\underline{l})\to(\Gamma,\underline{l})$ such that $\phi(F^1_i)=F^2_i$ for all $i=1,\dots,p$. Note that these spaces are not simplicial complexes in general. The space $QL_{n,k}$ is the moduli space of labeled (n,k)-graphs, that is (n,k)-graphs with cycles labeled $1,\dots,k$, whereas in $QL^{\Sigma}_{n,k}$ the cycles are unlabeled.

Proposition 3.3 *The action of $A^{\Sigma}_{n,k}$ (resp. $\mathrm{A}_{n,k}$) on $L_{n,k}$ is transitive on the sets of marked (n,k)-graphs with the same underlying unlabeled (n,k)-graph (resp. same underlying labeled (n,k)-graph), with finite simplex stabilizers, and the quotient of the action is $QL^{\Sigma}_{n,k}$ (resp. $QL_{n,k}$).*

We will prove the following theorem in the next two sections.

Theorem 3.4 *The space $L_{n,k}$ is contractible.*

We use this result to give a range for the virtual cohomological dimension of $\mathrm{A}_{n,k}$. We recall that it is already known that the vcd of $\mathrm{Aut}(F_n) = \mathrm{A}_{n,0}$ is $2n-2$ [5] and that the vcd of A_0^k is $k-1$ [4]. Since $A_{n,k}$ is a subgroup of $\mathrm{Aut}(F_{n+3k})$ (see Proposition 5.1) and the latter group is virtually torsion free and of finite vcd, the former group is as well [2, Example 1 on page 229].

Theorem 3.5 (1) $2n+2k-1 \leq vcd(\mathrm{A}_{n,k}) \leq 2n+3k-2 \quad \textit{if } k \geq 1.$

(2) $vcd(\mathrm{A}_{n,1}) = 2n+1.$

(3) $vcd(\mathrm{A}_{0,k}) = 2k-1.$

Proof of Theorem 3.5 For (1), we first note that there is a copy of $\mathbb{Z}^{2n+2k-1}$ in $\mathrm{A}_{n,k}$: take n generators sending x_i to $x_i y_k$ and fixing all other generators, then n generators sending x_i to $y_k x_i$, then k conjugating y_j by y_k for $1 \leq j \leq k$ and finally $k-1$ conjugating y_j by y_j for $j \neq k$.

For an upper bound, we calculate the dimension of a maximal simplex in $QL_{n,k}$. Suppose the cycles lie in the maximally blown up graph in c connected components and that the graph has v vertices. The $k+1$ vertices $*, \circ_1, \ldots, \circ_k$ all have valence 2. Since two cycles must meet in a valence 4 vertex and the dual graph of the cycles in the graph is a forest, there are $k-c$ vertices of valence 4 in the graph. The remaining vertices are valence 3. Hence there are $e = 3(v-2k+c-1)/2 + (k+1) + 2(k-c) = (3v-c-1)/2$ edges. Since $v - e = 1-(n+k)$, this yields $v = 2n+2k+c-1$ vertices. We can collapse these v vertices down to 1 by first collapsing all except one edge in each cycle and then collapsing some remaining maximal tree. This yields a simplex of dimension $2n+2k+c-2$. Since $c \leq k$, the dimension of a maximal simplex is less than or equal to $2n+3k-2$. The result follows from Theorem 3.4.

For (2), let $k=1$ in (1).

Finally for (3) if $n=0$ then $c=1$ and so $v = 2n+2k+c-1 = 2k$. Hence a maximal simplex has dimension $2k-1$. □

In Figure 3, we show an example of a maximal simplex of $QL_{1,2}$ given by a graph with 2 connected cycle components and a forest consisting of $6 > 5$ edges.

4 Initial segments of fixed point subcomplexes

Recall from [10] that the spine of Auter space X_n is a poset of pointed marked graphs of genus n, where the marking is given by a map from the rose $R_n :=$

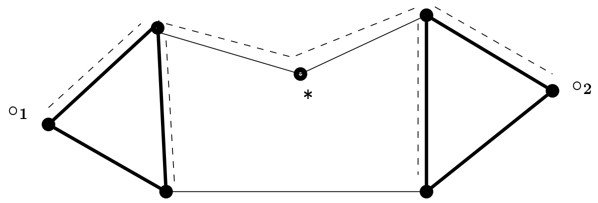

Figure 3: A maximal simplex in $QL_{1,2}$ with a 6 edge forest

$\vee_n S^1$. X_n is contractible and admits an action of $\mathrm{Aut}(F_n)$. Let G be a finite subgroup of $\mathrm{Aut}(F_n)$ and consider the fixed point subcomplex X_n^G. A point in X_n^G is a marked graph with a G-action. X_n^G is contractible [13, Thm. 1.1].

Edge collapses induce a poset structure on X_n^G with minimal elements the *reduced* marked graphs, i.e. those with no G-invariant subforest. An edge e of a marked G-graph Γ is *essential* if there exists a maximal G-invariant forest which does not contain e. The graph Γ is *essential* if all of its edges are essential. Let E_n^G be the full subcomplex of X_n^G spanned by essential marked graphs. Collapsing inessential edges induces a poset map $f : X_n^G \to E_n^G$ such that $f(x) \leq x$ for all x in X_n^G, and hence a deformation retraction on the corresponding spaces by the Poset Lemma (see [27] or [4, Lem. 4.1] for a direct proof of this special case).

Let $\mathcal{W}$ be the set of conjugacy classes of elements of F_n. Recall from [13, Section 2] that, given any well ordering of $\mathcal{W}$ and of F_n, we can define a norm

$$\|\cdot\|_{tot} = \|\cdot\|_{out} \times \|\cdot\|_{aut} \in \mathbb{Z}^{\mathcal{W}} \times \mathbb{Z}^{F_n}$$

on E_n^G which well orders the reduced marked graphs of E_n^G (using the corresponding lexicographic ordering). For $w \in \mathcal{W}$, the wth component of the norm is the sum of the cyclic lengths of the paths $\phi(xw)$ for all $x \in G$, where the cyclic length is the number of edges in the path reduced cyclically, i.e. the length of the shortest path in its unpointed homotopy class. And for $z \in F_n$, the zth component of the norm is the sum of the lengths of the paths $\phi(xz)$ for all $x \in G$, where we now consider the shortest paths in the pointed homotopy classes.

A nonempty set Λ of reduced marked graphs of E_n^G is called an *initial segment* if whenever two reduced marked graphs r_1, r_2 satisfy $\|r_1\|_{tot} \leq \|r_2\|_{tot}$ and $r_2 \in \Lambda$, then $r_1 \in \Lambda$. The *realization* of an initial segment Λ is the space

$$\|\Lambda\| = \bigcup_{r \in \Lambda} st(r),$$

the union of the ascending stars (in E_n^G) of the reduced marked graphs in Λ.

Let Λ be an initial segment and let Γ be a marked graph in $\|\Lambda\|$. An edge e of Γ is *essential in* Λ if there is a maximal G-invariant subforest $\mathcal{F}$ of Γ such that $e \notin \mathcal{F}$ and $\Gamma/\mathcal{F} \in \Lambda$. The graph Γ is *essential in* Λ if all of its edges are essential in Λ. The *essential realization* of an initial segment Λ is the full subcomplex $|\Lambda|$ of $\|\Lambda\|$ spanned by the essential marked graphs of Λ.

