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The boundary-Wecken classification of surfaces

Robert F. Brown
Michael R. Kelly

Abstract Let X be a compact 2-manifold with nonempty boundary ∂X and let $f\colon (X, \partial X) \to (X, \partial X)$ be a boundary-preserving map. Denote by $MF_\partial[f]$ the minimum number of fixed point among all boundary-preserving maps that are homotopic through boundary-preserving maps to f. The relative Nielsen number $N_\partial(f)$ is the sum of the number of essential fixed point classes of the restriction $\bar{f}\colon \partial X \to \partial X$ and the number of essential fixed point classes of f that do not contain essential fixed point classes of $\bar{f}$. We prove that if X is the Möbius band with one (open) disc removed, then $MF_\partial[f] - N_\partial(f) \leq 1$ for all maps $f\colon (X, \partial X) \to (X, \partial X)$. This result is the final step in the boundary-Wecken classification of surfaces, which is as follows. If X is the disc, annulus or Möbius band, then X is boundary-Wecken, that is, $MF_\partial[f] = N_\partial(f)$ for all boundary-preserving maps. If X is the disc with two discs removed or the Möbius band with one disc removed, then X is not boundary-Wecken, but $MF_\partial[f] - N_\partial(f) \leq 1$. All other surfaces are totally non-boundary-Wecken, that is, given an integer $k \geq 1$, there is a map $f_k\colon (X, \partial X) \to (X, \partial X)$ such that $MF_\partial[f_k] - N_\partial(f_k) \geq k$.

AMS Classification 55M20; 54H25, 57N05

Keywords Boundary-Wecken, relative Nielsen number, punctured Möbius band, boundary-preserving map

0 Introduction

Given a map $f\colon X \to X$ of a compact n-manifold with (possibly empty) boundary to itself, let $N(f)$ denote the Nielsen number of f, that is, the number of essential fixed point classes. For $MF[f]$ the minimum number of fixed points among all maps homotopic to f, we have $N(f) \leq MF[f]$. Wecken proved in [15] that if X is an n-manifold with $n \geq 3$, then $N(f) = MF[f]$ for all maps $f\colon X \to X$. Consequently, the property: $N(f) = MF[f]$ for every map $f\colon X \to X$ has come to be called the *Wecken property* of a manifold X and a manifold with this property is said to be a *Wecken manifold.* If X is a 2-manifold with Euler characteristic non-negative, then X is a Wecken manifold. The Wecken property of the Klein bottle is a consequence of Theorem 5.1 of [4] (see Corollary 8.3 of [5]), the property is verified for the torus in [1] and for the projective plane in [8]. The verification of the Wecken property for the four remaining such 2-manifolds (sphere, disc, annulus, Möbius band) is easy.

On the other hand, if X is a 2-manifold with negative Euler characteristic, then it is proved in [6] that X is *totally non-Wecken*, that is, given an integer $k \geq 1$, there is a map $f_k\colon X \to X$ such that $MF[f_k] - N(f_k) \geq k$. Thus we have

Theorem 1 (Wecken classification of surfaces) *Let X be a compact 2-manifold.*

(a) *If the Euler characteristic of X is non-negative, then X is Wecken.*

(b) *All other surfaces are totally non-Wecken.*

Now let X denote a compact n-manifold with nonempty boundary ∂X and consider boundary-preserving self-maps $f\colon (X, \partial X) \to (X, \partial X)$. We write the restriction of f to the boundary as $\bar{f}\colon \partial X \to \partial X$. The minimum number of fixed points among all maps of pairs homotopic to f through boundary-preserving maps is written as $MF_\partial[f]$. The relative Nielsen number, that we denote by $N_\partial(f)$, is defined to be the sum of the number of essential fixed point classes of $\bar{f}$ and the number of essential fixed point classes of f that do not contain essential fixed point classes of $\bar{f}$. We always have $N_\partial(f) \leq MF_\partial[f]$ and the n-manifold X is said to be *boundary-Wecken* if $N_\partial(f) = MF_\partial[f]$ for all maps of pairs $f\colon (X, \partial X) \to (X, \partial X)$. Schirmer [13] proved that all n-manifolds with nonempty boundary are boundary-Wecken if $n \geq 4$.

