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Parabolic isometries of CAT(0) spaces and CAT(0) dimensions

Koji Fujiwara
Takashi Shioya
Saeko Yamagata

Abstract We study discrete groups from the view point of a dimension gap in connection to CAT(0) geometry. Developing studies by Brady-Crisp and Bridson, we show that there exist finitely presented groups of geometric dimension 2 which do not act properly on any proper CAT(0) spaces of dimension 2 by isometries, although such actions exist on CAT(0) spaces of dimension 3.

Another example is the fundamental group, G, of a complete, non-compact, complex hyperbolic manifold M with finite volume, of complex-dimension $n \geq 2$. The group G is acting on the universal cover of M, which is isometric to $\mathbf{H}^n_{\mathbb{C}}$. It is a CAT$(-1)$ space of dimension $2n$. The geometric dimension of G is $2n-1$. We show that G does not act on any proper CAT(0) space of dimension $2n-1$ properly by isometries.

We also discuss the fundamental groups of a torus bundle over a circle, and solvable Baumslag-Solitar groups.

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1 Introduction and statement of results

As a generalization of simply connected Riemannian manifolds of non-positive sectional curvature, the notion of "CAT(0)" was introduced to geodesic spaces by M.Gromov. Many properties of simply connected Riemannian manifolds of non-positive sectional curvature remain true for CAT(0) spaces. We refer to [B] and also [BriH] for a thorough account of the subject. For example, if a space is CAT(0) then it is contractible, which is a Cartan-Hadamard theorem. This gives a powerful tool to show some spaces are contractible (for example,

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see [ChaD]). Another example is the classification of isometries of a complete CAT(0) space X: elliptic, hyperbolic, or parabolic. It seems harder to understand parabolic isometries than the other two, which give natural objects to look at: the set of fixed points for an elliptic isometry and the set of axes for a hyperbolic isometry. Moreover the union of the axes is isometric to the product of a convex set in X and a real line. "Flat torus theorem" (see Theorem 5.1) is an important consequence of the existence of axes. An isometry is called *semi-simple* if it is either elliptic or hyperbolic.

A metric space, X, is called *proper* if for any $x \in X$ and any $r > 0$, the closed ball in X of radius r centered at x is compact. If X is proper, then it is locally compact. If X is a complete, locally compact, geodesic space, then it is proper (cf. [BuBuI] Prop 2.5.22). Therefore, a complete CAT(0) space is proper if and only if it is locally compact.

If X is a proper CAT(0) space, then a parabolic isometry has at least one fixed point in the ideal boundary of X (see Proposition 3.1).

Suppose a discrete group G is acting on a topological space, X, by homeomorphisms. To give definitions of "proper actions", let's consider the following conditions.

(1) For any $x \in X$, there exists a neighborhood $U \subset X$ of x such that $\{g \mid U \cap gU \neq \emptyset\}$ is finite.

(2) For any $x, y \in X$, there exist neighborhoods U of x and V of y such that $\{g \mid V \cap gU \neq \emptyset\}$ is finite.

(3) For any compact subset $K \subset X$, $\{g \mid K \cap gK \neq \emptyset\}$ is finite.

(4) The map $G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is a proper map.

Recall that a continuous map, $f : X \to Y$, is called *proper* if it is closed and for each $y \in Y$, $f^{-1}(y)$ is compact. If X and Y are Hausdorff and Y is locally compact, then f is proper if and only if for each compact set $K \subset Y$, the preimage $f^{-1}(K)$ is compact ([Di] Ch1).

Clearly (2) implies (1). It is easy to see that (2) implies (3). If X is locally compact, then (3) implies (2). If X is Hausdorff, then (2) and (4) are equivalent ([Di] 3.22 Cor). If X is a metric space and the action is by isometries, then (1) and (2) are equivalent ([DeV] Thm1), so that (1),(2) and (4) are equivalent, and (3) follows from one of them.

Following [BriH], we say that an isometric action of a discrete group, G, on a metric space, X, is *proper* if the condition (1) is satisfied, in other words, for any x, there exists $r > 0$ such that $\{g \mid gB(x, r) \cap B(x, r) \neq \emptyset\}$ is finite, where

$B(x,r) = \{y \in X \mid d(x,y) \le r\}$. In the literature, proper actions are often called properly discontinuous. Most of the results in this paper are stated for proper actions, which is our main theme, but sometimes the weaker condition (3) is enough. Since we can not find a standard name, we decide to call an action with the condition (3) a *K-proper* action. As we said, if the space is locally compact, then those two properness are equivalent.

In this paper we study proper, isometric actions of a discrete group G on CAT(0) spaces X. We are interested in the minimal dimension of such X, which is called a *CAT(0) dimension* of G, denoted CAT(0)-dim G. If such actions do not exist, then CAT(0)-dim $= \infty$.

By the dimension of a topological (or metric) space, we mean the covering dimension (see subsection 4.1 for the precise definition).

If G is torsion-free, there is an obvious lower bound of CAT(0)-dim$G$. Recall that the *geometric dimension* $\le \infty$ of G, denoted $\operatorname{geom}\dim G$, is the minimal dimension of a $K(G,1)$-complex. If G is torsion-free, then a proper action of G on a CAT(0) space X is free, so that X/G is a $K(G,1)$-space, because $X \to X/G$ is a covering map. It follows that $\operatorname{geom}\dim G \le$ CAT(0)-dim$G$.

We are interested in the following problem.

Problem 1.1 Find a torsion-free group G such that

$$\operatorname{geom}\dim G < \text{CAT}(0)\text{-dim}\, G < \infty.$$

It seems an example of such G is not known, but a few potential examples of groups are studied by Brady-Crisp [BraC] and Bridson [Br] under an extra assumption on the actions. The following conditions on actions of G on a CAT(0) space X are natural.

(cc) The action is co-compact, i.e., X/G is compact.

(ss) The action is by semi-simple isometries.

(p) X is proper.

A group which acts properly on some CAT(0) space with the condition (cc) is called a *CAT(0) group*. The condition (cc) implies (ss).

If we choose one (or more) of the above conditions and only consider proper, isometric actions of G on CAT(0) spaces which satisfy the condition, we obtain another definition of a CAT(0) dimension of G, which is clearly not smaller than the original one. The condition (ss) is the one Brady-Crisp and Bridson imposed. Bridson denotes the CAT(0) dimension in this sense by CAT(0)-$\dim_{ss}$. As for this dimension, in other words, if we consider only actions of G on CAT(0) spaces by semi-simple isometries, it is easy to find G such that geom dim is finite and CAT(0)-$\dim_{ss}$ is ∞ (for example, see Proposition 5.2). The important part of their work is that they found groups with a finite gap, which is 1. Their examples are CAT(0) groups.

In this paper we study a group action on a CAT(0) space such that the action has a fixed point in the ideal boundary. It turns out that under certain circumstances, the condition (p) implies (ss). As an application, we replace the assumption (ss) by (p) in the results of Brady-Crisp and Bridson.

To analyze an action with a common fixed point in the ideal boundary, the following result, (2), on the dimension of horospheres is essential. Let $X(\infty)$ be the ideal boundary of a CAT(0) space, X, and $\bar{X} = X \cup X(\infty)$ with the cone topology (see section 2 for definitions). A metric sphere is defined by $S(x,r) = \{y \in X \mid d(x,y) = r\}$.

Theorem 4.1. *Let X be a proper* CAT(0) *space of dimension $n < \infty$. Then*

(1) *A metric sphere $S(x,r) \subset X$ has dimension at most $n-1$.*
(2) *A horosphere centered at a point $p \in X(\infty)$ has dimension at most $n-1$.*
(3) *$\bar{X}$ is an AR of dimension n. The ideal boundary $X(\infty)$ is a Z-set in $\bar{X}$, and the dimension is at most $n-1$. $\bar{X}$ is homotopy equivalent to X, hence contractible.*

Let $\operatorname{cd} G$ denote the cohomological dimension of G (the definition is in subsection 4.3). We show the next result using Theorem 4.1(2). Although the dimension of a horosphere is at most $n-1$ by (2), we need some extra work to show $\operatorname{cd} G \le n-1$ because the horosphere is not contractible in general.

Proposition 4.4. *Let X be a proper* CAT(0) *space of dimension $n < \infty$. Suppose a group G is acting on X properly by isometries. Assume G fixes a point $p \in X(\infty)$ and leaves each horosphere H_t centered at p invariant. If G has a finite $K(G,1)$-complex then $\operatorname{cd} G \le n-1$. In particular, if $n = 2$ then G is free.*

It has a useful consequence.

Proposition 5.1. *Let X be a proper* CAT(0) *space. Suppose a free abelian group of rank n, $\mathbb{Z}^n$, is acting properly on X by isometries. Then $\dim X \ge n$. If $\dim X = n$ then the action is by semi-simple isometries.*

Combining this proposition and the study by Brady-Crisp [BraC] or Bridson [Br] we obtain the following result, which is an immediate consequence of Theorems 5.5 and 5.6, or Theorems 5.7 and 5.8.

Theorem 1.1 *There exists a finitely presented group G of geometric dimension 2 such that*

(1) *G does not act properly on any proper $\mathrm{CAT}(0)$ space of dimension 2 by isometries.*

(2) *G acts properly and co-compactly on some proper $\mathrm{CAT}(0)$ space of dimension 3 by isometries. In particular, G is a $\mathrm{CAT}(0)$ group.*

Also we give a new class of examples which also have gaps between the geometric dimension and the $\mathrm{CAT}(0)$ dimension with the condition (p). Those groups are not $\mathrm{CAT}(0)$ groups. See Cor 5.1 for the class of examples of various dimensions and a proof. Let $\mathbf{H}^n_{\mathbb{C}}$ denote the complex hyperbolic space of complex-dimension n. It is a complete, simply connected Riemannian manifold of (real-) dimension $2n$ with the sectional curvature pinched by -1 and $-1/4$, hence a $\mathrm{CAT}(-1)$ space.