Let Γ be a marked graph in E_n^G and let v be a vertex of Γ. An *ideal edge* γ at the vertex v is a set of half edges of Γ terminating at v such that the *blow-up* Γ^γ is again in E_n^G, where Γ^γ is the graph obtained by G-equivariantly pulling the half edges in γ away from v (see [17, Section 5.A.] and [13, Section 2] for a characterization of ideal edges when Γ is reduced.) More precisely, one creates a new edge orbit $G\gamma$ and a new vertex orbit $Gv(\gamma)$. In Γ^γ, γ is an actual edge that goes from $v(\gamma)$ to v. In addition, each half edge $e \in \gamma$ is now attached to $v(\gamma)$ instead of v, and this process is done equivariantly on $G\gamma$. Note that the original graph Γ can be recovered from its blow up Γ^γ by collapsing $G\gamma$.

An *ideal forest* in a reduced marked graph is a sequence of ideal edges satisfying certain technical conditions, which make sure that by blowing up these ideal edges, one can construct a blow-up $\Gamma^{\mathcal{F}}$ in the star of Γ with an actual G-equivariant forest, whose partial collapses contain all the blow-ups of the ideal edges in the ideal forest. One can define a poset structure on the set of ideal forests of a reduced marked graph such that blowing up induces an isomorphism between this poset and the star of Γ in E_n^G [17, Prop. 5.9].

Let γ be an ideal edge of a reduced G-graph Γ. Define $D(\gamma)$ to be the set of edges c of γ such that Gc is a forest in Γ^γ. To a pair (γ, c) with $c \in D(\gamma)$ corresponds a *Whitehead move* that consists of first blowing up γ and then collapsing Gc. An ideal edge γ is *reductive* if there is a $c \in D(\gamma)$ such that performing the Whitehead move (γ, c) yields a graph of smaller norm.

Let Y be any of E_n^G, $\|\Lambda\|$ or $|\Lambda|$. The *reductive link* of a reduced marked graph Γ in Y is the part of the star of Γ in Y spanned by ideal forests with at least one reductive ideal edge. The *purely reductive link* of Γ in Y is the part of the star of Γ in Y spanned by nontrivial ideal forests with all edges reductive. By the Poset Lemma, there is a deformation retraction from the reductive link of Γ to the purely reductive link of Γ given by collapsing non-reductive ideal edges.

Theorem 4.1 *If Λ is an initial segment of E_n^G, then both its realization $\|\Lambda\|$ and its essential realization $|\Lambda|$ are contractible.*

Proof By the Poset Lemma, collapsing inessential edges induces a deformation retraction from $\|\Lambda\|$ to $|\Lambda|$. Thus it is enough to show that $\|\Lambda\|$ is contractible.

If r is a reduced marked graph in $\|\Lambda\|$, we have already observed that the reductive link of r in $\|\Lambda\|$ deformation retracts onto the purely reductive link of r in $\|\Lambda\|$. Since Λ is an initial segment, the purely reductive link of r in $\|\Lambda\|$ is the purely reductive link of r in E_n^G. From [13, Lem. 3.1 - 3.7], this reductive link (referred to as $S(\mathcal{R})$ in [13]) is contractible. As Λ is an initial segment, all Γ' such that $\|r'\| < \|r\|$ are also in Λ. The reductive link of a reduced marked graph r is the intersection

$$red(r) = st_{\|\Lambda\|}(r) \cap \bigcap_{\|r'\|<\|r\|} st_{\|\Lambda\|}(r').$$

Now the norm well orders the reduced marked graphs in $\|\Lambda\|$. We can build $\|\Lambda\|$ as the union of the stars of the elements of Λ, starting with the star of the smallest element and adding the stars successively. As the star of a graph is contractible and all the successive intersections are contractible, transfinite induction implies that every connected component of $\|\Lambda\|$ is contractible.

It remains to show that $\|\Lambda\|$ is connected. From [16, Thm. 2], we know that any two reduced marked graphs in E_n^G are connected by a sequence of Whitehead moves. The reduced marked graphs of E_n^G are ordered by the norm. Choose $I \subseteq \mathbb{Z}^{\mathcal{W}} \times \mathbb{Z}^{F_n}$ such that the ordered set of reduced marked graphs of E_n^G is $\{r_i : i \in I\}$ and denote by r_0 the least element. Note that $r_0 \in \Lambda$ because Λ is nonempty. Note also that the stars $st_{\|\Lambda\|}(r_i)$ and $st_{E_n^G}(r_i)$ of r_i in $\|\Lambda\|$ and E_n^G are equal. We will denote these stars by $st(r_i)$.

Suppose that $\|\Lambda\|$ is not connected. The star $st(r_0)$ is contractible and thus connected. Take the least index j such that $\cup_{i\leq j} st(r_i)$ is not connected. Then we can get from r_0 to any r_i with $i < j$ in $\|\Lambda\|$, but not from r_0 to r_j. Since E_n^G is connected and I is a well ordering, we can choose a least k such that there is a path from r_0 to r_j in $\cup_{i\leq k} st(r_i)$. This implies that the reductive link of r_k is not connected, however, since it has at least two components: one coming from r_0 and the other coming from r_j. This contradicts the fact that the reductive link is contractible. So $\|\Lambda\|$ is connected. $\square$

5 Contractibility of $L_{n,k}$

In this section, we prove Theorem 3.4 by identifying $L_{n,k}$ with the essential realization of an initial segment and applying Theorem 4.1.

Let $m = n + 3k$. Let $\{x_1, \dots, x_n, y_1, \dots, y_k, u_1, \dots, u_k, v_1, \dots, v_k\}$ denote the generators of F_m. For each $j = 1, \dots, k$, let $\Theta_j \cong \mathbb{Z}/3$ be the subgroup of $\mathrm{Aut}(F_m)$ generated by θ_j, where $\theta_j(u_j) = v_j$, $\theta_j(v_j) = v_j^{-1} u_j^{-1}$, and all other generators are fixed. This group can be realized as the group of rotations of the three edges of a theta-graph (see Figure 5). Let

$$\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_k.$$

be the direct product of these k groups. The normalizer $N_{\mathrm{Aut}(F_m)}(\Theta)$ acts on the fixed point subcomplex X_m^Θ and thus on E_m^Θ. Both spaces are contractible [13] (see also Section 4).

Note that a similar fixed point subcomplex was used in [15] to study the holomorph of free groups $F_n \rtimes \mathrm{Aut}(F_n)$. This last group is the subgroup of A_n^1 of automorphisms where y_1 does not occur in the images of the x_i's. One could work with circles and a $\mathbb{Z}/2$-action, instead of Θ-graphs and a $\mathbb{Z}/3$-action. This would however bring a few extra technical problems (having to introduce extra vertices to avoid acting on edges by inversions) while producing the same complex. (See also Section 6.)

