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Higher degree Galois covers of $\mathbb{CP}^1 \times T$

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Abstract Let T be a complex torus, and X the surface $\mathbb{CP}^1 \times T$. If T is embedded in $\mathbb{CP}^{n-1}$ then X may be embedded in $\mathbb{CP}^{2n-1}$. Let X_{Gal} be its Galois cover with respect to a generic projection to $\mathbb{CP}^2$. In this paper we compute the fundamental group of X_{Gal}, using the degeneration and regeneration techniques, the Moishezon–Teicher braid monodromy algorithm and group calculations. We show that $\pi_1(X_{\mathrm{Gal}}) = \mathbb{Z}^{4n-2}$.

AMS Classification 14Q10; 14J80, 32Q55

Keywords Galois cover, fundamental group, generic projection, Sieberg–Witten invariants

1 Overview

Let T be a complex torus embedded in $\mathbb{CP}^{n-1}$. The surface $X = \mathbb{CP}^1 \times T$ can be embedded into projective space using the Segre embedding from $\mathbb{CP}^1 \times \mathbb{CP}^{n-1} \to \mathbb{CP}^{2n-1}$. We compute the fundamental group of the Galois cover of X with respect to a generic projection f from $X \subset \mathbb{CP}^{2n-1}$ to $\mathbb{CP}^2$. This map has degree $2n$. The Galois cover can be defined as the closure of the $2n$–fold fibered product $X_{\mathrm{Gal}} = \overline{X \times_f \cdots \times_f X - \Delta}$ where Δ is the generalized diagonal. The closure is necessary because the branched fibers are excluded when Δ is omitted.

Since the induced map $X_{\mathrm{Gal}} \to \mathbb{CP}^2$ has the same branch curve S as $f\colon X \to \mathbb{CP}^2$, the fundamental group $\pi_1(X_{\mathrm{Gal}})$ is related to $\pi_1(\mathbb{CP}^2 - S)$. In fact it is a normal subgroup of $\widetilde{\Pi}_1$, the quotient of $\pi_1(\mathbb{CP}^2 - S)$ by the normal subgroup generated by the squares of the standard generators. In this paper we employ braid monodromy techniques, the van Kampen theorem and various computational methods of groups to compute a presentation for the quotient $\widetilde{\Pi}_1$ from which $\pi_1(X_{\mathrm{Gal}})$ can be derived. Our main result is that $\pi_1(X_{\mathrm{Gal}}) = \mathbb{Z}^{4n-2}$ (Theorem 4.17). This extends a previous result for $n = 3$ proven in [1].

The paper follows the structure of [1]. In Section 2 we describe the degeneration of the surface $X = \mathbb{CP}^1 \times T$ and the degenerated branch curve. In Section 3 we regenerate the branch curve and its braid monodromy factorization to get a presentation for $\pi_1(\mathbb{C}^2 - S)$, the fundamental group of the complement of the regenerated branch curve in $\mathbb{C}^2$. In Section 4 we compute $\pi_1(X_{\mathrm{Gal}})$ as the kernel of a permutation monodromy map using the Reidmeister–Schreier method.

2 Degeneration of $\mathbb{CP}^1 \times T$

To compute the braid monodromy of the branch curve S, we degenerate X to a union of projective planes X_0. The branch curve degenerates to a union of lines S_0 which are the images of the intersections of the planes of X_0. We use the following degeneration: Embedded as a degree n elliptic normal curve in $\mathbb{CP}^{n-1}$, the torus T degenerates to a cycle of n projective lines. Under the Segre embedding X degenerates to a cycle of n quadrics, $Q_i \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. Call this space X_1, see Figure 1.

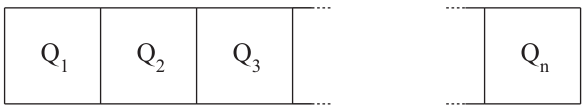

Figure 1: The space X_1

Each quadric in X_1 shares a projective line with its two neighbors. As a cycle of quadrics, it is understood that Q_1 and Q_n intersect as well, though it is not clearly indicated in Figure 1. Hence the left and right edges should be identified to make an n–prism. Each quadric in X_1 can be further degenerated to a union of two projective planes. In Figure 2 this is represented by a diagonal line which divides each square into two triangles, each representing $\mathbb{CP}^2$. We shall refer to this diagram as the simplicial complex of X_0.

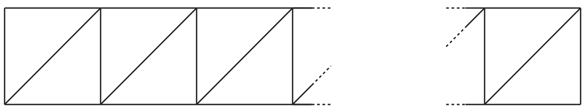

Figure 2: The simplicial complex X_0

A common edge between two triangles represents the intersection line of the two corresponding planes. The union of the intersection lines in X_0 is the ramification curve of $f_0\colon X_0 \to \mathbb{CP}^2$, denoted by R_0. Let $S_0 = f_0(R_0)$ be the degenerated branch curve. It is a line arrangement, composed of the images of all the intersection lines.

Each vertex of the simplicial complex represents an intersection point of three planes. These are the singular points of R_0. Each of these vertices is called a 3–point (reflecting the number of planes which meet there).

The vertices may be given any convenient enumeration. We have chosen left to right, bottom to top enumeration, see Figure 3. Because the left and right edges are identified, so are the corresponding vertices.

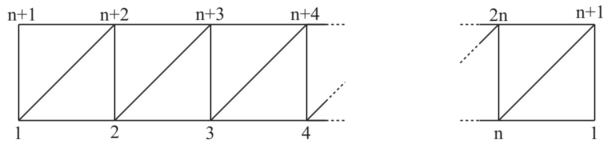

Figure 3: The enumeration of vertices

In order to use a result concerning the monodromy of dual to generic line arrangements [2] we number the edges based upon the enumeration of the vertices using reverse lexicographic ordering: if L_1 and L_2 are two lines with end points α_1, β_1 and α_2, β_2 respectively $(\alpha_1 < \beta_1, \alpha_2 < \beta_2)$, then $L_1 < L_2$ iff $\beta_1 < \beta_2$, or $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$. The resulting enumeration is shown in Figure 4. The horizontal lines at the top and bottom do not represent intersections of planes and hence are not numbered.

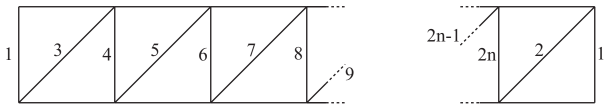

Figure 4: The enumeration of lines

The triangles can be numbered in any order, so we choose an enumeration which will simplify future computations.

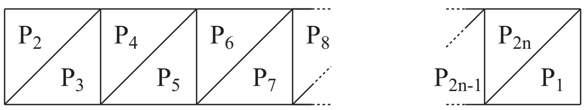

Figure 5: The enumeration of planes

The braid monodromy of S_0 is easily computed since it is a subset of a dual to generic line arrangement [2]. From this, the braid monodromy of the full branch curve S can be regenerated.