A compact n-manifold X with nonempty boundary is said to be *totally non-boundary-Wecken* if, given an integer $k \geq 1$, there is a map $f_k\colon (X, \partial X) \to (X, \partial X)$ such that $MF_\partial[f_k] - N_\partial(f_k) \geq k$. The manifold X is said to be *almost boundary-Wecken* if X is not boundary-Wecken but there exists an integer $B \geq 1$ such that $MF_\partial[f] - N_\partial(f) \leq B$ for all maps $f\colon (X, \partial X) \to (X, \partial X)$.

The purpose of this paper is to prove the following theorem:

Theorem 2 *Let X be the Möbius band with one (open) disc removed and let $f\colon (X, \partial X) \to (X, \partial X)$ be any map, then $MF_\partial[f] - N_\partial(f) \leq 1$.*

This result will allow us to complete the proof of the following:

Theorem 3 (Boundary-Wecken classification of surfaces) *Let X be a compact 2-manifold with nonempty boundary.*

(a) *If X is the disc, annulus or Möbius band, then X is boundary-Wecken.*

(b) *If X is the disc with two discs removed or the Möbius band with one disc removed, then X is almost boundary-Wecken, with $B = 1$.*

(c) *All other surfaces with boundary are totally non-boundary-Wecken.*

Proof Conclusion (a) is proved in [2]. With regard to conclusion (b), the proof that the disc with two discs removed is not boundary-Wecken is in [10] and it is proved in [2] that it is almost boundary-Wecken with $B = 1$. It is shown in [11] that the Möbius band with one disc removed is not boundary-Wecken so Theorem 2 will complete the proof of conclusion (b). The proof that the disc with three or more discs removed is totally non-boundary-Wecken is in [12]. For the Möbius band with two or more discs removed, the proof that it is totally non-boundary-Wecken is in [11]. For X any surface not yet listed, results from [2] and [10] imply that X is totally non-boundary-Wecken. Thus we have conclusion (c). □

To begin the proof of Theorem 2, we show in Section 1 that, for X the Möbius band with one disc removed, any map $f\colon (X, \partial X) \to (X, \partial X)$ can be homotoped as a map of pairs to a convenient "reduced form". For a map in that form, the inverse images of the arcs where handles are attached to a disc to form X are properly embedded 1-dimensional submanifolds of X. We classify the possible such submanifolds in Section 1 also.

Now let C_1 and C_2 be the components of ∂X and let $f_i\colon C_i \to \partial X$ be the restriction of f to C_i for $i = 1, 2$. The following terminology will be convenient for presenting the various cases to be checked in order to prove Theorem 2. If both the f_i are essential maps, we will say that f is *boundary essential.* If both f_i are inessential maps, then f is *boundary inessential.* If one of the maps f_1 or f_2 is essential and the other is inessential, we say that f is *boundary semi-essential.* Also, as in [2], we let $Im_\partial(f)$ denote the number of components of ∂X that contain points of $f(\partial X)$.

In Section 2, we prove Theorem 2 in the case that f is boundary inessential. Section 3 is devoted to the proof of Theorem 2 when f is boundary essential. We complete the proof of Theorem 2 in Section 4, by proving it for boundary semi-essential maps.

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1 Reduced form

Notation In Figure 1, we represent the punctured Möbius band X as a union of discs $X = D \cup H_1 \cup H_2$ where the handle H_1 is twisted, that is, $D \cup H_1$ is a Möbius band, and the handle H_2 is not twisted: $D \cup H_2$ is an annulus. We have $D \cap H_1 = A_1 \cup A_2$ and $D \cap H_2 = A_3 \cup A_4$ and we let A be the union of all the arcs A_j. We denote the arc that is the intersection of ∂D with the component C_2 of ∂X by T_2 and the three arcs that are the components of $\partial D \cap C_1$ are called T_1, T_3 and T_4, as the figure shows. We choose points $x_1 \in T_1$ and $x_2 \in T_2$ as well as $x_0 \in int\, X$ on the line segment, that we denote by $[x_1, x_2]$, with endpoints x_1 and x_2. The simple closed curves h_1 and h_2 generate the fundamental group of X based at x_0.

