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# Implications of the Ganea Condition

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**Abstract** Suppose the spaces $X$ and $X \times A$ have the same Lusternik-Schnirelmann category: $\mathrm{cat}(X \times A) = \mathrm{cat}(X)$. Then there is a *strict* inequality $\mathrm{cat}(X \times (A \rtimes B)) < \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$ for every space $B$, provided the connectivity of $A$ is large enough (depending only on $X$). This is applied to give a partial verification of a conjecture of Iwase on the category of products of spaces with spheres.

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## Introduction

The product formula $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$ [1] is one of the most basic relations of Lusternik-Schnirelmann category. Taking $Y = S^r$, it implies that $\mathrm{cat}(X \times S^r) \leq \mathrm{cat}(X) + 1$ for any $r > 0$. In [5], Ganea asked whether the inequality can ever be strict in this special case. The study of the 'Ganea condition' $\mathrm{cat}(X \times S^r) = \mathrm{cat}(X) + 1$ has been, and remains, a formidable challenge to all techniques for the calculation of Lusternik-Schnirelmann category. In fact, it was only recently that techniques were developed which were powerful enough to identify a space which does not satisfy the Ganea condition [8] (see also [9, 12]). It is still not well understood exactly which spaces $X$ do not satisfy the Ganea condition, although it has been conjectured that they are precisely those spaces for which $\mathrm{cat}(X)$ is not equal to the related invariant $\mathrm{Qcat}(X)$ (see [14, 17]).

Since the failure of the Ganea condition appears to be a strange property for a space to have, it is reasonable to expect that such failure would have useful and interesting implications. In this paper we explore some of the implications of the equation $\mathrm{cat}(X \times A) = \mathrm{cat}(X)$ for general spaces $A$, and for $A = S^r$ in particular.

A brief look at the method of the paper [8] will help to put our results into proper perspective. The new techniques begin with the following question: if $Y = X \cup_f e^{t+1}$, the cone on $f : S^t \to X$, then how can we tell if $\mathrm{cat}(Y) > \mathrm{cat}(X)$? It is shown (see [9, Thm. 5.2] and [12, Thm. 3.6]) that, if $t \geq \dim(X)$, then $\mathrm{cat}(Y) = \mathrm{cat}(X) + 1$ if and only if a certain Hopf invariant $\mathcal{H}_s(f)$ (which is a set of homotopy classes) does not contain the trivial map $*$. It is also shown [9, Thm. 3.8] that if $* \in \Sigma^r \mathcal{H}_s(f)$, then $\mathrm{cat}(Y \times S^r) \leq \mathrm{cat}(X) + 1$. Thus $Y$ does not satisfy Ganea's condition if $* \notin \mathcal{H}_s(f)$, but there is at least one $h \in \mathcal{H}_s(f)$ such that $\Sigma^r h \simeq *$.

Of course, if $\Sigma^r h \simeq *$, then $\Sigma^{r+1} h \simeq *$ as well, and this suggests the following conjecture (formulated in [8, Conj. 1.4]):

**Conjecture** *If* $\mathrm{cat}(X \times S^r) = \mathrm{cat}(X)$, *then* $\mathrm{cat}(X \times S^{r+1}) = \mathrm{cat}(X)$.

In this paper we prove that this conjecture is true, provided $r$ is large enough.

**Theorem 1** *Suppose $X$ is a $(c-1)$-connected space and let $r > \dim(X) - c \cdot \mathrm{cat}(X) + 2$. If $\mathrm{cat}(X \times S^r) = \mathrm{cat}(X)$, then*

$$\mathrm{cat}(X \times S^t) = \mathrm{cat}(X)$$

*for all $t \geq r$.*

The conjecture remains open for small values of $r$.

Our main result is much more general: it shows how the equation $\mathrm{cat}(X \times A) = \mathrm{cat}(X)$ governs the Lusternik-Schnirelmann category of products of $X$ with a vast collection of other spaces.