Although $\mathrm{A}_{n,k}$ is not a subgroup of $\mathrm{Aut}(F_{n+k})$, it is a subgroup of $\mathrm{Aut}(F_{n+3k})$:

Proposition 5.1 *There is an inclusion of groups $\beta : A_{n,k}^\Sigma \hookrightarrow N_{\mathrm{Aut}(F_{n+3k})}(\Theta) \leqslant \mathrm{Aut}(F_{n+3k})$.*

Proof The map β is induced geometrically by attaching two additional circles at the end of each of the k stems (see Figure 4). Algebraically, this map can be described as follows: for $\langle \sigma, \nu_i, w_j \rangle \in A_{n,k}^\Sigma = \Sigma_k \ltimes A_{n,k}$,

$$\phi = i\langle \sigma, \nu_i, w_j \rangle \quad \text{maps} \quad \begin{cases} x_i & \to \quad \nu_i \\ y_j & \to \quad w_j^{-1}\, y_{\sigma(j)}\, w_j \\ u_j & \to \quad w_j^{-1}\, u_{\sigma(j)}\, w_j \\ v_j & \to \quad w_j^{-1}\, v_{\sigma(j)}\, w_j \end{cases}$$

One then checks easily that $\phi = \beta\langle \sigma, \nu_i, w_j \rangle$ satisfies $\phi\, \theta_l\, \phi^{-1} = \theta_{\sigma(l)}$ for each $l = 1, \dots, k$. □

For each $j = 1, \dots, k$, let θ^j denote a graph with two vertices $\circ_j$ and v_j, and three edges e_1^j, e_2^j, e_3^j where each e_i^j goes from $\circ_j$ to v_j (See Figure 5). Consider the subcomplex Fix of X_m spanned by marked graphs $\phi : R_n \to \tilde{\Gamma}$ which satisfy the following conditions:

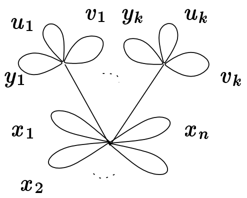

Figure 4: Graph giving the inclusion $\mathrm{A}_{n,k} \hookrightarrow \mathrm{Aut}(F_{n+3k})$

- The graph $\tilde{\Gamma}$ is obtained from a graph Γ of genus $n+k$ by attaching the graphs $\theta^1, \dots, \theta^k$ to Γ, identifying the vertex $\circ_j$ of θ^j with a vertex of Γ and remembering these —possibly multiple— labels.
- For each $j = 1, \dots, k$, there exists a path w_j in $\tilde{\Gamma}$ from $\circ_j$ to $*$ such that

$$\phi(u_j) = \bar{w}_j \star e_1^j \star \bar{e}_2^j \star w_j$$

and

$$\phi(v_j) = \bar{w}_j \star e_2^j \star \bar{e}_3^j \star w_j$$

where $\bar{p}$ denote the path p with the reverse orientation and $\star$ denotes the concatenation of paths.

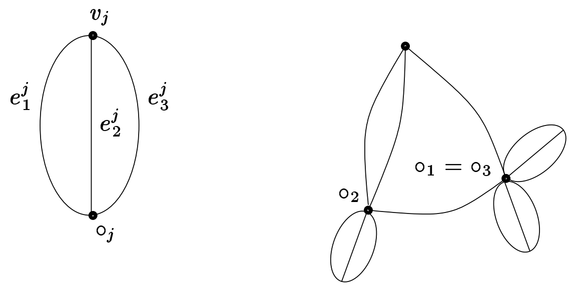

Figure 5: The graph θ^j and a graph of Fix

Proposition 5.2 $E_m^\Theta \cong Fix \cong X_m^\Theta$.

Proof Each marked graph in Fix is fixed by the action of Θ as θ_j cyclically rotates the edges of the attached θ^j. So $Fix \subset X_m^\Theta$.

Let $R_{n,k}^\Theta$ be the graph obtained by wedging the k graphs θ^j to the rose R_{n+k}. Recall that a graph is reduced if it has no invariant subforest. Note that all the reduced graphs in Fix have the same underlying graph $R_{n,k}^\Theta$.

Krstić showed that any two reduced Θ-graphs in X_m^Θ are connected by a sequence of Whitehead moves $(\{e,f\},f)$ where e,f have the same terminal vertex and $Stab_\Theta(e) \subseteq Stab_\Theta(f)$ (and $e \notin \Theta f$) [16, Prop. 4']. Pictorially the move $(\{e,f\},f)$ corresponds to pulling each edge in Θe along the corresponding edge in Θf. Because of the condition on stabilizers, we can only perform Whitehead moves on $R_{n,k}^\Theta$ of the form $(\{e,f\},f)$ or $(\{\bar{e}_i^j,f\},f)$ where e,f are edges of R_{n+k} and $\bar{e}_i^j$ is in θ^j. The first type of moves affect only the first $n+k$ petals, whereas the second type takes the subgraph θ^j around the edge f. The reduced graphs in Fix are closed under these moves.

Hence the reduced graphs in X_m^Θ are exactly the reduced graphs in Fix. Now the ascending star of a reduced graph in Fix is the same as the ascending star of the graph in X_m^Θ as equivariant blow-ups preserve the θ-graphs (see [12, Claim 6.6] for a similar computation). So $Fix = X_m^\Theta$. Note finally that all graphs in Fix are essential as no edge in a θ^j can be collapsed Θ-equivariantly and the other edges, being non-separating by hypothesis, are thus also essential. So $Fix = X_m^\Theta = E_m^\Theta$. □

We now choose a well ordering of $\mathbb{Z}^{\mathcal{W}} \times \mathbb{Z}^{F_n}$ such that the the space $L_{n,k}$ of marked (n,k)-graphs is the realization of an initial segment of the induced norm $\|\cdot\|_{tot}$. Well order the set of conjugacy classes $\mathcal{W}$ so that its first k elements are

$$u_1y_1u_1, u_2y_2u_2, \ldots, u_ky_ku_k$$

and well order F_n in any manner. Now define Λ to be the set of reduced Θ-graphs in E_m^Θ whose norm is minimal in the first k coordinates. This is an initial segment in the lexicographic ordering induced by the norm. Recall that all reduced marked graphs in E_m^Θ have underlying graph $R_{n,k}^\Theta$. The minimal norm on the component $u_jy_ju_j$ is thus $(2+1+2)\cdot|\Theta| = 5\cdot 3^k$. The marking ϕ of such a minimal reduced marked graph is determined on y_j, u_j and v_j by a loop w_j at $*$ and a single edge loop $\mathbb{S}_j$ at $*$ in the graph as follows:

$$\begin{aligned}
\phi(y_j) &= \bar{w}_j \star \mathbb{S}_j \star w_j \\
\phi(u_j) &= \bar{w}_j \star e_1^j \star \bar{e}_2^j \star w_j \\
\phi(v_j) &= \bar{w}_j \star e_2^j \star \bar{e}_3^j \star w_j,
\end{aligned}$$

Note that the edge loops $\mathbb{S}_j$ have to be different for ϕ to be a homotopy equivalence.

By Theorem 4.1, both $\|\Lambda\|$ and $|\Lambda|$ are contractible. So Theorem 3.4 follows directly from the following statement:

Theorem 5.3 *There is a poset equivalence between $L_{n,k}$ and $|\Lambda|$.*

Lemma 5.4 *Let e be an edge in a (n,k)-graph $(\Gamma,\underline{l})$. Then there exists a maximal forest in $\Gamma - e$, that is a forest $\mathcal{F}$ such that $\Gamma/\mathcal{F}$ is a (reduced) (n,k)-graph.*

Proof Let $\hat{\Gamma}$ be the graph of genus n obtained from Γ by collapsing the cycles. Take a maximal forest $\hat{\mathcal{F}}$ in $\hat{\Gamma}$. It can be extended to a maximal forest $\mathcal{F}$ of Γ by picking all but one edge in each cycle.