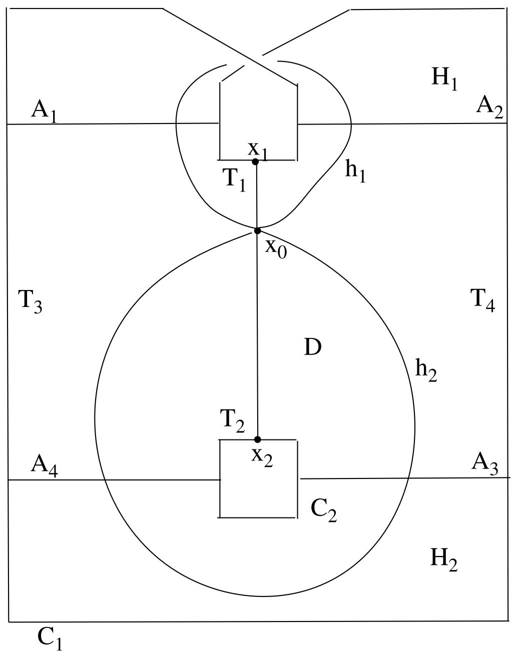

Figure 1

For a map $f\colon (X, \partial X) \to (X, \partial X)$, the restriction of f to the boundary component C_i will always be called f_i and we will use $C_{i\#}$ to denote the component of ∂X containing $f(C_i)$. Thus, corresponding to the map f, for $i = 1, 2$ we have the maps of simple closed curves $f_i\colon C_i \to C_{i\#}$. The absolute value of the degree of a map between simple closed curves is independent of the orientations of the curves, so it can be used without specifying orientations. For that reason, by the *degree* of a map between simple closed curves we will mean the absolute value. In particular, we denote the degrees of the maps f_i by $|d_i|$.

In order to analyze the boundary-preserving maps of X, we will need the classification of the simple closed curves in the interior of X that are transverse to A and have minimal intersection with it, up to isotopy. That is the purpose of the first two lemmas.

Lemma 1 *Let K be a simple closed curve in the interior of X that is transverse to A and such that the number of points in $K \cap A$ is minimal within the isotopy class of K. Then each of $K \cap H_1$ and $K \cap H_2$ has at most two components.*

Proof Write $K \cap H_1$ as a union of its components as $K \cap H_1 = J_1 \cup \ldots \cup J_{r_1}$. Each component J_j is an arc with its endpoints in A. If there is any components of $K \cap H_1$ with both endpoints in the same A_i, take the innermost such component, then it bounds a disc in $H_1 - K$ that can be used to reduce $K \cap A$ by two points, contrary to the assumption of minimal intersection. Thus we may assume that each component J_j has endpoints $p_{1j} \in A_1$ and $p_{2j} \in A_2$. Orienting the boundary of D in a clockwise direction induces an ordering on the A_i. Ordering the J_j so that $p_{1j} < p_{1,j+1}$ in the ordering of A_1 then, since the handle H_1 is twisted, $p_{2j} < p_{2,j+1}$ for all j. Let $L_1 \subset K \cap (D \cup H_2)$ be the component containing p_{11} and let x be the other endpoint of L_1. If $x = p_{21}$, then $J_1 \cup L_1$ is a simple closed curve so $r_1 = 1$. Now suppose $x \neq p_{21}$. If $x \neq p_{2r_1}$ then p_{2r_1} and the subarc of A_1 between p_{11} and p_{1r_1} are in different components of $(D \cup H_2) - L_1$ (which is disconnected because both endpoints of the arc L_1 are in the same component of the boundary of the annulus $D \cup H_2$). Since there could not then be an component of $K \cap (D \cup H_2)$ with endpoint p_{2r_1}, we conclude that the endpoints of L_1 are p_{11} and p_{2r_1}. A symmetric argument shows that if L_2 is the component of $K \cap (D \cup H_2)$ containing p_{21} then the other endpoint is p_{1r_1}. Since $J_1 \cup L_1 \cup J_2 \cup L_2$ is a simple closed curve, we conclude that $r_1 \leq 2$.