Theorem 1.2 *Let $G = \langle a, b, c \mid [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$. Then*

(1) $\mathrm{geom\,dim}\, G = 3$.

(2) *G does not act freely, hence not properly, on any $\mathrm{CAT}(0)$ space by semi-simple isometries, so that G is not a $\mathrm{CAT}(0)$ group.*

(3) *G does not act properly on any proper $\mathrm{CAT}(0)$ space of dimension 3 by isometries.*

(4) *G acts on $\mathbf{H}^2_{\mathbb{C}}$, which is a proper $\mathrm{CAT}(-1)$ space of dimension 4, properly by isometries.*

We also show the following.

Corollary 5.2. *Let G be the fundamental group of a non-compact, complete, complex hyperbolic manifold M of complex-dimension $n \geq 2$. If the volume of M is finite, then*

(1) $\mathrm{cd}\, G = \mathrm{geom\,dim}\, G = 2n - 1$.

(2) *G does not act properly on any proper $(2n-1)$-dimensional $\mathrm{CAT}(0)$ space by isometries.*

(3) *G acts properly on $\mathbf{H}^n_{\mathbb{C}}$, which is a proper $2n$-dimensional* $\mathrm{CAT}(-1)$ *space, by isometries.*

(4) *G does not act on any* $\mathrm{CAT}(0)$ *space freely, hence not properly, by semi-simple isometries, so that G is not a* $\mathrm{CAT}(0)$ *group.*

We ask several questions in the paper, including the following two (see discussions later).

Let $BS(1,m) = \langle a, b \,|\, aba^{-1} = b^m \rangle$. $BS(1,m)$ is called a solvable Baumslag-Solitar group.

Question 5.4 Does $BS(1,m)$ act properly on a $\mathrm{CAT}(0)$ space X of dimension 2?

Let $S = \langle a, b, c \,|\, ab = ba, cac^{-1} = a^2b, cbc^{-1} = ab \rangle$. S is the fundamental group of a closed 3-manifold which is a torus bundle over a circle.

Question 5.6 Does S act properly on some 3-dimensional $\mathrm{CAT}(0)$ space by isometries?

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2 A CAT(0) space and its ideal boundary

In this section we review basic definitions and facts on $\mathrm{CAT}(0)$ spaces. See [BriH] for details.

2.1 CAT(0) spaces

Let X be a geodesic space and $\Delta(x,y,z)$ a geodesic triangle in X, which is a union of three geodesics. A *comparison triangle* for Δ is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$

in $\mathbf{E}^2$ with the same side lengths as Δ. Let p be a point in the side of Δ between, say, x, y, which is often denoted by $[x, y]$. A *comparison point* in $\bar{\Delta}$ is a point $\bar{p} \in [\bar{x}, \bar{y}]$ with $d(x, p) = d_{\mathbf{E}^2}(\bar{x}, \bar{p})$. $\bar{\Delta}$ *satisfies the CAT(0) inequality* if for any $p, q \in \Delta$ and their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}$, $d(p, q) = d_{\mathbf{E}^2}(\bar{p}, \bar{q})$. X is a *CAT(0) space* if all geodesic triangles in X satisfy CAT(0) inequality.

Similarly, one defines a notion of $CAT(1)$ and $CAT(-1)$ spaces by comparing every geodesic triangle in X with its comparison triangle in the standard 2-sphere $\mathbf{S}^2$ and the real 2-dimensional hyperbolic space $\mathbf{H}^2$, respectively. In the case of CAT(1) we only consider geodesic triangles of total perimeter length less than 2π.

2.2 Ideal boundaries

Two geodesics $\gamma(t), \gamma'(t)$ in a complete CAT(0) space X are *asymptotic* if there exists a constant C such that for all $t \geq 0$, $d(\gamma(t), \gamma'(t)) \leq C$. This defines an equivalence relation, $\sim$. In this paper all geodesics have unit speed. The set of *points at infinity*, $X(\infty)$, or the *ideal boundary* of X is the set of equivalence classes of geodesics in X. The equivalence class of a geodesic $\gamma(t)$ is denoted by $\gamma(\infty)$. The equivalence class of a geodesic $\gamma(-t)$ is denoted by $\gamma(-\infty)$. It is a theorem that for any geodesic $\gamma(t)$ and a point $x \in X$, there exists a unique geodesic $\gamma'(t)$ such that $\gamma \sim \gamma'$ and $\gamma'(0) = x$.

2.3 The cone topology

Let X be a complete CAT(0) space and $\bar{X} = X \cup X(\infty)$. One standard topology on $\bar{X}$ is called the *cone topology*. The induced topology on X is the original one. We will give a basis for the cone topology. A typical basis for $x \in X$ is $B(x, r) \subset X$. To give a typical basis for a point $p \in X(\infty)$, fix $x \in X$ and $r > 0$. Let c be the geodesic such that $c(0) = x$ and $c(\infty) = p$. We extend the nearest point projection $\mathrm{pr} : X \to B(x, r)$ to $\bar{X}$. For any point $q \in X(\infty)$ let α_q be the geodesic with $\alpha_q(0) = x$ and $\alpha_q(\infty) = q$. Define $\mathrm{pr}(q) \in B(x, r)$ to be $\alpha_q(r)$. For $\epsilon > 0$ let

$$U(x, r, \epsilon; p) = \{x \in \bar{X} \mid d(c(r), pr(x)) < \epsilon\}.$$

There are two types of explicit neighborhood bases for the cone topology on $\bar{X}$: all $B(x, r)$ for $x \in X$ and $r > 0$, and all $U(x, r, \epsilon; p)$ for $x \in X, r > 0, \epsilon > 0, p \in X(\infty)$. We remark that not only $B(x, r)$ but also $U(x, r, \epsilon, p) \cap X$ is convex in X.

Let f be an isometry of X. Then since f acts on the set of geodesics on X leaving the equivalence relation $\sim$ invariant, it gives an action on $X(\infty)$. It is known this action of f on $\bar{X}$ is a homeomorphism.

If X is a proper CAT(0) space, then $\bar{X}$ is compact with respect to the cone topology. Also it is metrizable since it is a second-countable, normal space (Urysohn's Metrization Theorem).

There exists a (pseudo)metric on $X(\infty)$ which is called the *Tits metric*. The topology induced by the Tits metric is generally stronger than the cone topology. $\bar{X}$ may not be compact with respect to the Tits topology. This is one reason why we only use the cone topology in this paper.

2.4 Busemann functions and horospheres

Let X be a complete CAT(0) space and let $c : [0, \infty] \to X$ be a geodesic ray. The function $b_c : X \to \mathbb{R}$ defined by the following is called the *Busemann function* associated to c:

$$b_c(x) = \lim_{t \to \infty} (d(x, c(t)) - t).$$

b_c is a convex function such that for all $x, y \in X$, $|b_c(x) - b_c(y)| \leq d(x, y)$. It is known (see 8.20 Cor in [BriH]) that if $c \sim c'$ then there exists a constant C such that for all $x \in X$, $b_c(x) - b_{c'}(x) = C$. Because of this, level sets of b_c, $\{x \in X \mid b_c(x) = t\} \subset X, t \in \mathbb{R}$, depend only on the asymptotic class of c. They are called *horospheres* (centered) at $c(\infty)$. By construction of horospheres an isometry f of X which fixes $c(\infty)$ acts on the set of horospheres at $c(\infty)$. A *horoball* is defined by $\{x \in X \mid b_c(x) \leq t\}$. Horoballs are convex subsets in X.

3 Parabolic isometries

3.1 Classification of isometries

Let f be an isometry of a complete CAT(0) space (X, d). The *displacement function* $d_f : X \to \mathbb{R}$ is defined by

$$d_f(x) = d(x, f(x)).$$

This is a convex, f-invariant function. There is the following classification of isometries f (cf. [BriH]).

- If f fixes a point in X it is called *elliptic*.

- If there is a bi-infinite geodesic γ in X which is invariant by f and f acts on it by non-trivial translation, then f is called *hyperbolic*. γ is called *axis*. An axis may not be unique, but they are parallel to each other, i.e., there is a convex subspace between them which is isometric the product of an interval and $\mathbf{R}$, called a *strip*.
- Otherwise f is called *parabolic*.

If f is elliptic or hyperbolic, it is called *semi-simple*. It is known that f is hyperbolic if and only if $\inf d_f > 0$ and the infimum is attained. A point $x \in X$ is on an axis if and only if d_f attains its infimum at x. f is parabolic if and only if d_f does not attain its infimum. The infimum may be positive and sometimes f is called *strictly parabolic* if $\inf d_f = 0$.

In the case of the hyperbolic spaces $\mathbf{H}^n$ this classification is same as the standard one, and there are additional properties: an axis of a hyperbolic isometry is unique, a parabolic isometry has a fixed point in $X(\infty)$, which is unique.

3.2 Parabolic isometries

In general a parabolic isometry may not have any fixed point in $X(\infty)$, but if X is proper then there is always at least one. To see it we first quote a lemma from [BriH].

Lemma 3.1 *Let X be a proper* CAT(0) *space. Let $\delta : X \to \mathbb{R}$ be a continuous, convex function which does not attain its infimum. Then there is at least one point $p \in X(\infty)$ such that if g is an isometry of X with $\delta(g(x)) = \delta(x)$ for all $x \in X$, then g fixes p and leaves each horosphere centered at p invariant.*

For a proof we refer to 8.26 in Chap II.8 [BriH]. This claim is stronger than the lemma in 8.26 there, however the argument is same. As a consequence we obtain the following proposition.