3 Regeneration of the branch curve

The degeneration of X through X_1 to X_0 takes place in $\mathbb{CP}^{2n-1}$ so at every step of the process the generic projection $\mathbb{CP}^{2n-1} \to \mathbb{CP}^2$ restricts to a map $f_i\colon X_i \to \mathbb{CP}^2$. Each map f_i has its own branch curve S_i. Starting from the degenerated branch curve S_0, we reverse the steps of the degeneration of X to regenerate the braid monodromy of S.

3.1 The braid monodromy of S_0

We have enumerated the $2n$ planes $P_1, \dots, P_{2n}$ which comprise X_0, the $2n$ intersection lines $\hat{L}_1, \dots, \hat{L}_{2n}$ which comprise R_0, and their $2n$ intersection points $\hat{V}_1, \dots, \hat{V}_{2n}$. Let L_i and V_k denote the projections of $\hat{L}_i$ and $\hat{V}_k$ to $\mathbb{CP}^2$ by the map f_0. Clearly the degenerated branch curve $S_0 = \bigcup_{i=1}^{2n} L_i$. Each of the V_k is an intersection point of S_0. Additionally, every pair of lines $\hat{L}_i$ and $\hat{L}_j$ which do not intersect in R_0 must have a simple intersection when projected to $S_0 \subset \mathbb{CP}^2$. Thus the braid monodromy of S_0 consists of cycles Δ_k^2 corresponding to the V_k, and full twists D_{ij} corresponding to the simple intersection $L_i \cap L_j$.

Since S_0 is a sub-arrangement of a dual to generic arrangement the precise forms of the braids Δ_k^2 and D_{ij} can be found using Moishezon's result [2], summarized in Theorem IX.2.1 of [3].

The braids take place in a generic fiber $\mathbb{C}_u$ of the projection $\pi\colon \mathbb{CP}^2 \to \mathbb{CP}^1$. For each line we will refer to the intersection $L_i \cap \mathbb{C}_u$ as the point i. These are the points which are braided in $\mathbb{C}_u$. Since they are so closely related we may often use the concepts of the line L_i and the point i interchangeably.

Recall also that each line L_i is initially denoted by a pair of numbers indicating which two vertices it contains. The lines are sorted based on the second vertex then the first, producing the single index i. We will need to refer to the pairs of vertices associated to the lines L_i and L_j in defining the braids D_{ij}.

Assume $i < j$. From [2], [3] we know that each D_{ij} is a particular full twist of the points i and j in which i is brought next to j by passing over most of the intervening points, but under those points which share the same second vertex as j. The path described is denoted by $\tilde{z}_{ij}$ and the corresponding half-twist is called $\tilde{Z}_{ij}$ so we may say that $D_{ij} = \tilde{Z}_{ij}^2$. In the context of our enumeration of line this means that when j is a vertical (see Figure 4) then the path passes under the preceding diagonal. With the exception of lines 1 and 2, the odds are diagonal and the evens are vertical. Representative examples are shown in Figure 6.

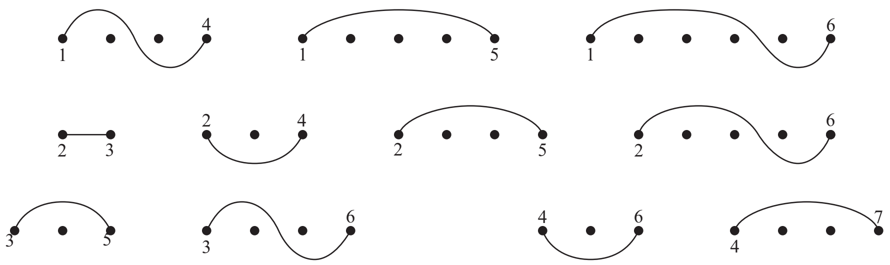

Figure 6: Full twists D_{ij}

Each Δ_k^2 is a particular full twist of all the points corresponding to lines through V_k in which the points pass under any intervening points. In our example each V_k is the intersection of exactly 2 lines. Let i and j be the indices of the two lines through V_k. Let $\underline{z}_{ij}$ denote the path from i to j under all the points in between and $\underline{Z}_{ij}$ the corresponding half-twist. In this terminology $\Delta_k^2 = \underline{Z}_{ij}^2$. Since most of the pairs of lines meeting at the V_k are consecutively numbered there aren't many intervening points to worry about. The exceptions are lines 1 and 3 intersecting at V_1 and lines 2 and $2n$ intersecting at V_n.

Using the braid monodromy of S_0 as a template, the braid monodromy of S can be obtained according to a few simple regeneration rules.

3.2 The braid monodromy of S

When X_0 is regenerated to X, each of the lines L_i in S_0 divides into two sheets of the branch curve S. So in the generic fiber $\mathbb{C}_u$ the point i divides into two points which we shall call i and i'. As each intersection point of S_0 splits into a collection of singularities of S, the associated braids D_{ij} and Δ_k^2 also split into collections of braids in predictable ways. (Basic regeneration rules are proven in [4]. Application to the specific types of singularities found here is as in [1].)

Assume $i < j$. For each pair of lines $\hat{L}_i$ and $\hat{L}_j$ which do not intersect in R_0 there is a simple node in S_0 with monodromy D_{ij} where L_i and L_j intersect. When L_i and L_j divide into two sheets the resulting figure has four nodes, each with its own monodromy. So each $D_{ij} = \tilde{Z}_{ij}^2$ becomes the four braids: $\tilde{Z}_{ij}^2$, $\tilde{Z}_{ij'}^2$, $\tilde{Z}_{i'j}^2$, and $\tilde{Z}_{i'j'}^2$, which we summarize with the symbol $\tilde{Z}_{ii',jj'}^2$. Figure 7 shows two representative examples.

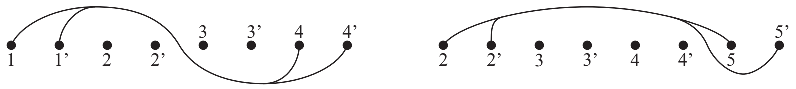

Figure 7: Regenerated collections of braids $\tilde{Z}^2_{11',44'}$ and $\tilde{Z}^2_{22',55'}$

Each $\hat{V}_k$ is the intersection of three planes in X_0. For that reason this type of singularity is called a 3–point. These points regenerate in two steps. First in X_1 pairs of planes are regenerated to quadrics so in the branch curve S_1 each diagonal line becomes a conic, tangent to the adjacent vertical lines. Near each 3–point this creates a branch point and a point of tangency. Then when X is regenerated, the vertical lines divides, and the points of tangency become three cusps each. Hence the braid Δ^2_k becomes braids. The branch point yields a half-twist, and each cusp yields a 3/2-twist. Let i and j respectively be the indices of the vertical and diagonal lines meeting at V_k. The symbol $Z_{jj'(i)}$ denotes the half-twist and $Z^3_{ii',j'}$ denotes the three 3/2-twists. An illustrative example is provided in Figure 8

Figure 8: Regenerated braids for V_1: $Z^3_{11',3'}$ and $Z_{33'(1)}$

These braids comprise the full braid monodromy of the branch curve S. To check this recall that the product, suitably ordered, of all the monodromy gives a factorization of Δ^2_S. Since S has degree $4n$, Δ^2_S has degree $4n(4n-1) = 16n^2 - 4n$. In the simplicial complex there are $2n$ different 3–points. Each produces 3 cusps and a branch point for a combined degree of $2n \cdot (3 \cdot 3 + 1) = 20n$. There are also $2n^2 - 3n$ incidental intersections which do not appear in the simplicial complex. Each produces 4 nodes for a combined degree of $(2n^2 - 3n)(4 \cdot 2) = 16n^2 - 24n$. So together they have the required total degree of $16n^2 - 4n$.