Writing $K \cap H_2 = J'_1 \cup \ldots \cup J'_{r_2}$, the arcs J'_j have endpoints $p_{3j} \in A_3$ and $p_{4j} \in A_4$. Ordering the J'_j so that $p_{3j} < p_{3,j+1}$ in the ordering of A_3 induced by the orientation of the boundary of D then, since the handle H_2 is not twisted, $p_{4j} > p_{4,j+1}$ for all j. Letting L'_1 denote the component of $K \cap (D \cup H_1)$ containing p_{31}, if the other endpoint is p_{41}, then $J'_1 \cup L'_1$ is a simple closed curve and $r_2 = 1$. Thus we suppose that endpoint is not p_{41}. If $L'_1 \subset D$, then p_{41} and the subarc of A_3 between p_{31} and p_{3r_2} are in different components of $D - L'_1$. Since there could not then be a component of $K \cap (D \cup H_1)$ with endpoint p_{41}, we conclude that $L'_1 \cap H_1$ is an arc. A symmetric argument shows that if $L'_{r_2} \subset D$ then no component of $K \cap (D \cup H_1)$ could have endpoint p_{4r_2} and so $L'_{r_2} \cap H_1$ is also an arc. Since we showed in the first part of the proof that there are at most two components of $K \cap H_1$, we conclude that $r_2 \leq 2$. $\square$

Let K be a simple closed curve in the interior of X that is transverse to A. If $K \cap H_j$ has r_j components for $j = 1, 2$, then we will say that K is *of type* (r_1, r_2). A simple closed curve in X is called *inessential* if it bounds a disc and *essential* otherwise.

Lemma 2 *Let K be an essential simple closed curve in the interior of X that is transverse to A and such that the number of points in $K \cap A$ is minimal within the isotopy class of K. Then the type (r_1, r_2) of K is one of the following: $(1,0), (2,0), (0,1), (1,1), (2,1)$ or $(2,2)$,*

Proof By Lemma 1, we know that the possible values of the r_j are $0, 1$ and 2. If $(r_1, r_2) = (0, 0)$ then K would bound a disc, in D, which is excluded since K is essential. We claim that K cannot be of type $(0, 2)$ or $(1, 2)$. We assume $r = 2$ and we establish a contradiction as follows. Let J'_1 be the component of K in H_2 with endpoints p_{j1} for $j = 3, 4$ and let L'_1 be the component in $D \cup H_1$ with endpoint p_{41}, then the other endpoint must be p_{32} (because if it were p_{31} then $J'_1 \cup L'_1$ would be a simple closed curve and if it were p_{42} that would contradict minimality because either L'_1 or L'_2 could be eliminated). If $L'_1 \subset D$, then p_{31} and p_{42} are in different components of $(D \cup H_1) - L'_1$ and so the case $(r_1, r_2) = (0, 2)$ cannot occur. If L'_1 intersects H_1, then, since p_{11} and p_{21} are in different components of $D - L'_1$, the case $(r_1, r_2) = (1, 2)$ also cannot occur. The six cases remaining are shown in Figure 2. $\square$

We will now describe a convenient form for a map $f\colon (X, \partial X) \to (X, \partial X)$ and then prove that we can always obtain it through a boundary-preserving homotopy. An arc in X is *properly embedded* if it intersects ∂X in its endpoints and at no other point. A compact 1-dimensional submanifold of X is said to be *properly embedded* if it is a union of simple closed curves in the interior of X and properly embedded arcs. A properly embedded arc K in X is *inessential* if there is an arc L in ∂X connecting the endpoints of K such that the simple closed curve $K \cup L$ bounds a disc in X. Otherwise, the properly embedded arc K is *essential.*

Definition A map $f\colon (X, \partial X) \to (X, \partial X)$ is said to be *in reduced form* if it has the following properties:

