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The $\mathbb{Z}$–graded symplectic Floer cohomology of monotone Lagrangian sub–manifolds

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Abstract We define an integer graded symplectic Floer cohomology and a Fintushel–Stern type spectral sequence which are new invariants for monotone Lagrangian sub–manifolds and exact isotopes. The $\mathbb{Z}$–graded symplectic Floer cohomology is an integral lifting of the usual $\mathbb{Z}_{\Sigma(L)}$–graded Floer–Oh cohomology. We prove the Künneth formula for the spectral sequence and an ring structure on it. The ring structure on the $\mathbb{Z}_{\Sigma(L)}$–graded Floer cohomology is induced from the ring structure of the cohomology of the Lagrangian sub–manifold via the spectral sequence. Using the $\mathbb{Z}$–graded symplectic Floer cohomology, we show some intertwining relations among the Hofer energy $e_H(L)$ of the embedded Lagrangian, the minimal symplectic action $\sigma(L)$, the minimal Maslov index $\Sigma(L)$ and the smallest integer $k(L,\phi)$ of the converging spectral sequence of the Lagrangian L.

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1 Introduction

In this paper, we construct a $\mathbb{Z}$–graded symplectic Floer cohomology of monotone Lagrangian sub-manifolds by a completely algebraic topology method. This is a local symplectic invariant in terms of symplectic diffeomorphisms. We show that there exists a spectral sequence which converges to a global symplectic invariant (the $\mathbb{Z}_{\Sigma(L)}$–graded Floer cohomology, where $\mathbb{Z}_{\Sigma(L)}$ is the minimal Maslov number of the monotone Lagrangian sub–manifold L). The $\mathbb{Z}$–graded symplectic Floer cohomology is an integral lifting of the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology. By exploiting the properties of our $\mathbb{Z}$–graded symplectic Floer cohomology, we show that there is a relation between the $\mathbb{Z}$–graded symplectic Floer cohomology and the restricted symplectic Floer cohomology constructed in [2] (see §5) via Hofer's symplectic energy. This may give an interesting way to understand the Hofer symplectic energy through the

$\mathbb{Z}$–graded symplectic Floer cohomology. We borrow some ideas from the instanton Floer theory, and construct the Fintushel-Stern type spectral sequence in [6] for monotone Lagrangian sub–manifolds. Our method is in the nature of algebraic topology (see [20]), and is very different from the method in [17] which is used local Darboux neighborhoods. We hope that there will be more algebraic cohomology operations which can be induced to the symplectic Floer cohomology through the quantum effects of higher differentials in our spectral sequence.

Let (P,ω) be a monotone symplectic manifold and L be a monotone Lagrangian sub-manifold in (P,ω). Let Z_ϕ be the critical point set of the symplectic action a_ϕ (see §2), where $\phi \in \mathrm{Symp}_0(P)$ is a symplectic diffeomorphism generated by a time–depended Hamiltonian function. The set $\mathrm{Im}\,(a_\phi)(Z_\phi)$ is discrete. For $r \in \mathbb{R}\backslash \mathrm{Im}\,(a_\phi)(Z_\phi) = \mathbb{R}_{L,\phi}$, we can associate the $\mathbb{Z}_2$–modules $I_*^{(r)}(L,\phi;P)$ with an integer grading. The $\mathbb{Z}$–graded symplectic Floer cohomology $I_*^{(r)}(L,\phi;P)$ depends on r:

(i) if $[r_0,r_1] \subset \mathbb{R}_{L,\phi}$, then $I_*^{(r_0)}(L,\phi;P) = I_*^{(r_1)}(L,\phi;P)$;

(ii) $I_{*+\Sigma(L)}^{(r)}(L,\phi;P) = I_*^{(r+\sigma(L))}(L,\phi;P)$, where $\Sigma(L)(>0)$ is the minimal Maslov number of L and $\sigma(L)(>0)$ is the minimal number in $\mathrm{Im}\, I_\omega|_{\pi_2(P,L)}$ (see Definition 2.2 for $\Sigma(L)$ and $\sigma(L)$).

Our main results are the following theorems.

Theorem A *Let L be a monotone Lagrangian sub–manifold in (P,ω). If $\Sigma(L) \geq 3$, then*

(1) *there exists an isomorphism*

$$\phi_{01}^n\colon\ I_n^{(r)}(L,\phi^0;P,J^0) \to I_n^{(r)}(L,\phi^1;P,J^1),$$

for $n \in \mathbb{Z}$, and a continuation $(J^\lambda,\phi^\lambda)_{0\le\lambda\le 1} \in \mathcal{P}_1$ which is regular at the ends.

(2) *there is a spectral sequence $(E_{n,j}^k, d^k)$ with*

$$E_{n,j}^1(L,\phi;P,J) \cong I_n^{(r)}(L,\phi;P,J), \quad n \equiv j \ \ (mod\ \ \Sigma(L)),$$

$$d^k\colon\ E_{n,j}^k(L,\phi;P,J) \to E_{n+\Sigma(L)k+1,j+1}^k(L,\phi;P,J),$$

and

$$E_{n,j}^\infty(L,\phi;P,J) \cong F_n^{(r)}HF^j(L,\phi;P,J)/F_{n+\Sigma(L)}^{(r)}HF^j(L,\phi;P,J).$$

Theorem B *For $\Sigma(L) \geq 3$ and $k \geq 1$, $E^k_{n,j}(L,\phi;P,J)$ is a symplectic invariant under continuous deformations of $(J^\lambda, \phi^\lambda)$ within the set of continuations.*

For $k \geq 1$ and $r \in \mathbb{R}_{L,\phi}$, all the $E^k_{*,*}(L,\phi;P,J)$ are new symplectic invariants. They provide potentially interesting invariants for the symplectic topology of L. Let $k(L,\phi)$ be the minimal k for which $E^k_{*,*}(L,\phi) = E^\infty_{*,*}(L,\phi)$. So $k(L,\phi)$ is an invariant of (L,ϕ). Using the $\mathbb{Z}$–graded symplectic Floer cohomology and the spectral sequence in Theorem A, we obtain the Künneth formulae for each term of the spectral sequence with $\mathbb{Z}_2$–coefficients. Since we work on the $\mathbb{Z}_2$–coefficients, there are Künneth formulae for the induced spectral sequence of the product monotone Lagrangian sub-manifold $(L_1 \times L_2, \phi_1 \times \phi_2)$. Then we study the Poincaré–Laurent polynomial for $L_1 \times L_2$ in terms of the Poincaré–Laurent polynomials for $(L_i, \phi_i)(i = 1, 2)$. For certain Lagrangian imbeddings, we obtain an internal cup–product structure on the spectral sequence which is descended from the usual cup product of the cohomology $H^*(L, \mathbb{Z}_2)$, by studying the $H^*(L, \mathbb{Z}_2)$–module structure on the spectral sequence and the $\mathbb{Z}_2$–coefficients. The index in the internal product (5.16) is unusual from the internal product structure due to the Maslov index shift. From the quantum effect aroused from the higher differentials on $I^{(r)}_*(L,\phi;P,J)$ in the spectral sequence in Theorem A, the ring $(HF^{*-m}(L,\phi;P), \cup_\infty)$ on the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology can be thought of as the quantum effect of the cohomology ring $(H^*(L;\mathbb{Z}_2), \cup)$ (see §5.2). Note that our cup-product structure is different with the multiplicative structure defined in [7, 14, 18]. In [14, 18], the Floer cohomology is the cohomology of the symplectic manifold, only the cup-product structure is deformed, i.e., the same cohomology group with different ring structures is studied in [14, 18]. Our induced cup–product on $E^k_{*,*}(L,\phi;P,J)$ may well have that the cohomology groups are different from the cohomology of the Lagrangian sub–manifolds (see [12] for instance).

Theorem C (1) *For the monotone Lagrangian $L_1 \times L_2$ in $(P_1 \times P_2, \omega_1 \oplus \omega_2)$ with $I_{\omega_i} = \lambda I_{\mu, L_i}$ and $\Sigma(L_i) = \Sigma(L) \geq 3$ $(i = 1, 2)$, we have, for $k \geq 1$,*

$$E^k_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \cong$$

$$\bigoplus_{n_1+n_2=n, j_1+j_2=j \pmod{\Sigma(L)}} E^k_{n_1,j_1}(L_1,\phi_1;P_1) \otimes E^k_{n_2,j_2}(L_2,\phi_2;P_2). \tag{1.1}$$

(2) *For the monotone Lagrangian $L \overset{i}{\hookrightarrow} P$ with $i^* \colon H^*(P;\mathbb{Z}_2) \to H^*(L;\mathbb{Z}_2)$ surjective and $\Sigma(L) \geq 3$, the spectral sequence $E^k_{*-m,*}(L,\phi;P)$ carries an ring structure which is descended from the cohomology ring $(H^*(L;\mathbb{Z}_2), \cup)$.*

As an easy consequence of Theorem C, we obtain a generalization of Theorem 1 and Theorem 3 of [4] as stated in Corollary 5.9. This proves the Arnold conjecture that the monotone Lagrangian intersections is bounded below by the $\mathbb{Z}_2$–cuplength of the Lagrangian sub-manifold (see §5.2). Using a result of Gromov and the Poincaré–Laurent polynomial associated to the spectral sequence, we show that the four invariants $\sigma(L), \Sigma(L), e_H(L)$ and $k(L, \phi)$ play important roles in the $\mathbb{Z}$–graded symplectic Floer cohomology and the study of Lagrangian embeddings in §5.1. We obtain Chekanov's result by using the $\mathbb{Z}$–graded symplectic Floer cohomology. Our study suggests a possible relation between the $\mathbb{Z}$–graded symplectic Floer cohomology and Hofer's symplectic energy for monotone Lagrangian sub-manifolds. In fact we conjecture that Hofer's symplectic energy of a monotone Lagrangian sub–manifold L with $\Sigma(L) \geq 3$ is a positive multiple of $\sigma(L)$ (More precisely, $e_H(L) = (k(L, \phi) - 1)\sigma(L))$. We will discuss this problem elsewhere. It would be also interesting to link the $\mathbb{Z}$–graded symplectic Floer cohomology with the (modified) Floer cohomology with Novikov ring coefficients in [7].

The paper is organized as follows. In §2, we define the $\mathbb{Z}$–graded symplectic Floer cohomology for monotone Lagrangian sub–manifolds. Its invariance under the symplectic continuations is given in §3. Theorem A (1) is proved in §3. Theorem A (2), Theorem B and Theorem C (1) (Theorem 4.13) are proved in §4. In §5, we give some applications related to Chekanov's construction and Lagrangian embeddings; at the last subsection §5.2, the proof of Theorem C (2) (Theorem 5.11) is given.

2 The $\mathbb{Z}$–graded Floer cohomology for Lagrangian intersections

In this section, we define the $\mathbb{Z}$–graded symplectic Floer cohomology, and discuss some basic properties.

Let (P, ω) be an oriented, connected and compact (or tamed) symplectic manifold with a closed non–degenerate 2-form ω. The 2–form ω defines the cohomology class $[\omega] \in H^2(P, \mathbb{R})$. By choosing an almost complex structure J on (P, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric, we have an integer valued cohomology class $c_1(P) \in H^2(P, \mathbb{Z})$ (the first Chern class). These two cohomology classes define two homomorphisms

$$I_\omega : \pi_2(P) \to \mathbb{R}; \qquad I_{c_1} : \pi_2(P) \to \mathbb{Z},$$

by $I_\omega(u)=\int_{S^2}u^*(\omega)$ and $I_{c_1}(u)=\int_{S^2}u^*(c_1)$ for $u\in\pi_2(P)$. If $u\colon(D^2,\partial D^2)\to(P,L)$ is a smooth map of pairs, up to homotopy, there is a unique trivialization of the pull-back bundle $u^*TP\cong D^2\times\mathbb{C}^m$. The trivialization of the symplectic vector bundle defines a map from $S^1=\partial D^2$ to the set $\Lambda(\mathbb{C}^m)$ of Lagrangians in $\mathbb{C}^m$. Let $\mu\in H^1(\Lambda(\mathbb{C}^m),\mathbb{Z})$ be the well–known Maslov class. Then the map $I_{\mu,L}\colon\pi_2(P,L)\to\mathbb{Z}$ is defined by $I_{\mu,L}(u)=\mu(\partial D^2)$. The Maslov index is invariant under any symplectic isotopy of P.

Definition 2.1 (i) The symplectic manifold (P,ω) is *monotone* if $I_\omega=\alpha I_{c_1}$ for some $\alpha\geq 0$.

(ii) A Lagrangian sub–manifold L in (P,ω) is *monotone* if $I_\omega=\lambda I_{\mu,L}$ for some $\lambda\geq 0$.

For $\alpha=0$ and $\lambda=0$ case, the manifold (P,ω) and L are monotone defined by Floer [3, 5]. The notion of monotone Lagrangian sub-manifolds is introduced by Oh [16]. The monotonicity is preserved under the exact deformations of L. By the canonical homomorphism $f\colon\pi_2(P)\to\pi_2(P,L)$,

$$I_\omega(x)=I_\omega(f(x)),\qquad I_{\mu,L}(f(x))=2I_{c_1}(x),$$

for $x\neq 0\in\pi_2(P)$. If the Lagrangian sub-manifold L is monotone, then (P,ω) is also a monotone symplectic manifold with $2\lambda=\alpha$. In fact, the constant λ does not depend on the Lagrangian L, but depends only on the (P,ω) if $I_\omega|_{\pi_2(P)}\neq 0$.

Definition 2.2 (i) Define $\sigma(L)$ to be the positive minimal number in the set $\operatorname{Im}I_\omega|_{\pi_2(P,L)}\subset\mathbb{R}$. Define $\Sigma(L)$ to be the positive generator for the subgroup $[\mu|_{\pi_2(P,L)}]=\operatorname{Im}I_{\mu,L}$ in $\mathbb{Z}$.

(ii) A Lagrangian sub–manifold L is called rational if $\operatorname{Im}I_\omega|_{\pi_2(P,L)}=\sigma(L)\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$ and $\sigma(L)>0$. For a monotone Lagrangian, we have $\sigma(L)=\lambda\Sigma(L)$ for some $\lambda\geq 0$.

Let $H\colon P\times\mathbb{R}\to\mathbb{R}$ be a smooth real valued function and let X_H be defined by $\omega(X_H,\cdot)=dH$. Then the ordinary differential equation

$$\frac{dx}{dt}=X_H(x(t)),\tag{2.1}$$

is called a Hamiltonian equation associated with the time–dependent Hamiltonian function H, or with the Hamiltonian vector field X_H. Equation (2.1) defines a family $\phi_{H,t}$ of diffeomorphisms of P such that $x(t)=\phi_{H,t}(x)$ is the solution of (2.1). The set $\mathcal{D}_\omega=\{\phi_{H,1}|H\in C^\infty(P\times\mathbb{R},\mathbb{R})\}$ of all diffeomorphisms arising in this way is a subgroup of the group of symplectic diffeomorphisms. An element in the set $\mathcal{D}_\omega$ of exact diffeomorphisms is called a (time–dependent) *exact isotopy.*

For an exact isotopy $\phi = \{\phi_t\}_{0\leq t\leq 1}$ on (P,ω), we define the space

$$\Omega_\phi = \{z\colon I \to P \ \mid \ z(0) \in L, z(1) \in \phi_1(L), [\phi_t^{-1} z(t)] = 0 \in \pi_1(P,L)\}.$$

Let L be a monotone Lagrangian sub–manifold, and let $\phi = \{\phi_t\}_{0\leq t\leq 1}$ be an exact isotopy on (P,ω). If u and v are two maps from $[0,1]\times[0,1]$ to Ω_ϕ such that

$$u(\tau,0), v(\tau,0) \in L, \quad u(\tau,1), v(\tau,1) \in \phi_1(L)$$

$$u(0,t) = v(0,t) \equiv x, \quad u(1,t) = v(1,t) \equiv y, \quad x,y \in L\cap\phi_1(L),$$

then we have

$$I_\omega(u) = I_\omega(v) \ \text{ if and only if } \ \mu_u(x,y) = \mu_v(x,y),$$

where $\mu_u(x,y) = I_{\mu,L}(u)$ is the Maslov–Viterbo index. In particular, if u and v are J–holomorphic curves with respect to the almost complex structure J (may vary with time t) compatible with ω, then

$$\int \|\nabla u\|_J^2 = \int \|\nabla v\|_J^2 \quad \text{if and only if} \quad I_{\mu,L}(u) = I_{\mu,L}(v).$$

Note that $\mu_u(x,y)$ is well–defined mod $\Sigma(L)$. The tangent space $T_z\Omega_\phi$ of Ω_ϕ consists of vector fields ξ of P along z which are tangent to L at 0 and to $\phi_1(L)$ at 1. Then ω induces a "1–form" on Ω_ϕ:

$$Da(z)\xi = \int_0^1 \omega(\frac{dz}{dt}, \xi(t))dt. \tag{2.2}$$

This form is closed in the sense that it can be integrated locally to a real function a on Ω_ϕ. The term $Da(z)\xi$ vanishes for all ξ if and only if z is a constant loop, i.e., $z(0)$ is a fixed point of ϕ_1. The critical point set Z_ϕ of the 1–form Da is the intersection point set $L\cap\phi_1(L)$. A critical point is non–degenerate if and only if the corresponding intersection is transversal.