**Theorem 2** *Let $X$ be a $(c-1)$-connected space and let $A$ be $(r-1)$-connected with $r > \dim(X) - c \cdot \mathrm{cat}(X) + 2$. If $\mathrm{cat}(X \times A) = \mathrm{cat}(X)$ then*

$$\mathrm{cat}(X \times (A \rtimes B)) < \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$$

*for every space $B$.*

Here $A \rtimes B = (A \times B)/B$ is the half-smash product of $A$ with $B$. When $A$ is a suspension, the half-smash product decomposes as $A \rtimes B \simeq A \vee (A \wedge B)$ (see, for example, [12, Lem. 5.9]), so we obtain the following.

**Corollary** *Under the conditions of Theorem 2, if $A$ is a suspension, then*

$$\mathrm{cat}(X \times (A \wedge B)) = \mathrm{cat}(X)$$

*for every space $B$.*

Our partial verification of the conjecture is an immediate consequence of this corollary: it the special case $A = S^r$ and $B = S^{t-r}$.

**Organization of the paper** In Section 1 we recall the necessary background information on homotopy pushouts, cone length and Lusternik-Schnirelmann category. We introduce an auxiliary space and establish its important properties in Section 2. The proof of Theorem 2 is presented in Section 3.

# 1 Preliminaries

In this paper all spaces are based and have the pointed homotopy type of CW complexes; maps and homotopies are also pointed. We denote by $*$ the one point space and any nullhomotopic map. Much of our exposition uses the language of homotopy pushouts; we refer to [11] for the definitions and basic properties.

## 1.1 Homotopy Pushouts

We begin by recalling some basic facts about homotopy pushout squares. We call a sequence $A \to B \to C$ a *cofiber sequence* if the associated square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$$

is a homotopy pushout square. The space $C$ is called the *cofiber* of the map $f$. One special case that we use frequently is the *half-smash product* $A \rtimes B$, which is the cofiber of the inclusion $B \to A \times B$.

Finally, we recall the following result on products and homotopy pushouts.

**Proposition 3** *Let $X$ be any space. Consider the squares*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times A \arrow[r] \arrow[d] & X \times B \arrow[d] \\ X \times C \arrow[r] & X \times D. \end{tikzcd}$$

*If the first square is a homotopy pushout, then so is the second.*

**Proof** This follows from Theorem 6.2 in [15]. □

## 1.2 Cone Length and Category

A *cone decomposition* of a space $Y$ is a diagram of the form

$$\begin{tikzcd} L_0 \arrow[d] & L_1 \arrow[d] & & L_{k-1} \arrow[d] & \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_{k-1} \arrow[r] & Y_k \end{tikzcd}$$

in which $Y_0 = *$, each sequence $L_i \to Y_i \to Y_{i+1}$ is a cofiber sequence, and $Y_k \simeq Y$; the displayed cone decomposition has *length* $k$. The *cone length* of $Y$, denoted $\mathrm{cl}(Y)$, is defined by

$$\mathrm{cl}(Y) = \begin{cases} 0 & \text{if } Y \simeq * \\ \infty & \text{if } Y \text{ has no cone decomposition, and} \\ k & \text{if the shortest cone decomposition of } Y \text{ has length } k. \end{cases}$$

The Lusternik-Schnirelmann category of $X$ may be defined in terms of the cone length of $X$ by the formula

$$\mathrm{cat}(X) = \inf\{\mathrm{cl}(Y) \mid X \text{ is a homotopy retract of Y}\}.$$

Berstein and Ganea proved this formula in [3, Prop. 1.7] with $\mathrm{cl}(Y)$ replaced by the strong category of $Y$; the formula above follows from another result of Ganea — strong category is equal to cone length [7]. It follows directly from this definition that if $X$ is a homotopy retract of $Y$, then $\mathrm{cat}(X) \leq \mathrm{cat}(Y)$. The reader may refer to [10] for a survey of Lusternik-Schnirelmann category.