3.3 The fundamental group of $\mathbb{C}^2 - S$

Let X^{Aff} denote $f^{-1}(\mathbb{C}^2)$, the portion of X lying above some affine $\mathbb{C}^2 \subset \mathbb{CP}^2$. We will use the van Kampen Theorem [5] to produce a presentation for the fundamental group of the affine complement of the branch curve $\pi_1(\mathbb{C}^2 - S)$ from the braid monodromy factorization computed in Subsection 3.2. Since the braid monodromy factorization is only defined up to Hurwitz equivalence of factorizations it is clear that equivalent factorization must give the same fundamental group. So any relations obtained from equivalent factorizations may be included without changing the group. To get extra relations we will be making liberal use of the following two invariance theorems.

Theorem 3.1 (Invariance Theorem) *The braid monodromy factorization of S is invariant under conjugation by any product of half-twists of the form $Z_{jj'}$.*

Note that the braids $Z_{ii'}$ and $Z_{jj'}$ commute for all i and j because the path from i to i' doesn't intersect the path from j to j'. Theorem 3.1 is proven in [4] and [1]. Philosophically it is true because going around a neighborhood of the degenerated surface X_0 causes the two sheets associated to L_j to interchange, which conjugates all braids by $Z_{jj'}$. The $Z_{jj'}$ can be used independently because in this figure the lines L_i can be regenerated independently.

Theorem 3.2 (Conjugation Theorem) *The braid monodromy factorization of S is invariant under complex conjugation in $\mathbb{C}_u$ if the order of factors is reversed.*

This theorem is proven in [4]. Philosophically it is true because complex conjugation in two dimensions is orientation preserving. Reversing the order of factors corresponds to complex conjugation in the other dimension.

By the van Kampen Theorem [5], there is a surjection from the fiber $\pi_1(\mathbb{C}_u - \{j,j'\})$ onto $\Pi_1 = \pi_1(\mathbb{C}^2 - S)$. The fundamental group of the fiber is freely generated by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{2n}$, where Γ_j and $\Gamma_{j'}$ are loops in $\mathbb{C}_u$ around j and j' respectively. These loops are explicitly constructed in [1]. Thus the images of $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{2n}$ are generators for Π_1. Without too much confusion we will refer to the images as $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{2n}$ as well. Each braid in the braid monodromy factorization of S induces a relation on Π_1 through its natural action on $\mathbb{C}_u - \{j,j'\}_{j=1}^{2n}$ [5].

Using the techniques described above we get a presentation for the affine complement of the branch curve S. For simplicity of notation we will be using the following shorthand:

$\Gamma_{(i)}$ stands for all of the conjugates of Γ_i by integer powers of $Z_{ii'}$. These include $...\Gamma_{i'}^{-1}\Gamma_i^{-1}\Gamma_{i'}\Gamma_i\Gamma_{i'},\ \Gamma_{i'}^{-1}\Gamma_i\Gamma_{i'},\ \Gamma_{i'},\ \Gamma_i,\ \Gamma_i\Gamma_{i'}\Gamma_i^{-1},\ \Gamma_i\Gamma_{i'}\Gamma_i\Gamma_{i'}^{-1}\Gamma_i^{-1}...$

$\Gamma_{ii'}$ stands for either Γ_i or $\Gamma_{i'}$.

$\Gamma_{\check{i}i'}$ stands for either $\Gamma_{i'}^{-1}\Gamma_i\Gamma_{i'}$ or $\Gamma_{i'}$.

$\Gamma_{j\hat{j}}$ stands for either Γ_j or $\Gamma_j\Gamma_{j'}\Gamma_j^{-1}$.

Theorem 3.3 *The group Π_1 is generated by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{2n}$ with the following relations:*

$$\begin{array}{rcll} [\Gamma_{(i)}, \Gamma_{(j)}] & = & 1 & \text{if the lines } i, j \text{ are disjoint in } X_0 \quad (1) \\ \Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} & = & \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)} & \text{if the lines } i, j \text{ intersect} \quad (2) \\ \Gamma_{j'}^{-1}\Gamma_i\Gamma_{i'}\Gamma_j\Gamma_{i'}^{-1}\Gamma_i^{-1} & = & 1 & \text{if vertical } i \text{ intersects diagonal } j. \quad (3) \end{array}$$

Proof The relations (3) come from the braids $Z_{jj'(i)}$. The relations from $Z^3_{ii',j'}$ include $\Gamma_{i'}\Gamma_{j'}\Gamma_{i'} = \Gamma_{j'}\Gamma_{i'}\Gamma_{j'}$ and two other variants. Using the Invariance Theorem 3.1 we get the triple relations (2) in their full generality. It remains to prove commutation relations (1).

For $i < j$, if j is odd then j is diagonal so $\tilde{Z}^2_{ij}$ only goes over the intervening points. As a result the complex conjugates of the regenerated braids give relations $[\Gamma_{ii'}, \Gamma_{j\hat{j}}] = 1$, see Figure 9. These four relations can easily be seen to generate the rest of $[\Gamma_{(i)}, \Gamma_{(j)}] = 1$.

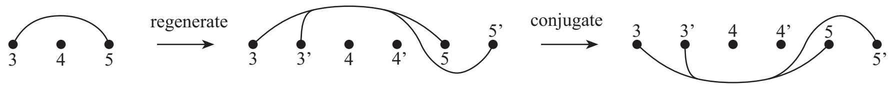

Figure 9

If j is even and $i = j-2$ then $\tilde{Z}^2_{ij}$ only goes under the intervening points. As a result the regenerated braids give relations $[\Gamma_{\check{i}i'}, \Gamma_{jj'}] = 1$, see Figure 10. Again these four relations generate all of $[\Gamma_{(i)}, \Gamma_{(j)}] = 1$.

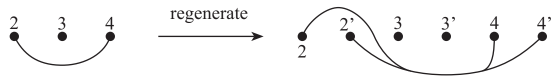

Figure 10

For the remainder of the commutators when j is even $\tilde{Z}^2_{ij}$ goes under $j-1$ and over all the other intervening points. The complex conjugates of the regenerated braids give relations $[\Gamma_{ii'}, \Gamma_{j-1}\Gamma_{j-1'}\Gamma_{j\hat{j}}\Gamma_{j-1'}^{-1}\Gamma_{j-1}^{-1}] = 1$, see Figure 11. Since $j-1$ is odd it normally follows from the arguments above that all $\Gamma_{(i)}$ commute with all $\Gamma_{(j-1)}$. Hence the commutator above simplifies to $[\Gamma_{ii'}, \Gamma_{j\hat{j}}] = 1$ which implies $[\Gamma_{(i)}, \Gamma_{(j)}] = 1$.