For a monotone Lagrangian sub–manifold L, an exact isotopy ϕ and $k > 2/p$, consider the space of L_k^p–paths

$$\mathcal{P}_{k,loc}^p(L,\phi;P) = \{u \in L_{k,loc}^p(\Theta,P) \ \mid \ u(\mathbb{R}\times\{0\}) \subset L, \ \ u(\mathbb{R}\times\{1\}) \subset \phi(L)\},$$

where $\Theta = \mathbb{R}\times[0,1] = \mathbb{R}\times iI \subset \mathbb{C}$. Let S_ω be the bundle of all $J \in \mathrm{End}(TP)$ whose fiber is given by

$$S_x = \{J \in \mathrm{End}(T_xP) \ \mid \ J^2 = -Id \ \text{ and } \ \omega(\cdot, J\cdot) \ \text{ is a Riemannian metric}\}.$$

Let $\mathcal{J} = C^\infty([0,1] \times S_\omega)$ be the set of time–dependent almost complex structures. Define

$$\overline{\partial}_J u(\tau, t) = \frac{\partial u(\tau, t)}{\partial \tau} + J_t \frac{\partial u(\tau, t)}{\partial t} \tag{2.3}$$

on $\mathcal{P}^p_{k,loc}(L, \phi; P)$. Then the equation $\overline{\partial}_J u = 0$ is translational invariant in the variable τ. Let $\mathcal{M}$ be the moduli space $\mathcal{M}_J(L,\phi) = \{u\colon \mathbb{R} \to \in \Omega_\phi | \int_{\mathbb{R}\times I} |\frac{\partial u}{\partial \tau}|^2 < \infty, \overline{\partial}_J u = 0\}$ of finite actions, and $\mathcal{M}_J(x,y) = \{u \in \mathcal{M} | \quad \lim_{\tau\to+\infty} u = x, \lim_{\tau\to-\infty} u = y; \quad x, y \in Z_\phi\}$. So the moduli space $\mathcal{M}$ is the union of J-holomorphic curves $\bigcup_{x,y\in L\cap\phi_1(L)} \mathcal{M}_J(x,y)$. If L intersects $\phi_1(L)$ transversely, then there exists a smooth Banach manifold $\mathcal{P}(x,y) = \mathcal{P}^p_k(x,y) \subset \mathcal{P}^p_{k,loc}$ for each $x, y \in Z_\phi$ such that (2.3) defines a smooth section $\overline{\partial}_J$ of the smooth Banach space bundle $\mathcal{L}$ over $\mathcal{P}(x,y)$ with fibers $\mathcal{L}_u = L^p_{k-1}(u^*TP)$. So $\mathcal{M}_J(x,y)$ is the zero set of $\overline{\partial}_J$. The tangent space $T_u\mathcal{P}$ consists of all elements $\xi \in L^p_k(u^*(TP))$ so that $\xi(\tau, 0) \in TL$ and $\xi(\tau, 1) \in T(\phi_1(L))$ for all $\tau \in \mathbb{R}$. The linearized operator of $\overline{\partial}_J u$, denoted by

$$E_u = D\overline{\partial}_J(u)\colon\ T_u\mathcal{P} \to \mathcal{L}_u,$$

is a Fredholm operator for $u \in \mathcal{M}_J(x,y)$. There is a dense set $\mathcal{J}_{reg}(L, \phi_1(L)) \subset \mathcal{J}$ so that if $J \in \mathcal{J}_{reg}(L, \phi_1(L))$, then E_u is surjective for all $u \in \mathcal{M}_J(x,y)$. Moreover the Fredholm index of the linearized operator E_u is the same as the Maslov index $\mu_u(x,y)$. In particular, the space $\mathcal{M}_J(x,y)$ is a smooth manifold with dimension $\mu_u(x,y)$ for $J \in \mathcal{J}_{reg}(L, \phi_1(L))$ (see Proposition 2.1 in [3]).

Theorem 2.3 [3, 16] *Let L be a monotone Lagrangian sub–manifold in P, $\Sigma(L) \geq 3$ and $\phi = \{\phi_t\}_{0\leq t\leq 1}$ be an exact isotopy such that L intersects $\phi_1(L)$ transversely. Then there is a dense subset $\mathcal{J}_*(L,\phi) \subset \mathcal{J}_{reg}(L,\phi)$ of $\mathcal{J}$ such that (1) the zero dimensional component of $\hat{\mathcal{M}}_J(x,y) = \mathcal{M}_J(x,y)/\mathbb{R}$ is compact and (2) the one dimensional component of $\hat{\mathcal{M}}_J(x', y') = \mathcal{M}_J(x', y')/\mathbb{R}$ is compact up to the splitting of two isolated trajectories for $J \in \mathcal{J}_*(L,\phi)$. Let $C_*(L,\phi; P, J)$ be the free module over $\mathbb{Z}_2$ generated by Z_ϕ. Moreover, there exists a homomorphism*

$$\delta\colon\ C_*(L,\phi; P, J) \to C_*(L,\phi; P, J) \tag{2.4}$$

with $\delta \circ \delta = 0$ for $J \in \mathcal{J}_{reg}(L, \phi_1(L))$. The $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology $HF^*(L,\phi; P, J)$ is defined to be the cohomology of the complex $(C_*(L,\phi; P, J), \delta)$, and $HF^*(L,\phi; P, J)$ is invariant under the continuation of (J, ϕ), denoted by $HF^*(L,\phi; P)$ with $* \in \mathbb{Z}_{\Sigma(L)}$.

In order to extend the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer–Oh cohomology to a $\mathbb{Z}$–graded symplectic Floer cohomology, we make use of the infinite cyclic cover $\tilde{\Omega}^*_\phi$

of Ω_ϕ. By (2.2), the functional a on Ω_ϕ is only defined as $a\colon \Omega_\phi \to \mathbb{R}/\sigma(L)\mathbb{Z}$, for different topology classes in $\pi_2(P,L)$. The symplectic action on $\tilde{\Omega}^*_\phi$ and the Maslov index function on $\tilde{Z}^*_\phi$ are well–defined: $a\colon \tilde{\Omega}^*_\phi \to \mathbb{R}$ and $\mu\colon \tilde{Z}^*_\phi \to \mathbb{Z}$. The $\mathbb{Z}$–graded symplectic Floer cohomology is constructed from the lifted symplectic action and the lifted Maslov index. The functional a on Ω_ϕ and its lift on $\tilde{\Omega}^*_\phi$ are clearly distinguished from the context.

Lemma 2.4 *There exists a universal covering space $\tilde{\Omega}_\phi$ of Ω_ϕ with transformation group $\pi_2(P,L)$.*

Proof By theorem 3.1 in [15], there is a universal covering space $\tilde{\Omega}_\phi$ of Ω_ϕ since the space Ω_ϕ has the homotopy type of a CW complex. For $[u] \in \pi_1(\Omega_\phi)$, we have a representative $u\colon I \to \Omega_\phi$ such that there is a homotopy $F(\tau,t)$ of $\phi_t^{-1}u(\tau,t)$ to a constant path in (P,L) by the definition of Ω_ϕ. Thus we can reformulate the map u to yield a map $\overline{u}(\tau,t) = u(2\tau,t)$ for $0 \le \tau \le 1/2$; $\overline{u}(\tau,t) = F(2\tau-1,t)$ for $1/2 \le \tau \le 1$. Such a map $\overline{u}\colon (D^2,\partial D^2) \to (P,L)$ defines an element in $\pi_2(P,L)$. It is easy to check that $u \to \overline{u}$ is a bijective homomorphism between $\pi_1(\Omega_\phi)$ and $\pi_2(P,L)$ (see also Proposition 2.3 in [3]). So the result follows. □

Now the closed 1–form $Da(z)$ has a function $a\colon \tilde{\Omega}_\phi \to \mathbb{R}$ which is well–defined up to a constant. Pick a point $z_0 \in L \cap \phi_1(L)$ such that $a(z_0) = 0$ by adding a constant. For $g \in \pi_1(\Omega_\phi) = \pi_2(P,L)$, we have

$$a(g(x)) = a(x) + \deg(g)\sigma(L), \tag{2.5}$$

where $\deg(g)$ is defined by $I_\omega(g) = \deg(g)\sigma(L)$. Let $\mathrm{Im}\,(a)(Z_\phi)$ be the image of a of Z_ϕ; modulo $\sigma(L)\mathbb{Z}$, the set $\mathrm{Im}\,(a)(Z_\phi)$ is finite. Thus the set $\mathbb{R}_{L,\phi} = \mathbb{R} \setminus \mathrm{Im}\,(a)(Z_\phi)$ consists of the regular values of the symplectic action a on $\tilde{\Omega}_\phi$. From the map $a\colon \Omega_\phi \to \mathbb{R}/\sigma(L)\mathbb{Z}$, we pullback the universal covering space $\mathbb{R} \to \mathbb{R}/\sigma(L)\mathbb{Z}$ over Ω_ϕ. Let $\tilde{\Omega}^*_\phi$ be the pullback $a^*(\mathbb{R}) \to \Omega_\phi$. The space $\tilde{\Omega}^*_\phi$ is an infinite cyclic sub–covering space of the covering space $\tilde{\Omega}_\phi$ of Ω_ϕ.

Given $x \in Z_\phi \subset \Omega_\phi$, let $x^{(r)} \in \tilde{Z}^*_\phi \subset \tilde{\Omega}^*_\phi$ be the unique lift of x such that $a(x^{(r)}) \in (r, r+\sigma(L))$. Let $\mu^{(r)}(x) = \mu_{\tilde{u}}(x^{(r)}, z_0) \in \mathbb{Z}$ for $\tilde{u} = g \circ u$ with $g \in \pi_2(P,L)$ and $x^{(r)} = g(x)$. We define the $\mathbb{Z}$–graded symplectic Floer cochain group by

$$C_n^{(r)}(L,\phi;P,J) = \mathbb{Z}_2\{x \in Z_\phi \quad | \quad \mu^{(r)}(x) = n \in \mathbb{Z}\}. \tag{2.6}$$

The group $C_n^{(r)}(L,\phi;P,J)$ is a free module over $\mathbb{Z}_2$ generated by $x \in Z_\phi$ with $\mu(x^{(r)}, z_0) = n$. The grading is independent of the choice of z_0. If $\overline{z}_0$ is another choice of the based point and $g(z_0) = \overline{z}_0$ for some covering transformation g, then the corresponding choice of a lift $\overline{x}^{(r)}$ of x is just $g(x^{(r)})$ by (2.5). Note that the Maslov index $\mu^{(r)}(x)$ is independent of the choice of the based point z_0 used in the definition of a. The following lemma shows that the lift of the functional a is compatible with the universal lift of the circle $\mathbb{R}/\sigma(L)\mathbb{Z}$.

Lemma 2.5 *The lift of the symplectic action over $\tilde{\Omega}^*_\phi$ is compatible with the one of the Maslov index: for $g \in \pi_2(P, L)$ with $\deg(g) = n$,*

$$a(g(z_0)) = n\sigma(L) \quad \textit{if and only if} \quad \mu^{(r)}(g(z_0), z_0) = n\Sigma(L).$$

Proof Let J be a compatible almost complex structure and $\omega(\cdot, J\cdot)$ be the corresponding Riemannian metric on P. Denote ∇ be the Levi–Civita connection of the metric $\omega(\cdot, J\cdot)$. Then T_xL is an orthogonal complement of JT_xL. One can represent J_x to be the standard J for suitable orthonormal basis in T_xL. Let h be the parallel transport along the path $u(\tau)$ ($u(\tau, t)$ for each fixed $t \in I$) in Ω_ϕ. Then we get an isometry

$$h_{\tau,t}\colon\, T_xP \to T_{u(\tau,t)}P.$$

Define $J_{\tau,t} = h^{-1}_{\tau,t} \circ J_{u(\tau,t)} \circ h_{\tau,t}$. Then we have a smooth map $f\colon\, I \times I \to SO(2m)$ such that $f^{-1}_{\tau,t} \circ J_{\tau,t} \circ f_{\tau,t} = J_x$. Set

$$\tilde{L}(\tau) = h^{-1}_{\tau,0}(T_{u(\tau,0)}L), \quad \widetilde{\phi_1(L)}(\tau) = h^{-1}_{\tau,1}(T_{u(\tau,1)}\phi_1(L)).$$

Thus $f_{\tau,0}(\tilde{L}(\tau)) = L(\tau)$ and $f_{\tau,1}(\phi_1 \tilde{(L)}(\tau)) = \phi_1(L)(\tau)$. The trivialization of u^*TP by using the parallel transportation $\{h_{\tau,t}\}$ is given by

$$u^*(T_xP) = I \times I \times T_xP = I \times I \times \mathbb{C}^m.$$

Then there are two paths of Lagrangian subspaces $\tilde{L}(\tau)$ and $\widetilde{\phi_1(L)}(\tau)$ in $T_xP = \mathbb{C}^m$. Note that these two Lagrangian paths intersect transversely at end points $\tau = 0$ and $\tau = 1$. There is a map f_{ϕ_1} from the space Ω_ϕ to the space $\Lambda(m)$ of pairs of Lagrangian subspaces in $\mathbb{C}^m$ defined by

$$f_{\phi_1}(\{u(\tau)\}) = \{L(\tau), \phi_1(L)(\tau)\}, 0 \leq \tau \leq 1.$$

The Lagrangian Grassmannian $\Lambda(m)$ has a universal covering $\tilde{\Lambda}(m)$ [1]. For the map $f_{\phi_1}\colon\, \Omega_\phi \to \Lambda(m)$, there is a map from the CW complex Ω_ϕ to $\Lambda(m)$ from the obstruction theory. Hence there exists a corresponding map F between the covering space $\tilde{\Omega}^*_\phi$ and the universal covering space $\tilde{\Lambda}(m)$. From the choice of z_0, $a(g(z_0)) = n\sigma(L)$. Note that $u_g(0) = z_0, u_g(1) = g(z_0)$, and $\{u_g(\tau)\}_{0 \leq \tau \leq 1}$ corresponds to an element $g \in \pi_2(P, L)$.