The category of $X$ can be defined in another way that is essential to our work. Begin by defining the $0^{\text{th}}$ Ganea fibration sequence $F_0(X) \longrightarrow G_0(X) \xrightarrow{p_0} X$ to be the familiar path-loop fibration sequence $\Omega(X) \longrightarrow \mathcal{P}(X) \longrightarrow X$. Given the $n^{\text{th}}$ Ganea fibration sequence

$$F_n(X) \longrightarrow G_n(X) \xrightarrow{p_n} X,$$

let $\overline{G}_{n+1}(X) = G_n(X) \cup CF_n(X)$ be the cofiber of $p_n$ and define $\overline{p}_{n+1} : \overline{G}_{n+1}(X) \to X$ by sending the cone to the base point of $X$. The $(n+1)^{\text{st}}$ Ganea fibration $p_{n+1} : G_{n+1}(X) \to X$ results from converting the map $\overline{p}_{n+1}$ to a fibration. The following result is due to Ganea (cf. Svarc).

**Theorem 4** *For any space $X$,*

(a) $\mathrm{cl}(G_n(X)) \leq n$,

(b) *the map $p_n : G_n(X) \to X$ has a section if and only if* $\mathrm{cat}(X) \leq n$, *and*

(c) $F_n(X) \simeq (\Omega(X))^{*(n+1)}$, *the* $(n+1)$*-fold join of* $\Omega X$ *with itself.*

**Proof** Assertion (a) follows immediately from the construction. For parts (b) and (c), see [6]; these results also appear, from a different point of view, in [16]. □

## 2 An Auxilliary Space

Let $\widetilde{G}_n$ denote the homotopy pushout in the square

$$\begin{tikzcd} G_{n-1}(X) \arrow[r, hook, "i_1"] \arrow[d] & G_{n-1}(X) \times A \arrow[d] \\ G_n(X) \arrow[r] & \widetilde{G}_n. \end{tikzcd}$$

The maps $p_n : G_n(X) \to X$ and $1_A : A \to A$ piece together to give a map $\widetilde{p}_n : \widetilde{G}_n \to X \times A$. The space $\widetilde{G}_n$ and the map $\widetilde{p}_n$ play key roles in the forthcoming constructions; this section is devoted to establishing some of their properties.

### 2.1 Category Properties of $\widetilde{G}_n$

We begin by estimating the category of $\widetilde{G}_n$.

**Proposition 5** *For any noncontractible* $A$ *and* $n > 0$, $\mathrm{cat}(\widetilde{G}_n) < n + \mathrm{cat}(A)$.

**Proof** For simplicity in this proof, we write $F_i$ for $F_i(X)$ and $G_i$ for $G_i(X)$. Let $\mathrm{cat}(A) = k$, so $A$ is a retract of another space $A'$ with $\mathrm{cl}(A') = k$. Let $\widetilde{G}'_n = G_n \cup G_{n-1} \times A'$; clearly $\widetilde{G}_n$ is a homotopy retract of $\widetilde{G}'_n$ and so it suffices to show that $\mathrm{cl}(\widetilde{G}'_n) < n + k$. Let

$$\begin{tikzcd} L_0 \arrow[d] & L_1 \arrow[d] & & L_{k-1} \arrow[d] & \\ A'_0 \arrow[r] & A'_1 \arrow[r] & \cdots \arrow[r] & A'_{k-1} \arrow[r] & A'_k \end{tikzcd}$$

be a cone decomposition of $A'$. We will also use the cone decomposition of $G_n$ given by the cofiber sequences $F_{i-1} \to G_{i-1} \to G_i$. According to a result of Baues [2] (see also [13, Prop. 2.9]), for each $i$ and $j$ there is a cofiber sequence

$$F_{i-1} * L_{j-1} \longrightarrow G_i \times A'_{j-1} \cup G_{i-1} \times A'_j \longrightarrow G_i \times A'_j.$$