Proposition 3.1 *Let X be a proper* CAT(0) *space. Suppose f is a parabolic isometry on X. Then there is a point $p \in X(\infty)$ such that any isometry g of X with $gf = fg$, in particular, f, fixes p and leaves each horosphere centered at p invariant.*

Again this is similar but stronger than the proposition in 8.25 in Chap II.8 [BriH], where the claim is only on f itself and not about g commuting with f. The proof is identical too, but we put it since it is short. We remark this claim is essentially same as Lemma 7.3 (2) on p87 in [BGS], where they consider only manifolds of non-positive curvature.

Proof Consider the displacement function d_f of f. It is a convex function which does not attain its infimum. Since $gf = fg$, d_f is g-invariant. Now we apply Lemma 3.1 putting $\delta = d_f$. □

A parabolic isometry of CAT(0) space may have more than one fixed point in the ideal boundary. For example, let T be a tree and put $X = T \times \mathbf{H}^2$. Let g be a parabolic isometry on $\mathbf{H}^2$ and put $f = \mathrm{id}_T \times g$, where id_T is the identity map on T. Then f is a parabolic isometry on X and the set of fixed points in $X(\infty)$, denoted by $X_f(\infty)$, is the join of $T(\infty)$ and the (only) fixed point of g in $\mathbf{H}^2(\infty)$. We remark that the conclusion of Proposition 3.1 may not apply to all points p in $X_f(\infty)$.

4 Horospheres

4.1 Covering dimension

We recall the definition of the covering dimension (cf. [Na]). Let $\mathcal{U}$ be a covering of a topological space. The *order* of $\mathcal{U}$ at a point $p \in X$, $\mathrm{ord}_p(\mathcal{U})$, is the number (possibly ∞) of the members in $\mathcal{U}$ which contain p. The *order* of $\mathcal{U}$, $\mathrm{ord}(\mathcal{U})$, is $\sup_{p\in X} \mathrm{ord}_p(\mathcal{U})$.

A covering $\mathcal{V}$ is a *refinement* of a covering $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. If for any finite open covering $\mathcal{U}$ of a topological space X there exists an open covering $\mathcal{V}$ which is a refinement of $\mathcal{U}$ such that $\mathrm{ord}(\mathcal{V}) \leq n+1$, then X has covering dimension $\leq n$. X has *covering dimension* $= n$ if the covering dimension is $\leq n$ but not $\leq n-1$.

It is a theorem that in the case that X is a metric space one can replace "any finite open covering" by "any open covering" in the definition of covering dimension (see II 5 in [Na]).

4.2 A Z-set in an ANR

We state a theorem which is a consequence of a general result by Bestvina-Mess, [BeM]. They are interested mainly in the ideal boundary of a word-hyperbolic group.

Recall that a closed subset Z in a compact ANR Y is a *Z-set* if for every open set U in Y the inclusion $U - Z \to U$ is a homotopy equivalence. It is known ([BeM]) that each of the following properties characterizes Z-sets:

(i) For every $\epsilon > 0$ there is a map $Y \to Y - Z$, which is ϵ-close to the identity.

(ii) For every closed $A \subset Z$ there is a homotopy $H : Y \times [0,1] \to Y$ such that $H_0 =$ identity, $H_t|A =$ inclusion, and $H_t(Y - A) \subset Y - Z$ for $t > 0$.

We first quote one result from their paper (Proposition 2.6 in [BeM]).

Proposition 4.1 *Suppose* Y *is a finite-dimensional* ANR *and* $Z \subset Y$ *a* Z*-set. Then* $\dim Z < \dim Y$*, and hence* $\dim Z < \dim(Y - Z)$.

We show the following.

Theorem 4.1 *Let* X *be a proper* CAT(0) *space of dimension* $n < \infty$*. Then*

(1) *A metric sphere* $S(x, r) \subset X$ *for* $x \in X, r > 0$ *has dimension at most* $n - 1$.

(2) *A horosphere centered at a point* $p \in X(\infty)$ *has dimension at most* $n-1$.

(3) $\bar{X}$ *is an AR of dimension* n*. The ideal boundary* $X(\infty)$ *is a* Z*-set in* $\bar{X}$*, and the dimension is at most* $n - 1$*.* $\bar{X}$ *is homotopy equivalent to* X*, hence contractible.*

Proof First of all, for a metric space of finite dimension to be an ANR is equivalent to being locally contractible. Therefore a CAT(0) space X of finite dimension as well as any convex subset in X is an ANR.

(1) A (closed) metric ball $B = B(x, r)$ in X is an ANR since it is a convex subset. And its boundary, namely, the metric sphere $S = S(x, r) \subset B$ is a Z-set. To see it one verifies (i) in the above. Indeed for any $\epsilon > 0$ one can construct, using a unique geodesic from each point in B to the center of B, a map $B \to B - S$ which is ϵ-close to the identity. It follows that $\dim S < \dim B \le \dim X = n$.

(2) A closed horoball B in X is an ANR since it is convex, and its boundary, namely, a horosphere, $H \subset B$ is a Z-set. As in (1) one can construct a map which satisfies the property (i) using a unique geodesic from each point in B to p.

(3) This is more complicated than (1) and (2). One can construct a homotopy to show $X(\infty) \subset \bar{X}$ is a Z-set, but we will show it as one of the conclusions of the proposition below.

We quote another proposition from [BeM].

Proposition 4.2 *Suppose Y is a compactum (i.e., compact, metrizable space) and $Z \subset Y$ a closed subset such that*

(1) $\mathrm{int}(Z) = \emptyset$.

(2) $\dim Y = n < \infty$.

(3) *For every $k = 0, 1, \cdots, n$, every point $z \in Z$, and every neighborhood U of z, there is a neighborhood V of z such that every map $\alpha : S^k \to V - Z$ extends to $\tilde{\alpha} : B^{k+1} \to U - Z$.*

(4) *$Y - Z$ is an* ANR.

Then Y is an ANR *and $Z \subset Y$ is a Z-set.*

Let $Y = \bar{X}$ and $Z = X(\infty)$. $\bar{X}$ is a compact metrizable space because X is proper. We check (1)-(4).

(1) is clear. To show (2), fix $x \in X$. Consider a system of metric balls centered at x, $B(x,r), r > 0$, and a system of metric spheres, $S(x,r), r > 0$. For $0 < r \leq s$, define $p_{rs} : B(x,s) \to B(x,r)$ by $p_{rs}(y) = y$ if $y \in B(x,r)$ and $p_{rs}(y) = [x,y](r)$ if $y \in B(x,s) - B(x,r)$, where $[x,y]$ is the geodesic from x to y. Remark in the latter case $p_{rs}(y) \in S(x,r)$, so that it gives $p_{r,s}|S(x,s) : S(x,s) \to S(x,r)$. We obtained inverse systems $\{B(x,r), p_{rs}\}_{r,s>0}$ and $\{S(x,r), p_{rs}\}_{r,s>0}$. Since X is proper they are compact Hausdorff spaces.

The following is more or less direct from the definition of the cone topology. We leave a precise argument to readers.

Claim (a) $\bar{X}$ with the cone topology is the inverse limit of the inverse system $\{B(x,r), p_{rs}\}_{r,s>0}$.

(b) $X(\infty)$ with the cone topology is the inverse limit of the inverse system $\{S(x,r), p_{rs}\}_{r,s>0}$.

Since the dimension of the inverse limit of compact Hausdorff spaces of $\dim \leq d$ is also at most d (e.g., see 1.7 Cor, Ch8 in [P]), $\dim \bar{X} \leq n$ and $\dim X(\infty) \leq n-1$. Therefore $\dim Y \leq n$.

(3) There exist $x \in X, r > 0, \epsilon > 0$ such that $U(x,r,\epsilon;z) \subset U$. Let $V = U(x,r,\epsilon;z)$. Then $V - Z = V \cap X$ is convex, therefore one can just cone off α in $V - Z$ to extend it to $\tilde{\alpha}$, which is still in $V - Z$.

(4) Since $Y - Z$ is X itself, it is an ANR.

Now by Proposition 4.2, $\bar{X}$ is a finite-dimensional (in fact n-dim) ANR and $X(\infty) \subset \bar{X}$ is a Z-set. Therefore $X \subset \bar{X}$ is homotopy equivalent, so that $\bar{X}$ is contractible, hence AR. □

4.3 The cohomological dimension of a parabolic subgroup

Let $\mathrm{cd}\, G$ denote the *cohomological dimension* of G. It is defined (cf. p185 [Br]) by

$$\mathrm{cd}\, G = \sup\{n : H^n(G, M) \neq 0 \text{ for some } G\text{-module } M\}.$$

It is known that $\mathrm{cd}\, G = \mathrm{geom}\dim G$ if $\mathrm{cd}\, G \geq 3$ (7.2 Corollary p205 [Br]). Also G is free if and only if $\mathrm{geom}\dim G = 1$, as well as if and only if $\mathrm{cd}\, G = 1$. It is known (6.7 Proposition on p202 [Br]) that if G is of type FP, in particular, if there is a finite $K(G,1)$, then

$$\mathrm{cd}\, G = \max\{n : H^n(G, \mathbb{Z}G) \neq 0\}.$$

We use Theorem 4.1 to bound the cd of a group leaving horospheres invariant. We remark that horospheres are not connected, not simply connected and not locally contractible in general. But we can at least “approximate” them with locally finite simplicial complexes (maybe not connected), equivariantly in terms of group actions.

Proposition 4.3 *Let X be a proper* $\mathrm{CAT}(0)$ *space of dimension $n < \infty$. Let $p \in X(\infty)$ and $\{H_t\}_{t\in\mathbb{R}}$ be the family of horospheres centered at p. Suppose a group G is acting properly on X by isometries fixing $p \in X(\infty)$, leaving each horosphere H_t invariant.*

If the action of G on a horosphere H_t is free then there exists a locally finite simplicial complex of dimension $\leq n-1$, L_t, such that G acts freely on L_t by simplicial isomorphisms and there is a G-equivariant continuous map $f_t : H_t \to L_t$.