The one exception is when $i = 1$ and $j = 4$ because $\Gamma_{(1)}$ and $\Gamma_{(3)}$ satisfy triple relations rather than commutators. In this case, the regenerated braids from $\tilde{Z}^2_{14}$ without complex conjugation give $[\Gamma_{2'}^{-1}\Gamma_2^{-1}\Gamma_{\check{1}1'}\Gamma_2\Gamma_{2'}, \Gamma_{44'}] = 1$. Now the fact that $\Gamma_{(2)}$ and $\Gamma_{(4)}$ commute will finish the proof. □

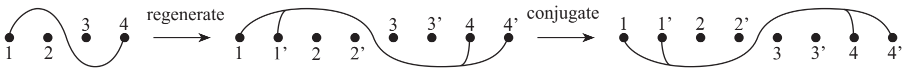

Figure 11

4 The Galois cover of X

4.1 The homomorphism ψ

$X^{\mathrm{Aff}} - f^{-1}(S)$ is a degree $2n$ covering space of $\mathbb{C}^2 - S$. Let

$$\psi\colon\ \pi_1(\mathbb{C}^2 - S) \to S_{2n}$$

be the permutation monodromy of this cover. We can compute ψ precisely by considering the degeneration of X to X_0. In X_0 the sheets are the planes P_k which we numbered in Figure 5. In X_0 there is no monodromy since $X_0 - R_0$ breaks up into a disjoint union so it is impossible to get from one sheet to another.

For a regeneration of X near X_0 the sheets of X are very close to the planes of X_0, so we can use the same numbering. As we have seen already, the ramification curve R locally has two pieces near each line $\hat{L}_i$ of R_0. The regeneration of the monodromy here looks very much like a node pulling apart into two branch points. In fact when restricted to the generic fiber $\mathbb{C}_u$ it is precisely the node i pulling apart into the two branch points i and i' with identical monodromy. If $\hat{L}_i$ is the intersection of P_k and P_ℓ then it is clear that $\psi(\Gamma_i) = \psi(\Gamma_{i'}) = (k\ \ell)$. Based on the enumerations of lines and planes in Figures 4 and 5 we have:

Definition 4.1 *The map $\psi\colon\ \pi_1(\mathbb{C}^2 - S) \to S_{2n}$ is given by*

$$\begin{aligned}
\psi(\Gamma_1) = \psi(\Gamma_{1'}) &= (1\ 2), \\
\psi(\Gamma_i) = \psi(\Gamma_{i'}) &= (i{-}1\ i) \qquad \text{for } 3 \le i \le 2n, \\
\psi(\Gamma_2) = \psi(\Gamma_{2'}) &= (2n\ 1)
\end{aligned}$$

The reader may wish to check that ψ is well defined by testing the relations given in Theorem 3.3), but this is of course guaranteed by the theory. From the definitions 4.1 we see that ψ is surjective. In fact the images of $\Gamma_2, \cdots, \Gamma_{2n}$ generate S_{2n}.

4.2 The fundamental groups of $X_{\mathrm{Gal}}^{\mathrm{Aff}}$ and X_{Gal} in terms of ψ

The sheets of the Galois cover of X are labeled by permutations of the $2n$ sheets of X. Therefore an element of $\pi_1(\mathbb{C}^2 - S)$ lifts to a closed path in $X_{\mathrm{Gal}}^{\mathrm{Aff}}$ if and only if it has no permutation monodromy, which means it is in the kernel of ψ. Among these are elements Γ_j^2 and $\Gamma_{j'}^2$ which lift to double loops around simple ramification curves. These double loops vanish in $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$. We may define a quotient

$$\widetilde{\Pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S)}{\left\langle \Gamma_j^2, \Gamma_{j'}^2 \right\rangle} \tag{4}$$

in which the elements Γ_j^2 and $\Gamma_{j'}^2$ are forcibly killed. Since these elements are already in the kernel, ψ remains well defined on $\widetilde{\Pi}_1$. Let $\mathcal{A}$ be the kernel of $\psi\colon \widetilde{\Pi}_1 \to S_{2n}$. We have a short exact sequence sequence

$$1 \longrightarrow \mathcal{A} \longrightarrow \widetilde{\Pi}_1 \xrightarrow{\psi} S_{2n} \to 1. \tag{5}$$

With these observations it is clear that the fundamental group $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$ is isomorphic to $\mathcal{A}$. Based on the presentation of Π_1 given in Theorem 3.3 we can write a presentation for $\widetilde{\Pi}_1$

Theorem 4.2 *The group $\widetilde{\Pi}_1$ is generated by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{2n}$ with the following relations:*

$$\begin{aligned}
\Gamma_i^2 &= 1 && && (6)\\
\Gamma_{i'}^2 &= 1 && && (7)\\
[\Gamma_{(i)}, \Gamma_{(j)}] &= 1 && \textit{if the lines } i,j \textit{ are disjoint in } X_0 && (8)\\
\Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} &= \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)} && \textit{if the lines } i,j \textit{ intersect} && (9)\\
\Gamma_{j'}\Gamma_i\Gamma_{i'}\Gamma_j\Gamma_{i'}\Gamma_i &= 1 && \textit{if vertical } i \textit{ intersects diagonal } j. && (10)
\end{aligned}$$

$\Gamma_{(i)}$ stands for any odd length word in the infinite dihedral group $\langle \Gamma_i, \Gamma_{i'} \rangle$.

This result can be extended to the whole surface X_{Gal} by including the projective relation. $\pi_1(\mathbb{CP}^2 - S)$ has the same generators and relations as the affine $\pi_1(\mathbb{C}^2 - S)$ with one additional relation:

$$\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\cdots\Gamma_{2n}\Gamma_{2n'} = 1.$$

This relation is in the kernel of ψ so it does not interfere with the definition of ψ. If we add the projective relation to the presentation of $\widetilde{\Pi}_1$ then we can compute $\pi_1(X_{\mathrm{Gal}})$ as the kernel of ψ as before.

4.3 The splitting φ and the Reidmeister–Schreier method

It will be useful to have a splitting of the short exact sequence (5). Define the splitting map $\varphi\colon\ S_{2n} \to \widetilde{\Pi}_1$ as follows:

Definition 4.3 *The map* $\varphi\colon\ S_{2n} \to \widetilde{\Pi}_1$ *is given by*

$$\begin{aligned} \varphi(2n\ 1) &= \Gamma_2, \\ \varphi(i{-}1\ i) &= \Gamma_i \qquad \text{for } 3 \le i \le 2n. \end{aligned}$$

From the definitions of ψ and φ it is clear that $\psi \circ \varphi$ is the identity on S_{2n}, so it remains to check that φ is well defined. The set of generators $(2\ 3)$, $(3\ 4)$, $\cdots$, $(2n{-}1\ 2n)$, $(2n\ 1)$ used in the definition of φ satisfy the following relations: The square of each generator is the identity; Non intersecting generator commute; Intersecting generators satisfy triple relations. These relations are suffice (it is well known, as it was proven in [1]). Note that all of these relations are also satisfied by Γ_3, Γ_4, $\cdots$, Γ_{2n}, Γ_2 in $\widetilde{\Pi}_1$. In fact they are in the presentation as relations (6), (8), and (9).