Now define subspaces $W_s \subseteq \widetilde{G}'_n$ by the formula

$$W_s = \begin{cases} \bigcup_{i+j=s} G_i \times A'_j & \text{if } s \leq n \\ G_n \times A'_0 \cup \left( \bigcup_{\substack{i+j=s \\ i<n}} G_i \times A'_j \right) & \text{if } s > n \end{cases}$$

with the understanding that $A'_j = A'_k$ for all $j \geq k$. The cofiber sequences guaranteed by Baues' theorem can be pieced together with the given cone decompositions of $A'$ and $G_n$ to give the cofiber sequences

$$F_s \vee L_s \vee \left( \bigvee_{\substack{i+j=s-1 \\ i<n-1}} F_i * L_j \right) \longrightarrow W_s \longrightarrow W_{s+1}$$

for each $s < \min\{n, k\}$; when $s \geq n$ we alter the cobase of the cofiber sequence by removing the $F_s$ summand, and when $s \geq k$ we must remove the summand $L_s$. Since $\widetilde{G}'_n = W_{n+k-1}$, we have the result. □

Next, we show that the map $\widetilde{p}_n : \widetilde{G}_n \to X \times A$ has one of the category-detecting properties of $p_n : G_n(X \times A) \to X \times A$.

**Proposition 6** *If* $\mathrm{cat}(X \times A) = \mathrm{cat}(X) = n$, *then* $\widetilde{p}_n$ *has a homotopy section.*

**Proof** We follow [4] (see also [8, Thm. 2.7]) and define

$$\widehat{G}'_n(X \times A) = \bigcup_{i+j=n} G_i(X) \times G_j(A).$$

There is a natural map $h : \widehat{G}'_n(X \times A) \to X \times A$ induced by the Ganea fibrations over $X$ and $A$. According to [4, Thm. 2.3], $\mathrm{cat}(X \times A) = n$ if and only if $h$ has a homotopy section.

Each map $G_i(X) \times G_j(A) \to X \times A$ (with $j > 0$) factors through $G_i(X) \times A$ and these factorizations are compatible because $p_{i+1}$ extends $p_i$. So $h$ factors as $\widehat{G}'_n(X \times A) \to \widetilde{G}_n \to X \times A$. Therefore, if $\mathrm{cat}(X \times A) = n$, then $h$, and hence $\widetilde{p}_n$, has a section. □

## 2.2 Comparison of $\widetilde{G}_n$ with $G_n(X) \times A$

Let $j : \widetilde{G}_n \to G_n(X) \times A$ denote the natural inclusion map.

**Proposition 7** *Assume that* $X$ *is* $(c-1)$*-connected and that* $A$ *is* $(r-1)$*-connected. Then the homotopy fiber* $F$ *of the map* $j$ *is* $(nc+r-2)$*-connected.*

**Proof** There is a cofiber sequence

$$\widetilde{G}_n \xrightarrow{j} G_n(X) \times A \longrightarrow \Sigma F_{n-1}(X) \wedge A.$$

Therefore the homotopy fiber of $j$ has the same connectivity as the space $\Omega(\Sigma F_{n-1}(X) \wedge A) \simeq \Omega(\Omega(X)^{*n} * A)$, namely $nc + r - 2$. □

**Corollary 8** *Assume* $\dim(Z) < nc + r - 2$ *and let* $f, g : Z \to \widetilde{G}_n$. *Then* $f \simeq g$ *if and only if* $jf \simeq jg$.

The proof is standard, and we omit it.

## 2.3 New Sections from Old Ones

Suppose that $\mathrm{cat}(X) = \mathrm{cat}(X \times A) = n$. By Proposition 6 there is a section $\sigma : X \times A \to \widetilde{G}_n$ of the map $\widetilde{p}_n : \widetilde{G}_n \to X \times A$. Define a new map $\sigma' : X \to G_n(X)$ by the diagram

$$\begin{tikzcd} X \arrow[rr, "\sigma'"] \arrow[d, "i_1"'] & & G_n(X) \\ X \times A \arrow[r, "\sigma"] & \widetilde{G}_n \arrow[r, hook, "j"] & G_n(X) \times A. \arrow[u, "\mathrm{pr}_1"'] \end{tikzcd}$$

We need the following basic properties of $\sigma'$.