Proof By Theorem 4.1, the dimension of H_t is at most $n-1$. Since the action of G on H_t is free and proper, we have a covering $\pi : H_t \to H_t/G$. In this proof, let $B(x,r)$ denote the open metric ball in H_t centered at $x \in H_t$ and of radius $r > 0$ with respect to the metric on X. Since the action of G is proper, for each $x \in H_t$ there exists a number $\rho(x) > 0$ such that $B_x := B(x, \rho(x))$ does not intersect gB_x whenever $g \in G \setminus \{e\}$. We may assume that for all $x \in X, g \in G$, $\rho(gx) = \rho(x)$, so that $B_{gx} = gB_x$. Since H_t/G has covering dimension $\leq n-1$ and since $\{\pi(B_x)\}_{x\in H_t}$ is an open covering of H_t/G, according to Corollary of II.6 in [Na] we can choose a locally finite refinement $\{U\}$ of it with order $\leq n$, i.e., for any compact set in H_t/G, there are only finitely many U's which intersect the compact set, and any distinct $n+1$ elements in $\{U\}$ has empty intersection.

We take a lift $\hat{U}$ of each U in H_t. Then, by the properness of the action of G, the family $\mathcal{U} := \{g\hat{U}\}_{U,g\in G}$ is a locally finite open covering of H_t. Let us now prove that $\mathcal{U}$ has order $\leq n$. In fact, if not, there would exist $n+1$ different open sets $g_i\hat{U}_i$, $i = 1,2,\cdots,n+1$, in $\mathcal{U}$ that contain a common point. We can find $x_i \in H_t$ and $g_i' \in G$ for each i such that $g_i\hat{U}_i$ is contained in $g_i' B_{x_i}$. Therefore, by recalling the definition of ρ, $g_i\hat{U}_i$ does not intersect $g\hat{U}_i$ for any $g \in G$ with $g \neq g_i$, and in particular, if $i \neq j$, then $\hat{U}_j$ is different from $g\hat{U}_i$ for any $g \in G$. Thus $\pi(\hat{U}_i)$ for $i = 1,2,\cdots,n+1$ are all different, but have a common intersection point, which contradicts that the order of $\{U\}$ is $\leq n$. Thus $\mathcal{U}$ has order $\leq n$.

Let L_t be the nerve of $\mathcal{U}$, which is a locally finite simplicial complex of dimension $\leq n-1$ since $\mathrm{ord}(\mathcal{U}) \leq n$. Denote its geometric realization also by L_t. L_t (and H_t) may not be connected. Since G acts freely on the set $\mathcal{U}$, we have an induced free action of G on L_t. We shall construct a G-equivariant continuous map $f : H_t \to L_t$. Let $w_V(x) := d(x, H_t \setminus V)$ for $x \in H_t$ and $V \in \mathcal{U}$. Note that w_V is a continuous function with the property that $w_V > 0$ on V and $w_V = 0$ on $H_t \setminus V$. Using this set of functions as "coordinates" we define a map $f_t : H_t \to L_t$ as follows. For a point $x \in H_t$, let $V_1, V_2, \cdots, V_m \in \mathcal{U}$ be the open sets which contain x, so that $1 \leq m \leq n$. By the definition of L_t there is a simplex, σ, of dimension $m-1$ which corresponds to the sets V_i's. Since L_t is a simplicial complex we can put a system of linear coordinates on all simplexes so that they are canonical in terms of restricting them to faces of each simplex. We define $f_t(x) \in \sigma$ to be the point with the coordinates

$$\left(\frac{w_{V_1}(x)}{w(x)}, \cdots, \frac{w_{V_m}(x)}{w(x)}\right),$$

where $w(x) = \sum_{i=1}^{m} w_{V_i}(x)$. By construction f_t is a (well-defined) continuous and G-equivariant map. □

Proposition 4.4 *Let X be a proper* CAT(0) *space of dimension $n < \infty$. Suppose a group G is acting properly on X by isometries. Assume G fixes a point $p \in X(\infty)$ and leaves each horosphere H_t centered at p invariant. If G has a finite $K(G,1)$-complex then $\mathrm{cd}\, G \leq n-1$. In particular, if $n = 2$ then G is free.*

Proof Since G has a finite $K(G,1)$, G is torsion-free, so that the action on X is free. Fix a Busemann function b_p associated to p, and let H_t and B_t be the set of horospheres and horoballs defined by

$$H_t = \{x \in X \mid b_p(x) = t\}, B_t = \{x \in X \mid b_p(x) \leq t\}.$$

Each of them is G-invariant. Let K be the universal covering of a finite $K(G,1)$ complex. We claim that there exists u such that there is a G-equivariant continuous map $h: K \to H_u$.

We first construct a map f from K to X such that the image of K^0 is in one horosphere. Fix s and give a G-equivariant map $f: K^0 \to H_s$ arbitrarily. Since X is contractible, K/G is a finite complex and the action of G on K is free, one can extend this map continuously to a G-equivariant map $f: K \to X$.

We now "project" the map f so that the whole image is contained in one horosphere (maybe different from H_s). By construction, there is a constant C such that $f(K) \subset B_{s+C} - B_{s-C}$. Define a projection

$$\mathrm{pr}: B_{s+C} - B_{s-C} \to H_{s-C}$$

by $\mathrm{pr}(x) = [x,p] \cap H_{s-C}$, where $[x,p]$ is the unique geodesic from x to p. This projection is G-equivariant. Let

$$h = \mathrm{pr} \circ f : K \to H_{s-C}.$$

This is G-equivariant as well. Put $u = s - C$. We obtained a desired map h.

By Proposition 4.3, for each t there is a G-equivariant continuous map $f_t: H_t \to L_t$, where L_t is a locally finite simplicial complex of dim $\leq n-1$ on which G acts freely by simplicial isomorphisms. Let's write L_u as H (because this is a substitution to a horosphere) from now on and consider the composition $k = f_u \circ h : K \to H$. This is a continuous, G-equivariant map.

Since K is contractible, and H is a simplicial complex on which G acts freely by simplicial isomorphisms, there exists a G-equivariant homotopy inverse of k, $g: H \to K$, i.e., $g \circ k$ is G-equivariantly homotopic to the identity of K. We will show $H^i(G, \mathbb{Z}G) = 0$ for $i > n-1$, which implies $\operatorname{cd} G \leq n-1$.

In general we have

$$H^*(G, M) = H^*(K(G,1); \mathcal{M}),$$

where M is a G-module and $\mathcal{M}$ is the local coefficient system on $K(G,1)$ associated to M (p59 [Br]). Therefore since K/G is $K(G,1)$, it suffices to show $H^i(K/G; \mathbb{Z}G) = 0$ for $i > n-1$, where we abuse a notation and $\mathbb{Z}G$ denotes the local coefficient system on K/G with the natural action of $G = \pi_1(K/G)$ on $\mathbb{Z}G$.

Let k, g denote the continuous maps induced by k, g on $K/G, H/G$ as well. Consider the induced homomorphisms $g^*: H^i(K/G; \mathbb{Z}G) \to H^i(H/G; \mathbb{Z}G), k^*: H^i(H/G; \mathbb{Z}G) \to H^i(K/G; \mathbb{Z}G)$. Then the composition $k^* \circ g^*$ is an isomorphism of $H^i(K/G; \mathbb{Z}G)$ because the composition $g_* \circ k_*$ is the identity on

$\pi_1(K/G) = G$, so that the action of G on $\mathbb{Z}G$ is the original one. For $i > n-1$, clearly $H^i(H/G;\mathbb{Z}G) = 0$ because $\dim(H/G) = \dim H \le n-1$, so that $H^i(K/G;\mathbb{Z}G) = 0$ as well. □

5 Applications

In this section we first apply Proposition 4.4 to free abelian groups and prove Proposition 5.1. Then we discuss the example by Brady-Crisp [BraC], the example by Bridson [Bri] and new examples in connection to Problem 1.1. An interesting and new aspect about the new examples is that although they do not act properly, nor even freely, on any CAT(0) spaces by semi-simple isometries at all, they act properly on some proper CAT(0) spaces (indeed even CAT(-1) spaces) if we allow parabolic isometries. However, the minimal dimensions of such CAT(0) spaces are strictly bigger than the geometric dimensions. See Corollary 5.1 for the precise statement.

5.1 Flat torus theorems

The following is called the flat torus theorem. See 7.1 in Chap II, [BriH] for a proof.

Theorem 5.1 (Flat torus theorem [BriH]) *Let G be a free abelian group of rank n acting properly on a complete CAT(0) space X by semi-simple isometries. Then there is a G-invariant, convex subspace in X which is isometric to the Euclidean space $\mathbf{E}^n$ of dimension n such that the quotient of the action by G is an n-torus.*

Remark 5.1 In the above theorem, in fact, it suffices to assume that the action is only K-proper. An argument is same as the one in [BriH]. Of course, the freeness of the action is not enough. Think of a free action of $\mathbb{Z}^2$ on $\mathbf{R}$ by isometries.

The following is a main result of the paper. We discuss several applications of this proposition in the rest of the paper.

Proposition 5.1 *Let X be a proper* CAT(0) *space. Suppose a free abelian group of rank n, $\mathbb{Z}^n$, is acting properly on X by isometries. Then* $\dim X \ge n$*. If* $\dim X = n$ *then the action is by semi-simple isometries.*

Proof First of all, the dimension of X is at least n since $X/\mathbb{Z}^n$ is a $K(\pi,1)$ space and $\operatorname{cd}\mathbb{Z}^n = \operatorname{geom}\dim\mathbb{Z}^n = n$.