We use the Reidmeister–Schreier method to find a presentation for the kernel $\mathcal{A}$ of the map $\psi\colon\ \widetilde{\Pi}_1 \to S_{2n}$. Since ψ is split by φ we can write the generators of $\mathcal{A}$ as follows:

$$\gamma(\sigma,\Gamma) = \sigma\Gamma(\varphi\psi(\Gamma))^{-1}\sigma^{-1} \quad \forall\sigma \in \varphi(S_{2n}), \quad \forall\Gamma \text{ generator of } \widetilde{\Pi}_1.$$

To simplify notation let $\overline{\Gamma} = \varphi\psi(\Gamma)$ denote the projection of Γ onto the image of φ. Using this notation we get generators:

$$\gamma(\sigma,\Gamma) = \sigma\Gamma\overline{\Gamma}^{-1}\sigma^{-1}.$$

Since $S_{2n} \cong \varphi(S_{2n}) \subset \widetilde{\Pi}_1$ we will not distinguish between the two groups. We will think of S_{2n} as a subgroup of $\widetilde{\Pi}_1$, so $\sigma \in S_{2n}$ above are permutations.

The relations of $\widetilde{\Pi}_1$ can be translated into expressions in these generators by the following process. If the word $\omega = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_t}$ represents an element of $\operatorname{Ker}\psi$ then ω can be rewritten as the product

$$\tau(\omega) = \gamma(1,\Gamma_{i_1})\gamma(\overline{\Gamma}_{i_1},\Gamma_{i_2})\cdots\gamma(\overline{\Gamma_{i_1}\cdots\Gamma_{i_{t-1}}},\Gamma_{i_t}).$$

Theorem 4.4 (Reidmeister–Schreier) *Let* $\{R\}$ *be a complete set of relations for* $\widetilde{\Pi}_1$. *Then* $\mathcal{A} = \operatorname{Ker}\psi$ *is generated by the* $\gamma(\sigma,\Gamma)$ *with the relations* $\{\tau(\sigma r\sigma^{-1})\}_{r\in R,\sigma\in S_{2n}}$.

We use this method to find generators and relations for the kernel.

4.4 Generators for $\mathcal{A} = \operatorname{Ker}\psi$

By Theorem 4.4, $\mathcal{A}$ is generated by the elements $\sigma\Gamma_j\overline{\Gamma}_j^{-1}\sigma^{-1}$ and $\sigma\Gamma_{j'}\overline{\Gamma}_{j'}^{-1}\sigma^{-1}$, $1 \leq j \leq 2n$, $\sigma \in S_{2n}$. We compute $\overline{\Gamma}_j$ and $\overline{\Gamma}_{j'}$. Recall that $\langle\Gamma_2, \dots, \Gamma_{2n}\rangle = S_{2n}$ is the image of φ, so for $j \neq 1$ we get $\overline{\Gamma}_j = \overline{\Gamma}_{j'} = \Gamma_j$, and the associated generators are

$$A_{\sigma,j} = \sigma\Gamma_j\Gamma_{j'}\sigma^{-1}. \tag{11}$$

The permutation $(1\,2)$ can be expressed in terms of the generators of S_{2n} corresponding to $\Gamma_2, \dots, \Gamma_{2n}$ as follows:

$$(1\,2) = (2n\,1)\cdots(4\,5)(3\,4)(2\,3)(3\,4)(4\,5)\cdots(2n\,1).$$

So for $j = 1$ we have that $\overline{\Gamma}_1 = \overline{\Gamma}_{1'} = \Gamma_2\Gamma_{2n}\cdots\Gamma_4\Gamma_3\Gamma_4\cdots\Gamma_{2n}\Gamma_2$.

Since we are identifying S_{2n} with $\varphi(S_{2n}) = \langle\Gamma_2, \dots, \Gamma_{2n}\rangle$ it is reasonable to write the above as simply

$$\overline{\Gamma}_1 = \overline{\Gamma}_{1'} = (1\,2).$$

So we get generators

$$X_\sigma = \sigma(1\,2)\Gamma_1\sigma^{-1} \tag{12}$$

$$B_\sigma = \sigma(1\,2)\Gamma_{1'}\sigma^{-1}. \tag{13}$$

Since $X_\sigma^{-1}B_\sigma = \sigma\Gamma_1\Gamma_{1'}\sigma^{-1}$ we can define $A_{\sigma,1} = X_\sigma^{-1}B_\sigma$ for $j = 1$ to get the following result:

Corollary 4.5 *The group $\mathcal{A} = \operatorname{Ker}\psi$ is generated by $A_{\sigma,j}$, X_σ, for $\sigma \in S_{2n}$ and $j = 1, \dots, 2n$.*

4.5 Reducing the set of generators for $\mathcal{A}$

First we show that the $A_{\sigma,j}$ are not needed for $j = 2, \dots, 2n$.

Theorem 4.6 *$\mathcal{A}$ is generated by $\{A_{\sigma,1}, X_\sigma\}$.*

Proof The following relations are translations of the relations (10) from Theorem 4.2. Together they show that all of the $A_{\sigma,j}$ for $j \neq 1$ can be written in terms of the $A_{\sigma,1}$. Derivations follow the table.

Table 4.7 *Translations of the branch point relations:*

$$\begin{array}{rcll}
A_{\sigma,3} & = & A_{\sigma(23),1}A_{\sigma,1}^{-1} & (14)\\
A_{\sigma,3} & = & A_{\sigma(234),4} & (15)\\
A_{\sigma,4} & = & A_{\sigma(345),5} & (16)\\
& \vdots & & \\
A_{\sigma,2n-1} & = & A_{\sigma(2n-2\,2n-1\,2n),2n} & (17)\\
A_{\sigma,2n} & = & A_{\sigma(2n-1\,2n\,1),2} & (18)\\
A_{\sigma,2} & = & A_{\sigma(2n\,1),1}A_{\sigma,1}^{-1}. & (19)
\end{array}$$

We use the relations (10) of Theorem 4.2. Let I denote the identity element of S_{2n}, so that by definition $A_{I,j} = \Gamma_j\Gamma_{j'}$.

To prove (14) we use V_1 which has diagonal $j = 3$ and vertical $i = 1$. The branch point relation is $1 = \Gamma_{3'}\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1 = (\Gamma_{3'}\Gamma_3)(\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3)(\Gamma_{1'}\Gamma_1) = A_{I,3}^{-1}A_{(23),1}A_{I,1}^{-1}$ so we get $A_{I,3} = A_{(23),1}A_{I,1}^{-1}$.

To prove (19) we use V_{n+1} which has diagonal $j = 2$ and vertical $i = 1$. The relation is $1 = \Gamma_{2'}\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_1 = (\Gamma_{2'}\Gamma_2)(\Gamma_2\Gamma_1\Gamma_{1'}\Gamma_2)(\Gamma_{1'}\Gamma_1) = A_{I,2}^{-1}A_{(2n\,1),1}A_{I,1}^{-1}$; so we get $A_{I,2} = A_{(2n\,1),1}A_{I,1}^{-1}$.