**Proposition 9** *If* $\mathrm{cat}(X \times A) = \mathrm{cat}(X) = n$, *then*

(a) $\sigma'$ *is a homotopy section of the projection* $p_n : G_n(X) \to X$, *and*

(b) *if* $X$ *is* $(c-1)$*-connected and* $A$ *is* $(r-1)$*-connected with* $r > \dim(X) - nc + 2$, *then the diagram*

$$\begin{tikzcd} X \arrow[r, "\sigma'"] \arrow[d, "i_1"'] & G_n(X) \arrow[d, hook, "k"] \\ X \times A \arrow[r, "\sigma"] & \widetilde{G}_n \end{tikzcd}$$

*commutes up to homotopy.*

**Proof** First consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma'"] \arrow[d, "i_1"'] & G_n(X) \arrow[r, equal] \arrow[d, dotted, "k"'] & G_n(X) \arrow[r, "p_n"] & X \\
X \times A \arrow[r, "\sigma"] \arrow[drr, "1_{X\times A}"'] & \widetilde{G}_n \arrow[r, "j"] & G_n(X) \times A \arrow[r, "\mathrm{pr}_1"] \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "p_n \times 1_A"] & G_n(X) \arrow[u, "p_n"'] \arrow[d, "p_n"] \\
 & & X \times A \arrow[r, "\mathrm{pr}_1"] & X.
\end{tikzcd}$$

The diagram of solid arrows is evidently commutative. Therefore, we have $p_n \circ \sigma' \simeq \mathrm{pr}_1 \circ 1_{X\times A} \circ i_1 \simeq 1_X$, proving (a).

To prove (b) we have to show that two maps $X \to \widetilde{G}_n$ are homotopic. Since $\dim(X) < nc + r - 2$, it suffices by Corollary 8 to show that $j \circ (\sigma \circ i_1) \simeq j \circ (k \circ \sigma')$. Since $\mathrm{pr}_2 \circ j \circ (\sigma \circ i_1) \simeq * \simeq \mathrm{pr}_2 \circ j \circ (k \circ \sigma')$, it remains to show that $\mathrm{pr}_1 \circ j \circ (\sigma \circ i_1) \simeq \mathrm{pr}_1 \circ j \circ (k \circ \sigma')$. But both of these maps are homotopic to $\sigma'$. □

# 3 Proof of the Main Theorem

**Proof of Theorem 2** We have $n = \mathrm{cat}(X) = \mathrm{cat}(X \times A)$ by hypothesis. It follows from Proposition 6 that there is a section $\sigma : X \times A \to \widetilde{G}_n$ of the map $\widetilde{p}_n : \widetilde{G}_n \to X \times A$. We then get the section $\sigma' : X \to G_n(X)$ that was constructed and studied in Section 2.3.

Consider the following diagram and the induced sequence of maps on the homotopy pushouts of the rows

$$\begin{tikzcd}
(X \times A) \times B \arrow[d, "\sigma\times 1_B"', "\simeq s"] & X \times B \arrow[l, "i_1 \times 1_B"'] \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\sigma' \times 1_B"] & X \arrow[d, "\sigma'"] & & Y \arrow[d] \\
\widetilde{G}_n \times B \arrow[d, "\widetilde{p}_n \times 1_B"'] & G_n(X) \times B \arrow[l, "k \times 1_B"'] \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p_n \times 1_B"] & G_n(X) \arrow[d, "p_n"] & {} \arrow[r, dotted, "\text{homotopy}", "\text{pushout}"'] & P \arrow[d] \\
(X \times A) \times B & X \times B \arrow[l, "i_1 \times 1_B"'] \arrow[r, "\mathrm{pr}_1"] & X & & Y.
\end{tikzcd}$$

Proposition 9 implies that the upper left square commutes up to homotopy. Since $i_1 \times 1_B$ is a cofibration, we can apply homotopy extension and replace the map $\sigma \times 1_B : (X \times A) \times B \to \widetilde{G}_n \times B$ with a homotopic map $s$ which makes that square strictly commute. All other squares are strictly commutative as they stand.