By the definitions of $\sigma(L)$ and $\Sigma(L)$, we have $\deg\colon \pi_1(\Omega_\phi) \to \sigma(L)\mathbb{Z}$ and $Mas\colon \pi_1(\Lambda(m)) \cong \Sigma(L)\mathbb{Z}$. So there is a $g_1 \in \pi_1(\Lambda(m))$ induced by g such that $g_1 \circ F = f_{\phi_1} \circ g$. Note that $\mu^{(r)}(z_0, z_0) = 0$. The following diagram is commutative:

$$\begin{array}{ccc} \pi_1(\Omega_\phi) & \xrightarrow{\pi_1(f_\phi)} & \pi_1(\Lambda(m)) \\ \Big\downarrow{\scriptstyle \deg(g)} & & \Big\downarrow{\scriptstyle \deg(g_1)} \\ \sigma(L)\mathbb{Z} & \xrightarrow{F_*} & \Sigma(L)\mathbb{Z}. \end{array}$$

So $I_\omega(u_g) = n\sigma(L)$ and $I_{\mu,L}(u_g) = \deg(g_1)\Sigma(L)$ by the definitions of $\Sigma(L)$ and the Maslov index. Thus the result follows from $\sigma(L) = \lambda\Sigma(L)$ and the monotonicity of L. □

The index $\mu_u(x)$ depends on the trivialization over $I \times I$, only the relative index does not depend on the trivialization. So the choice of a single z_0 fixes the shifting in the $\mathbb{Z}$–graded symplectic Floer cochain complex. For $g \in \pi_2(P, L)$ with $x^{(r)} = g(x)$, we have $\mu^{(r)}(x) = \mu(x) + \deg g \cdot \Sigma(L)$.

Proposition 2.6 (Lemma 2.5 and Proposition 2.4 [3]) *If $u \in \mathcal{P}(x, y)$ for $x, y \in Z_\phi$ and $\tilde{u}$ is any lift of u, then $\mu_{\tilde{u}}(x^{(r)}, y^{(r)}) = \mu^{(r)}(y) - \mu^{(r)}(x) = \mu(y^{(r)}, z_0) - \mu(x^{(r)}, z_0)$.*

Definition 2.7 The $\mathbb{Z}$–graded symplectic Floer coboundary map is defined by

$$\partial^{(r)}\colon C^{(r)}_{n-1}(L, \phi; P, J) \to C^{(r)}_n(L, \phi; P, J)$$
$$\partial^{(r)} x = \sum_{y \in C^{(r)}_n(L,\phi;P,J)} \# \hat{\mathcal{M}}_J(x, y) \cdot y,$$

where $\mathcal{M}_J(x, y)$ is the union of the components of 1–dimensional moduli space of J–holomorphic curves, and $\hat{\mathcal{M}}_J(x, y) = \mathcal{M}_J(x, y)/\mathbb{R}$ is the zero–dimensional moduli space modulo τ–translational invariant. The number $\#\hat{\mathcal{M}}_J(x, y)$ counts the points modulo 2.

Remark The condition $\Sigma(L) \geq 3$, rather than $\Sigma(L) \geq 2$, enters only in proving that $\langle \delta \circ \delta x, x \rangle = 0$. For $\Sigma(L) = 2$, Oh evaluated a number $\pmod 2$ of J–holomorphic disks with Maslov index 2 that pass through $x \in L \subset P$, and verified that the number is always even. Hence $\langle \delta \circ \delta x, x \rangle = 0 \pmod 2$. In our case, this reflects to understand the two lifts $x^{(r)}$ and $g(x^{(r)})$ of x with $\deg(g) = \pm 1$. Note that $x^{(r)} \in (r, r + \sigma(L))$ and $g(x^{(r)}) \in (r + \deg(g)\sigma(L), r +$

$(\deg(g)+1)\sigma(L))$. So the $\mathbb{Z}$–graded symplectic coboundary is not well–defined in this case. We leave it to future study.

The coboundary map $\partial^{(r)}$ only counts part of Floer's coboundary map in (2.4). Next task is to verify $\partial^{(r)} \circ \partial^{(r)} = 0$ in the following.

Lemma 2.8 *Under the same hypothesis in Theorem 2.3, we have* $\partial^{(r)} \circ \partial^{(r)} = 0$.

Proof If $x \in C^{(r)}_{n-1}(L, \phi; P, J)$ $(\mu(x^{(r)}, z_0) = n-1)$, by the definition of $\partial^{(r)}$, then the coefficient of $z \in C^{(r)}_{n+1}(L, \phi; P, J)$ in $\partial^{(r)} \circ \partial^{(r)}(x)$ is given by

$$\sum_{y \in C^{(r)}_n(L,\phi;P,J)} \#\hat{\mathcal{M}}_J(x,y) \cdot \#\hat{\mathcal{M}}_J(y,z). \tag{2.7}$$

By Proposition 2.6, the boundary of the 1–dimensional manifold $\hat{\mathcal{M}}_J(x,z) = \mathcal{M}_J(x,z)/\mathbb{R}$ corresponds to two isolated trajectories $\mathcal{M}_J(x,y) \times \mathcal{M}_J(y,z)$. Each term $\#\hat{\mathcal{M}}_J(x,y) \cdot \#\hat{\mathcal{M}}_J(y,z)$ is the number of the 2–cusp trajectories of $\hat{\mathcal{M}}_J(x,z)$ with $y \in C^{(r)}_n(L, \phi; P, J)$. For any such y, there are J–holomorphic curves $u \in \mathcal{M}_J(x,y)$ and $v \in \mathcal{M}_J(y,z)$. The other end of the corresponding component of $\hat{\mathcal{M}}_J(x,z)$ corresponds to the space $\mathcal{M}_J(x,y^{'}) \times \mathcal{M}_J(y',z)$ with $u^{'} \in \mathcal{M}_J(x,y^{'})$ and $v^{'} \in \mathcal{M}_J(y',z)$. Then $\hat{\mathcal{M}}_J(x,z)$ has an 1–parameter family of paths from x to z with ends $u \# v$ and $u^{'} \# v^{'}$ for appropriate grafting (see [3] section 4). If we lift u to $\tilde{u} \in \tilde{\mathcal{M}}_J(x^{(r)}, \tilde{y})$ the moduli space of J–holomorphic curves in $\tilde{\Omega}_\phi$ with asymptotics $x^{(r)}$ and $\tilde{y}$, then

$$1 = I_{\mu,\tilde{L}}(\tilde{u}) = \mu(\tilde{y}, z_0) - \mu^{(r)}(x) = \mu(\tilde{y}, z_0) - (n-1). \tag{2.8}$$

So $\mu(\tilde{y}, z_0) = n$, and $\tilde{y} = y^{(r)}$ is the preferred lift. So $\mu^{(r)}(y) = \mu(\tilde{y}, z_0) = n$. Thus $\tilde{u} \in \tilde{\mathcal{M}}_J(x^{(r)}, y^{(r)})$. Similarly $\tilde{v} \in \tilde{\mathcal{M}}_J(y^{(r)}, z^{(r)})$. Since $u^{'} \# v^{'}$ is homotopic to $u \# v$ rel $(x^{(r)}, z^{(r)})$, the lift $\tilde{u}^{'} \# \tilde{v}^{'}$ is also a path with ends $(x^{(r)}, z^{(r)})$. Using the fact of the symplectic action a is non-increasing along any gradient trajectory $\tilde{u}^{'}$, we have

$$r < a(z^{(r)}) \leq a(\tilde{y}^{'}) \leq a(x^{(r)}) < r + \sigma(L). \tag{2.9}$$

By the uniqueness, we have $\tilde{y}^{'} = (y^{'})^{(r)}$. By (2.8) for $u^{'}$, we have $\mu^{(r)}((y^{'})^{(r)}) = \mu^{(r)}(x^{(r)}) + 1 = n$. So $y^{'} \in C^{(r)}_n(L, \phi; P, J)$. Thus the number of two–cusp trajectories connecting $x^{(r)}$ and $z^{(r)}$ with index 2 is always even. Hence we obtain $\partial^{(r)} \circ \partial^{(r)} = 0$. □

The complex $(C_n^{(r)}(L,\phi;P,J),\partial_n^{(r)})_{n\in\mathbb{Z}}$ is indeed a $\mathbb{Z}$–graded symplectic Floer cochain complex. We call its cohomology to be an $\mathbb{Z}$*–graded symplectic Floer cohomology*, denoted by

$$I_*^{(r)}(L,\phi;P,J)=H^*(C_*^{(r)}(L,\phi;P,J),\partial^{(r)}),\quad *\in\mathbb{Z}. \tag{2.10}$$

By the construction of $I_*^{(r)}(L,\phi;P,J)$, if $[r,s]\subset\mathbb{R}_{L,\phi}$, then $I_*^{(r)}(L,\phi;P,J)=I_*^{(s)}(L,\phi;P,J)$. The relation between $I_*^{(r)}(L,\phi;P,J)$ and $HF^*(L,\phi;P)$ will be discussed in §4.

3 Invariance property of the $\mathbb{Z}$–graded symplectic Floer cohomology

In this section we show that the $\mathbb{Z}$–graded symplectic Floer cohomology in (2.10) is invariant under the changes of J and under the exact deformations ϕ_1 of the Lagrangian sub–manifold L.

Let $\{(J^\lambda,\phi^\lambda)\}_{\lambda\in\mathbb{R}}$ be an 1–parameter family which interpolates from (J^0,ϕ^0) to (J^1,ϕ^1). The family (J^λ,ϕ^λ) is constant in λ outside $[0,1]$. We also assume that ϕ_1^λ is exact under the change of λ. Let $J_t^\lambda=J(\lambda,t)$ be a 2-parameter family of almost complex structures compatible to ω, and $\phi_t^\lambda=\phi(\lambda,t)$ with $\phi(\lambda,0)=\mathrm{Id}$ is the 2–parameter family of exact isotopies contractible to the identity. Such ϕ_t^λ connecting ϕ_t^0,ϕ_t^1 does exist. Floer [5] discussed the invariance of the symplectic Floer cohomology under the change of (J,ϕ) for (J^0,ϕ^0) C^∞–close to (J^1,ϕ^1). Let H^λ be the Hamiltonian function generated by $\phi^\lambda=\{\phi_t^\lambda\}_{0\leq t\leq 1}$. Then the deformed gradient flow of a_H is

$$\overline{\partial}_{J^\lambda}u_\lambda(\tau,t)+\nabla_J H_t^\lambda(u_\lambda(\tau,t))=\frac{\partial u_\lambda}{\partial\tau}+J_t^\lambda\frac{\partial u_\lambda}{\partial t}+\nabla_J H_t(u_\lambda(\tau,t))=0, \tag{3.1}$$

with the moving Lagrangian boundary conditions

$$u_\lambda(\tau,0)\in L, u_\lambda(\tau,1)\in\phi_1^\lambda(L). \tag{3.2}$$

We define $C_{J,L}^{(r)}=\min\{a(x^{(r)})-r,\sigma(L)+r-a(x^{(r)})|x\in Z_\phi\}$. For each $x\in Z_\phi$, there is an open neighborhood U_x in Ω_ϕ such that (1) U_x is evenly covered in $\tilde{\Omega}_\phi^*$, (2) for each $z\in U_x$, $|a(z)-a(x)|<C_{J,L}^{(r)}/8$. There are finite sub–cover $\{U_{x_1},\cdots,U_{x_k}\}$ of Z_ϕ, and by Gromov's compactness theorem [5] we have $\varepsilon_1>0$ such that if $\|Da(z)\|_{L_1^3}<\varepsilon_1$ then $z\in\bigcup_{i=1}^k U_{x_i}$. Let $\varepsilon=\min\{\varepsilon_1,C_{L_1,L_2}^{(r)}/8\}$. We set a deformation $\{J,\phi\}$ satisfying the usual perturbation requirements in [5], and also satisfying

$$\text{(i)}\ \ |H_t^\lambda(z)|<\varepsilon/2,\qquad \text{(ii)}\ \ \|\nabla_J H_t^\lambda(z)\|_{L_1^3}<\varepsilon/2, \tag{3.3}$$

for all $z \in \Omega_\phi$. These deformation conditions can be achieved by the density statement in [5].

Let $\mathcal{P}_{1,\varepsilon/2}$ be the set of $\{J, \phi\}$ which satisfies these extra conditions (3.3).

This directly generalizes the J-holomorphic curve equation in the cases of (J^0, ϕ^0) and (J^1, ϕ^1). The moduli space $\mathcal{M}_{J^\lambda}(x, y)$ of (3.1) and (3.2) has the same analytic properties as the moduli space $\mathcal{M}_J(x, y)$ except for the translational invariance (see Proposition 3.2 in [3]). Hofer analyzed the compactness property for a similar moving Lagrangian coboundary condition, Oh [16] determined that the bubbling–off spheres or disks can not occur in the components of $\mathcal{M}_{J^\lambda}(x, y)$ for the monotone Lagrangian sub–manifold L with $\Sigma(L) \geq 3$. The index of u_λ can be proved to be the same as a topological index for the moduli space of perturbed J–holomorphic curves. The proof of the invariance under the changes of (J, ϕ) is the same as in [3, 5, 16]. It is sufficient for us to verify that the cochain map is well-defined for the $\mathbb{Z}$–graded symplectic Floer cochain complexes.

Lemma 3.1 *If $u_\lambda \in \mathcal{M}_{J^\lambda}(x_0, x_1)$, $(J^\lambda_t, \phi^\lambda_t) \in \mathcal{P}_{1,\varepsilon/2}$, and $\tilde{u}_\lambda \in \mathcal{P}(\tilde{x}_0, \tilde{x}_1)$ is any lift of u_λ, then*

$$a_{(J^1,\phi^1)}(\tilde{x}_1) < a_{(J^0,\phi^0)}(\tilde{x}_0) + \varepsilon.$$

Proof Note that the path $\{u_\lambda(\tau) | \tau \in (-\infty, 0)\}$ is a gradient trajectory for (J^0, ϕ^0) and $\{u_\lambda(\tau) | \tau \in (1, \infty)\}$ is a gradient trajectory for (J^1, ϕ^1). So

$$a_{(J^0,\phi^0)}(\tilde{u}(0)) \leq a_{(J^0,\phi^0)}(\tilde{x}_0), \quad a_{(J^1,\phi^1)}(\tilde{x}_1) \leq a_{(J^1,\phi^1)}(\tilde{u}(1)). \tag{3.4}$$

Since $u_\lambda \in \mathcal{M}_{J^\lambda}(x_0, x_1)$, by the property of $\mathcal{P}_{1,\varepsilon/2}$, we have

$$\begin{aligned} \|\overline{\partial}_{J^\lambda} u_\lambda(\tau, t)\|_{L^3_1} &= \|\frac{\partial u_\lambda}{\partial \tau} + J^\lambda_t \frac{\partial u_\lambda}{\partial t}\|_{L^3_1} \\ &= \|\nabla_J H^\lambda_t(u_\lambda(\tau, t))\|_{L^3_1} \\ &< \varepsilon/2. \end{aligned}$$

By the Sobolev embedding $L^3_1 \hookrightarrow L^2$,

$$\begin{aligned} I_\omega(u_\lambda)|_{\Theta\times[0,1]} &= \|\partial_{J^\lambda} u_\lambda\|^2_{L^2(\Theta\times[0,1])} - \|\overline{\partial}_{J^\lambda} u_\lambda\|^2_{L^2(\Theta\times[0,1])} \\ &\geq -\|\overline{\partial}_{J^\lambda} u_\lambda\|^2_{L^2(\Theta\times[0,1])} \\ &\geq -\varepsilon/2 \end{aligned}$$

Thus we obtain the following.

$$\begin{aligned} a_{(J^0,\phi^0)}(\tilde{u}_\lambda(0)) &= a_{(J^1,\phi^1)}(\tilde{x}_1) + I_\omega(u_\lambda)|_{\Theta\times[0,1]} \\ &> a_{(J^1,\phi^1)}(\tilde{x}_1) - \varepsilon/2. \end{aligned} \tag{3.5}$$

$$\begin{aligned} a_{(J^0,\phi^0)}(\tilde{x}_0) &\geq a_{(J^0,\phi^0)}(u_\lambda(0)) \\ &= a_{(J^0,\phi^0)}(u_\lambda(0)) + H_0(u_\lambda(0)) \\ &> (a_{(J^1,\phi^1)}(\tilde{x}_1) - \varepsilon/2) - \varepsilon/2, \end{aligned}$$

by (3.3) and (3.5). Then the result follows. □

Definition 3.2 For $n \in \mathbb{Z}$, define a homomorphism $\phi_{01}^n\colon C_n^{(r)}(L,\phi^0;P,J^0) \to C_n^{(r)}(L,\phi^1;P,J^1)$ by

$$\phi_{01}^n(x_0) = \sum_{x_1 \in C_n^{(r)}(L,\phi^1;P,J^1)} \#\mathcal{M}_{J\lambda}^0(x_0,x_1)\cdot x_1,$$

where $\mathcal{M}_{J\lambda}^0(x_0,x_1)$ is a 0–dimensional moduli space of J–holomorphic curves satisfying (3.1) and (3.2).