Suppose there is a parabolic isometry $f \in \mathbb{Z}^n$ on X. Then since X is proper, by Proposition 3.1 there exists $p \in X(\infty)$ such that $\mathbb{Z}^n$ fixes p and leaves each horosphere centered at p invariant. Since $\mathbb{Z}^n$ has a finite $K(\pi,1)$ it follows from Proposition 4.4 that $n = \operatorname{cd}\mathbb{Z}^n \leq \dim X - 1$. Therefore $n+1 \leq \dim X$. □

We obtain another torus theorem.

Theorem 5.2 (Flat torus theorem) *Suppose X is a proper geodesic space of dimension $n \geq 2$ whose fundamental group is $\mathbb{Z}^n$. If the universal covering of X is* CAT(0) *then X contains a totally geodesic flat n-torus which is a deformation retract.*

Proof Consider the proper isometric action of $G = \mathbb{Z}^n$ on the universal cover Y of X. By Proposition 5.1 the action is by semi-simple isometries. Then by Theorem 5.1, there is $P = \mathbf{E}^n \subset Y$ which is invariant by G. The flat n-torus $P/G \subset X$ is a deformation retract. The retraction is given by the projection of Y to P, which is G-equivariant. □

5.2 Nilpotent groups, Heisenberg groups and complex hyperbolic manifolds

Theorem 5.3 *Let G be a torsion-free, nilpotent group.*

(1) *If G acts freely on a* CAT(0) *space by semi-simple isometries, then G is abelian.*

(2) *Suppose G is finitely generated, not abelian, and acting properly on a proper* CAT(0) *space X by isometries. Then $\operatorname{cd} G \leq \dim X - 1$.*

(3) *Suppose G is finitely generated, and acting properly on a proper* CAT(0) *space X. If $\operatorname{cd} G = \dim X$ then G is abelian and the action is by semi-simple isometries.*

Proof (1) This claim is stated for manifolds of non-positive curvature in 7.4 Lemma (3) in [BGS]. Our argument is essentially same. Suppose G is not abelian. Let $C(G)$ be the center. There exists a non-trivial element $g \in C(G) \cap [G,G]$. Then g generates $\mathbb{Z}$ in G. If G is acting on a CAT(0) space X freely by isometries then g has to be parabolic. This is because if g was hyperbolic then the union of the axes of g in X would be isometric to $Y \times \mathbf{R}$, which is G-invariant. Moreover each $h \in G$ acts on $Y \times \mathbf{R}$ as a product of isometries on Y and $\mathbf{R}$. The induced isometric actions of G on $\mathbf{R}$ is by translations because g is a non-trivial translation and in the center. But since $g \in [G,G]$ the action of g on $\mathbf{R}$ must be trivial, which is a contradiction.

(2) Note that the action is free because G is torsion-free. Take g as in (1), so that g is parabolic. Since $g \in C(G)$, by Proposition 3.1, there is a point $p \in X(\infty)$ such that $G(p) = p$ and each horosphere H at p is G-invariant. It is a theorem of Malcev that there exists a simply connected nilpotent Lie group L which contains G as a co-compact lattice, so that L/G is a closed manifold of finite dimension with $\pi_1 \simeq G$. It follows that G has a finite $K(\pi,1)$-complex (for example take the nerve of a covering of L/G). Therefore by Proposition 4.4, $\operatorname{cd} G \leq \dim H \leq \dim X - 1$.

(3) By (2), G is abelian, and its rank is $\operatorname{cd} G$. Then we already know that the action has to be by semi-simple isometries by Proposition 5.1. □

Let $\mathbf{H}^n_{\mathbb{C}}$ denote the complex hyperbolic space of complex-dimension n (cf. [CheGr]). $\mathbf{H}^n_{\mathbb{C}}$ is a complete, $2n$-dimensional Riemannian manifold of sectional curvature bounded between -1 and $-1/4$. It is a proper $\mathrm{CAT}(-1)$ space of dimension $2n$.

Corollary 5.1 *For an integer $n \geq 1$, let G_n be the finitely presented group defined by a set of generators $\{a_1, b_1, \cdots, a_n, b_n, c\}$ and a set of relators as follows: all commutators in generators $= 1$ except for $[a_i, b_i] = c$ for all i.*

Then G_n is a finitely generated, torsion-free, nilpotent group with the following properties.

(1) $\operatorname{geom\,dim} G_n = 2n + 1$.

(2) *G_n does not act properly on any proper $\mathrm{CAT}(0)$ space of dimension $2n+1$ by isometries.*

(3) *G_n acts properly on $\mathbf{H}^{n+1}_{\mathbb{C}}$, which is a proper $\mathrm{CAT}(-1)$ space of dimension $2n+2$, by isometries such that all non-trivial elements are strictly parabolic.*

(4) *If G_n acts freely on a $\mathrm{CAT}(0)$ space by isometries, then c has to be parabolic. In particular, G_n does not act freely, hence not properly, on a $\mathrm{CAT}(0)$ space by semi-simple isometries, so that G_n is not a $\mathrm{CAT}(0)$ group.*

Proof It is known that G_n is a finitely generated, torsion free, nilpotent group such that $[G_n, G_n] = \langle c \rangle = C(G_n) \simeq \mathbb{Z}$, and $G_n/[G_n, G_n] \simeq \mathbb{Z}^{2n}$. They are called *(discrete) Heisenberg groups.*

(1) See p186 in [Br]. It is known that the *rank* (or *Hirsch number*) of G_n is equal to $\mathrm{cd}\, G_n$. And the rank of G_n is $2n+1$ because G_n has a central series $G_n > [G_n, G_n] \simeq \mathbb{Z} > 1$ with $G_n/[G_n, G_n] \simeq \mathbb{Z}^{2n}$.

Remark that if $n = 1$ then G is the fundamental group of a closed 3-manifold, which is a non-trivial circle bundle over a torus.

(2) Combine (1) and Theorem 5.3 (2).

(3) It is known that one can embed G in $\mathrm{Isom}(\mathbf{H}^{n+1}_{\mathbb{C}})$ as a discrete subgroup, so that G acts properly on $\mathbf{H}^{n+1}_{\mathbb{C}}$ by strictly parabolic isometries. To see it, fix a point in the ideal boundary and let $P_n < \mathrm{Isom}(\mathbf{H}^{n+1}_{\mathbb{C}})$ be the subgroup of all strictly parabolic isometries (and the trivial element) fixing this point. Then P_n is a nilpotent Lie group which is a central extension of $\mathbb{R}$ by $\mathbb{C}^n$. P_n is called the Heisenberg Lie group. If we take "integer points" in P_n we obtain $G_n < P_n$ as a discrete subgroup. One finds details on this paragraph in Proposition 4.1.1 in [CheGr].

(4) Suppose c is semi-simple on X. Let $Y \subset X$ be the union of axes of c. Y is isometric to $A \times \mathbf{R}$. But since c is in the center, as we saw in the proof of Theorem 5.3 (1), c has to act trivially on $\mathbf{R}$ because $c = [a_1, b_1]$, a contradiction. □

Question 5.1 Does G_n act properly on a CAT(0) space of dimension $2n+1$ by isometries? (No if the space is proper).

Corollary 5.2 *Let G be the fundamental group of a non-compact, complete, complex hyperbolic manifold M of complex-dimension $n \geq 2$. If the volume of M is finite, then*

(1) $\mathrm{cd}\, G = \mathrm{geom\,dim}\, G = 2n-1$.

(2) *G does not act properly on any proper $(2n-1)$-dimensional* CAT(0) *space by isometries.*

(3) *G acts properly on $\mathbf{H}^n_{\mathbb{C}}$, which is a proper $2n$-dimensional* CAT(−1) *space, by isometries.*

(4) *G does not act freely, hence not properly, on any* CAT(0) *space by semi-simple isometries, so that G is not a* CAT(0) *group.*

Proof (1) First of all $\dim M = 2n$. Since M is not compact, by 8.1 Proposition [Br], $\operatorname{cd} G < 2n$. Because the volume of M is finite, it has cusps. Let N be a cross-section of one of the cusps, and P its fundamental group. P is torsion-free. N is a closed, infra-nil manifold (i.e., finitely covered by a nil manifold) of dimension $2n-1$. The universal cover of N is diffeomorphic to $\mathbf{R}^{2n-1}$ since it is a horosphere of $\mathbf{H}^n_{\mathbb{C}}$. Thus $\operatorname{cd} P = 2n-1$, so that $\operatorname{cd} G = 2n-1$ as well because $P < G$ and $\operatorname{cd} G < 2n$. In general $\operatorname{cd} G = \operatorname{geom}\dim G$ if $\operatorname{cd} G \geq 3$.

(2) Because N is an infra-nil manifold, P is virtually nilpotent. But P is not virtually abelian. Let $P' < P$ be a nilpotent group of finite index. Then $\operatorname{cd} P' = 2n-1$ and P' is finitely generated. To argue by contradiction suppose G acts on a proper $\mathrm{CAT}(0)$ space X of dimension $2n-1$ properly by isometries. We apply Theorem 5.3 (3) to the action of $P' < G$ on X and conclude that P' is abelian. Impossible since P is not virtually abelian.

(3) The universal cover of M is isometric to $\mathbf{H}^n_{\mathbb{C}}$, which is a proper $\mathrm{CAT}(-1)$ space of dimension $2n$. The action of G on the universal cover of M is a desired one.