If e is an edge in a cycle $\mathbb{S}_j$, we can choose such a maximal forest containing $\mathbb{S}_j - \{e\}$ to obtain the result. If e is not in a cycle, e is non-separating in Γ by assumption and hence non-separating in $\hat{\Gamma}$, so we can choose a forest $\hat{\mathcal{F}}$ in $\hat{\Gamma}$ which does not contain e and extend it to Γ as above. □

Proof of the Theorem Let (Γ,ϕ) be a marked (n,k)-graph. Its underlying (n,k)-graph has $\mathbb{S}_j = \phi(C_j)$ with basepoint $\circ_j$, and let w_j be the path from $\circ_j$ to $*$ in Γ, image of the stem of the jth cycle in $G_{n,k}$, so that $\phi(y_j) = \bar{w}_j \star \mathbb{S}_j \star w_j$. Define a graph $\tilde{\Gamma}$ by attaching θ^j to $\circ_j$ in Γ for each j and define a marking $\tilde{\phi}: R_{n+3k} \to \tilde{\Gamma}$ by

$$\begin{array}{rcl}\tilde{\phi}(x_i) & = & \phi(x_i)\\ \tilde{\phi}(y_j) & = & \bar{w}_j \star \mathbb{S}_j \star w_j\\ \tilde{\phi}(u_j) & = & \bar{w}_j \star e_1^j \star \bar{e}_2^j \star w_j\\ \tilde{\phi}(v_j) & = & \bar{w}_j \star e_2^j \star \bar{e}_3^j \star w_j.\end{array}$$

This defines an inclusion of $L_{n,k}$ into $\|\Lambda\|$. Indeed, the poset minimal elements of $L_{n,k}$ are mapped to Λ, so their stars are in $\|\Lambda\|$.

To show that it maps into the smaller subcomplex $|\Lambda|$, we need to show that for each (Γ,ϕ) in $L_{n,k}$, the pair $(\tilde{\Gamma},\tilde{\phi})$ is essential in Λ, that is for each edge e of $\tilde{\Gamma}$, there is an invariant forest $\mathcal{F} \in \tilde{\Gamma} - e$ such that $\tilde{\Gamma}/\mathcal{F} \in \Lambda$. Note first that a maximal invariant forest of $\tilde{\Gamma}$ has to lie in Γ. If e lies in θ^j for some j, choose a maximal forest $\mathcal{F}$ in Γ such that $\Gamma/\mathcal{F}$ is a reduced (n,k)-graph and hence $\tilde{\Gamma}/\mathcal{F}$ is in Λ. On the other hand, if e lies in Γ, we know that such a forest exists in Γ/e by Lemma 5.4.

We claim that the image of $L_{n,k}$ equals $|\Lambda|$. To show this, we need to show that the marking of any graph in $|\Lambda|$ is induced by a marking from $G_{n,k}$ to the graph minus the theta subgraphs, that is we need to show how that the image of y_j can be decomposed as a cycle with a stem for each j with at most a point in the intersection of any two different cycles, and we need to show that the u_j's and v_j's are mapped accordingly. This is true for the poset minimal elements, i.e. the elements of Λ: the cycles are the $\mathbb{S}_j$'s with basepoint $\circ_j = *$

and with stem w_j. So we need to show that the cycle structure is carried by the admissible blow-ups (that is the blow-ups which stay within $|\Lambda|$).

Let $(\Gamma, \psi) \in |\Lambda|$ be a graph which has a cycle structure and let γ be an ideal edge at a vertex v of Γ such that Γ^γ is still in $|\Lambda|$. We want to show that Γ^γ has a cycle structure.

Case 1 Suppose that $v \neq \circ_j$ for all j, so there is no θ^j attached to v. Each cycle has at most two edges ending at v. If γ has either 0 or 2 half edges from $\mathbb{S}_j$ for each j except for at most one j, then the induced marking on Γ^γ still has embedded circles, with intersection possibly smaller than in Γ.

Suppose now that there is $j \neq j'$ such that both j and j' have exactly one half edge in γ. Then the induced cycles in Γ^γ are such that $\mathbb{S}_j \cap \mathbb{S}_{j'} = \gamma$, where γ is now the new edge in Γ^γ. But there is no forest $\mathcal{F}$ in $\Gamma^\gamma - \{\gamma\}$ such that $\Gamma^\gamma/\mathcal{F}$ is in Λ. Indeed, if we do not collapse γ again, the induced marking will have the jth and j'th cycles intersecting in γ. Hence Γ^γ is not in $|\Lambda|$.

Case 2 Suppose that there is at least one j such that $v = \circ_j$. In particular, there is at least one θ-graph attached to v. Note first that the equivariance conditions on γ imply that at most one half edge of a graph θ^j may be in γ, and if there is such an edge, the whole θ-graph is moved along the new edge γ in Γ^γ. Moreover, by the same argument as in Case 1, their can be at most one j such that $\mathbb{S}_j$ has exactly one half edge in γ.

Suppose that there is a j such that $\mathbb{S}_j$ has two half edges in γ. Then e_i^j must also be in γ for some i, otherwise any forest collapse which does not collapse γ will all induce a marking with $\bar{w}_j \star \gamma \star \mathbb{S}_j \star \bar{\gamma} \star w_j$ for y_j but still $\bar{w}_j \star e_1^j \star \bar{e}_2^j \star w_j$ for u_j. Similarly, if a half edge e_i^j is in γ for some j, then the two half edges of $\mathbb{S}_j$ terminating at v must also be in γ. This shows that a blow up cannot disconnect θ_j from $\mathbb{S}_j$.

Lastly, if there is a j such that only one half edge is in γ, then the induced marking will move $\circ_j$ and prolong w_j if there is also an e_i^j in Γ. □

6 A space L_n^k for the group A_n^k

We recall from Section 2 that there is a central extension

$$\mathbb{Z}^k \to A_{n,k} \to A_n^k,$$

where A_n^k is a subgroup of $\mathrm{Aut}(F_{n+k})$. To this group extension, corresponds a sequence of contractible spaces

$$F_{n,k} \to L_{n,k} \to L_n^k$$

where the spaces are realizations of posets, the maps are poset maps and $F_{n,k} \cong \mathbb{R}^k$. We describe here the spaces L_n^k and $F_{n,k}$.

As A_n^k is a subgroup of $\mathrm{Aut}(F_{n+k})$, a natural way to construct a contractible space L_n^k with an action of A_n^k with finite stabilizers and finite quotient, is to consider a subcomplex of Auter space X_{n+k}, as small as possible to get as good an approximation as we can of the vcd. In the same spirit as the construction of $L_{n,k}$, one can consider the essential realization $|\Lambda'|$ of the initial segment Λ' of reduced graphs of minimal norm with respect to the n-tuple of cyclic words $(y_1, \ldots, y_k)$. We claim that $|\Lambda'|$ is homeomorphic to the following space L_n^k:

Definition 6.1 An $\binom{k}{n}$*-graph* $(\Gamma, \underline{l})$ is a graph Γ of genus $(n+k)$ with vertices of valence at least 3, except the basepoint which has valence at least 2, with no separating edges, and equipped with embeddings $l_j : S^1 \to \Gamma$ for $j = 1, \ldots, k$ such that any two $\mathbb{S}_j := l_j(S^1)$ meet in at most a point and the dual graph of the $\mathbb{S}_j$'s is a forest.