We show that the homomorphism ϕ_{01}^n is a cochain map with respect to the integral lifts.

Lemma 3.3 *The homomorphism $\{\phi_{01}^*\}_{*\in Z}$ is a cochain map:*

$$\partial_{n,1}^{(r)} \circ \phi_{01}^n = \phi_{01}^n \circ \partial_{n,0}^{(r)}, \qquad n \in \mathbb{Z}.$$

Proof For $x_0 \in C_n^{(r)}(L,\phi^0;P,J^0)$ and $y_1 \in C_{n+1}^{(r)}(L,\phi^1;P,J^1)$, the coefficient of y_1 in $(\partial_{n,1}^{(r)} \circ \phi_{01}^n - \phi_{01}^n \circ \partial_{n,0}^{(r)})(x_0)$ is the modulo 2 number of the set:

$$\bigcup_{y_0 \in C_{n+1}^{(r)}(L,\phi^0;P,J^0)} \hat{\mathcal{M}}_{J^0}(x_0,y_0) \times \mathcal{M}_{J\lambda}^0(y_0,y_1) \tag{3.6}$$

$$\coprod \bigcup_{x_1 \in C_n^{(r)}(L,\phi^1;P,J^1)} \mathcal{M}_{J\lambda}^0(x_0,x_1) \times \hat{\mathcal{M}}_{J^1}(x_1,y_1).$$

The ends of the 1–dimensional manifold $\mathcal{M}_{J\lambda}^1(x_0,y_1)$ are in one–to–one correspondence with the set

$$(\bigcup_{y \in Z_{\phi^0}} \hat{\mathcal{M}}_{J^0}(x_0,y) \times \mathcal{M}_{J\lambda}^0(y,y_1)) \cup (\bigcup_{x \in Z_{\phi^1}} \mathcal{M}_{J\lambda}^0(x_0,x) \times \hat{\mathcal{M}}_{J^1}(x,y_1)). \tag{3.7}$$

For an end $u\#v$ of $\mathcal{M}_{J\lambda}^1(x_0,y_1)$ corresponding to an element in (3.6), the other end $u^{'}\#v^{'}$ of the same component corresponds to an element in (3.7) (see [3] section 4 for the gluing construction on $u\#v$). For $u \in \mathcal{M}_{J\lambda}^0(x_0,y)$ and $v \in \hat{\mathcal{M}}_{J^1}(y,y_1)$, the space $\mathcal{M}_{J\lambda}^1(x_0,y_1)$ gives a 1-parameter family of paths in

$\mathcal{P}(L, \phi^\lambda; P)$ with fixed end points x_0 and y_1. The 1–parameter family gives the homotopy of paths from $u\#_\rho v$ to $u^{'}\#_\rho v^{'}$ rel end points. The lift of $u\#_\rho v$ starts at $x_0^{(r)}$ and ends at $y_1^{(r)}$, so does the lift of $u^{'}\#_\rho v^{'}$. Suppose $u^{'}$ lifts to an element in $\mathcal{M}^0_{J^\lambda}(x_0^{(r)}, \tilde{y})$. By Lemma 3.1, we have

$$a_{(J^1,\phi^1)}(\tilde{y}) < a_{(J^0,\phi^0)}(x_0^{(r)}) + \varepsilon < r + \sigma(L). \tag{3.8}$$

Using the fact that trajectory decreases the symplectic action, we obtain

$$a_{(J^1,\phi^1)}(\tilde{y}) > a_{(J^1,\phi^1)}(y_1^{(r)}) > r. \tag{3.9}$$

So $a_{(J^1,\phi^1)}(\tilde{y}) \in (r, r + \sigma(L))$. Inequalities (3.8) and (3.9) give the preferred lift $\tilde{y} = y^{(r)}$ of y. By Proposition 2.6 (1),

$$1 = \mu^{(r)}(y_1) - \mu^{(r)}(y) = (n+1) - \mu^{(r)}(y).$$

So $\mu^{(r)}(y) = n$ and $y \in C_n^{(r)}(L, \phi^1; P, J^1)$. This shows that the $u^{'}\#_\rho v^{'}$ in (3.7) actually corresponds to an element in (3.6). So the cardinality is always even. □

For $(J^i, \phi^i) \in \mathcal{P}_{1,\varepsilon/2}$ $(i = 0, 1, 2)$, we define a class $\mathcal{P}_{2,\varepsilon}$ of perturbations consisting of

$$(J^\lambda, \phi^\lambda) = \begin{cases} (J^0, \phi^0) & \text{if } \lambda \leq -T, \\ (J^1, \phi^1) & \text{if } -T+1 \leq \lambda \leq T-1, \\ (J^2, \phi^2) & \text{if } \lambda \geq T, \end{cases}$$

for a fixed number $T(>2)$ with $(J^\lambda, \phi^\lambda)_{0\leq\lambda\leq 1} \in \mathcal{P}_{1,\varepsilon/2}$ and $(J^\lambda, \phi^\lambda)_{1\leq\lambda\leq 2} \in \mathcal{P}_{1,\varepsilon/2}$. If both perturbations $(J^\lambda, \phi^\lambda)_{0\leq\lambda\leq 1} \in \mathcal{P}_{1,\varepsilon/2}((J^0, \phi^0), (J^1, \phi^1))$ and $(J^\lambda, \phi^\lambda)_{1\leq\lambda\leq 2} \in \mathcal{P}_{1,\varepsilon/2}((J^1, \phi^1), (J^2, \phi^2))$, then we can compose $(J^\lambda, \phi^\lambda)_{0\leq\lambda\leq 1}$ with $(J^\lambda, \phi^\lambda)_{1\leq\lambda\leq 2}$ to get $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$. Let $(J^\lambda, \phi^\lambda) = (J^\lambda, \phi^\lambda)_{0\leq\lambda\leq 1}\#_T(J^\lambda, \phi^\lambda)_{1\leq\lambda\leq 2}$ be the composition. Then for a large fixed T and each compact set K in $\mathcal{M}_{J^\lambda_{0\leq\lambda\leq 1}}(x, y) \times \mathcal{M}_{J^\lambda_{1\leq\lambda\leq 2}}(y, z)$, there is a $\rho_T > 0$ and for all $\rho > \rho_T$ a local diffeomorphism

$$\#_{\rho_T} \colon K \to \mathcal{M}_{(J^\lambda,\phi^\lambda)_{0\leq\lambda\leq 1}\#_{T,\rho_T}(J^\lambda,\phi^\lambda)_{1\leq\lambda\leq 2}}(x, z). \tag{3.10}$$

See Proposition 2d.1 in [5].

Lemma 3.4 *For $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{2,\varepsilon}$, and $\rho > \rho_T$, we have*

$$\phi^n_{02} = \phi^n_{12} \circ \phi^n_{01}, \qquad \text{for } n \in \mathbb{Z}.$$

Proof For $x_0 \in C_n^{(r)}(L, \phi^0; P, J^0)$, we have

$$\phi_{02}^n(x_0) = \sum_{y_0 \in C_n^{(r)}(L,\phi^2;P,J^2)} \#\mathcal{M}^0_{J^\lambda}(x_0, y_0) \cdot y_0,$$

where the summation $\sum$ runs over $y_0 \in C_n^{(r)}(L, \phi^2; P, J^2)$. Also we have

$$\phi_{12}^n \circ \phi_{01}^n(x_0) = \sum \#(\mathcal{M}^0_{J^\lambda_{0\le\lambda\le1}}(x_0, y) \times \mathcal{M}_{J^\lambda_{1\le\lambda\le2}}(y, y_0)) \cdot y_0,$$

where the summation $\sum$ runs over $y \in C_n^{(r)}(L, \phi^1; P, J^1)$. The local diffeomorphism $\#_{\rho_T}$ in (3.10) determines the following:

$$\#\mathcal{M}^0_{J^\lambda}(x_0, y_0) = \#(\mathcal{M}^0_{J^\lambda_{0\le\lambda\le1}}(x_0, y) \times \mathcal{M}^0_{J^\lambda_{1\le\lambda\le2}}(y, y_0)).$$

All we need to check is that $y \in C_n^{(r)}(L, \phi^1; P, J^1)$. This can be verified by the same argument in the Lemma 3.3. □

For two classes $(J^\lambda, \phi^\lambda)$ and $(\overline{J}^\lambda, \overline{\phi}^\lambda)$ in $\mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$, the following lemma shows that the induced cochain maps ϕ_{02}^n and $\overline{\phi}_{02}^n$ are cochain homotopic to each other.

Lemma 3.5 *If $(J^\lambda, \phi^\lambda)$, $(\overline{J}^\lambda, \overline{\phi}^\lambda) \in \mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$ can be smoothly deformed from one to another by a 1-parameter family $(J^\lambda_s, \phi^\lambda_s)$ of $s \in [0, 1]$: $(J^\lambda_s, \phi^\lambda_s) = (J^\lambda, \phi^\lambda)$ for $s \le 0$, and $(J^\lambda_s, \phi^\lambda_s) = (\overline{J}^\lambda, \overline{\phi}^\lambda)$ for $s \ge 1$. Then ϕ_{02}^* and $\overline{\phi}_{02}^*$ are cochain homotopic to each other.*

Proof It suffices to construct a homomorphism

$$H\colon C_*^{(r)}(L, \phi^0; P, J^0) \to C_*^{(r)}(L, \phi^2; P, J^2),$$

of degree -1 with the property

$$\phi_{02}^n - \overline{\phi}_{02}^n = H\partial_{n,0}^{(r)} + \partial_{n,2}^{(r)} H, \quad \text{for } n \in \mathbb{Z}. \tag{3.11}$$

Associated to the family $(J^\lambda_s, \phi^\lambda_s)$, there is a moduli space $H\mathcal{M}(x_0, y_0) = \cup_{s\in[0,1]}\mathcal{M}^0_{(J^\lambda_s,\phi^\lambda_s)}(x_0, y_0)$:

$$H\mathcal{M}(x_0, y_0) = \{(u, s) \in \mathcal{M}^0_{(J^\lambda_s,\phi^\lambda_s)}(x_0, y_0) \times [0, 1]\} \subset \mathcal{P}(L, \phi^\lambda_s; P)(x_0, y_0) \times [0, 1].$$

The space $H\mathcal{M}(x_0, y_0)$ is the regular zero set of $\overline{\partial}_{J(J^\lambda_s,\phi^\lambda_s)}$, and is smooth manifolds of dimension $\mu^{(r)}(y_0) - \mu^{(r)}(x_0) + 1$. For the case of $\mu^{(r)}(x_0) = \mu^{(r)}(y_0) = n$, the boundaries of the 1-dimensional manifold $H\mathcal{M}(x_0, y_0)$ of $\mathcal{P}(L, \phi^\lambda_s; P)(x_0, y_0) \times [0, 1]$ consist of

- $\mathcal{M}^0_{(J^\lambda,\phi^\lambda)}(x_0,y_0)\times\{0\}\cup\mathcal{M}^0_{(\overline{J}^\lambda,\overline{\phi}^\lambda)}(x_0,y_0)\times\{1\}$,
- $\overline{\mathcal{M}}^0_{(J^\lambda_s,\phi^\lambda_s)}(x_0,y)\times\mathcal{M}^0_{(J^2,\phi^2)}(y,y_0)$ for $y\in C^{(r)}_{n-1}(L,\phi^2;P,J^2)$,
- $\mathcal{M}^0_{(J^0,\phi^0)}(x_0,x)\times\overline{\mathcal{M}}^0_{(J^\lambda_s,\phi^\lambda_s)}(x,y)$ for $x\in C^{(r)}_{n-1}(L,\phi^0;P,J^0)$.

Note that $\overline{\mathcal{M}}^0_{(J^\lambda_s,\phi^\lambda_s)}(x_0,y)$ and $\overline{\mathcal{M}}^0_{(J^\lambda_s,\phi^\lambda_s)}(x,y)$ are moduli spaces of solutions of (u,s) of J-holomorphic equations lying in virtual dimension -1 ($\mu_u=-1$), they can only occur for $0<s<1$. Define $H\colon C^{(r)}_n(L,\phi^0;P,J^0)\to C^{(r)}_{n-1}(L,\phi^2;P,J^2)$ by

$$H(x_0)=\sum_{y\in C^{(r)}_{n-1}(L,\phi^2;P,J^2)}\#\overline{\mathcal{M}}^0_{(J^\lambda_s,\phi^\lambda_s)}(x_0,y)\cdot y, \tag{3.12}$$

for $0<s<1$. Similar to Lemma 3.3, by checking the corresponding preferred lifts and the integral Maslov indexes, we get the desired cochain homotopy H between ϕ^n_{02} and $\overline{\phi}^n_{02}$ such that H satisfies (3.11). □

By Lemma 3.5, the homomorphism ϕ^*_{02} induced from (J^λ,ϕ^λ) is the same homomorphism $\overline{\phi}^*_{02}$ induced from $(\overline{J}^\lambda,\overline{\phi}^\lambda)$ on the $\mathbb{Z}$-graded symplectic Floer cohomology. So the $\mathbb{Z}$-graded symplectic Floer cohomology is invariant under the continuation of (J,ϕ). The following is Theorem A (1).

Theorem 3.6 *For any continuation $(J^\lambda,\phi^\lambda)\in\mathcal{P}_{1,\varepsilon/2}$ which is regular at the ends, there exists an isomorphism*

$$\phi^n_{01}\colon I^{(r)}_n(L,\phi^0;P,J^0)\to I^{(r)}_n(L,\phi^1;P,J^1),\qquad \textit{for } n\in\mathbb{Z}.$$

Proof Let $(J^{-\lambda},\phi^{-\lambda})$ be the reversed family of (J^λ,ϕ^λ) by setting $\tau=-\tau'$. So we can form a family of composition $(J^\lambda,\phi^\lambda)_{0\le\lambda\le1}\#_T(J^{-\lambda},\phi^{-\lambda})_{1\le\lambda\le2}$ in $\mathcal{P}_{2,\varepsilon}$ for some fixed $T(>2)$. By Lemma 3.4,

$$\phi_{(J^\lambda,\phi^\lambda)_{0\le\lambda\le1}\#_T(J^{-\lambda},\phi^{-\lambda})_{1\le\lambda\le2}}=\phi^*_{10}\circ\phi^*_{01}.$$

One can deform $(J^\lambda,\phi^\lambda)_{0\le\lambda\le1}\#_T(J^{-\lambda},\phi^{-\lambda})_{1\le\lambda\le2}$ into the trivial continuation (J^0,ϕ^0) for all $\tau\in\mathbb{R}$. Then by Lemma 3.5, we have

$$\phi^*_{10}\circ\phi^*_{01}=\phi^*_{00}=id: I^{(r)}_*(L,\phi^0;P,J^0)\to I^{(r)}_*(L,\phi^0;P,J^0).$$

Similarly, $\phi^*_{01}\circ\phi^*_{10}=\phi^*_{11}=id$ on $I^{(r)}_*(L,\phi^1;P,J^1)$. The result follows. □

The $\mathbb{Z}$–graded symplectic Floer cohomology $I^{(r)}_*$ is functorial with respect to compositions of continuations (J^λ,ϕ^λ), and invariant under continuous deformations of (J^λ,ϕ^λ) within the set of continuations $\mathcal{P}_{1,\varepsilon/2}$.