(4) This is similar to (2). To argue by contradiction suppose we had such action. The action is free. Let $P' < G$ be as in (2). Apply Theorem 5.3 (1) to the action of P' and get a contradiction. □

A parallel result holds for quaternionic hyperbolic spaces as well. Let $\mathbf{H}^n_{\mathbb{H}}$ denote the quoternionic hyperbolic space of quoternionic-dimension n (cf. [CheGr] for a unified treatment of real, complex and quoternionic hyperbolic spaces). $\mathbf{H}^n_{\mathbb{H}}$ is a complete, $4n$-dimensional Riemannian manifold of sectional curvature bounded between -1 and $-1/4$. It is a proper $\mathrm{CAT}(-1)$ space of dimension $4n$.

Corollary 5.3 *Let G be the fundamental group of a non-compact, complete, quoternionic hyperbolic manifold M of quoternionic dimension $n \geq 2$. If the volume of M is finite, then*

(1) $\operatorname{cd} G = \operatorname{geom}\dim G = 4n-1$.

(2) *G does not act properly on any proper $(4n-1)$-dimensional $\mathrm{CAT}(0)$ space by isometries.*

(3) *G acts properly on $\mathbf{H}^n_{\mathbb{H}}$, which is a proper $4n$-dimensional $\mathrm{CAT}(-1)$ space, by isometries.*

(4) *G does not act freely, hence not properly, on any $\mathrm{CAT}(0)$ space by semi-simple isometries, so that G is not a $\mathrm{CAT}(0)$ group.*

Proof An argument is nearly identical to the proof of Cor 5.2. Just replace $\mathbf{H}^n_{\mathbb{C}}$ by $\mathbf{H}^n_{\mathbb{H}}$, and $2n$ by $4n$ appropriately. □

Interestingly, there are different phenomena in the real hyperbolic case. Let M be the mapping torus of a homeomorphism, f, of the once puncture torus. $G = \pi_1(M)$ is an extension of F_2 by $\mathbb{Z}$ such that $\mathbb{Z}$ acts on F_2 by an automorphism, ϕ.

$$1 \to F_2 \to G \to \mathbb{Z} \to 0.$$

T.Brady, [Brad], showed the following.

Theorem 5.4 *Suppose G does not contain the direct product $F_2 \times \mathbb{Z}$ as a subgroup of finite index. Then G acts properly and co-compactly on some proper CAT(0) space of dimension 2 by isometries.*

He takes a 2-dimensional spine, X, of M, which is a 2-complex, and put a metric which makes X into a piecewise Euclidean 2-complex. He shows that the universal cover of X is a CAT(0) space. Since X is a retraction of M, the fundamental group is G, so that its action on the universal cover is a desired one. Note that the action is by semi-simple isometries.

On the other hand, it is known that if (and only if) f is a pseudo Anosov map then M admits a complete hyperbolic metric such that the volume is finite, with one cusp. A general result is due to Thurston (cf. Thm15.18, [Kapo]). There is a construction of a hyperbolic metric using ideal hyperbolic tetrahedra, which is originated by Jørgensen (cf. [Brad]). If M is a hyperbolic manifold, then its universal cover of M is isometric to $\mathbf{H}^3$, on which G acts properly by isometries. Note that a cusp subgroup is isomorphic to $\mathbb{Z}^2$, and acts on $\mathbf{H}^3$ by parabolic isometries.

5.3 Artin groups

The Artin group $A(m,n,p)$ is defined by the following presentation.

$$A(m,n,p) = \langle a,b,c \,|\, (a,b)_m = (b,a)_m, (b,c)_n = (c,b)_n, (a,c)_p = (c,a)_p \rangle,$$

where $(a,b)_m$ is the alternating product of length m of a's and b's starting with a. Brady and Crisp ([BraC] Thm3.1) showed the following theorem.

Theorem 5.5 (Brady-Crisp) *Let $A = A(m,n,2)$. Then we have the following.*

(1) *Let $m, n \geq 3$ be odd integers. Then A does not act properly on any complete* $\mathrm{CAT}(0)$ *space of dimension 2 by semi-simple isometries.*

(2) *With finitely many exceptions, for each A, there is a proper 3-dimensional* $\mathrm{CAT}(0)$ *space X on which A acts properly by isometries such that X/A is compact, hence the action is by semi-simple isometries, and in particular, A is a* $\mathrm{CAT}(0)$ *group.*

Remark 5.2 Theorem 5.5 (1) remains true if we weaken "properly" to "K-properly" in the statement. In their argument [BraC], a certain properness of the action is used in Th 1.3 (flat torus theorem), Lemma 1.4, Prop 2.2, where the K-properness is enough. Mostly it is about the properness of an action on a line. Combining those, they obtain flats for several $\mathbb{Z}^2$ subgroups in the beginning of §3, where the theorem is proven, then they do not use any properness of the action any more.

It is known that $A(m,n,p)$ has a 2-dimensional finite $K(A,1)$-complex if $\frac{1}{m}+\frac{1}{n}+\frac{1}{p} \leq 1$ ([ChaD]), so that $\operatorname{geom}\dim A = 2$ if $\frac{1}{m}+\frac{1}{n} \leq \frac{1}{2}$. Therefore there are infinitely many $A = A(m,n,2)$ such that $\operatorname{geom}\dim A = 2$ but $\mathrm{CAT}(0)\text{-}\dim_{ss} A = 3$.

Brady-Crisp asked if the conclusion of Theorem 5.5 (1) remains valid if one drops the assumption that the action is by semi-simple isometries. Using Proposition 5.1 we give an answer to this question in the case that X is proper and of dimension 2.

Theorem 5.6 *Let $m, n \geq 3$ be odd integers. The Artin group $A = A(m,n,2)$ does not act properly on any proper* $\mathrm{CAT}(0)$ *space of dimension 2 by isometries.*

Proof Brady-Crisp showed that if A is acting on a complete 2-dimensional $\mathrm{CAT}(0)$ space X properly by isometries then A contains a parabolic isometry, g. In fact their argument shows an additional property on g that we may assume that the element g is in a subgroup isomorphic to $\mathbb{Z}^2$.

To prove our theorem, let's assume that such action of A exists. Then as we have just said there is a subgroup $G < A$ isomorphic to $\mathbb{Z}^2$ which contains a non semi-simple isometry. But this is impossible by Proposition 5.1. □

Theorem 5.5 (2) and Theorem 5.6 imply Theorem 1.1.

Question 5.2 Does A in Theorem 5.5 act properly on a complete $\mathrm{CAT}(0)$ space of dimension 2 by isometries? (No if the space is proper).

5.4 An example by Bridson

Let B be a group with the following presentation.

$$B = \langle a, b, \gamma, s, t \mid \gamma a \gamma^{-1} = a^{-1}, \gamma b \gamma^{-1} = b^{-1}, sas^{-1} = [a, b] = tbt^{-1} \rangle.$$

A geodesic space is *locally* $CAT(0)$ if its universal cover is CAT(0). Bridson [Bri] showed the following theorem.

Theorem 5.7 (Bridson)

(1) $\operatorname{geom\,dim} B = 2$.

(2) *B does not act properly on any complete 2-dimensional* CAT(0) *space by semi-simple isometries.*

(3) *B is the fundamental group of a compact, locally* CAT(0), *3-dimensional cubed complex. In particular, B is a* CAT(0) *group.*

Remark 5.3 As in Theorem 5.5, one may wonder if one can weaken the condition "properly" to "K-properly" in the point (2) of this theorem. We do not know the answer. The only place he may really need the properness, not just the K-properness of an action in his argument is Prop 1.2, where he finds a minimal tree of an action by F_2 on some $\mathbf{R}$-tree. He uses a fact that if the action if proper, then the minimal tree is simplicial.

In (3), B acts on the universal cover of the cubed complex by semi-simple isometries, so that taking (2) into account we conclude that CAT(0)-$\dim_{ss} B = 3$. We show the following.

Theorem 5.8 *B does not act properly on any proper CAT(0) space of dimension 2 by isometries.*

Proof Suppose it did. In the proof of Theorem 5.7 (2), (cf. Proposition 3.1 in [Bri]), he in fact showed that if B acts properly by isometries on a complete CAT(0) space of dimension 2 then γ^2 must be parabolic. On the other hand, the subgroup generated by γ^2 and a is isomorphic to $\mathbb{Z}^2$, so that each element in this subgroup is semi-simple by Proposition 5.1, a contradiction. $\square$

We remark that Theorem 5.7 (3) and Theorem 5.8 imply Theorem 1.1 as well.

Question 5.3 Does B act on a complete CAT(0) space of dimension 2 properly by isometries? (No if the space is proper).

5.5 Solvable Baumslag-Solitar groups

A Baumslag-Solitar group is defined for non zero integers n, m by

$$BS(n,m) = \langle a, b \mid ab^n a^{-1} = b^m \rangle.$$

Suppose $|m| \geq 2$ in the following discussion.

(1) $BS(1,m)$ is torsion-free and solvable, and its geometric dimension is 2. One can construct a finite CW-complex of dimension 2 which is a $K(\pi,1)$-space for $BS(1,m)$ as follows. Take an annulus and glue one of its two boundary components to the other one such that it wraps m-times.

(2) Suppose $BS(1,m)$ is acting on a complete CAT(0) space freely by isometries. Then b is parabolic, because if not, then the relation $aba^{-1} = b^m$ would force the translation length of b, $l = \min d_b$, to satisfy $l = |m|l$, so that $l = 0$. This is impossible since b has an infinite order.

(3) $BS(1,m)$ acts on $\mathbf{H}^2$ by isometries freely. Define a presentation ϕ : $BS(1,m) \to SL(2,\mathbb{R})$ by

$$\phi(a) = \begin{pmatrix} \sqrt{|m|} & 0 \\ 0 & \sqrt{|m|}^{-1} \end{pmatrix}, \phi(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $PSL(2,\mathbb{R}) = \mathrm{Isom}(\mathbf{H}^2)$, ϕ gives a desired action, such that a acts as a hyperbolic isometry and b is parabolic.