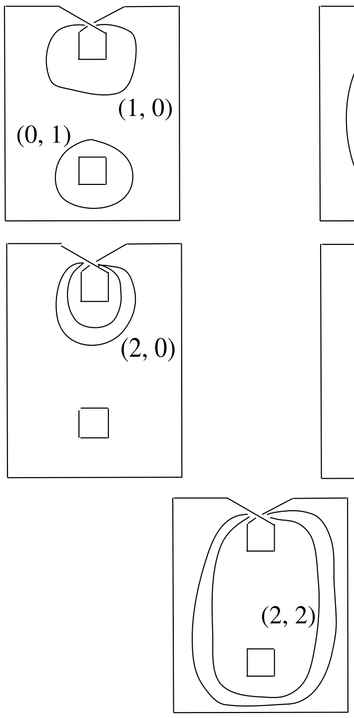

Figure 2

(1) for $i = 1, 2$, the map $f_i\colon C_i \to C_{i\#}$ is the constant map to $x_{i\#}$ if $|d_i| = 0$ and a $|d_i|$-to-one map otherwise,

(2) $f^{-1}(A)$ is a properly embedded 1-dimensional submanifold transverse to A such that each component of $f^{-1}(A)$ has minimal intersection with A within its proper isotopy class,

(3) all the components of $f^{-1}(A)$ are essential and

(4) if λ is an arc in A with endpoints in $f^{-1}(A)$ and interior disjoint from $f^{-1}(A)$, then the endpoints of λ are mapped to distinct components of A.

Lemma 3 *Given a map $f\colon (X, \partial X) \to (X, \partial X)$, there is a map g in reduced form that is homotopic, as a map of pairs, to f.*

Proof The restriction of f to ∂X can be made to satisfy condition (1) by the homotopy extension theorem. Without changing the map on the boundary, and thus condition (1) is retained, the map can then be modified, using the techniques of [9], so that the inverse image of A is a properly embedded 1-submanifold whose components make minimal intersection, within their proper homotopy class, with A. In this way, (2) is also satisfied. For condition (3), components that are inessential simple closed curves can be eliminated by an "innermost simple closed curve" argument and, similarly, inessential properly embedded arcs can be eliminated by an "outermost arc" argument. To simplify the notation, let us suppose that f already satisfies conditions (1), (2) and (3) of the definition of reduced form, then it remains to homotope f so that, if λ is an arc in A with endpoints in $f^{-1}(A)$ and interior disjoint from $f^{-1}(A)$, then the endpoints of λ are mapped to distinct components of A. To accomplish this, suppose the endpoints of such an arc λ are mapped to the same component of A. Since the interior of λ does not contain points of $f^{-1}(A)$, the path $f(\lambda)$ either lies entirely in D or entirely in one of the handles H_j. If $f(\lambda)$ lies in D then, since $f^{-1}(A)$ is transverse to A, there are neighborhoods of the endpoints of λ in A whose image lie in some handle H_j. We can homotope f so that the image of a neighborhood of λ in A lies entirely in that handle. Similarly, if $f(\lambda)$ lies in some handle H_j, then we can homotope f so that the neighborhood of λ is mapped to D. Repeating this procedure a finite number of times we obtain a map in reduced form homotopic to f. □

Theorem 4 *Let $f\colon (X, \partial X) \to (X, \partial X)$ be a map in reduced form. Then f may be homotoped to a map in reduced form with the property that every simple closed curve in the inverse image of A is of type $(0,1)$.*