4 The spectral sequence for the symplectic Floer cohomology

In this section, we show that $I_*^{(r)}(L,\phi;P)$ (the $\mathbb{Z}$–graded symplectic Floer cohomology) for $r\in\mathbb{R}_{L,\phi}$ and $*\in\mathbb{Z}$ determines the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology $HF^*(L,\phi;P)$. The way to link them together is to filter the $\mathbb{Z}$-graded symplectic Floer cochain complex. Then by a standard method in algebraic topology (see [20]), the filtration gives arise a spectral sequence which converges to the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology $HF^*(L,\phi;P)$. The Künneth formula (Theorem 4.13) for the spectral sequence is obtained by the analysis of higher differentials and Maslov indexes.

Definition 4.1 For $r\in\mathbb{R}_{L,\phi}, j\in\mathbb{Z}_{\Sigma(L)}$ and $n\equiv j \ \ (\ mod \ \ \Sigma(L))$, we define the free module over $\mathbb{Z}_2$:

$$F_n^{(r)}C_j(L,\phi;P,J)=\sum_{k\geq 0}C^{(r)}_{n+\Sigma(L)k}(L,\phi;P,J).$$

The free module $F_*^{(r)}C_*(L,\phi;P,J)$ gives a natural decreasing filtration on the symplectic Floer cochain groups $C_*(L,\phi;P,J)\,(*\in\mathbb{Z}_{\Sigma(L)})$.

There is a finite length decreasing filtration of $C_j(L,\phi;P,J)$, $j\in\mathbb{Z}_{\Sigma(L)}$:

$$\cdots F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J)\subset F_n^{(r)}C_j(L,\phi;P,J)\subset\cdots\subset C_j(L,\phi;P,J). \quad (4.1)$$

$$C_j(L,\phi;P,J)=\bigcup_{n\equiv j(mod \ \ \Sigma(L))}F_n^{(r)}C_j(L,\phi;P,J). \quad (4.2)$$

Note that the symplectic action is non–increasing along the gradient trajectories. The coboundary map $\delta\colon F_n^{(r)}C_j(L,\phi;P,J)\to F_{n+1}^{(r)}C_{j+1}(L,\phi;P,J)$ in the Theorem 2.3 preserves the filtration in Definition 4.1. Thus the $\mathbb{Z}_{\Sigma(L)}$-graded symplectic Floer cochain complex $(C_j(L,\phi;P,J),\delta)_{j\in\mathbb{Z}_{\Sigma(L)}}$ has a decreasing bounded filtration $(F_n^{(r)}C_*(L,\phi;P,J),\delta)$:

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow\\
\cdots F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J) & \subset F_n^{(r)}C_j(L,\phi;P,J) & \cdots\subset C_j(L,\phi;P,J)\\
\downarrow\partial^{(r)} & \downarrow\partial^{(r)} & \downarrow\delta\\
\cdots F^{(r)}_{n+\Sigma(L)+1}C_{j+1}(L,\phi;P,J) & \subset F_{n+1}^{(r)}C_{j+1}(L,\phi;P,J) & \cdots\subset C_{j+1}(L,\phi;P,J)\\
\downarrow & \downarrow & \downarrow
\end{array} \quad (4.3)$$

The cohomology of the vertical cochain subcomplex $F_n^{(r)}C_*(L,\phi;P,J)$ in the filtration (4.3) is $F_n^{(r)}I_j^{(r)}(L,\phi;P,J)$.

Lemma 4.2 *There is a filtration for the $\mathbb{Z}$–graded symplectic Floer cohomology $\{I_*^{(r)}(L,\phi;P,J)\}_{*\in Z}$,*

$$\cdots F^{(r)}_{n+\Sigma(L)}HF^j(L,\phi;P,J)\subset F_n^{(r)}HF^j(L,\phi;P,J)\subset\cdots\subset I_j^{(r)}(L,\phi;P,J),$$

where $F_n^{(r)}HF^j(L,\phi;P,J)=\ker(I_j^{(r)}(L,\phi;P,J)\to F^{(r)}_{n-\Sigma(L)}I_j^{(r)}(L,\phi;P,J))$.

Proof The results follows from Definition 4.1 and standard results in [20] Chapter 9. □

Theorem 4.3 *For $\Sigma(L)\geq 3$, there is a spectral sequence $(E^k_{n,j},d^k)$ with*

$$E^1_{n,j}(L,\phi;P,J)\cong I_n^{(r)}(L,\phi;P,J),\quad n\equiv j\ \ (mod\ \ \Sigma(L)),$$
$$d^k:\ E^k_{n,j}(L,\phi;P,J)\to E^k_{n+\Sigma(L)k+1,j+1}(L,\phi;P,J),$$

and

$$E^\infty_{n,j}(L,\phi;P,J)\cong F_n^{(r)}HF^j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}HF^j(L,\phi;P,J).$$

In other words the spectral sequence $(E^k_{n,j},d^k)$ converges to the $\mathbb{Z}_{\Sigma(L)}$-graded symplectic Floer cohomology $HF^*(L,\phi;P)$.

Proof Note that

$$F_n^{(r)}C_j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J)=C_n^{(r)}(L,\phi;P,J).$$

It is well–known from [20] that there exists a spectral sequence $(E^k_{n,j},d^k)$ with E^1– term given by the cohomology of $F_n^{(r)}C_j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J)$. So $E^1_{n,j}(L,\phi;P,J)\cong I_n^{(r)}(L,\phi;P,J)$ and $E^\infty_{n,j}(L,\phi;P,J)$ is isomorphic to the bi-graded $\mathbb{Z}_2$–module associated to the filtration $F_*^{(r)}$ of the $\mathbb{Z}$–graded symplectic Floer cohomology $I_n^{(r)}(L,\phi;P,J)$. Note that the grading is unusual (jumping by $\Sigma(L)$ in each step), we list the terms for $Z^k_{*,*}$ and $E^k_{*,*}$.

$$Z^k_{n,j}(L,\phi;P,J)=\{x\in F_n^{(r)}C_j(L,\phi;P,J)|\delta x\in F^{(r)}_{n+1+\Sigma(L)k}C_{j+1}(L,\phi;P,J)\},$$

$$E^k_{n,j}(L,\phi;P,J)=$$

$$Z^k_{n,j}(L,\phi;P,J)/\{Z^{k+1}_{n+\Sigma(L),j}(L,\phi;P,J)+\delta Z^{k-1}_{n+(k-1)\Sigma(L)-1,j-1}(L,\phi;P,J)\},$$

$$Z^{\infty}_{n,j}(L,\phi;P,J) = \{x \in F^{(r)}_n C_j(L,\phi;P,J) | \delta x = 0\},$$

$$E^{\infty}_{n,j}(L,\phi;P,J) =$$

$$Z^{\infty}_{n,j}(L,\phi;P,J)/\{Z^{\infty}_{n+\Sigma(L),j}(L,\phi;P,J) + dZ^{\infty}_{n+(k-1)\Sigma(L)-1,j-1}(L,\phi;P,J)\}.$$

Thus $d^k\colon\ E^k_{n,j}(L,\phi;P,J) \to E^k_{n+\Sigma(L)k+1,j+1}(L,\phi;P,J)$ is induced from δ.

Since the Lagrangian intersections are transverse, and $Z_\phi = L \cap \phi_1(L)$ is a finite set, so the filtration $F^{(r)}_*$ is bounded and complete from (4.2). Thus the spectral sequence converges to the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology. □

Remark For monotone Lagrangians L with $\Sigma(L) \geq 3$ and $\phi \in Symp_0(P,\omega)$, the spectral sequence in Theorem 4.3 gives a precise information on $E^1_{*,*}$–term and all higher differentials d^k (see also later lemmae). The spectral sequence for monotone Lagrangian L with $\Sigma(L) \geq 2$ and $\phi \in Ham(P,\omega)$ C^1-sufficiently close to Id in Theorem IV of [17] gives a filtration from the Morse index of L. Our filtration is given by the integral lifting of the Maslov index.

Theorem 4.3 gives Theorem A (2). The following theorem is Theorem B.

Theorem 4.4 *For $\Sigma(L) \geq 3$,*

(1) *there exists an isomorphism*

$$E^1_{n,j}(L,\phi^0;P,J^0) \cong E^1_{n,j}(L,\phi^1;P,J^1),$$

for any continuation $(J^\lambda,\phi^\lambda) \in \mathcal{P}_{1,\varepsilon/2}$ which is regular at ends.

(2) *for $k \geq 1$, $E^k_{n,j}(L,\phi;P,J)$ is the symplectic invariant under continuous deformations of (J^λ,ϕ^λ) within the set of continuations.*

Proof Clearly (2) follows from (1) by the Theorem 1 in [20] page 468. By Theorem 4.3, there is an isomorphism:

$$E^1_{n,j}(L,\phi^0;P,J^0) \cong I^{(r)}_n(L,\phi^0;P,J^0).$$

By Theorem 3.6, we have the isomorphism

$$\phi^n_{0,1} : I^{(r)}_n(L,\phi^0;P,J^0) \to I^{(r)}_n(L,\phi^1;P,J^1),$$

which is compatible with the filtration. The isomorphism $\phi^n_{0,1}$ induces an isomorphism on the E^1 term. So we obtain the result. □

By Theorem 4.4, $E^k_{n,j}(L,\phi;P,J)=E^k_{n,j}(L,\phi;P)$, $E^1_{n,j}(L,\phi;P)=I^{(r)}_n(L,\phi;P)$ are new symplectic invariants provided $\Sigma(L)\geq 3$. All these new symplectic invariants should contain more information on $(P,\omega;L,\phi)$. They are finer than the usual symplectic Floer cohomology $HF^*(L,\phi;P)(*\in\mathbb{Z}_{\Sigma(L)})$. In particular, the minimal k for which $E^k(L,\phi)=E^\infty(L,\phi)$ should be meaningful, denoted by $k(L,\phi)$. The number $k(L,\phi)$ is certainly a numerical invariant for the monotone Lagrangian sub–manifold L and $\phi\in Symp_0(P)$.

Corollary 4.5 *For $\Sigma(L)\geq 3$ and $j\in\mathbb{Z}_{\Sigma(L)}$,*
$$\sum_{k\in Z} I^{(r)}_{j+\Sigma(L)k}(L,\phi;P)=HF^j(L,\phi;P)$$
if and only if all the differentials d^k in the spectral sequence $(E^k_{n,j},d^k)$ are trivial, if and only if $k(L,\phi)=1$.

□

In general, $\sum_{k\in Z} I^{(r)}_{j+\Sigma(L)k}(L,\phi;P)\neq HF^j(L,\phi;P)$ for $j\in\mathbb{Z}_{\Sigma(L)}$. The $\mathbb{Z}$–graded symplectic Floer cohomology $I^{(r)}_*(L,\phi;P)$ can be thought as an integral lifting of the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology $HF^*(L,\phi;P)$. From our construction of the $\mathbb{Z}$–graded symplectic Floer cohomology, we have that

(i) if $[r_0,r_1]\subset\mathbb{R}_{L,\phi}$, then $E^{k,(r_0)}_{*,*}(L,\phi;P)=E^{k,(r_1)}_{*,*}(L,\phi;P)$;

(ii) $E^{k,(r)}_{*+\Sigma(L),*}(L,\phi;P)=E^{k,(r+\sigma(L))}_{*,*}(L,\phi;P)$.

Since we work on field coefficients (over $\mathbb{Z}_2$), we give a description of d^k in terms of generators.

Lemma 4.6 *Modulo $E^{k-1}_{*,*}$, the differential $d^k\colon E^k_{n,j}\to E^k_{n+k\Sigma(L)+1,j+1}$ is given by*
$$d^k x=\sum_{-\mu^{(r)}(x)+\mu^{(r)}(y)=k\Sigma(L)+1}\#\hat{\mathcal{M}}_J(x,y)\cdot y. \tag{4.4}$$
The differential d^k extends linearly over $E^k_{n,j}$.

Proof For $x\in E^k_{n,j}$, we have, by definition, that x is survived from all previous differentials. Since the coefficients are in a field $\mathbb{Z}_2$, there are no torsion elements. So $x\in Z^k_{n,j}$, $\mu^{(r)}(x)=n$ and $\delta x\in F^{(r)}_{n+k\Sigma(L)+1}C_{j+1}(L,\phi;P,J)$. Thus
$$d^k x=\delta x=\sum_{y\in F^{(r)}_{n+k\Sigma(L)+1}C_{j+1}(L,\phi;P,J)}\#\hat{\mathcal{M}}_J(x,y)\cdot y.$$
The result follows. □

Lemma 4.7 *If $\mu_u^{(r)}(x,y) < (p+1)\Sigma(L)$ for any u in 1–dimensional moduli spaces of J–holomorphic curves with $x, y \in C_*^{(r)}(L, \phi; P, J)$, then $d^k = 0$ for $k \geq p+1$. So the spectral sequence $E^*_{*,*}(L, \phi; P, J)$ collapses at least $(p+1)$th term.*

Proof Suppose the contrary. There is $x \in E^k_{n,j}$ such that $d^k x \neq 0$ for $k \geq p+1$. By Lemma 4.6,

$$\sum_{-\mu^{(r)}(x)+\mu^{(r)}(y)=k\Sigma(L)+1} \#\hat{\mathcal{M}}_J(x,y) \cdot y \neq 0.$$

So there exist y and $u \in \hat{\mathcal{M}}_J(x,y)$ such that there is a nontrivial J–holomorphic curve u with $\mu_u^{(r)}(x,y) = k\Sigma(L)+1$. On the other hand, there exists a J–holomorphic curve u with $\mu_u^{(r)}(x,y) \geq (p+1)\Sigma(L)+1$. Hence the result follows from the contradiction. □

By Lemma 4.7, the lifted symplectic action actually measures the how large the J–holomorphic curves by the integral lifted Maslov index. If $a(x^{(r)}) - a(y^{(r)}) < (p+1)\sigma(L)$ for all $x, y \in C_*^{(r)}(L, \phi; P, J)$, then $k(L, \phi) \leq (p+1)$.

Proposition 4.8 *For any compact monotone Lagrangian sub–manifold L in (P, ω) with $\Sigma(L) \geq m+1, (m \geq 2)$, then*

(1) *all the differentials d^k are trivial for $k \geq 0$,*

(2) *we have the following relation:*

$$\sum_{k \in Z} I^{(r_0)}_{j+\Sigma(L)k}(L, \phi; P) = HF^j(L, \phi; P).$$

Proof For the $\mathbb{Z}$–graded symplectic Floer cochain complex $C_*^{(r_0)}(L, \phi_s; P, J)$, by Proposition 5.4, the Maslov index satisfies the following:

$$0 < \max \mu^{(r_0)}(y) - \min \mu^{(r_0)}(x) = \mu_{H_s}(y) - \mu_{H_s}(x) \leq m.$$

The result follows from the definition of d^k, $\Sigma(L) \geq m+1$ and Corollary 4.5. □

For any compact monotone Lagrangian embedding $L \subset \mathbb{C}^m$, we have $1 \leq \Sigma(L) \leq m$ and the inequality is optimal based on Polterovich's examples.

Definition 4.9 (1) The associated Poincaré–Laurent polynomial $P(E^k,t)$ $(k \geq 1)$ of the spectral sequence is defined as:

$$P^{(r)}(E^k,t) = \sum_{n\in\mathbb{Z}} (\dim_{\mathbb{Z}_2} E^k_{n,j}) t^n.$$

(2) The Euler number of the spectral sequence is defined to be the number $\chi(E^k_{*,*}) = P^{(r)}(E^k,-1)$.

By Theorem 4.3, $P^{(r)}(E^1,t) = \sum_{n\in Z}(\dim_{Z_2} I_n^{(r)}(L,\phi;P,J))t^n$. From Remark 5.b (ii), we have

$$P^{(r+\sigma(L))}(E^k,t)t^{\Sigma(L)} = P^{(r)}(E^k,t). \tag{4.5}$$

We can compare two Poincaré–Laurent polynomials $P^{(r)}(E^k,t)$ and $P^{(r)}(E^\infty,t)$ in the following.