Since the composites $(\widetilde{p}_n \times 1_B) \circ (\sigma' \times 1_B)$ and $p_n \circ \sigma'$ are the identity maps and $(\widetilde{p}_n \times 1_B) \circ s$ is a homotopy equivalence, each vertical composite in the modified diagram is a homotopy equivalence. Thus $Y$ is a homotopy retract of $P$, and consequently $\mathrm{cat}(Y) \leq \mathrm{cat}(P)$.

The space $Y$ is the homotopy pushout of the top row in the diagram, which is the product of the homotopy pushout diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & * \arrow[d] \\ A \times B \arrow[r] & A \rtimes B \end{tikzcd}$$

with the space $X$. Therefore $Y \simeq X \times (A \rtimes B)$ by Proposition 3. Since $Y$ is a homotopy retract of $P$, it follows that

$$\mathrm{cat}(X \times (A \rtimes B)) \leq \mathrm{cat}(P),$$

the proof will be complete once we establish that $\mathrm{cat}(P) < \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$. This is accomplished in Lemma 10, which is proved below. □

**Lemma 10** *The space $P$ constructed in the proof of Theorem 2 satisfies* $\mathrm{cat}(P) \leq \mathrm{cl}(P) < \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$.

**Proof** The space $\widetilde{G}_n$ is defined by the homotopy pushout square

$$\begin{tikzcd} G_{n-1}(X) \arrow[r] \arrow[d] & G_n(X) \arrow[d] \\ G_{n-1}(X) \times A \arrow[r] & \widetilde{G}_n. \end{tikzcd}$$

Take the product of this square with the space $B$ and adjoin the homotopy pushout square that defines $P$ to obtain the diagram

$$\begin{tikzcd} G_{n-1}(X) \times B \arrow[r] \arrow[d] & G_n(X) \times B \arrow[r] \arrow[d] & G_n(X) \arrow[d] \\ G_{n-1}(X) \times A \times B \arrow[r] & \widetilde{G}_n \times B \arrow[r] & P. \end{tikzcd}$$

By [11, Lem. 13], the outer square

$$\begin{tikzcd} G_{n-1}(X) \times B \arrow[r] \arrow[d] & G_n(X) \arrow[d] \\ G_{n-1}(X) \times A \times B \arrow[r] & P \end{tikzcd}$$

is also a homotopy pushout square. The top map is the composite

$$G_{n-1}(X) \times B \xrightarrow{\mathrm{pr}_1} G_{n-1}(X) \hookrightarrow G_n(X),$$

and so we have a new factorization into homotopy pushout squares:

$$\begin{tikzcd}
G_{n-1}(X) \times B \arrow[r, "\mathrm{pr}_1"] \arrow[d] & G_{n-1}(X) \arrow[r] \arrow[d] & G_n(X) \arrow[d] \\
G_{n-1}(X) \times A \times B \arrow[r] & L \arrow[r] & P.
\end{tikzcd}$$

To identify the space $L$, observe that the left square is simply the product of the space $G_{n-1}(X)$ with the homotopy pushout square

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & * \arrow[d] \\
A \times B \arrow[r] & A \rtimes B.
\end{tikzcd}$$

By Proposition 3, $L \simeq G_{n-1}(X) \times (A \rtimes B)$. Hence the right-hand square is the homotopy pushout square

$$\begin{tikzcd}
G_{n-1}(X) \arrow[r] \arrow[d] & G_n(X) \arrow[d] \\
G_{n-1}(X) \times (A \rtimes B) \arrow[r] & P.
\end{tikzcd}$$

Therefore $\mathrm{cl}(P) \leq \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$ by Proposition 5. □

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