But G does not act on $\mathbf{H}^n$ properly by isometries. To see this, suppose it did. Then b has to be parabolic by (2). Let $\alpha \in \mathbf{H}^n(\infty)$ be a point fixed by b. Since α is a unique fixed point of b and b^m as well, a has to fix α, so that G fixes α. G acts on the set of horospheres centered at α. Using a Busemann function associated to α we obtain an action of G on $\mathbf{R}$ by translations. b is in the kernel of the action. The action of G factors through the abelianization of G. Thus the kernel of the abelianization, which is isomorphic to $\mathbb{Z}[\frac{1}{m}]$, leaves each of the horospheres invariant, and the action on each horosphere is proper and by isometries. Each horosphere is isometric to $\mathbf{E}^{n-1}$, so that only a virtually free abelian group of finite rank can act on it properly by isometries by the Bieberbach theorem. But $\mathbb{Z}[\frac{1}{m}]$ is not a virtually free abelian group of finite rank, a contradiction.

(4) $BS(1,m)$ is an HNN-extension of $\mathbb{Z}$ as follows: $BS(1,m) = \mathbb{Z}*_{\mathbb{Z},f}$ such that the injective homomorphism f is given by $z \mapsto mz, z \in \mathbb{Z}$. The other injective homomorphism is the identity map. On the Bass-Serre tree, T, the element b is elliptic and a is hyperbolic. We made T into a CAT(0) space such

that each edge has length one. The tree T is a regular tree of index $m+1$. Note that for any vertex $v \in T$, the stabilizer is generated by some conjugate of b.

Set $X = \mathbf{H}^2 \times T$. This is a proper CAT(0) space of dimension three. Using the action on $\mathbf{H}^2$ given in (3) and the Bass-Serre action on T, we obtain the product isometric action of $BS(1,m)$ on X. The action is proper. To see this, let $x = (y,v) \in X$ be any point. We may assume that $v \in T$ is a vertex. Moreover, since the action of $BS(1,m)$ on T is transitive on the set of vertices, we may assume that the stabilizer subgroup of v is generated by b. Set $r = 1/2$. Then there are only finitely many elements $g \in G$ with $d(x, gx) \le r$. Indeed, if the inequality holds for g, then $d(v, gv) \le 1/2$ on T, so that g has to fix v. So, $g \in \langle b \rangle$. On the other hand, $d(y, gy) \le 1/2$ on $\mathbf{H}^2$, but there are only finitely many elements of the form b^n with $d(y, b^n y) \le 1/2$.

(5) Suppose $BS(1,m)$ acts on a proper CAT(0) space X properly by isometries. We recall a theorem by Adams-Ballmann. Remark that $\mathbf{E}^0$ means a point.

Theorem 5.9 (Adams-Ballmann [AB]) *Let X be a proper* CAT(0) *space. Suppose an amenable group G acts on X by isometries, then either*

(i) *there is a point $\alpha \in X(\infty)$ fixed by G, or*

(ii) *there is a G-invariant convex subspace in X which is isometric to a Euclidean space $\mathbf{E}^n, (n \ge 0)$.*

Since $G = BS(1,m)$ is solvable, hence amenable, by this theorem there is a point $\alpha \in X(\infty)$ fixed by G. This is because the case (ii) does not occur by the Bieberbach theorem. G acts on the set of horospheres centered at α. As in (3), this gives a homomorphism h from G to $\mathbb{R}$. Clearly $h(b) = 0$.

Now suppose $\dim X = 2$ (we do not know if such actions exist. See the questions below). Then the homomorphism h has to be non-trivial, since otherwise G leaves a horosphere invariant, which would imply that G is free by Proposition 4.4. It follows that $h(a) \neq 0$. Therefore the kernel of h is isomorphic to $\mathbb{Z}[\frac{1}{m}]$. This subgroup is acting on a horosphere properly by isometries.

We record our discussion on $BS(1,m) = \langle a, b \mid aba^{-1} = b^m \rangle$ as follows.

Proposition 5.2 *Let $G = BS(1,m)$ such that $|m| \ge 2$.*

(1) $\operatorname{geom}\dim G = 2$.

(2) *If G acts freely on a* CAT(0) *space by isometries then b is parabolic. In particular, G does not act freely, hence not properly, on any* CAT(0) *space by semi-simple isometries, so that G is not a* CAT(0) *group.*

(3) *G acts freely on $\mathbf{H}^2$ by isometries. But G does not act properly on $\mathbf{H}^n$ for any $n \geq 1$ by isometries.*

(4) *G acts properly on $\mathbf{H}^2 \times T$ by isometries where T is a regular tree of index $m+1$. $\mathbf{H}^2 \times T$ is a proper CAT(0) space of dimension 3.*

(5) *If G acts properly on a proper* CAT(0) *space X by isometries, then there exists a point $\alpha \in X(\infty)$ which is fixed by G. The action of G on the set of horospheres centered at α gives a homomorphism from G to $\mathbb{R}$. b is in the kernel. If $\dim X = 2$ then the kernel of the homomorphism is isomorphic to $\mathbb{Z}[\frac{1}{m}]$.*

Question 5.4 Does $BS(1,m)$ act properly on a CAT(0) space X of dimension 2?

If the answer is yes, then $\mathbb{Z}[\frac{1}{m}]$ acts on a horosphere of X. We do not know the answer even for proper spaces. See Prop 5.5 for the case of visible CAT(0) spaces. In connection to Theorem 5.9 we ask another question.

Question 5.5 Suppose an amenable group G acts properly on a proper, finite-dimensional CAT(0) space X by isometries. Suppose $\operatorname{cd} G = \dim X$. Then is G virtually abelian?

We know the answer is yes if G is a finitely generated, nilpotent group (Theorem 5.3). For example, we do not know the answer for $BS(1,2)$, which is solvable, hence amenable.

5.6 Torus bundle over a circle

We discuss another example, which is suggested by A.Casson. Let S be the group given by the following presentation.

$$S = \langle a, b, c \mid ab = ba, cac^{-1} = a^2 b, cbc^{-1} = ab \rangle.$$

S is solvable (but not virtually nilpotent), torsion free, and of cohomological dimension 3 because it is the fundamental group of a closed three-manifold, M, which is a torus bundle over a circle. M is a $K(\pi,1)$ space of S. The subgroup generated by a, b, which is isomorphic to $\mathbb{Z}^2$, is the fundamental group of the torus fiber, and the base circle gives the element c.

S is an HNN extension as $\mathbb{Z}^2 *_{\mathbb{Z}^2,f}$ such that the self monomorphism f of $\mathbb{Z}^2$ is the linear map given by a matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$. The other monomorphism to define the HNN extension is the identity.

S is a semi-direct product with the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ given by A:

$$0 \to \mathbb{Z}^2 \to S \to \mathbb{Z} \to 0.$$

Using the same matrix A, we produce a semi-direct product of $\mathbb{R}^2$ and $\mathbb{Z}$ as follows. S is a subgroup of G.

$$0 \to \mathbb{R}^2 \to G \to \mathbb{Z} \to 0.$$

The group multiplication in G for $(x,n),(y,m) \in \mathbb{R}^2 \times \mathbb{Z}$ is

$$(x,n)(y,m) = (A^m x + y, n+m).$$

To define a monomorphism $\sigma : G \to SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, let $p, 1/p$ be the (positive, real) eigenvalues of A, with eigenvectors $\alpha, \beta \in \mathbb{R}^2$. $p = (3+\sqrt{5})/2$. For $x \in \mathbb{R}^2$, write it uniquely as $x = a_x\alpha + b_x\beta, a_x, b_x \in \mathbb{R}$, and define

$$\sigma(x,0) = \left(\begin{pmatrix} 1 & a_x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b_x \\ 0 & 1 \end{pmatrix} \right).$$

For $n \in \mathbb{Z}$, define

$$\sigma(0,n) = \left(\begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix}^n, \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1/\sqrt{p} \end{pmatrix}^n \right).$$

Since $(x,n) \in \mathbb{R}^2 \times \mathbb{Z}$ is written as a product $(x,n) = (0,n)(x,0)$ in G, we define

$$\sigma(x,n) = \sigma(0,n)\sigma(x,0) = \left(\begin{pmatrix} 1/\sqrt{p}^n & a_x/\sqrt{p}^n \\ 0 & \sqrt{p}^n \end{pmatrix}, \begin{pmatrix} \sqrt{p}^n & b_x\sqrt{p}^n \\ 0 & 1/\sqrt{p}^n \end{pmatrix} \right).$$

Clearly σ is a one to one map on G. The map σ is a homomorphism, namely $\sigma(x,n)\sigma(y,m) = \sigma(A^m x + y, n+m)$. One can check this using

$$A^m x = a_x p^m \alpha + \frac{b_x}{p^m}\beta.$$

We have used that α, β are eigenvectors of A.

If we restrict σ to S, the image $\sigma(S)$ is a discrete subgroup in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, because if a sequence $\sigma(x_i, n_i)$ converges, then n_i eventually becomes constant, so that a_{x_i}, b_{x_i} become constant as well since they are discrete in $\mathbb{R}^2$. $PSL(2,\mathbb{R})$ is by definition $SL(2,\mathbb{R})/\pm 1$, which is isomorphic to $\mathrm{Isom}_+(\mathbf{H}^2)$, the group of orientation preserving isometries of $\mathbf{H}^2$. Naturally we obtain a homomorphism, which we also denote by σ, $\sigma : S \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. This is also a monomorphism, so that we obtain a proper, isometric action of S on $X = \mathbf{H}^2 \times \mathbf{H}^2$. The action is free because S is torsion-free. Therefore we obtain a non-compact, complete 4-dimensional Riemannian manifold $N = (\mathbf{H}^2 \times \mathbf{H}^2)/S$.