Proposition 4.10 *For any monotone Lagrangian L with $\Sigma(L) \geq 3$,*

$$P^{(r)}(E^1,t) = \sum_{i=1}^{k}(1+t^{-i\Sigma(L)-1})\overline{Q}_i(t) + P^{(r)}(HF^*,t),$$

where $k+1 = k(L,\phi)$ and $\overline{Q}_i(t)$ $(i = 1,2,\cdots,k)$ are the Poincaré–Laurent polynomials of nonnegative integer coefficients.

Proof Let $Z^1_{n,j} = \ker\{d^1 : E^1_{n,j} \to E^1_{n+\Sigma(L)+1,j+1}\}$ and $B^1_{n,j} = \operatorname{Im} d^1 \cap E^1_{n,j}$. We have two short exact sequences:

$$0 \to Z^1_{n,j} \to E^1_{n,j} \to B^1_{n+\Sigma(L)+1,j+1} \to 0, \tag{4.6}$$

$$0 \to B^1_{n,j} \to Z^1_{n,j} \to E^2_{n,j} \to 0. \tag{4.7}$$

So the degree $\Sigma(L)+1$ of the differential d^1 derives the following.

$$P^{(r)}(E^1,t) = P^{(r)}(E^2,t) + (1+t^{-\Sigma(L)-1})P^{(r)}(B^1,t). \tag{4.8}$$

Since the higher differential d^i has degree $i\Sigma(L)+1$, we can repeat (4.8) for E^2 and so on. Let $\overline{Q}_i(t) = P^{(r)}(B^i,t)$. Note that $\oplus E^\infty \cong HF^*(L,\phi;P)$ by Theorem 3.6 and Theorem 4.3. Thus we obtain the desired result. □

From the proof of Proposition 4.10, we have

$$P^{(r)}(E^l,t) = \sum_{i=l}^{k}(1+t^{-i\Sigma(L)-1})\overline{Q}_i(t) + P^{(r)}(HF^*,t), \tag{4.9}$$

for $1 \leq l \leq k$. Since $\Sigma(L)$ is even, so we have

$$\begin{aligned}\chi(E^k_{*,*}) &= \sum_{n\in Z}(-1)^n dim_{Z_2} E^k_{n,j} \\ &= \sum_{j=0}^{\Sigma(L)-1}(-1)^j dim_{Z_2}(\bigoplus_{n\equiv j \pmod{\Sigma(L)}} E^k_{n,j}).\end{aligned}$$

In particular, by Proposition 4.10,

$$\chi(E^k_{*,*}) = \chi(E^\infty_{*,*}) = \chi(HF^*), \quad \text{for all } k \geq 1. \tag{4.10}$$

For an oriented monotone Lagrangian sub–manifold L_i in (P_i, ω_i), we have $I_{\omega_i} = \lambda I_{\mu,L_i}$ for the same $\lambda \geq 0$. So the product $L_1 \times L_2 \times \cdots \times L_s$ is also an oriented Lagrangian in $(\prod_{i=1}^s P_i, \omega_1 \oplus \omega_2 \cdots \oplus \omega_s)$. For $u : (D^2, \partial D^2) \to (\prod_{i=1}^s P_i, \prod_{i=1}^s L_i)$, let $p_i : \prod_{i=1}^s P_i \to P_i$ be the projection on the i-th factor $(i = 1, 2, \cdots, s)$. Now we obtain

$$I_{\mu\oplus\cdots\oplus\mu, L_1\times\cdots\times L_s}(u) = \sum_{i=1}^s I_{\mu,L_i}(p_i u),$$

which follows from the product symplectic form and the Künneth formula for the Maslov class in $\Lambda(\mathbb{C}^{m_1}) \times \cdots \times \Lambda(\mathbb{C}^{m_s})$, where $\mathbb{C}^{m_i} = (p_i \circ u)^*(T_x P_i)$ (see [1]). Hence we have

$$\begin{aligned}I_{\omega_1\oplus\cdots\oplus\omega_s}(u) &= \sum_{i=1}^s I_{\omega_i}(p_i \circ u) \\ &= \sum_{i=1}^s \lambda I_{\mu,L_i}(p_i \circ u) \\ &= \lambda I_{\mu\oplus\cdots\oplus\mu, L_1\times\cdots\times L_s}(u).\end{aligned}$$

So the product Lagrangian $L_1 \times \cdots \times L_s$ is also *monotone* in $(\prod_{i=1}^s P_i, \omega_1 \oplus \omega_2 \cdots \oplus \omega_s)$. Note that $\langle \Sigma(L_1 \times \cdots \times L_s)\rangle = \langle g.c.d(\Sigma(L_i) : 1 \leq i \leq s)\rangle$ (as an ideal in $\mathbb{Z}$) by the additivity of the Maslov index. For simplicity, we will assume that $\Sigma(L_i) = \Sigma(L) \geq 3$ for each $1 \leq i \leq s$.

Since $\Sigma(L_1 \times L_2) = \Sigma(L) \geq 3$ and $L_1 \times L_2$ is an oriented monotone Lagrangian, we use the symplectic diffeomorphism $\phi_1 \times \phi_2 \in Symp_0(P_1 \times P_2)$. In particular, by Lemma 5.3 and [2] we have

$$I_*^{(r_0)}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \cong H^{*+m_1+m_2}(L_1 \times L_2; \mathbb{Z}_2).$$

So there is a Künneth formula for the $\mathbb{Z}$–graded symplectic Floer cohomology.

$$I_*^{(r_0)}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \cong I_*^{(r_0)}(L_1, \phi_1; P_1) \otimes I_*^{(r_0)}(L_2, \phi_2; P_2). \tag{4.11}$$

The torsion terms are irrelevant in this case due to the field $\mathbb{Z}_2$–coefficients. In terms of the filtration and our spectral sequence, we have

$$E^1_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \cong$$

$$\bigoplus_{n_1+n_2=n, j_1+j_2=j \pmod{\Sigma(L)}} E^1_{n_1,j_1}(L_1, \phi_1; P_1) \otimes E^1_{n_2,j_2}(L_2, \phi_2; P_2). \tag{4.12}$$

Proposition 4.11 *For the monotone Lagrangian sub–manifold $L_1 \times L_2$, there exists a spectral sequence $E^k_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2)$ which converges to the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology $HF^*(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2)$ with*

$$E^1_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) = I^{(r_0)}_*(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2).$$

Proof The result follows from the monotonicity of $L_1 \times L_2$, $\Sigma(L_1 \times L_2) = \Sigma(L) \geq 3$ and Theorem 4.3. □

Lemma 4.12 *For the spectral sequence $E^k_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2)$ in Proposition 4.11, the higher differential d^k is given by*

$$d^k|_{E^k_{n_1,j_1}(L_1,\phi_1;P_1)} \otimes 1 \pm 1 \otimes d^k|_{E^k_{n_2,j_2}(L_2,\phi_2;P_2)}.$$

Proof It is true for $k = 0$ by (4.11). By the definition of d^k (see the proof of Theorem 4.3),

$$d^k\colon\ E^k_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \to E^k_{n+2Nk+1,j+1}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2),$$

the term $E^k_{n,j}$ is generated by cocycles in $F^{(r,r)}_n C_j(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2)$ (modulo $E^{k-1}_{n,j}$ term) since we use the field $\mathbb{Z}_2$–coefficients. After modulo $E^{k-1}_{*,*}$ we have the higher differential

$$d^k\colon\ F^{(r)}_{n_1} C^{j_1}(L_1, \phi_1; P_1) \otimes F^{(r)}_{n_2} C^{j_2}(L_2, \phi_2; P_2) \to$$

$$\{F^{(r)}_{n_1+2Nk+1} C^{j_1+1}(L_1, \phi_1; P_1) \otimes F^{(r)}_{n_2} C^{j_2}(L_2, \phi_2; P_2)\}$$

$$\oplus \{F^{(r)}_{n_1} C^{j_1}(L_1, \phi_1; P_1) \otimes F^{(r)}_{n_2+2Nk+1} C^{j_2+1}(L_2, \phi_2; P_2)\}.$$

Note that $d^k \circ d^k = 0$ by the monotonicity and Proposition 4.11. Thus the result follows from the very definition of $d^k|_{E^k(L_i,\phi_i;P_i)}$ for $i = 1, 2$. The sign is not important since the coefficients are in $\mathbb{Z}_2$. □

In general we can not expect the Künneth formulae for the tensor product of two spectral sequences due to the torsions (see [12, 20]). For our case with **the field** $\mathbb{Z}_2$–coefficients, we do have such a Künneth formula for the spectral sequence.

Theorem 4.13 *For the monotone Lagrangian $L_1 \times L_2$ in $(P_1 \times P_2, \omega_1 \oplus \omega_2)$ with $I_{\omega_i} = \lambda I_{\mu, L_i}$ and $\Sigma(L_i) = \Sigma(L) \geq 3$ $(i = 1, 2)$, we have, for $k \geq 1$,*

$$E^k_{n,j}(L_1 \times L_2, \phi_1 \times \phi_2; P_1 \times P_2) \cong$$

$$\bigoplus_{n_1+n_2=n, j_1+j_2=j \pmod{\Sigma(L)}} E^k_{n_1,j_1}(L_1, \phi_1; P_1) \otimes E^k_{n_2,j_2}(L_2, \phi_2; P_2).$$

Proof The case of $k = 1$ is the usual Künneth formula for the cohomology of $L_1 \times L_2$ (see (4.11 and (4.12). The result follows by induction on k (see [12, 20]) for the field $\mathbb{Z}_2$–coefficients. □

Corollary 4.14 (1) *For a monotone Lagrangian $L \subset (P, \omega)$, for $k \geq 1$ and $s \geq 2$, $E^k_{n,j}(L \times L \times \cdots \times L, \phi \times \cdots \times \phi; P^s) \cong$*

$$\bigoplus_{n_1+\cdots+n_s=n, j_1+\cdots+j_s=j \pmod{\Sigma(L)}} \{\otimes_{i=1}^s E^k_{n_i,j_i}(L, \phi; P)\}.$$

(2) *The Poincaré–Laurent polynomial for $E^k_{n,j}(L \times L \times \cdots \times L, \phi \times \cdots \times \phi; P^s)$ satisfies*

$$P^{(\mathbf{r})}(E^k(L \times L \times \cdots \times L, \phi \times \cdots \times \phi; P^s), t) = \{P^{(r)}(E^k(L, \phi; P), t)\}^s.$$

Proof (1) follows from the induction proof of Theorem 4.13 with $L_i = L (1 \leq i \leq s)$ and the field $\mathbb{Z}_2$–coefficients, and (2) follows from the definition of the Poincaré–Laurent polynomial and (1). □

5 Applications

5.1 Hofer's energy and Chekanov's construction

In this subsection, we relate our $\mathbb{Z}$–graded symplectic Floer cohomology with the one constructed by Chekanov [2]. We also show some results to illustrate the interactions among the Hofer energy $e_H(L)$, the minimal symplectic action $\sigma(L)$ and the minimal Maslov number $\Sigma(L)$.

Hofer [9] introduced the notion of the disjunction energy or the displacement energy associated with a subset of symplectic manifold. Hofer's symplectic energy measures how large a variation of a (compactly supported) Hamiltonian function must be in order to push the subset off itself by a time–one map of corresponding Hamiltonian flow. Hofer showed that the symplectic energy of every open subset in the standard symplectic vector space is nontrivial. See [10] for more geometric study of the Hofer energy.

Definition 5.1 Let $\mathcal{H}$ be the space of compactly supported functions on $[0,1]\times P$. The Hofer symplectic energy of a symplectic diffeomorphism $\phi_1\colon P \to P$ is defined by

$$\begin{aligned} E(\phi_1) &= \inf\{\int_0^1 (\max_{x\in P} H(s,x) - \min_{x\in P} H(s,x))ds| \\ &\qquad \phi_1 \text{ is a time one flow generated by } H \in \mathcal{H}\}. \\ e_H(L) &= \inf\{E(\phi_1) : \phi_1 \in Ham(P), L\cap\phi_1(L) = \emptyset \quad \text{empty set}\}. \end{aligned}$$

Theorem 5.2 [2] *If $E(\phi_1) < \sigma(L)$, L is rational, and L intersects $\phi_1(L)$ transversely, then*

$$\#(L\cap\phi_1(L)) \geq SB(L; Z_2),$$

where $SB(L;Z_2)$ is the sum of Betti numbers of L with $\mathbb{Z}_2$–coefficients.

Remark 5.a (1) Polterovich used Gromov's figure 8 trick and a refinement of Gromov's existence scheme of the J–holomorphic disk to show that $e_H(L) \geq \frac{\sigma(L)}{2}$. Chekanov extends the result to $e_H(L) \geq \sigma(L)$ which is optimal for general Lagrangian sub-manifolds (see [2] §1). It is unclear whether Theorem 5.2 remains true for the case of $E(\phi_1) = \sigma(L)$.

(2) Sikorav showed that $e_H(T^m) \geq \sigma(T^m)$. The Theorem 5.2 generalizes Sikorav's result to all rational Lagrangian sub–manifolds.

Chekanov [2] used a restricted symplectic Floer cohomology in the study of Hofer's symplectic energy of rational Lagrangian sub–manifolds. Denoted by

$$\begin{aligned} \Omega_s &= \{\gamma\in C^\infty([0,1],P)|\gamma(0)\in L, \gamma(1)\in\phi_s(L)\}, \\ \Omega &= \bigcup_{s\in[0,1]} \Omega_s \subset [0,1]\times C^\infty([0,1],P). \end{aligned}$$

Assume that ϕ_s is generic so that $D = \{s\in[0,1] : \phi_s(L)\cap L \text{ transversely}\}$ is dense in $[0,1]$. One may choose the anti–derivative of $Da_s(z)\xi$ as $a_s\colon \Omega_s \to$

$\mathbb{R}/\sigma(L)\mathbb{Z}$, and fix a_0 with critical value 0. Pick $z_s \in L \cap \phi_s(L)$ such that modulo $\sigma(L)\mathbb{Z}$,

$$0 < a_s(z_s) = \min\{a_s(x) \pmod{\sigma(L)\mathbb{Z}} | x \in Z_{\phi_s}\} < \sigma(L). \tag{5.1}$$

So it is possible to have $a_s(z_s)$ in (5.1) for $s \in D$ due to the constant factor for a_s. Let $r_0 \neq 0$ be sufficiently small positive number in $\mathbb{R}_{L,\phi_s} \cap (0, \sigma(L))$, say $0 < r_0 < \frac{1}{16}a_s(z_s)$. The condition $E(\phi_1) < \sigma(L)$ provides that there is a unique $x \in Z_{\phi_s}$ (the $x^{(r_0)}$) which corresponds to the unique lift in $(r_0, r_0 + \sigma(L))$,

$$0 < a_s(x) - a_s(z_s) < \sigma(L). \tag{5.2}$$

Under these restrictions, define the free module C_s over $\mathbb{Z}_2$ generated by $L \cap \phi_s(L)$ and the coboundary map $\partial_s \in \mathrm{End}(C_s)$ (see below) such that $\partial_s \circ \partial_s = 0$ [2]. Thus $H^*(C_s, \partial_s)$ is well–defined for every $s \in [0,1]$. With the unique liftings of x and y in $(r_0, r_0 + \sigma(L))$, we can identify Chekanov's restricted symplectic Floer cochain complex with our $\mathbb{Z}$–graded symplectic Floer cochain complex.