Proof We claim that all the types of simple closed curves listed in Lemma 2 except $(0,1)$ may be excluded as a component of the inverse image of A for some map in reduced form homotopic to the given map. Suppose K is a component of $f^{-1}(A)$ of type $(1,0)$ or type $(1,1)$. Then there is a closed neighborhood N of K such that $N \cap f^{-1}(A) = K$ and $N - K$ is connected. Since $f(N - K)$ is a connected subset of $X - A$, it lies in one of the three components of $X - A$, each of which is contractible. We denote the component of $X - A$ that contains $f(N - K)$ by B. Since f maps the boundary ∂N of N into the contractible set B, we can extend $f|\partial N$ to a map $g\colon N \to B$. Extend g to all of X by letting it equal f outside of N. Then g is a reduced form of f with $g^{-1}(A) = f^{-1}(A) - K$. Repeating this procedure a finite number of times, we may assume we have a reduced form for f (which we still call f) in which no component of $f^{-1}(A)$ is a simple closed curve of type $(1,0)$ or type $(1,1)$. Since f is in reduced form, if λ is an arc in A with endpoints in $f^{-1}(A)$ and interior disjoint from $f^{-1}(A)$, then the endpoints of λ must be mapped to distinct components of A. We claim that this property implies that simple closed curves of types $(2,0),(2,1)$ and $(2,2)$ cannot be components of $f^{-1}(A)$. To prove it, let $K \subseteq f^{-1}(A)$ be an innermost simple closed curve of type $(2,n)$ for any $n = 0,1,2$, then $f(K)$ is connected so it lies entirely inside one of the A_j. Taking the two points of intersection of K with A_1, the arc $\lambda \subseteq A_1$ between them would have no points of the inverse image of A in its interior because there are now no simple closed curves of types $(1,0)$ or $(1,1)$. But the endpoints of λ map to the same component of A, contrary to the definition of reduced form. Since we have eliminated from the inverse image of A all simple closed curves of types $(0,1),(1,1),(2,0),(2,1)$ and $(2,2)$, we conclude that, if there are any simple closed curves, they must be of type $(0,1)$. □

In order to complete the analysis of the components of $f^{-1}(A)$ for f in reduced form, we turn now to the isotopy classification of properly embedded arcs in X. Recall that the components of $\partial D - (H_1 \cup H_2)$ are $T_1 \cup T_2 \cup T_3 \cup T_4$ where $T_2 \subset C_2$, as in Figure 1. If a properly embedded arc K is isotopic to a line segment in D with endpoints in T_i and T_j, then we will call K an arc *of type* $[i,j]$.

We begin with the classification of arcs K whose endpoints lie in different components of ∂X.

Theorem 5 *Let K be an arc in X that has minimal intersection with A in its isotopy class and whose endpoints lie in different components of ∂X. Then K is of type $[1,2]$ or of type $[2,3]$ and K lies entirely in D.*

Proof We can isotope K so that one endpoint lies in $T_1 \cup T_3 \cup T_4$ and the other in T_2 and K has minimal intersection with A within its isotopy class. It will be convenient to orient K so that we view K as ending in T_2. We will first show that the first intersection of K with $H_1 \cup H_2$ cannot be with H_1. The intersections of K with ∂D cannot begin $T_1A_1A_2$ because K could be isotoped to begin at T_4, so the intersection with A is not minimal. In the same way, the intersection $T_1A_2A_1$ can be isotoped to begin at T_3. If the intersections began with $T_3A_1A_2$ we can isotope to begin at T_1 whereas, if it began with $T_3A_2A_1$, the arc could not reach T_2. Checking the two corresponding cases for K beginning in T_4, we conclude that if K intersects $H_1 \cup H_2$, it must first intersect H_2. If K starts in T_1, intersects H_2 and then H_1, we can assume without loss of generality that the intersections are $T_1A_3A_4A_1A_2$, but then K cannot end in T_2. On the other hand if, after intersecting H_2, the arc K does not next intersect H_1, then it must have its last intersection with H_2. But it is clear that, if the last intersection is with H_2, we can isotope K to reduce the intersection with A, so that intersection was not minimal. Thus, if K starts at T_1 and has minimal intersection with A, then it does not intersect A at all. If K starts at T_3 and intersects A in the order $T_3A_3A_4$, it also would have its last intersection with H_2 and thus not have minimal intersection with A. The intersection with A is also not minimal if K begins $T_3A_4A_3$ because the first intersection can be reduced by isotoping K to begin in T_4. Symmetric arguments when K begins in T_4 lead us to conclude that, no matter where K begins, it cannot intersect $H_1 \cup H_2$ and so K lies entirely in D. Therefore K is isotopic to a line segment ending in T_2 and beginning in either T_1, T_3 or T_4. Since the line segment between T_2 and T_4 is isotopic to the line segment between T_2 and T_3, there are just the two possibilities given. □