To prove (15) we use V_{n+2} which has diagonal j=3 and vertical i=4. The corresponding relation is $1 = \Gamma_{3'}\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_4 = (\Gamma_{3'}\Gamma_3)\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_4 = (\Gamma_{3'}\Gamma_3)\Gamma_3\Gamma_4\Gamma_3\Gamma_{4'}\Gamma_3\Gamma_4 = (\Gamma_{3'}\Gamma_3)\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_4 = A_{I,3}^{-1}A_{(34)(23),4}$. As a result $A_{I,3} = A_{(234),4}$.

The rest of Table 4.7 follows in much the same manner as (15). □

Combining all of the relations in Table 4.7, we obtain one relation among the $A_{\sigma,1}$:

$$\begin{array}{rcl}
A_{\sigma(23),1}A_{\sigma,1}^{-1} & = & A_{\sigma,3}\\
& = & A_{\sigma(234),4}\\
& = & A_{\sigma(234)(345),5}\\
& = & A_{\sigma(234)(345)(456),6}\\
& \vdots & \\
& = & A_{\sigma(234)(345)(456)\cdots(2n-1\,2n\,1),2}\\
& = & A_{\sigma(234)\cdots(2n-1\,2n\,1)(2n\,1),1}A^{-1}_{\sigma(234)\cdots(2n-1\,2n\,1),1}
\end{array}$$

which may be rewritten as

$$A_{\sigma(23),1}A_{\sigma,1}^{-1} = A_{\sigma(1\,2n-1\ldots5\,3\,2n\ldots4\,2),1}A^{-1}_{\sigma(2n-1\ldots3\,1)(2n\ldots4\,2),1} \tag{20}$$

Now we further reduce the set of generators by recognizing that the X_σ and $A_{\sigma,1}$ are redundant.

Lemma 4.8 *For every $\sigma \in S_{2n}$, the generators $A_{\sigma,1}$, B_σ, and X_σ depend only on $\sigma^{-1}(1)$ and $\sigma^{-1}(2)$.*

Proof Consider τ in the stabilizer of $1,2$. Clearly τ commutes with $(1\,2)$. In $\varphi(S_{2n})$ the stabilizer is generated by $\{\Gamma_4,\Gamma_5,\ldots\Gamma_{2n}\}$ all of which commute with Γ_1 and $\Gamma_{1'}$. So τ commutes with Γ_1 and $\Gamma_{1'}$ as well. By definition $A_{\tau,1}$, B_τ, and X_τ are given by $\tau\Gamma_1\Gamma_{1'}\tau^{-1}$, $\tau(12)\Gamma_{1'}\tau^{-1}$, and $\tau(12)\Gamma_1\tau^{-1}$ respectively. Since τ commutes with all of these things we have $A_{\tau,1} = A_{I,1}$, $B_\tau = B_I$, and $X_\tau = X_I$. Suppose $\sigma_1^{-1}(1) = \sigma_2^{-1}(1)$ and $\sigma_1^{-1}(2) = \sigma_2^{-1}(2)$. Then $\sigma_2 = \sigma_1\tau$ for some τ in the stabilizer of $1,2$. Hence $A_{\sigma_2,1} = A_{\sigma_1,1}$, $B_{\sigma_2} = B_{\sigma_1}$, and $X_{\sigma_2} = X_{\sigma_1}$. □

Lemma 4.8 suggests a convenient trio of definitions.

Definition 4.9 *For distinct $k,\ell \in \{1,\ldots,2n\}$, $A_{k\ell}$, $B_{k\ell}$, and $X_{k\ell}$ can be defined by*

$$A_{k\ell} = \sigma\Gamma_1\Gamma_{1'}\sigma^{-1} \tag{21}$$

$$B_{k\ell} = \sigma(12)\Gamma_{1'}\sigma^{-1} \tag{22}$$

$$X_{k\ell} = \sigma(12)\Gamma_1\sigma^{-1} \tag{23}$$

where $\sigma \in S_{2n} = \langle\Gamma_2,\ldots,\Gamma_{2n}\rangle$ is any permutation such that $\sigma(k) = 1$ and $\sigma(\ell) = 2$.

Now we investigate the behavior of these generators under conjugation by elements $\sigma \in S_{2n}$.

Proposition 4.10 *For every $\sigma \in S_{2n} = \langle\Gamma_2,\ldots,\Gamma_{2n}\rangle$ and distinct $k,\ell \in \{1,\ldots,2n\}$, we have that*

$$\sigma^{-1}A_{k\ell}\sigma = A_{\sigma(k),\sigma(\ell)} \tag{24}$$

$$\sigma^{-1}B_{k\ell}\sigma = B_{\sigma(k),\sigma(\ell)} \tag{25}$$

$$\sigma^{-1}X_{k\ell}\sigma = X_{\sigma(k),\sigma(\ell)} \tag{26}$$

Proof Let $\tau \in S_{2n}$ be such that $\tau(k) = 1$ and $\tau(\ell) = 2$. Since $A_{12} = \Gamma_1\Gamma_{1'}$ by Definition 4.9, we have $A_{k\ell} = \tau\Gamma_1\Gamma_{1'}\tau^{-1}$ and $\sigma^{-1}A_{k\ell}\sigma = \sigma^{-1}\tau\Gamma_1\Gamma_{1'}\tau^{-1}\sigma = A_{\tau^{-1}\sigma(1),\tau^{-1}\sigma(2)} = A_{\sigma(k)\sigma(\ell)}$. The same proof works for $B_{k\ell}$ and $X_{k\ell}$. □

From Theorem 4.6 (with a little help from Lemma 4.8), we obtain

Corollary 4.11 *The group* $\mathcal{A}$ *is generated by* $\{A_{k\ell}, X_{k\ell}\}_{1\le k,\ell\le 2n}$.

In terms of the A_{ik} the relation (20) can be written (for $\sigma = I$) as $A_{13}A_{12}^{-1} = A_{24}A_{34}^{-1}$. Then conjugating by any σ we get

$$A_{ij}A_{ik}^{-1} = A_{k\ell}A_{j\ell}^{-1} \tag{27}$$

for any four distinct indices i, j, k, ℓ. With two applications of relation (27) we see that $A_{ij}A_{ik}^{-1} = A_{mj}A_{mk}^{-1} = A_{\ell j}A_{\ell k}^{-1}$ and so

$$A_{ij}A_{ik}^{-1} = A_{\ell j}A_{\ell k}^{-1} \tag{28}$$

for any distinct indices i, j, k and $\ell \neq j, k$. Using one more application of (27) we can also allow $\ell = i$ in (27) because $A_{ij}A_{ik}^{-1} = A_{\ell j}A_{\ell k}^{-1} = A_{ki}A_{ji}^{-1}$. In view of (27) and (28) and Table 4.7 we can write a translation table for the generators $A_{\sigma,j}$ in terms of the new generators $A_{k\ell}$.