A *marked* $\binom{n}{k}$*-graph* (Γ, ϕ) is a graph Γ equipped with a pointed homotopy equivalence $\phi : G_{n,k} \to \Gamma$ such that $(\Gamma, \phi \circ \underline{\kappa})$ is an $\binom{n}{k}$-graph.

Two marked $\binom{n}{k}$-graphs (Γ^1, ϕ^1) and (Γ^2, ϕ^2) are *equivalent* if there is a graph isomorphism $\psi : \Gamma^1 \to \Gamma^2$ such that $\psi \circ \phi^1(C_j) = \phi^2(C_j)$ and $\psi \circ \phi^1 \simeq \phi^2$ (rel. C_j for all j, up to a rotation).

Define now L_n^k to be the poset of equivalence classes of marked $\binom{n}{k}$-graphs (or its realization) where the poset structure is given by the same admissible edge collapses as for $L_{n,k}$ (see p6).

Theorem 6.2 *There is a poset equivalence $L_n^k \to |\Lambda'|$ and hence L_n^k is contractible.*

The proof is totally analogous to the proof of Theorem 5.3, although a little bit simpler as we do not need to carry around Θ-graphs. The poset map $L_n^k \to X_{n+k}$ takes the minimal poset elements of L_n^k to Λ', and L_n^k maps into the essential realization of Λ' by Lemma 5.4. On the other hand, we have $\Lambda' \subset L_n^k$ and the inclusion of $|\Lambda'|$ reduced to the case 1 in the proof of Theorem 5.3. Finally, $|\Lambda'|$ is contractible by Theorem 4.1 with G the trivial group.

Proposition 6.3 *The quotient space $L_n^k / A_n^k := QL_n^k$ has dimension $2n+2k-2$ if $n \geq 1$ and dimension $k-1$ if $n = 0$.*

Again, the proof is essentially the same as for $QL_{n,k}$ (in Theorem 3.5).

Note that Collins constructed a space for A_0^k which is isomorphic to our space L_0^k (up to basepoint questions) [4]. In this particular case when $n = 0$, the dimension is equal to the vcd.

There is a forgetful map $L_{n,k} \to L_n^k$. Let $F_{n,k}$ be the fiber over the rose (R_{n+k}, id) with the $\mathbb{Z}^k$-action induced by the action of $A_{n,k}$ on $L_{n,k}$.

Proposition 6.4 *There is a poset equivalence $F_{n,k} \to \mathbb{Z}^k \times (\mathbb{Z}/2)^k$, where the poset structure on $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$ is defined by $(\bar{z}, \bar{p}) \le (\bar{z}', \bar{p}')$ iff for all i we have*
$p_i = p_i'$ and $z_i = z_i'$ or
$p_i < p_i'$ and $z_i = z_i'$ or $z_i = z_i' + 1$.

Moreover, the realization of the poset $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$ is homeomorphic to $\mathbb{R}^k$ and the $\mathbb{Z}^k$-action on $F_{n,k}$ translates to the canonical $\mathbb{Z}^k$-action on $\mathbb{R}^k$.

Proof Suppose $(\Gamma, \phi) \in F_{n,k}$. Then Γ is the rose R_{n+k} with additional basepoints for the cycles, which are either equal to the basepoint of the graph, or a valence two vertex on the petal. Up to (n,k)-graph isomorphisms, there are $(\mathbb{Z}/2)^k$ possibilities. We identify $\bar{p} \in (\mathbb{Z}/2)^k$ with the rose which has a valence two vertex on the $(n+j)$th petal if $p_j = 1$ and no such vertex if $p_j = 0$.

Now $\phi : G_{n,k} \to R_{n+k}$ is homotopic to the identity if the cycles are allowed to rotate. So there are $\mathbb{Z}^k$ possibilities for ϕ, where $\bar{z} \in \mathbb{Z}^k$ for a rose $(R, \bar{p})$ represents the map ϕ which maps the cycles via the identity, according to the basepoints parametrized by $\bar{p}$, and wraps the jth stem z_j times around the jth cycle.

If $p_j = 0$ for all j, then the graph is reduced. If there exists a j such that $p_j = 1$, then there are two possible collapses on the jth petal, one which leaves z_j constant, and one which increases z_j by 1. (We choose the identification $\phi \rightsquigarrow \bar{z}$ so that this is the case.) This corresponds to the poset structure of $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$.

We are left to show that the realization of the poset $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$ is homeomorphic to $\mathbb{R}^k$ and that the induced $\mathbb{Z}^k$ action is the canonical action. Embed the set $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$ in $\mathbb{R}^k$ by mapping $(\bar{z}, \bar{p})$ to $(\bar{z} + \frac{\bar{p}}{2}) = (z_1 + \frac{p_1}{2}, \ldots, z_k + \frac{p_k}{2}) \in \mathbb{R}^k$. Let $C_{\bar{z}}$ be the cube of length 1 in $\mathbb{R}^k$ with bottom corner $\bar{z}$ and top corner $\bar{z} + \bar{1}$. Let $c_{\bar{z}} = (z_1 + \frac{1}{2}, \ldots, z_k + \frac{1}{2})$ be the center of the cube. Then we have $C_{\bar{z}} \cap F_{n,k} = \{(\bar{z}, \bar{p}) \in F_{n,k} | (\bar{z}, \bar{p}) \le c_{\bar{z}} = (\bar{z}, \bar{1})\}$. Similarly, the center of each face is the maximal element of the face in the poset. It follows that the realization of $\mathbb{Z}^k \times (\mathbb{Z}/2)^k$ is $\mathbb{R}^k$. The $\mathbb{Z}^k$-action, which is by addition on the $\bar{z}$-coordinate, is then by translation. □

The contractibility of L_n^k and $F_{n,k}$ leads to an alternative proof of the contractibility of $L_{n,k}$. This second approach, suggested to us by the referee, makes more apparent the fact that the Θ-graphs in the original proof of the contractibility of $L_{n,k}$ only play the role of adding basepoints.

7 Presentation

We denote by $P_{i,j}$ the automorphism of the free group $F_n = \langle x_1, \dots, x_n\rangle$ which permutes x_i and x_j and leaves the other generators fixed, and we denote by I_i the automorphism that maps x_i to its inverse and fixes the other generators.

For Whitehead moves, we use the following shortened notation: for $\epsilon, \eta = \pm 1$,

- $(z_i^\epsilon; z_j^\eta)$ denotes the automorphism which maps z_i to $z_i z_j^\eta$ if $\epsilon = 1$ and to $z_j^{-\eta} z_i$ if $\epsilon = -1$, and fixes the other generators;
- $(z_i^\pm; z_j^\eta)$ denotes the automorphism which maps z_i to $z_j^{-\eta} z_i z_j^\eta$ and fixes the other generators.