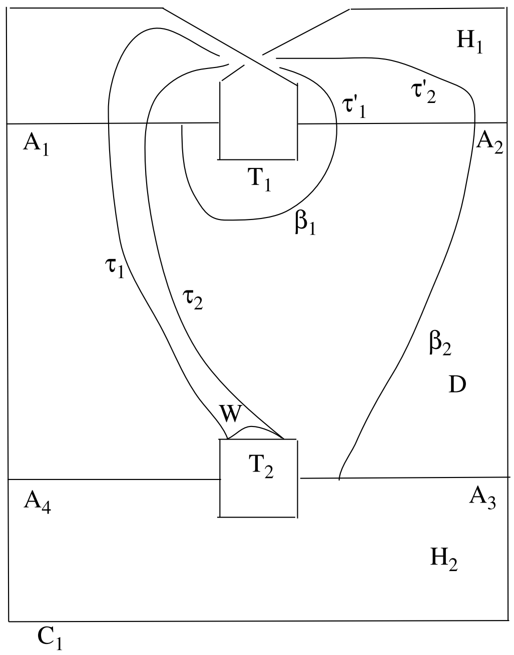

Figure 3

Lemma 4 *Let K be an arc in X, both of whose endpoints lie in C_2, such that the number of points in $K \cap A$ is minimal in the isotopy class of K. Let W be an arc in D connecting the endpoints of K and otherwise disjoint from K. Then the number of points in $(K \cup W) \cap A$ is minimal in the isotopy class of the simple closed curve $K \cup W$.*

Proof Let τ_1 and τ_2 be the two components of $K \cap D$ that intersect C_2 (see

Figure 3). Assuming, without loss of generality, that τ_1 intersects A_1. We note that the minimality condition on $K \cap A$ also assures us that τ_2 cannot intersect A_3 or A_4. Suppose τ_2 also intersects A_1. Consider those portions of K, which we denote by τ_i' for $i = 1, 2$, that start at the endpoints in A_1 of the respective τ_i and that intersect the A_i in the same order, and that are maximal with respect to this property. Figure 3 illustrates the case in which in the τ_i' intersect in the order $A_1 A_2$. The other possible case, in which the order is $A_1 A_2 A_3 A_4 A_1 A_2$, can be described by a similar figure. Let β_i denote the components of $K - A$ whose closures intersect the τ_i'. Since, by definition of the τ_i', the arcs β_1 and β_2 cannot have their endpoints in the same A_i, one must have an endpoint in A_1 and the other an endpoint in A_3, as we illustrated in Figure 3. Relative to those endpoints, we can isotope the portion of $K \cup W$ consisting of $W \cup \tau_1 \cup \tau_2 \cup \tau_1' \cup \tau_2' \cup \beta_1 \cup \beta_2$ to an arc in D with endpoints in A_1 and A_3. Thus, if we isotope $K \cup W$ so that its intersection with A is minimal, we obtain a simple closed curve that contains such an arc. However, as we can see in Figure 2, none of the simple closed curves that have minimal intersection with A can contain an arc in D connecting A_1 and A_3. Thus, assuming that τ_2 intersects A_1 has lead to a contradiction and we conclude that τ_2 intersects A_2, which implies that the number of points in $(K \cup W) \cap A$ is minimal in the isotopy class of the simple closed curve $K \cup W$. $\square$

The punctured Möbius band X can be constructed from a surface $\widetilde{X}$ that consists of the two-dimensional sphere with four symmetrically-placed open discs deleted, by identifying antipodal points. Consider a 90-degree rotation of $\widetilde{X}$ that is equivariant with respect to the antipodal action. Then, for $\phi\colon X \to X$ the corresponding homeomorphism, we have $\phi(C_2) = C_1$ and $\phi(C_1) = C_2$. Moreover, we may assume that $\phi(x_1) = x_2$ then, since ϕ^2 is the identity map, we have $\phi(x_2) = x_1$ as well.