Table 4.12 $A_{\sigma,j}$ *in terms of* $A_{k\ell}$

$$\begin{aligned}
A_{\sigma,1} &= \sigma A_{12}\sigma^{-1} && && (29)\\
A_{\sigma,3} &= \sigma A_{x3}A_{x2}^{-1}\sigma^{-1} = \sigma A_{2x}A_{3x}^{-1}\sigma^{-1} && \textit{where } x \neq 2,3 && (30)\\
A_{\sigma,4} &= \sigma A_{x4}A_{x3}^{-1}\sigma^{-1} = \sigma A_{3x}A_{4x}^{-1}\sigma^{-1} && \textit{where } x \neq 3,4 && (31)\\
&\ \vdots && &&\\
A_{\sigma,2} &= \sigma A_{x1}A_{x\,2n}^{-1}\sigma^{-1} = \sigma A_{2n\,x}A_{1x}^{-1}\sigma^{-1} && \textit{where } x \neq 1,2n && (32)
\end{aligned}$$

We have reduced the generating set for $\mathcal{A}$ to $\{X_{k\ell}, A_{k\ell}\}_{k\neq\ell}$. Now we use the Reidmeister–Schreier rewriting process to translate all of the relations of $\widetilde{\Pi}_1$. Using the notation of Subsection 4.3, $\gamma(\sigma,\Gamma_j) = I$ and $\gamma(\sigma,\Gamma_{j'}) = A_{\sigma,j}^{-1}$ for $j \neq 1$. For $j = 1$, $\gamma(\sigma,\Gamma_1) = X_\sigma^{-1}$, and $\gamma(\sigma,\Gamma_{1'}) = B_\sigma^{-1}$. We begin by translating some of the relations which involve Γ_1 but not $\Gamma_{1'}$.

$\Gamma_1\Gamma_1 \xmapsto{\tau} \gamma(I,\Gamma_1)\gamma(\overline{\Gamma_1},\Gamma_1) = X_I^{-1}X_{(12)}^{-1} = X_{12}^{-1}X_{21}^{-1}$, so we deduce that $X_{21} = X_{12}^{-1}$ and conjugating we get

$$X_{\ell k} = X_{k\ell}^{-1}. \tag{33}$$

The relations $[\Gamma_1, \Gamma_j]$ in $\widetilde{\Pi}_1$ all produce the same relations on $\langle X_{k\ell}\rangle$ as above. Now we translate the triple relations (9) for i, j adjacent. We start for example with $\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1\Gamma_2$. The relation $\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1\Gamma_2$ translates through τ to the expression

$$\gamma(I,\Gamma_1)\gamma(\overline{\Gamma_1},\Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2},\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1},\Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2},\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1},\Gamma_2).$$

But since $\gamma(\sigma,\Gamma_2) = I$ we get $\gamma(I,\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2},\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2},\Gamma_1)$, and since $\overline{\Gamma_1\Gamma_2} = (1\,2\,2n)$ we can further simplify to

$$X_I^{-1}X_{(1\,2\,2n)}^{-1}X_{(2n\,2\,1)}^{-1} = X_{12}^{-1}X_{2n\,1}^{-1}X_{2\,2n}^{-1}.$$

Thus $X_{2\,2n}X_{2n\,1}X_{12} = 1$ and including all conjugates we have

$$X_{k\ell}X_{\ell m}X_{mk} = 1. \tag{34}$$

Similarly the relation $\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1\Gamma_3$ translates through τ to the expression

$$\gamma(I,\Gamma_1)\gamma(\overline{\Gamma_1},\Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3},\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1},\Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3},\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1},\Gamma_3)$$

which equals $X_I^{-1}X_{(321)}^{-1}X_{(123)}^{-1} = X_{12}^{-1}X_{23}^{-1}X_{31}^{-1}$. Thus $X_{31}X_{23}X_{12} = 1$, and conjugating we obtain

$$X_{\ell m}X_{k\ell}X_{mk} = 1. \tag{35}$$

Together the relations (33)–(35) show that $\langle X_{\ell m}\rangle$ is generated by the $2n-1$ commuting elements $X_{12}, \ldots, X_{1\,2n}$ so that $\langle X_{k\ell}\rangle \cong \mathbb{Z}^{2n-1}$.

Next, taking the above relations and replacing Γ_1 with $\Gamma_{1'}$ it is easy to see that $\Gamma_{1'}\Gamma_{1'}$, $[\Gamma_{1'}, \Gamma_j]$, $\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2$, and $\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3$ yield identical relations among the $B_{k\ell}$. So $\langle B_{k\ell}\rangle \cong \mathbb{Z}^{2n-1}$ as well,

$$B_{\ell k} = B_{k\ell}^{-1} \tag{36}$$

$$B_{k\ell}B_{\ell m}B_{mk} = 1 \tag{37}$$

$$B_{\ell m}B_{k\ell}B_{mk} = 1. \tag{38}$$

We finish with the last necessary triple relations. Note that $\overline{\Gamma_{1'}\Gamma_1\Gamma_{1'}} = \overline{\Gamma}_1$ and $\tau(\Gamma_{1'}\Gamma_1\Gamma_{1'}) = B_I^{-1}X_{(12)}^{-1}B_I^{-1} = B_{12}^{-1}X_{21}^{-1}B_{12}^{-1} = B_{12}^{-1}X_{12}B_{12}^{-1}$. So if we define $C_{k\ell}$ to be $B_{k\ell}X_{k\ell}^{-1}B_{k\ell} = X_{k\ell}A_{k\ell}^2$ then the additional relations are $C_{\ell k} = C_{k\ell}^{-1}$, $C_{k\ell}C_{\ell m}C_{mk} = 1$, and $C_{\ell m}C_{k\ell}C_{mk} = 1$. By the arguments above, the $\{C_{k\ell}\}$ generate another copy $\mathbb{Z}^{2n-1} \subset \mathcal{A}$. In fact for each exponent n the elements $X_{k\ell}A_{k\ell}^n$ generate a subgroup isomorphic to $\mathbb{Z}^{2n-1}$.

The relations computed thus far turn out to be all of the relations in $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$. Computations showing that the remaining relations translated from $\widetilde{\Pi}_1$ are consequences of the relations above are identical to computations in [1] and are omitted here. We have therefore proven the following theorem.

Theorem 4.13 *The fundamental group $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$ is generated by elements $\{X_{ij}, A_{ij}\}$ with the relations*

$$\begin{aligned} X_{ji}A_{ji}^n &= (X_{ij}A_{ij}^n)^{-1}, && (39)\\ (X_{ij}A_{ij}^n)(X_{jk}A_{jk}^n)(X_{ki}A_{ki}^n) &= 1, && (40)\\ (X_{jk}A_{jk}^n)(X_{ij}A_{ij}^n)(X_{ki}A_{ki}^n) &= 1, && (41)\\ A_{ij}A_{ik}^{-1} &= A_{k\ell}A_{j\ell}^{-1} && (42) \end{aligned}$$

for every $n \in \mathbb{Z}$ and distinct i, j, k, ℓ.