In terms of the discussion of Whitehead moves at the beginning of Section 4 before Theorem 4.1, $(z_i^\epsilon; z_j^\eta) := (\{z_i^\epsilon, z_j^\eta\}; z_j^\eta)$ and $(z_i^\pm; z_j^\eta) := (\{z_i, z_i^{-1}, z_j^\eta\}; z_j^\eta)$.

Note that $(z_i^\epsilon; z_j^\eta)^{-1} = (z_i^\epsilon; z_j^{-\eta})$ and $(z_i^\pm; z_j^\eta)^{-1} = (z_i^\pm; z_j^{-\eta})$. We will use the second notation when writing relations for simplicity.

Theorem 7.1 *The following gives a presentation for* A_n^k*:*

Generators:

$P_{i,j}$	*for* $1 \le i, j \le n$ *and* $i \ne j$
I_i	*for* $1 \le i \le n$
$(x_i^\epsilon; z)$	*for* $1 \le i \le n$, $\epsilon = \pm 1$ *and* $x_i \ne z \in \{x_1, \dots, x_n, y_1, \dots, y_k\}$
$(y_i^\pm; z)$	*for* $1 \le i \le k$ *and* $y_i \ne z \in \{x_1, \dots, x_n, y_1, \dots, y_k\}$

Relations: For $z, z_i \in \{x_1, \dots, x_n, y_1, \dots, y_k\}$ *and* $w, w_i = x_{j_i}^{\delta_i}$ *or* $y_{j_i}^\pm$,

Q1	*Relations in* $\mathrm{Aut}(F_n)$ *for* $\{(x_i^\epsilon; x_j), P_{i,j}, I_j\}$
Q2	$(w_1; z_1)(w_2; z_2) = (w_2; z_2)(w_1; z_1)$ *for* $w_1 \ne w_2$ *and* $z_i^{\pm 1} \notin \{w_1, w_2\}$
Q3	(1) $(y_i^\pm; x_j)\ P_{j,l} = P_{j,l}\ (y_i^\pm; x_l)$
	(2) $(y_i^\pm; x_j)\ I_j = I_j\ (y_i^\pm; x_j^{-1})$
Q4	(1) $(w; x_j^\eta)(x_j^\eta; z)(w; x_j^{-\eta}) = (w; z)(x_j^\eta; z)$
	(2) $(y_i^\pm; z^\epsilon)(w; y_i)(y_i^\pm; z^{-\epsilon}) = (w; z^{-\epsilon})(w; y_i)(w; z^\epsilon)$
Q5	$(y_i^\pm; x_j^\eta)(x_j^{-\eta}; y_i) = (x_j^\eta; y_i^{-1})(y_i^\pm; x_j^\eta)$

whenever these symbols denote generators or their inverses.

In particular, for $n = 0$ we recover McCool's presentation of the pure symmetric (or basis-conjugating) automorphisms of free groups [25], which has generators $\{(y_i^{\pm}; y_j) \mid 1 \leq i, j \leq k, i \neq j\}$ and relations

Z1 $(y_i^{\pm}; y_j)(y_l^{\pm}; y_j) = (y_l^{\pm}; y_j)(y_i^{\pm}; y_j)$
Z2 $(y_i^{\pm}; y_j)(y_l^{\pm}; y_m) = (y_l^{\pm}; y_m)(y_i^{\pm}; y_j)$
Z3 $(y_i^{\pm}; y_j)(y_l^{\pm}; y_j)(y_i^{\pm}; y_l) = (y_i^{\pm}; y_l)(y_i^{\pm}; y_j)(y_l^{\pm}; y_j)$

where the indices are assumed to be different when given by different letters.

Theorem 7.2 *The following gives a presentation for* $\mathrm{A}_{n,k}$*:*

Generators: same as for A_n^k *with* k *additional generators* $(y_i^{\pm}; y_i)$ *for* $1 \leq i \leq k$*.*

Relations: Q1, Q2, Q3, Q4 (never allowing a symbol $(y_i^{\pm}; y_i)$*), and*

Q2′ $(y_i^{\pm}; y_i)a = a(y_i^{\pm}; y_i)$ *with* $a = P_{i,j}, I_j$ *or* $(w; z)$
Q5′ $(y_i^{\pm}; x_j^{\eta})(x_j^{-\eta}; y_i)(y_i^{\pm}; x_j^{-\eta})(x_j^{\eta}; y_i) = (y_i^{\pm}; y_i^{-1})$

whenever these symbols denote generators or their inverses.

The generator $(y_i^{\pm}; y_i) := (x_1, \ldots, x_n, 1, \ldots, y_i, \ldots, 1) \in \mathrm{A}_{n,k}$ should be thought of as the conjugation of y_i by itself, even though this is trivial as an automorphism of the free group.

Proof of Theorem 7.1 The group $\mathrm{Aut}(F_{n+k})$ acts on the set of cyclic words with letters in $L_{n+k} := \{x_1, x_1^{-1}, \ldots, y_k, y_k^{-1}\}$ and A_n^k is the stabilizer of the k-tuple of cyclic words $U = (y_1, \ldots, y_k)$. Theorem 1.1 in [24] says that such a group is finitely presented. As remarked in [24, Section 4(1)], the proof actually constructs a finite 2-dimensional complex K whose fundamental group is this stabilizer and hence gives a presentation. We analyze this complex.

The vertices of the complex in our case are

$$K_0 := \{V \text{ minimal tuple } | V = UP \text{ for some } P \in \mathrm{Aut}(F_{n+k})\} = \{U\sigma | \sigma \in \Omega_k\}$$

where a minimal tuple is a tuple of reduced cyclic words of minimal length in the orbit of U and $\Omega_k \cong \Sigma_k \wr \mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}(F_{n+k})$ of permutations and inverses of the y_i's. So $K_0 \cong \Omega_k$.

Let $\mathcal{W}_{n+k}$ denote the set of Whitehead automorphisms for $\mathrm{Aut}(F_{n+k})$. We recall that $\mathcal{W}_{n+k}$ is the finite set $T_1 \cup T_2$ where $T_1 = \Omega_{n+k}$ is the set of permutations and inverses and $T_2 = \{(A; a) | A \subset L_{n+k}, a \in A \text{ and } a^{-1} \notin A\}$ with $(A; a)$ denoting the automorphism mapping $z \in L_{n+k}$ to za (and z^{-1} to

$a^{-1}z^{-1}$) if $a \neq z \in A$ and $z^{-1} \notin A$, mapping z to $a^{-1}za$ (and z^{-1} to $a^{-1}z^{-1}a$) if $z, z^{-1} \in A$ and fixing z otherwise. The set of edges in the complex is

$$K_1 := \{(V_1, V_2; P) | P \in \mathcal{W}_{n+k} \text{ and } V_1 P = V_2\} / \sim .$$

Note that if $P \in \mathcal{W}$, then P^{-1} is also in $\mathcal{W}$. The directed edge $e = (V_1, V_2; P)$, from V_1 to V_2, is identified with $\bar{e} = (V_2, V_1; P^{-1})$ in K_1. Finally the 2-cells are all the cells coming from the relations R1 to R10 in [23], attached to directed paths. We recall these relations below.