Lemma 5.3 *For r_0 as above, $C_*^{(r_0)}(L, \phi_s; P, J) = C_s$. Let $\mathcal{M}_{J_s}(x,y)$ be the restricted moduli space of J–holomorphic curves $\{u \in \mathcal{M}(L, \phi_s(L)) | u^*(\omega) = a_s(x) - a_s(y)\}$. So*

$$\partial^{(r_0)}x = \partial_s x = \sum \#\hat{\mathcal{M}}_{J_s}(x,y)y.$$

Proof Note that $\mu_u = \mu^{(r_0)}(y) - \mu^{(r_0)}(x) = dim\mathcal{M}_{J_s}(x,y)$ for $u \in \mathcal{M}_{J_s}(x,y)$. So y in $\partial_s x$ is the element in $C_{n+1}^{(r_0)}(L, \phi_s; P, J)$; for any $u \in \mathcal{M}_{J_s}(x,y)$ contributing in the coboundary of $\partial^{(r_0)}x$, we have $u^*(\omega) = a_s(x) - a_s(y)$ by the Proposition 2.3 in [3]. For the unique lifts in $(r_0, r_0 + \sigma(L))$ of Z_{ϕ_s}, the choice of a_s gives arise to the one–to–one correspondence between $\mathcal{M}_{J_s}(x,y)$ and $\mathcal{M}_J(x,y)$ for $\mu^{(r_0)}(y) - \mu^{(r_0)}(x) = 1$. Therefore the coboundary maps agree on the $\mathbb{Z}_2$–coefficients. □

For s sufficiently small, the point $x \in L \cap \phi_s(L)$ is also a critical point of the Hamiltonian function H_s of ϕ_s. The Maslov index is related to the usual Morse index of the time–independent Hamiltonian H with sufficiently small second derivatives:

$$\mu^{(r_0)}(x) = \mu_H(x) - m. \tag{5.3}$$

Proposition 5.4 *For the r_0 as above, we assume that (i) $\Sigma(L) \geq 3$ and $E(\phi_1) < \sigma(L)$, (ii) L is monotone Lagrangian sub–manifold in P, (iii) L intersects $\phi_1(L)$ transversely. Then there is a natural isomorphism between*

$$I_*^{(r_0)}(L, \phi_s; P) \cong H^{*+m}(L; Z_2) \quad \text{for } * \in Z \text{ and } s \in [0,1].$$

Proof For any $s, s^{'} \in [0,1]$ sufficiently close, by Theorem 3.6 we have

$$I_*^{(r_0)}(L, \phi_s; P) \cong I_*^{(r_0)}(L, \phi_{s^{'}}; P). \tag{5.4}$$

For $s \in [0,1]$ sufficiently small, we have

$$I_*^{(r_0)}(L, \phi_s; P) \cong H^*(C_s, \partial_s) \cong H^{*+m}(L; Z_2). \tag{5.5}$$

The first isomorphism is given by Lemma 5.3 and the second by Lemma 3 of [2] and Theorem 2.1 of [19]. Then the result follows by applying (5.4) and (5.5). □

Remark 5.b (i) Proposition 5.4 provides the Arnold conjecture for monotone Lagrangian sub–manifold with $\Sigma(L) \geq 3$ and $E(\phi_1) < \sigma(L)$. Chekanov's result does not require the assumptions of the monotonicity of L and $\Sigma(L) \geq 3$.

(ii) There is a natural relation $I_*^{(r_0+\sigma(L))}(L, \phi_s; P) = I_{*+\Sigma(L)}^{(r_0)}(L, \phi_s; P)$ for different choices of a_s and $a_s + \sigma(L)$. In fact, for $s, s^{'} \in [0,1]$,

$$I_*^{(r_0)}(L, \phi_s; P) \cong I_*^{(r_0)}(L, \phi_{s^{'}}; P).$$

For the Lagrangian sub–manifolds L in $(\mathbb{C}^m, \omega_0)$ with the standard symplectic structure $\omega_0 = -d\lambda$. The 1–form λ is called *Liouville form*, which has the corresponding Liouville class $[\lambda|_L] \in H^1(L, \mathbb{R})$. One of the fundamental results in [8] is the non–triviality of the Liouville class.

Theorem 5.5 (Gromov [8]) *For any compact Lagrangian embedding L in $\mathbb{C}^m$, the Liouville class $[\lambda|_L] \neq 0 \in H^1(L, \mathbb{R})$.*

Lemma 5.6 *If $E(\phi_1) < (p+1)\sigma(L)$, then $k(L, \phi) \leq (p+1)$ for the spectral sequence $E_{n,j}^k(L, \phi; P, J)$.*

Proof By definition of $E(\phi)$, we have

$$a_{+,H} = \int_0^1 \max_{x \in P} H(s,x), \qquad a_{-,H} = \int_0^1 \min_{x \in P} H(s,x),$$

for ϕ_1 a time–one flow generated by $H(s,x)$. For any nontrivial $u \in \hat{\mathcal{M}}_J(x,y)$ which contributes in d^k, we have

$$I_\omega(u) = a_u(x^{(r)}) - a_u(y^{(r)}) \leq a_{+,H} - a_{-,H}.$$

Thus for any $x, y \in C_*^{(r)}(L, \phi; P, J)$, we have $a_u(x^{(r)}) - a_u(y^{(r)}) \leq a_{+,H} - a_{-,H}$. So

$$\begin{aligned} a_u(x^{(r)}) - a_u(y^{(r)}) &\leq \inf_H (a_{+,H} - a_{-,H}) \\ &= E(\phi_1) \\ &< (p+1)\sigma(L) \end{aligned} \tag{5.6}$$

By the monotonicity, $\lambda I_{\mu,\tilde{L}}(u) = I_\omega(u) < (p+1)\sigma(L)$ $(\mu^{(r)}(x,y) = I_{\mu,\tilde{L}}(u) < (p+1)\Sigma(L))$. By Lemma 4.7, we obtain the desired result. □

Remark 5.c Note that $k(L, \phi) = p + 1$, one can not claim that $E(\phi) < (p+1)\sigma(L)$. For $k \geq p+1$, the property $d^k = 0$ is the zero $\pmod 2$ of the 1–dimensional moduli space of J–holomorphic curves. There are possible pairs $u_\pm$ of nontrivial J–holomorphic curves which have $a_{u_\pm}(x^{(r)}) - a_{u_\pm}(y^{(r)}) \geq (p+1)\sigma(L)$.

There are four interesting numbers $\Sigma(L), \sigma(L), e_H(L)$ and $k(L, \phi)$ of a monotone Lagrangian manifold L. Both of them intertwine and link with the $\mathbb{Z}$–graded symplectic Floer cohomology and its derived spectral sequence. It would be interesting to study further relations among them.

5.2 Internal cup–product structures

For the monotone Lagrangian sub–manifold L in (P, ω) with $\Sigma(L) \geq 3$, we have obtained an external product structure (cross product) in Theorem 4.13. In this subsection, we show that there is an internal product structure on the spectral sequence and the symplectic Floer cohomology of the Lagrangian sub–manifold L.

By a result of Chekanov [2] and Proposition 5.4, we can identify $I_*^{(r)}(L, \phi; P) \cong H^{*+m}(L; \mathbb{Z}_2)$. Let a be a cohomology class in $H^p(L; \mathbb{Z}_2)$. Define a map

$$a\cup \colon C_n^{(r)}(L, \phi; P, J) \to C_{n+p}^{(r)}(L, \phi; P, J) \tag{5.7}$$

$$x \mapsto \sum_{-\mu^{(r)}(x) + \mu^{(r)}(y) = p} \#(\mathcal{M}_J(x, y) \cap i_*(PD_L(a))) \cdot y,$$

where $i \colon L \hookrightarrow P$ is the Lagrangian imbedding, $\#(\mathcal{M}_J(x, y) \cap i_*(PD_L(a)))$ is the algebraic number of $i_*(PD_L(a))$ intersecting $\mathcal{M}_J(x, y)$, and $PD_L(a)$ is the Poincaré dual of a in L.

Proposition 5.7 *The map $a\cup$ in (5.7) is well–defined for the monotone Lagrangian embedding, and $\partial^{(r)}_{n+p}\circ(a\cup)=(a\cup)\circ\partial^{(r)}_n$.*

Proof Note that $i_*(PD_L(a))$ is a divisor in P where the intersection $\mathcal{M}_J(x,y)$ $\cap\, i_*(PD_L(a))$ can be made transversally without holomorphic bubblings (see [3, 5] Theorem 6). In fact, we can apply the similar argument in Proposition 4.1 of [13] to the $\mathbb{Z}$–graded symplectic Floer cohomology of the Lagrangian L in order to avoid the bubbling issue. Now it suffices to check the map $a\cup$ commutes with the $\mathbb{Z}$–graded differential.

Without holomorphic bubblings, the partial compactification of $\mathcal{M}_J(x,z)$ with only k–tuple holomorphic curves can be described as

$$\overline{\mathcal{M}_J(x,z)}=\cup(\times_{i=0}^{k-1}\mathcal{M}_J(c_i,c_{i+1})),$$

the union over all sequence $x=c_0,c_1,\cdots,c_k=z$ such that $\mathcal{M}_J(c_i,c_{i+1})$ is nonempty for all $0\le i\le k-1$. For any sequence $c_0,c_1,\cdots,c_k\in\mathrm{Fix}\,(\phi)$, there is a gluing map

$$G\colon\ \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}^k\to\overline{\mathcal{M}_J(x,z)},$$

where $\mathbf{D}^k=\{(\lambda_1,\cdots,\lambda_k)\in[-\infty,\infty]^k: 1+\lambda_i<\lambda_{i+1}, 1\le i\le k-1\}$.

(1) The image of G is a neighborhood of $\times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})$ in the compactification with only k–tuple holomorphic curves.

(2) The restriction of G to $\times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathrm{Int}\,(\mathbf{D}^k)$ is a diffeomorphism onto its image.

(3) The extension of the gluing map is independent of $u_1,\cdots,u_{j-1}$ and $u_{j+p+1},\cdots,u_k$ provided $\lambda_{j-1}=-\infty$ and $\lambda_{j+p+1}=+\infty$.

$$\begin{aligned}&G(u_1,\cdots,u_k,-\infty,\cdots,-\infty,\lambda_j,\cdots,\lambda_{j+p},+\infty,\cdots,+\infty)\\&\qquad=G(u_j,\cdots,u_{j+p},\lambda_j,\cdots,\lambda_{j+p}).\end{aligned}$$

For $x\in C^{(r)}_n(L,\phi;P,J)$ and $z\in C^{(r)}_{n+p+1}(L,\phi;P,J)$, the space $K=\mathcal{M}_J(x,z)\cap i_*(PD(a))$ is a 1–dimensional manifold in Ω_ϕ with $p=\deg(a)$. We have

$$\begin{aligned}0&=\int_{\overline{\mathcal{M}_J(x,z)}}dPD_P^{-1}(i_*(PD_L(a)))\\&=\int_{\partial\overline{\mathcal{M}_J(x,z)}}PD_P^{-1}(i_*(PD_L(a)))\\&=\partial\overline{\mathcal{M}_J(x,z)}\cap i_*(PD_L(a)),\end{aligned}$$

where PD_X stands for the Poincaré dual of the space X. Since G is a local diffeomorphism near $\partial\overline{\mathcal{M}_J(x,z)}$, we can integrate $PD_P^{-1}(i_*(PD_L(a)))$ (by changing variables) over $\times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times \mathbf{D}^k$. We have

$$\partial(\times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times \mathbf{D}^k) = \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times \partial\mathbf{D}^k.$$

From the definition of $\mathbf{D}^k$, we get $\partial(\mathbf{D}^k) = \mathbf{D}_+^{k-1}\coprod -\mathbf{D}_-^{k-1}$, where $\mathbf{D}_-^{k-1} = \{(-\infty,\lambda_2,\cdots,\lambda_k)\in\mathbf{D}^k\}$ and $\mathbf{D}_+^{k-1} = \{(\lambda_1,\cdots,\lambda_{k-1},+\infty)\in\mathbf{D}^k\}$. Thus

$$\begin{aligned}
0 &= \int_{\partial\overline{\mathcal{M}_J(x,z)}} PD_P^{-1}(i_*(PD_L(a)))\\
&= \langle \partial G^{-1}(\times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}^k), PD_P^{-1}(i_*(PD_L(a)))\rangle\\
&= \langle \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\partial(\mathbf{D}^k), G^*(PD_P^{-1}(i_*(PD_L(a))))\rangle\\
&= \langle \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}_+^{k-1}, G^*(PD_P^{-1}(i_*(PD_L(a))))\rangle\\
&- \langle \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}_-^{k-1}, G^*(PD_P^{-1}(i_*(PD_L(a))))\rangle.
\end{aligned}\tag{5.8}$$

The image of G is independent of u_1 in $\mathbf{D}_-^{k-1}$ and u_k in $\mathbf{D}_+^{k-1}$ respectively. For $\lambda_1\to-\infty$,

$$G(\times_{i=1}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}_-^{k-1}) = \overline{\mathcal{M}_J(c_1,z)}.$$

We have the dimension counting as follows from $\sum_{i=0}^{k-1}(\mu^{(r)}(c_i)-\mu^{(r)}(c_{i+1})) = p+1$,

$$\dim\overline{\mathcal{M}_J(c_1,z)} = (p+1)-(\mu^{(r)}(x)-\mu^{(r)}(c_1)).$$

By the transversal of the intersection with $i_*(PD_L(a))$ in Ω_ϕ, the only possible nontrivial contribution of $\overline{\mathcal{M}_J(c_1,z)}\cap i_*(PD_L(a))$ is from $\dim\overline{\mathcal{M}_J(c_1,z)} = p$. Hence $(p+1)-(\mu^{(r)}(x)-\mu^{(r)}(c_1)) = p$ if and only if $\mu^{(r)}(x)-\mu^{(r)}(c_1) = 1$. Therefore we obtain

$$\begin{aligned}
&\langle \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}_-^{k-1}, G^*(PD_P^{-1}(i_*(PD_L(a))))\rangle\\
&\qquad = \#\hat{\mathcal{M}}_J^1(x,c_1)\cdot\#\mathcal{M}_J(c_1,z)\cap i_*(PD_L(a)),
\end{aligned}$$

this gives the term $(a\cup)\circ\partial_n^{(r)}(x)$. Similarly for $\lambda_k\to+\infty$,

$$\langle \times_{i=0}^{k-1}\hat{\mathcal{M}}_J(c_i,c_{i+1})\times\mathbf{D}_+^{k-1}, G^*(PD_P^{-1}(i_*(PD_L(a))))\rangle = \partial_{n+p}^{(r)}\circ(a\cup)(x). \tag{5.9}$$

Hence the result follows. □

Now the map $a\cup$ defined in (5.7) induces a map (still denoted by $a\cup$) on the $\mathbb{Z}$–graded symplectic Floer cohomology by Proposition 5.7,

$$a\cup\colon\ I_n^{(r)}(L,\phi;P,J)\to I_{n+p}^{(r)}(L,\phi;P,J). \tag{5.10}$$

Since $H^*(L;\mathbb{Z}_2)$ is a graded algebra with cup product as multiplication, the $H^*(L;\mathbb{Z}_2)$–module structure of $I_*^{(r)}(L,\phi;P,J)$ is given by the following commutative diagram:

$$\begin{array}{ccc} H^*(L;\mathbb{Z}_2)\otimes H^*(L;\mathbb{Z}_2)\otimes I_*^{(r)}(L,\phi;P,J) & \xrightarrow{1\otimes\psi} & H^*(L;\mathbb{Z}_2)\otimes I_*^{(r)}(L,\phi;P,J) \\ \Big\downarrow \cup\otimes 1 & & \Big\downarrow \psi \\ H^*(L;\mathbb{Z}_2)\otimes I_*^{(r)}(L,\phi;P,J) & \xrightarrow{\psi} & I_*^{(r)}(L,\phi;P,J) \end{array}$$

where $\psi\colon H^*(L;\mathbb{Z}_2)\otimes I_*^{(r)}(L,\phi;P,J)\to I_*^{(r)}(L,\phi;P,J)$ is given by $\psi(a,\cdot)=a\cup\cdot$ in (5.10). Thus we have $\{(a\cup)\circ(b\cup)\}(x)$ equals

$$\sum_{y\in C^{(r)}_{n+\deg(b)}}\#\mathcal{M}_J(x,y)\cap i_*(PD_L(b))\cdot\sum_{z\in C^{(r)}_{n+\deg(b)+\deg(a)}}\#\mathcal{M}_J(y,z)\cap i_*(PD_L(a))\cdot z.$$