Before adding the projective relation to compute $\pi_1(X_{\mathrm{Gal}})$ we prove a useful lemma, showing that some of the $A_{k\ell}$ commute in $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$. We shall frequently use the fact that $X_{k\ell}X_{\ell m} = X_{\ell m}X_{k\ell} = X_{km}$ which is a consequence of (33)–(35). $B_{k\ell}$ and $C_{k\ell}$ satisfy this as well.

Lemma 4.14 *In $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$ we have $[A_{ij}, A_{ik}] = 1$ and $[A_{ji}, A_{ki}] = 1$ for distinct i, j, k.*

Proof Starting with $1 = C_{ki}C_{jk}C_{ij}$ and use the definition of C_{ij} to rewrite it as

$$\begin{aligned} 1 &= (B_{ki}X_{ik}B_{ki})(B_{jk}X_{kj}B_{jk})(B_{ij}X_{ji}B_{ij}) = B_{ki}X_{ik}B_{ji}X_{kj}B_{ik}X_{ji}B_{ij}\\ &= B_{ki}X_{ik})(B_{ji}X_{ij})(X_{ki}B_{ik})(X_{ji}B_{ij}) = A_{ik}^{-1}A_{ij}^{-1}A_{ik}A_{ij}. \end{aligned}$$

Thus the commutator $[A_{ij}, A_{ik}] = 1$. The relation (27) can be used to show that the commutator $[A_{ji}, A_{ki}] = 1$ as well. □

4.6 The projective relation

To complete the computation of $\pi_1(X_{\mathrm{Gal}})$ we need only to add the projective relation

$$\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\cdots\Gamma_{2n}\Gamma_{2n'} = 1.$$

This relation translates in $\mathcal{A}$ as the product $P = A_{I,1}A_{I,2}\cdots A_{I,2n}$. We must translate the $A_{I,j}$ in terms of the $A_{k\ell}$, using Table 4.12. P translates to

$$A_{12}(A_{31}A_{3\,2n}^{-1})(A_{21}A_{31}^{-1})(A_{31}A_{41}^{-1})\cdots(A_{2n-2\,1}A_{2n-1\,1}^{-1})(A_{2n-1\,1}A_{2n\,1}^{-1}).$$

All but five terms cancel in the expression above, leaving $A_{12}A_{31}A_{3\,2n}^{-1}A_{21}A_{2n\,1}^{-1}$. Using Equation (27), we get $A_{12}A_{31}A_{3\,2n}^{-1}A_{3\,2n}A_{32}^{-1}$. Thus the projective relation may be written as $A_{12}A_{31}A_{32}^{-1} = 1$ or equivalently $A_{32} = A_{12}A_{31}$. Conjugating, this becomes

$$A_{ij} = A_{kj}A_{ik}. \qquad (43)$$

Substituting back into (27), writing $A_{ij} = A_{kj}A_{ik}$ and $A_{k\ell} = A_{j\ell}A_{kj}$, we obtain

$$A_{kj}A_{j\ell} = A_{j\ell}A_{kj}. \tag{44}$$

Lemma 4.15 *The subgroup $\langle A_{k\ell}\rangle$ of $\pi_1(X_{\mathrm{Gal}})$ is commutative of rank of at most $2n-1$.*

Proof We will compute the centralizer of A_{ij} for fixed i,j. Let i,j,k,ℓ be four distinct indices. We already know from Lemma 4.14 that A_{ij} commutes with A_{ik} and $A_{\ell j}$. By equation (44) it also commutes with A_{ki} and $A_{j\ell}$. Now equation (43) allows us to write $A_{k\ell} = A_{i\ell}A_{\ell j}$, both of which commute with A_{ij}, so $\langle A_{k\ell}\rangle$ is commutative.

Now, since $A_{jk}A_{ij} = A_{ik}$ and $A_{ik}A_{ji} = A_{jk}$, we have $A_{jk}A_{ij}A_{ji} = A_{ik}A_{ji} = A_{jk}$, so that $A_{ji} = A_{ij}^{-1}$, the group is generated by the A_{1k} $(k = 2,\dots,2n)$, and the rank is at most $2n-1$. □

We see that $\pi_1(X_{\mathrm{Gal}}) = \langle A_{ij}, X_{ij}\rangle$ with the two subgroups $\langle A_{ij}\rangle, \langle X_{ij}\rangle$ each isomorphic to $\mathbb{Z}^{2n-1}$. The only question left is how these two subgroups interact.

Lemma 4.16 *In $\pi_1(X_{\mathrm{Gal}})$ the A_{ij} and $X_{k\ell}$ commute.*

Proof We need only consider the commutators of A_{12} and X_{ij} since all others are merely conjugates of these. First consider the commutator $[X_{12}, A_{12}]$. Since $X_{12} = (12)\Gamma_1$ and $A_{12} = \Gamma_1\Gamma_{1'}$ the commutator $X_{12}A_{12}X_{12}^{-1}A_{12}^{-1}$ becomes $(12)\Gamma_1(\Gamma_1\Gamma_{1'})\Gamma_1(12)A_{12}^{-1} = (12)\Gamma_{1'}\Gamma_1(12)A_{12}^{-1} = A_{21}^{-1}A_{12}^{-1} = A_{12}A_{12}^{-1} = 1$. So X_{12} and A_{12} commute.

Next consider the commutator $X_{13}A_{12}X_{13}^{-1}A_{12}^{-1}$. By definition we have that $X_{13} = (23)X_{12}(23) = (23)(12)\Gamma_1(23) = (123)\Gamma_1\Gamma_3$. We are using the fact that $\Gamma_3 = (23)$ in the image $\varphi(S_{2n})$. Thus the commutator can be written as $(123)\Gamma_1\Gamma_3(\Gamma_1\Gamma_{1'})\Gamma_3\Gamma_1(321)A_{12}^{-1}$. We use the triple relations

$$\Gamma_{(1)}\Gamma_{(3)}\Gamma_{(1)} = \Gamma_{(3)}\Gamma_{(1)}\Gamma_{(3)}$$

to rewrite it as

$$\begin{aligned}(123)\Gamma_3\Gamma_1\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_1(321)A_{12}^{-1} &= (123)\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1(321)A_{12}^{-1}\\ &= (123)(23)\Gamma_1\Gamma_{1'}(23)(321)(123)\Gamma_{1'}\Gamma_1(321)A_{12}^{-1}\end{aligned}$$

which is equal to $A_{(123)(23),1}A_{(123),1}^{-1}A_{12}^{-1} = A_{(13),1}A_{(123),1}^{-1}A_{12}^{-1} = A_{32}A_{31}^{-1}A_{12}^{-1} = A_{32}A_{32}^{-1} = 1$, proving that X_{13} and A_{12} commute. Conjugating by $(3j)$ we see that X_{1j} commutes with A_{12} and since $X_{ij} = X_{1i}^{-1}X_{1j}$ we see that every X_{ij} commutes with A_{12}. □

Theorem 4.17 *The fundamental group* $\pi_1(X_{\mathrm{Gal}}) \cong \mathbb{Z}^{4n-2}$.

Proof $\pi_1(X_{\mathrm{Gal}})$ is generated by A_{1j} and X_{1j} which all commute. Hence the group they generate is $\mathbb{Z}^{4n-2}$. □

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