There are two types of edges in K, coming from the two types of Whitehead automorphisms:

Type I edges $(U\sigma, U\sigma\tau; \gamma)$ for $\sigma, \tau \in \Omega_k$ and $\gamma \in \Omega_{n+k}$ such that $(U\sigma)\gamma = U\sigma\tau$. In particular, γ is only allowed to permute the y_i's among themselves. Hence $\gamma \in \Omega_n \times \Omega_k \subset \Omega_{n+k}$ and $\gamma = (\overline{\gamma}, \tau)$. So the set of type I edges between any two vertices is isomorphic to Ω_n.

Type II edges Type II edges do not involve permutations so they can exist only from one vertex to itself in our complex. Moreover, they have to preserve the cyclic words $y_1, \ldots, y_k$, so they have the form

$$(U\sigma, U\sigma; (X \sqcup \{y_{i_1}, y_{i_1}^{-1}, \ldots, y_{i_r}, y_{i_r}^{-1}\} \sqcup \{a\}; a)$$

where $X \subset L_x = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$ and $a \in L_{n+k}$.

Choose a maximal tree T in K_1 with edges of type $(U\sigma, U\sigma\tau; \tau)$ for some $\tau \in \Omega_k \cong \{1\} \times \Omega_k$. The generators of $\pi_1 K \cong \mathrm{A}_n^k$ are given by the edges in $K_1 - T$.

If we consider the subcomplex of K formed by taking all the edges of the form $(U\sigma, U\sigma\tau; \tau)$ for $\tau \in \Omega_k$ and the 2-cells given by the relations R7 (which is a set of relations for Ω_{n+k}) restricted to these edges, we obtain a retraction of the 2-skeleton of $E\Omega_k$, which is simply connected. It follows that all edges of type $(U\sigma, U\sigma\tau; \tau)$ give trivial generators in A_n^k.

Moreover, $(U\sigma, U\sigma\tau; (\gamma, \tau)) \sim (U, U; (\gamma, 1))$ as $(1, \sigma)(\gamma, \tau) = (\gamma, 1)(1, \sigma\tau)$ in Ω_{n+k} (giving thus a relation in R7) and $(U\sigma, U\sigma; (A; a)) \sim (U, U; (A\sigma^{-1}; a\sigma^{-1}))$ as $(1, \sigma)(A; a) = (A\sigma^{-1}; a\sigma^{-1})(1, \sigma)$ by R6. (R6 says that $T^{-1}(A; a)T = (AT; aT)$ for any $T \in T_1$.) So the loops at the vertex U generate the group. Also, the relations given by 2-cells in the complex all follow from the relations R1-R10 interpreted on the generators $(U, U; \gamma)$ and $(U, U; (A; a))$. Indeed, the other relations are conjugated to these (except for the relations in R6 and R7 already used to identify generators) and thus do not bring anything new by R6, used in the case $T = (1, \sigma)$.

R1 identifies the inverses in T_2 and R2 says that $(A;a)(B;a) = (A \cup B;a)$ when $A \cap B = \{a\}$, showing thus that the generators we propose are indeed generators for the group. Note that R2 also implies the commutation relations of Q2 when $z_1 = z_2$. As the relations in the Theorem are easily verified, we are left to show that they imply the relations R3-R10.

R3 is a set of commutation relations and follows from Q2 when $z_1 \neq z_2$.

R4 says that $(B;b)^{-1}(A;a)(B;b) = (A + B - b;a)$ when $A \cap B = \emptyset$, $a^{-1} \notin B$ and $b^{-1} \in A$. We need to deduce it from our relations when $(A;a)$ and $(B;b)$ define loops at U in K. In particular, we need $b = x_i^\epsilon \in L_x$ as $b^{-1} \in A$ but $b \notin A$ and $b \neq a^{-1}$. Using Q2, we can isolate a minimal case on the left hand side: $(B - w;x_i^\epsilon)^{-1}(w;x_i^{-\epsilon})(x_i^{-\epsilon};a)(w;x_i^\epsilon)(A - x_i^{-\epsilon};a)(B - w;x_i^\epsilon)$. We then use Q4 (1) and repeat the procedure with all the elements of B.

R5 says that $(A;a)(A-a+a^{-1};b) = \begin{pmatrix} a & b \\ \downarrow & \downarrow \\ b^{-1} & a \end{pmatrix} (A-b+b^{-1};a)$ when $b \in A$, $b^{-1} \notin A$ and $a \neq b$. In our case, we need to have $a, b \in L_x$. The relation follows from Q1 when only x's are involved. Using R2, we isolate $(b;a)(a^{-1};b)$ in the left hand side and use R5 in this minimal case (that is use Q1). We then use Q3 to move the matrix (element of T1) in front of the left hand side and finally use R4.

R6, which was recalled above, follows from Q1 and Q3 (using Q2 as usual).

R7 gives a set of relations for Ω_n, which is included in Q1.

R8 says that $(A;a) = (L_{n+k} - a^{-1};a)(L_{n+k} - A;a^{-1}) = (L_{n+k} - A;a^{-1})(L_{n+k} - a^{-1};a)$. This just follows from R1-R2.

R9 says that $(L_{n+k} - b;b^{-1})(A;a)(L_{n+k} - b^{-1};b) = (A;a)$ if $b, b^{-1} \notin A$. In particular, $a \neq b$. This relation follows from Q4. Indeed, if $a = x_j^\delta \in L_x$, isolate $(x_j^{-\delta};b^{-1})$ from of the left hand side and use Q4 (1) to pass it over $(A;x_j^\delta)$, which gives $(L_{n+k} - b - x_j^{-\delta};b^{-1})(A;x_j^\delta)(A - x_j^\delta + b^{-1};b^{-1})(x_j^{-\delta};b^{-1})(L_{n+k} - b^{-1};b)$.

Now simplify and isolate $(x_j^\delta;b)$ and use again Q4(1) to pass it over $(A;x_j^\delta)$, which gives $(L_{n+k} - b - x_j^{-\delta};b^{-1})(x_j^\delta;b)(A - x_j^\delta + b;b)(A;x_j^\delta)(L_{n+k} - A - x_j^{-\delta} - b^{-1};b)$.

This simplifies to $(A;x_j^\delta)$ as required.

If on the other hand $a = y_i^\epsilon \in L_y$, isolating $(y_i^\pm;b^{-1})$ in the left hand side and using Q4 (2) to pass it over $(A;y_i^\epsilon)$ gives

$(L_{n+k}-b-y_i^{\pm};b^{-1})(A-y_i^{\epsilon}+b;b)(A;y_i^{\epsilon})(A-y_i^{\epsilon}+b^{-1};b^{-1})(y_i^{\pm};b^{-1})(L_{n+k}-b^{-1};b)$, which simplifies to $(A;y_i^{\epsilon})$.