The manifold N is homeomorphic to $M \times \mathbf{R}$. To see this, let γ be the geodesic in $\mathbf{H}^2$ which is the positive part of the y-axis in the upper half plane model of $\mathbf{H}^2$. Suppose $\gamma(t)$ is parametrized by arc length, upwards, i.e., the y-coordinate increases as t increases. Note that γ is the common unique axis of hyperbolic isometries $\sigma(0,n), n \neq 0$, and $v = \gamma(\infty)$ is the common unique fixed point of $\sigma(x,0), x \neq 0$, so that v is also the unique fixed point of the action of S on $\mathbf{H}^2(\infty)$. The ideal boundary of $\mathbf{H}^2 \times \mathbf{H}^2$ is the spherical join of two copies of $\mathbf{H}^2(\infty)$. The spherical join of two v's in $(\mathbf{H}^2 \times \mathbf{H}^2)(\infty)$ is the set of fixed point of S, which is a segment of length $\pi/2$. Let w be the midpoint of this segment, which is $(\gamma \times \gamma)(\infty)$. Then each horosphere in $X = \mathbf{H}^2 \times \mathbf{H}^2$, $H_s, s \in \mathbb{R}$, centered at w is invariant by S. One can check this by the definition of σ. We remark that if we take a point different from w in the segment of the fixed points of S, the horospheres are permuted by c, while the elements a, b leaves each of them invariant.

Each H_s is diffeomorphic to $\mathbf{R}^3$, and moreover foliated by planes each of which is invariant by the subgroup generated by a, b in S, which is isomorphic to $\mathbb{Z}^2$. Indeed a leaf, L_u, of the foliation is the product of a horosphere, which is diffeomorphic to $\mathbf{R}$, in the first $\mathbf{H}^2$ in X and a horosphere in the second $\mathbf{H}^2$. By the definition of σ, L_u is invariant by $\langle a, b\rangle$. It follows that for each s, H_s/S is homeomorphic to M, and $N = X/S$ is homeomorphic to $M \times \mathbf{R}$. N is foliated by $H_s/S, s \in \mathbf{R}$, which is a product.

A simply connected, complete Riemannian manifold whose sectional curvature is non-positive is called an *Hadamard manifold.* Hadamard manifolds are proper CAT(0) spaces. It is easy to see that S does not act properly on any Hadamard manifold, X, of dimension 3 by isometries. Indeed if it did, then the action would be free and the quotient X/S is a manifold. It has to be closed since otherwise, the cohomological dimension of S would be less than 3, the dimension of X, which is impossible. But if the action is co-compact, then the action has to be by semi-simple isometries, which is impossible because the elements a, b are forced to be parabolic from the presentation of S.

We record the discussion.

Proposition 5.3 *Let*

$$S = \langle a, b, c \mid ab = ba, cac^{-1} = a^2 b, cbc^{-1} = ab\rangle.$$

S is the fundamental group of a closed 3-manifold, M, which is a torus bundle over a circle. Then

(1) $\operatorname{geom}\dim(S) = 3$.

(2) *S acts properly, hence freely, by isometries on $X = \mathbf{H}^2 \times \mathbf{H}^2$ such that X/S is homeomorphic to $M \times \mathbf{R}$.*

(3) *S does not act properly on any Hadamard manifold of dimension 3 by isometries.*

Question 5.6 Does S act properly on some 3-dimensional CAT(0) space by isometries?

Remark 5.4 If S acts properly on a proper CAT(0) space, X, by isometries, then by Theorem 5.9, there exists a point $v \in X(\infty)$ with $S(v) = v$ because S is solvable, hence amenable. Look at the action of S on the set, $\mathcal{H}$, of horospheres centered at v. Since the action factors through the abelianization of S, both a and b act trivially. Suppose that the dimension of X is 3. Then c has to act non-trivially on $\mathcal{H}$ since otherwise S would act trivially on $\mathcal{H}$, so that we can apply Prop 4.4 and get $\mathrm{cd}S \leq 2$, a contradiction.

A CAT(0) space, X, is called *visible* if for any two distinct points, p, q, in the ideal boundary, there exists a geodesic, γ, in X with $\gamma(\infty) = p, \gamma(-\infty) = q$. We know a partial answer to Question 5.6.

Proposition 5.4 *S does not act properly on any 3-dimensional proper CAT(0) space which is visible.*

Proof Suppose such an action did exist on X which is visible. Then, as we said in Remark 5.4, there is a point, $v \in X(\infty)$, with $S(v) = v$. Remark that since X is visible, a parabolic isometry has a unique fixed point in $X(\infty)$. This is a standard fact (cf. [FNS]).

We first show that c can not be parabolic. Indeed, if it was parabolic, a fixed point of c given by Prop 3.1 is the point v, so that c has to leave each horosphere centered at v invariant. Therefore, looking at the action of S on the set of the horospheres, we get a contradiction as in Remark 5.4. Thus, c is hyperbolic. Let γ be an axis of c. Then, either $\gamma(\infty)$ or $\gamma(-\infty)$ is v, since otherwise γ would bound a flat half plane in X because $c(v) = c$, which is impossible since X is visible.

In the following argument, without loss of generality, we assume that the action is a right action, and denote the result of the action by a group element g of a point y by yg, instead of gy or $g(y)$. For example, the point yg is mapped by h to ygh.

Suppose $\gamma(\infty) = v$. There is a constant R such that the displacement function of a on X, d_a, is at most R on $\gamma(t), t \geq 0$, because $a(\gamma(\infty)) = \gamma(\infty)$.

Set $x = \gamma(0)$. Let K be the set of elements $g \in B$ such that $d(x, xg) \leq R$. Then, $d(x, xcac^{-1}) = d(xc, xca) \leq R$, because xc is on $\gamma(t), t \geq 0$. Therefore $cac^{-1} \in K$. By the same reason, for all $n \geq 1$, $c^n a c^{-n} \in K$. On the other hand, they are all different elements. One can see it by rewriting each of them as a (positive) word in $\langle a, b \rangle$ using the relators in the presentation of S.

Suppose $\gamma(-\infty) = v$. Fix an integer $N > 0$. Let $x_N = xc^{-N}$. For each $0 \leq n \leq N$, $d(x_N, x_N c^n a c^{-n}) \leq R$, because $x_N c^n$ is on $\gamma(t), t \geq 0$. Therefore, there are at least $N+1$ group elements, g, with $d(x_N, x_N g) \leq R$. However, the number of those elements for each x_N does not depend on N, and also finite, because the points x_N are in one orbit of the S-action, which contains x. Since N is arbitrary, we get a contradiction. □

By the same argument, we can show the following proposition. In the proof, the roles of the elements c and a of S in the proof of Prop 5.4 are replaced by the elements a and b of $BS(1,m)$, respectively. We omit details. This is a partial answer to Question 5.4

Proposition 5.5 *$BS(1,m), m \geq 2$, does not act on a proper, visible CAT(0) space of dimension 2.*

5.7 Products

If one looks for a group with the gap between the cohomological dimension and a CAT(0) dimension to be 2, or bigger, natural candidates to be considered are the product of groups with the gap 1, which we already found. To prove a theorem in this regard using induction, one may want to use a splitting theorem. Among several versions of those(e.g., 6.21 in [BriH]), we quote the following one, which seems least restrictive.

Theorem 5.10 (Monod, [Mo]) *Suppose X is a proper CAT(0) space. Suppose $G = G_1 \times \cdots \times G_n$ is acting on X by isometries.*

Then, either there is a point in $X(\infty)$ fixed by G, or there is a non-empty closed convex G-invariant subset $Z \subset X$ which splits G-equivariantly, isometrically as $Z = Z_1 \times \cdots \times Z_n$ with an isometric action of G_i on each Z_i.

Although we do not know how to show a desired result, let's see what we can say. For example, let G be either the group A in Theorem 5.6 or the group B in Theorem 5.7. Set $H = G \times G$. Then $\mathrm{cd}\, H = 4$. Suppose H acts properly on a proper CAT(0) space, X, by isometries. Since $\mathrm{cd}\, H = 4$, $\dim X \geq 4$. We would like to show that $\dim X \geq 6$.

It is easy to see that $\dim X \neq 4$. Suppose $\dim X = 4$. Recall that G contains a subgroup, K, isomorphic to $\mathbb{Z}^2$. Consider the action of $K \times K$ on X. Since $K \times K \simeq \mathbb{Z}^4$, by Proposition 5.1, the action is by semi-simple isometries. Then, by the flat torus theorem (Theorem 5.1), there is a convex subspace in X which is isometric to $\mathbf{E}^2$ and invariant by $K \times \{e\}$. It is a standard fact (cf. 7.1, II in BrH) that the union of those flats in X is isometric to $\mathbf{E}^2 \times Y$ such that Y is a proper CAT(0) space. Since $\dim X = 4$, $\dim Y \leq 2$. The group G is acting properly on Y by isometries, because $\{e\} \times G$ centralizes $K \times \{e\}$. But this is impossible by Theorems 5.6, 5.7.

Therefore, the critical case is when $\dim X = 5$. Suppose $\dim X = 5$. Then we can show that H has to fix a point in $X(\infty)$, because, otherwise, by Theorem 5.10, H has to act on some product space $Z = Z_1 \times Z_2$ whose dimension is at most 5. Therefore either Z_1 or Z_2 has dimension less than 2, but it is impossible since G acts properly on it by isometries. In conclusion, to show that $\dim X \geq 6$, we are left with the case that H has a common fixed point in $X(\infty)$. Probably we also need to analyze a group action with a fixed point in the ideal boundary to deal with Questions 5.4, 5.6.

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Mathematics Institute, Tohoku University, Sendai 980-8578, Japan

Email: `fujiwara@math.tohoku.ac.jp`, `shioya@math.tohoku.ac.jp`, `sa1m28@math.tohoku.ac.jp`

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