Proposition 5.8 *If $i\colon L\hookrightarrow P$ induces a surjective map $i^*\colon H^*(P;\mathbb{Z}_2)\to H^*(L;\mathbb{Z}_2)$, then the $\mathbb{Z}$–graded symplectic Floer cohomology $I_*^{(r)}(\phi,P)$ has an $H^*(L;\mathbb{Z}_2)$–module structure.*

Proof If $i^*\colon H^*(P;\mathbb{Z}_2)\to H^*(L;\mathbb{Z}_2)$ is surjective, then $i_*\circ PD_L\circ i^*=PD_P$. Thus $i_*(PD_L(a\cup b))=i^*(PD_L(i^*(a_P\cup b_P)))=PD_P(a_P\cup b_P)$ for some classes $a_P,b_P\in H^*(P;\mathbb{Z}_2)$. Therefore

$$\begin{aligned}\#\mathcal{M}_J(x,z)\cap i_*(PD_L(a\cup b)) &= \langle PD_P^{-1}(i_*(PD_L(a\cup b))),\mathcal{M}_J(x,z)\rangle\\ &=\langle a_P\cup b_P,\mathcal{M}_J(x,z)\rangle\\ &=\langle D^*(a_P\times b_P),\mathcal{M}_J(x,z)\rangle\\ &=\langle a_P\times b_P,\mathbf{D}_*\mathcal{M}_J(x,z)\rangle,\end{aligned}\tag{5.11}$$

where $\mathbf{D}_*$ is the unique chain map up to chain homotopy induced from the diagonal map $\mathbf{D}\colon\Omega_\phi\to\Omega_\phi\times\Omega_\phi$, and $H^*(P;\mathbb{Z}_2)$ is viewed as a subring of $H^*(\Omega_\phi;\mathbb{Z}_2)$ (see [5] §1c or [13]). By the monotonicity and the energy formula, for any elements in $\mathcal{M}_J(x,y)\times\mathcal{M}_J(y,z)$, we have

$$r<\tilde a_J(x^{(r)})\le\tilde a_J(y^{(r)});\qquad \tilde a_J(y^{(r)})\le\tilde a_J(z^{(r)})<r+2\alpha N.$$

So by the uniqueness of the lifting $y^{(r)}$, we have $y\in C_*^{(r)}(L,\phi;P,J)$. For $\mu^{(r)}(y)\ne n+deg(b)$, then, by the dimension counting,

$$\langle a_P\times b_P,\mathcal{M}_J(x,y)\times\mathcal{M}_J(y,z)\rangle=0.$$

Thus we obtain $\langle a_P \times b_P, \cup_{y \in C^{(r)}_{n+\deg(b)}} \mathcal{M}_J(x,y) \times \mathcal{M}_J(y,z)\rangle$

$$
\begin{aligned}
&= \sum_{\mu^{(r)}(y)=n+deg(b)} \langle b_P, \mathcal{M}_J(x,y)\rangle \cdot \langle a_P, \mathcal{M}_J(y,z)\rangle \\
&= \sum_{\mu^{(r)}(y)=n+deg(b)} \langle PD_P^{-1}(i_* PD_L(b)), \mathcal{M}_J(x,y)\rangle \cdot \langle PD_P^{-1}(i_* PD_L(a)), \mathcal{M}_J(y,z)\rangle \\
&= \sum_{y \in C^{(r)}_{n+\deg(b)}} \#(\mathcal{M}_J(x,y) \cap i_*(PD_L(b))) \cdot \#(\mathcal{M}_J(y,z) \cap i_*(PD_L(a)))z.
\end{aligned}
\tag{5.12}
$$

This equals to $\langle (a\cup) \circ (b\cup)(x), z\rangle$. □

One can verify that the module structure is invariant under the compact continuation $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{1,\varepsilon/2}$ by the standard method in [5, 13]. Note that one can define the action of $H^*(L;\mathbb{Z}_2)$ for any monotone Lagrangian sub–manifold L, but the module structure requires for the special property of the Lagrangian embedding $i\colon L \hookrightarrow P$ in Proposition 5.8.

Hence we obtain a $H^*(L;\mathbb{Z}_2)$–module structure on $I^{(r)}_*(L,\phi;P)$. Now we associate the internal product structure on the $\mathbb{Z}$–graded symplectic Floer cohomology:

$$I^{(r)}_{n_1-m}(L,\phi;P) \times I^{(r)}_{n_2-m}(L,\phi;P) \stackrel{\cup}{\to} I^{(r)}_{n_1+n_2-m}(L,\phi;P), \tag{5.13}$$

as a bilinear form defined by $I^{(r)}_{n_1-m}(L,\phi;P) \cong H^{n_1}(L;\mathbb{Z}_2)$ and (5.10). Note that the index–shifting makes the compatibility of the usual cup–product of the cohomology ring on $H^*(L;\mathbb{Z}_2)$: $H^{n_1}(L;\mathbb{Z}_2) \times H^{n_2}(L;\mathbb{Z}_2) \stackrel{\cup}{\to} H^{n_1+n_2}(L;\mathbb{Z}_2)$.

Corollary 5.9 *If $i\colon L \hookrightarrow P$ induces a surjective map $i^*\colon H^*(P;\mathbb{Z}_2) \to H^*(L;\mathbb{Z}_2)$ and $\Sigma(L) \geq 3$, then*

$$H^*(L;\mathbb{Z}_2) \to End(I^{(r)}_*(L,\phi,P;\mathbb{Z}_2))$$

is an injective homomorphism.

Corollary 5.9 generalizes Theorem 3 of Floer [4] of $\pi_2(P,L) = 0$ to the case of monotone Lagrangian sub-manifolds L with $\Sigma(L) \geq 3$ if the intersection L with $\phi(L)$ is transverse, where $\phi \in Symp_0(P)$ not necessary $\phi \in Ham(P)$. Note that one may combine our construction in §2 - §4 with the one in §3 of [4] for the non-transverse points in $L \cap \phi(L)$. With the gluing result of trajectories along those degenerate points in $L \cap \phi(L)$, we obtain that for any exact Hamiltonian

ϕ on the monotone pair $(P, L; \omega)$ with $\Sigma(L) \geq 3$, the number $\# L \cap \phi(L)$ of intersections of $\phi(L)$ with L is greater than or equal to the $\mathbb{Z}_2$-cuplength of L via (5.13) and Corollary 5.9. I.e., a generalization of Theorem 1 of [4] is achieved for a symplectic manifold (P, ω) and its monotone Lagrangian sub-manifold L without the hypothesis $\pi_2(P, L) = 0$.

Lemma 5.10 *For the Lagrangian embedding in Proposition 5.8, the map $a\cup$ defined in (5.7) preserves the filtration in Definition 4.1, and $d^k \circ (a\cup) = (a\cup) \circ d^k$, where d^k is the higher differential in the spectral sequence of Theorem 4.3.*

Proof Recall $F_n^{(r)} C^j(L, \phi, P; J) = \sum_{k \geq 0} C^{(r)}_{n+\Sigma(L)k}(L, \phi, P; J)$. Thus by our definition $a\cup$ in (5.7), we have

$$a\cup : \sum_{k \geq 0} C^{(r)}_{n+\Sigma(L)k}(L, \phi, P; J) \to \sum_{k \geq 0} C^{(r)}_{n+\Sigma(L)k+\deg(a)}(L, \phi, P; J),$$

which induces a map $a\cup$: $F_n^{(r)} C^j(L, \phi, P; J) \to F^{(r)}_{n+\deg(a)} C^{j_a}(L, \phi, P; J)$ with $j_a \equiv j + \deg(a) \pmod{\Sigma(L)}$. Thus we obtain a filtration preserving homomorphism with degree $\deg(a)$. Element in $E^k_{n,j}$ is a survivor from previous differentials, and $x \in E^k_{n,j}$ is an element in $F_n^{(r)} C^j(L, \phi, P, J)$ with $\delta x \in F^{(r)}_{n+1+\Sigma(L)k} C^{j+1}(L, \phi, P, J)$. The differential d^k (induced from δ) is counting the signed one–dimensional moduli space of J–holomorphic curves from $x \in E^k_{n,j}$ to $y \in E^k_{n+1+\Sigma(L)k, j+1}$. The diagram

$$\begin{array}{ccc} E^k_{n,j} & \xrightarrow{d^k} & E^k_{n+1+\Sigma(L)k, j+1} \\ \downarrow a\cup & & \downarrow a\cup \\ E^k_{n+\deg(a), j_a} & \xrightarrow{d^k} & E^k_{n+1+\Sigma(L)k+\deg(a), j_a+1} \end{array} \tag{5.14}$$

is commutative by the same method of the proof in Proposition 5.7. □

Theorem 5.11 *For the monotone Lagrangian $L \overset{i}{\hookrightarrow} P$ with i^*: $H^*(P; \mathbb{Z}_2) \to H^*(L; \mathbb{Z}_2)$ surjective and $\Sigma(L) \geq 3$, the spectral sequence $E^k_{*-m,*}(L, \phi; P)$ has an ring structure which is descended from the cohomology ring $(H^*(L; \mathbb{Z}_2), \cup)$.*

Proof As we see from the above, there is an internal product structure on $E^1_{*-m,*}$: $H^{n_1}(L; \mathbb{Z}_2) \times E^1_{n_2-m, j_2}(L, \phi; P) =$

$$E^1_{n_1-m, j_1}(L, \phi; P) \times E^1_{n_2-m, j_2}(L, \phi; P) \to E^1_{n_1+n_2-m, j}(L, \phi; P), \tag{5.15}$$

where $E^1_{n_1-m,j_1}(L,\phi;P)\cong H^{n_1}(L;\mathbb{Z}_2)$ and $j\equiv j_1+j_2+m \pmod{\Sigma(L)}$. Hence the module structure in Proposition 5.8 shows that the internal product is associative through the identifications. Furthermore the module structure descends to the spectral sequence by Lemma 5.10:

$$H^{n_1}(L;\mathbb{Z}_2)\times E^k_{n_2-m,j_2}(L,\phi;P)\overset{\cup}{\to} E^k_{n_1+n_2-m,j}(L,\phi;P).$$

As a subgroup $E^k_{n_1-m,j_1}(L,\phi;P)\subset H^{n_1}(L;\mathbb{Z}_2)$, elements in $E^k_{n_1-m,j_1}(L,\phi;P)$ are the survivors which are neither an image of some d^k nor the source of the nontrivial d^k due to the $\mathbb{Z}_2$–coefficients. Hence they are exactly the cohomology classes in $H^{n_1}(L;\mathbb{Z}_2)$. In general, the k-th term $E^k_{n_1-m,j_1}(L,\phi;P)$ lies in the quotient of $E^{k-1}_{n_1-m,j_1}(L,\phi;P)$ which is not necessary a subgroup of $H^{n_1}(L;\mathbb{Z}_2)$. Therefore there is an associate (from the module structure) internal product structure on $E^k_{*-m,*}(L,\phi;P)$:

$$E^k_{n_1-m,j_1}(L,\phi;P)\times E^k_{n_2-m,j_2}(L,\phi;P)\overset{\cup_k}{\to} E^k_{n_1+n_2-m,j}(L,\phi;P), \tag{5.16}$$

via the identification of the first factor $E^k_{n_1-m,j_1}(L,\phi;P)\subset H^{n_1}(L;\mathbb{Z}_2)$. □

In particular, $(E^\infty_{*-m,*}(L,\phi;P),\cup_\infty)=(HF^{*-m}(L,\phi;P),\cup_\infty)$ is an associative ring on the $\mathbb{Z}_{\Sigma(L)}$–graded symplectic Floer cohomology which is induced from the cohomology ring of the imbedded monotone Lagrangian sub–manifold L. Note that the index shifting (5.16) in the internal product is not the usual one due to the relation between the Maslov index and the Morse index.

For $\phi\in \mathrm{Symp}_0(P)$ and the monotone Lagrangian embedding $i\colon L\hookrightarrow P$ with $i^*\colon H^*(P;\mathbb{Z}_2)\to H^*(L;\mathbb{Z}_2)$ surjective and $\Sigma(L)\geq 3$, we obtain the ring structure on $E^k_{*-m,*}(L,\phi;P)$ for every $k\geq 1$. From the quantum effect arisen from the higher differentials in the spectral sequence in Theorem 4.3, the ring $(HF^{*-m}(L,\phi;P),\cup_\infty)$ can be thought of as the quantum effect of the regular cohomology ring $(H^*(L;\mathbb{Z}_2),\cup)=(E^1_{*-m,*}(L,\phi;P),\cup_1)$ of the monotone Lagrangian sub–manifold L with embedding $i\colon L\to P$.

Remark There is a general approach for the A_∞–structure on the symplectic Floer cohomology of Lagrangians with Novikov ring coefficients in [7]. The A_m–multiplicative structure is defined by pair–of–pants construction which is quite a complicated and hard in terms of computation. Our cup–product is induced from the usual cup-product of $H^*(L;\mathbb{Z}_2)$ and incorporated with those higher differentials. Our construction and the cup-product structure have more algebraic topology techniques in terms of computation.

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References

[1] S. Cappell, R. Lee and E. Miller, *On the Maslov index*, Comm. Pure. Appl. Math. **47**, 121–186 (1994).

[2] Y. Chekanov, *Hofer's symplectic energy and Lagrangian intersections*, Contact and Symplectic Geometry (Cambridge 1994), 296–306, Publ. Newton Inst. **8** (1996).

[3] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28**, 513–547, (1988).

[4] A. Floer, *Cuplength estimates on Lagrangian intersections*, Comm. Pure Appl. Math. **42**, 335–356, (1989).

[5] A. Floer, *Symplectic fixed points and holomorphic spheres*, Comm. in Math. Phys. **120**, 575–611 (1989).

[6] R. Fintushel, R. Stern, *Integer graded instanton homology groups for homology three spheres*, Topology, **31**, 589–604, (1992).

[7] K. Fukaya, Y. Oh, Ohta and K. Ono, *Lagrangian intersection Floer theory – anomaly and obstruction*, Kyoto University preprint 2000.

[8] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82**, 307–347, (1985).

[9] H. Hofer, *On the topological properties of symplectic maps*, Proc. of the Royal Soc. Edingburgh, **115**, 25–38 (1990).

[10] F. Lalonde and D. McDuff, *The geometry of symplectic energy*, Ann. Math, **141**, 349–371 (1995).

[11] R. Lee and W. Li, *Floer homologies for Lagrangian intersections and instantons*, submitted.

[12] W. Li, *Künneth formulae and Cross products for the symplectic Floer cohomology*, Topology Appl. **110**, 211–236, (2001).

[13] W. Li, *A module structure on the symplectic Floer cohomology*, Comm. Math. Phys. **211**, 137–151, (2000).

[14] G. Liu and G. Tian, *On the equivalence of multiplicative structures in Floer homology and quantum homology*, Acta Math. Sin. (Engl. Ser.) **15**, 53–80, (1999).

[15] J. Milnor, *Construction of Universal bundles, I*, Annals of Math. Vol **63**, 272–284, (1956).

[16] Y. G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I*, Comm. Pure Appl. Math. **46**, 949–993, (1993).

[17] Y. G. Oh, *Floer cohomology, spectral sequences and the Maslov class of Lagrangian embeddings*, Internat. Math. Res. Notices, 1996, No. 7, 305–346.

[18] Y. Ruan and G. Tian, *Bott-type symplectic Floer cohomology and its multiplication structures*, Math. Res. Lett. **2**, 203–219, (1995).

[19] D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45**, 1303–1360, (1992).

[20] E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

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