R10 says that $(L_{n+k}-b;b^{-1})(A;a)(L_{n+k}-b^{-1};b)=(L_{n+k}-A;a^{-1})$ if $b\neq a$, $b\in A$ and $b^{-1}\notin A$. So $b\in L_x$ in our case. If $a\in L_x$, we can follow [23] and show that the relation follows from R9 and R5. We do the case $a\in L_y$. So we take $a=y_j^{\eta}$, $b=x_i^{\epsilon}$ The left hand side in the equation equals

$$(L_{n+k}-x_i^{\epsilon}-y_j^{\pm};x_i^{-\epsilon})(y_j^{\pm};x_i^{-\epsilon})(x_i^{\epsilon};y_j^{\eta})(A-x_i^{\epsilon};y_j^{\eta})(L_{n+k}-x_i^{-\epsilon};x_i^{\epsilon})$$

which, using Q5, gives

$$(L_{n+k}-x_i^{\epsilon}-y_j^{\pm};x_i^{-\epsilon})(x_i^{-\epsilon};y_j^{-\eta})(y_j^{\pm};x_i^{-\epsilon})(A-x_i^{\epsilon};y_j^{\eta})(L_{n+k}-x_i^{-\epsilon};x_i^{\epsilon}).$$

Now we use Q4 (1) to pass $(x_i^{-\epsilon};y_j^{-\eta})$ to the left, and Q4(2) to pass $(y_j^{\pm};x_i^{-\epsilon})$ over $(A-x_i^{\epsilon};y_j^{\eta})$, which gives

$$(x_i^{-\epsilon};y_j^{-\eta})(L_{n+k}-x_i^{\epsilon}-y_j^{\eta};y_j^{-\eta})(L_{n+k}-x_i^{\epsilon}-y_j^{\pm};x_i^{-\epsilon})(A-y_j^{\eta};x_i^{\epsilon})(A-x_i^{\epsilon};y_j^{\eta})$$
$$(A-x_i^{\epsilon}-y_j^{\eta}+x_i^{-\epsilon};x_i^{-\epsilon})(y_j^{\pm};x_i^{-\epsilon})(L_{n+k}-x_i^{-\epsilon};x_i^{\epsilon}).$$

This simplifies to $(L_{n+k}-A;y_j^{-\eta})$ as required. □

To prove Theorem 7.2, we need a lemma about group extensions.

Let $1\to A\xrightarrow{i}E\xrightarrow{\pi}G\to 1$ be a central extension of a group G by an abelian group A, with a section of sets $s:G\to E$ (so $\pi\circ s=id$) satisfying $s(1)=1$ and $f:G\times G\to A$ the classifying cocycle of the extension, defined by

$$s(g)\ s(h)=i\circ f(g,h)\ s(gh).$$

Note that if $r_1\dots r_m=1$ in G, then $s(r_1\dots r_m)=1$ in E and hence

$$s(r_1)\dots s(r_m)=f(r_1,r_2)f(r_1r_2,r_3)\dots f(r_1\dots r_{m-1},r_m).$$

We denote by s^f the operator lifting the relations of G to relations in E this way.

Lemma 7.3 *Let $1\to A\xrightarrow{i}E\xrightarrow{\pi}G\to 1$ be a central extension as above. If A and G are finitely presented, with presentation*

$$A=\langle a_1,\dots,a_k\rangle/R_A$$
$$G=\langle g_1,\dots,g_n\rangle/R_G.$$

Then E is finitely presented, with presentation

$$E=\langle i(a_1)\dots,i(a_k),s(g_1),\dots,s(g_n)\rangle/[A,G]\cup i(R_A)\cup s^f(R_G)$$

where $[A,G]$ is the set of commutation relations $i(a_j)s(g_i)=s(g_i)i(a_j)$ and $s^f(R_G)$ is the lift of the relations R_G to E twisted by f as defined above.

Proof Consider the free group $F := \langle i(a_1) \dots, i(a_k), s(g_1), \dots, s(g_n)\rangle$. The natural map $h : F \to E$ is surjective as $i \times s : A \times G \to E$ is a bijection of sets (since $s(G)$ is a set of coset representatives). The relations $[A, G]$, $i(R_A)$ and $s^f(R_G)$ are clearly satisfied in E, and so define elements in the kernel of h.

Let $e = i(a_{j_1}) \dots i(a_{j_p}) s(g_{i_1}) \dots s(g_{i_q}) \in F$ and suppose that $h(e) = 1$. (It is enough to consider this case as $[A, G]$ is satisfied in E.) Then $\pi \circ h(e) = 1$ in G, that is $g_{i_1} \dots g_{i_q} = 1$. This relation in G lifts to a relation $s(g_{i_1}) \dots s(g_{i_q}) = i(a)$ in E for some $a \in A$. So $i(a_{j_1}) \dots i(a_{j_p}) s(g_{i_1}) \dots s(g_{i_q}) = i(a_{j_1}) \dots i(a_{j_p}) i(a)$. As $h(e) = 1$, we have $a_{j_1} \dots a_{j_p} a = 1$ is a relation in A (as i is an injective homomorphism). So the relation $h(e) = 1$ follows from a relation in $s^f(R_G)$ and a relation in $i(R_A)$. □

Proof of Theorem 7.2 There is a central extension $1 \to \mathbb{Z}^k \to \mathrm{A}_{n,k} \to \mathrm{A}_n^k \to 1$. By the Lemma, the generators of $\mathrm{A}_{n,k}$ are the union of the generators of A_n^k with those of $\mathbb{Z}^k$. The last generators are denoted $(y_i^{\pm}; y_i)$, for $1 \le i \le k$, in the Theorem. Then relations for $\mathrm{A}_{n,k}$ are given by the relations for $\mathbb{Z}^k$, the commutations $[\mathbb{Z}^k, \mathrm{A}_n^k]$ and the lifts of the relations from A_n^k. Q2' takes care of the first two sets of relations and Q1,Q2,Q3,Q4 and Q5' is the lift of the relations from A_n^k. □

In [3], the group A_0^k is studied (denoted C_k in the paper). They show that

$$H_1(\mathrm{A}_0^k) = \bigoplus_{1 \le i,j \le k,\ i \ne j} \mathbb{Z}[y_i^{\pm}; y_j]$$

where $\mathbb{Z}[y_i^{\pm}; y_j]$ is the free abelian group generated by $[y_i^{\pm}; y_j]$. Recall that A_0^k can be thought of as an extended pure braid group (which is the mapping class group of a punctured sphere). Note in particular that, as for the pure braid group, H_1 is not stable with respect to k. The presentation given above allows us to calculate H_1 in the higher genus cases.

Theorem 7.4 *$H_1(\mathrm{A}_{n,k}) \cong \mathbb{Z}_2 \cong H_1(\mathrm{A}_n^k)$ if $n > 2$*

Proof If $n > 2$, we have $H_1(\mathrm{Aut}(F_n)) = \mathbb{Z}_2$ generated by $[P_{i,j}] = [I_k]$ (the generators $(x_i^{\delta}; x_j)$ being trivial in the abelianization).

For A_n^k, Q4 (1) implies that $(w; z)$ is trivial in the abelianization for any $w = x_i^{\delta}$ or $y_i^{\pm}$ and $z \in \{x_1, x_2, \dots, y_k\}$. The only relations involving $P_{i,j}$ or I_k which are not in $\mathrm{Aut}(F_n)$ give trivial relations in the abelianization, hence $H_1(\mathrm{A}_n^k) = \mathbb{Z}_2$.

For $\mathrm{A}_{n,k}$, the additional relation Q5' implies that the generators $(y_i^{\pm}; y_i)$ are also trivial in the abelianization and hence we get the same result. □

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