Lemma 5 *Let K be an arc in X both of whose endpoints lie in the same component of ∂X. If both endpoints are in C_2, then K is isotopic to an arc obtained by removing an arc from a simple closed curve of type $(1,0), (2,0)$ or $(2,1)$ and connecting the endpoints of the arc that remains to C_2 by parallel arcs. Such arcs are said to be of types $\{1,0\}, \{2,0\}$ and $\{2,1\}$, respectively. If both endpoints are in C_1, then either K is of type $[1,3]$ or of type $[3,4]$, or it is isotopic to an arc with both endpoints in T_1 that passes once through H_2 and does not intersect H_1. Such an arc is said to be of type $\{0,1\}$.*

Proof Suppose both endpoints of K are in C_2. Isotope K so that the number of points in $K \cap A$ is minimal in the isotopy class of K. Let W be an arc in

D connecting the endpoints of K and otherwise disjoint from K. By Lemma 4, the number of points in $(K \cup W) \cap A$ is minimal in the isotopy class of the simple closed curve $K \cup W$ and therefore $K \cup W$ is of one of the six types of simple closed curves described by Lemma 2. Thus, once the arc K has been isotoped so that the number of points in $K \cap A$ is minimal in the isotopy class, it must have the same intersections with H_1 and H_2 as in that lemma. We can describe K itself by deleting an arc in D from a simple closed curve of Lemma 2 and connecting the endpoints of what remains with C_2 by parallel arcs. It is easy to check that, if we carry out this construction with the types $(0,1), (1,1)$ or $(2,2)$, then $K \cap A$ is not minimal in the isotopy class of K. That leaves the three types $(1,0), (2,0)$ and $(2,1)$; see Figure 4.

Now suppose both endpoints of K are in C_1. Then the endpoints of $\phi(K)$ are in C_2 and so $\phi(K)$ is isotopic to some arc K' that is one of the three types just described. Applying $\phi = \phi^{-1}$ once more, we have K isotopic to $\phi(K')$ and thus there are three isotopy types of arcs with both endpoints in C_1. To show that those three isotopy types are the ones described in the statement of the theorem, it suffices to prove that no two such arcs are boundary-preserving isotopic. Proper arcs are relative 1-cycles that represent classes of $H_1(X, \partial X)$. If two properly embedded arcs are boundary-preserving isotopic, then the corresponding cycles are homologous. However, the arcs of type $[1,3], [3,4]$ and $\{0,1\}$ represent three different elements of $H_1(X, \partial X)$. □

In the proof of Lemma 5, we used the homeomorphism ϕ, that interchanges the components of ∂X, in order to apply our analysis of the isotopy classes of arcs with endpoints in C_2 to arcs whose endpoints lie in C_1. We will also use ϕ so that information about minimizing the number of fixed points through a homotopy can be extended to additional cases by means of the following, easily verified, result.

Lemma 6 *Let $f\colon (X, \partial X) \to (X, \partial X)$ be a map and suppose that $g\colon (X, \partial X) \to (X, \partial X)$ is homotopic, as a map of pairs, to $\phi f \phi$, then $\phi g \phi$ is homotopic to f and the set of fixed points of $\phi g \phi$ is homeomophic to the set of fixed points of g.* □

If a map $f\colon (X, \partial X) \to (X, \partial X)$ is in reduced form, then for $i = 1, 2$, the map $f_i\colon C_i \to C_{i\#}$ is either the constant map at $x_{i\#} \notin A$ if $|d_i| = 0$ or, otherwise, f_i is a $|d_i|$-to-one map. If $|d_i| = 0$, then $f_i^{-1}(A) \cap C_i$ is the empty set.

Definition Suppose that $|d_i| \neq 0$ then, for each point of the finite set $f_i^{-1}(A) \cap C_i$ we record the integer $1, 2, 3$ or 4 corresponding to the component of A to which it is sent by f. As we go once around the simple closed curve C_i, we obtain an ordered set made up of the integers 1 through 4 which we call the *pattern* of f_i. Of course the pattern depends on where on C_i we begin our circuit of the simple closed curve and in which direction we go.

Lemma 7 *Given a map $f\colon (X, \partial X) \to (X, \partial X)$ in reduced form, if the degree $|d_i|$ of $f_i\colon C_i \to C_{i\#}$ is non-zero, then we may choose the starting point and direction so that the pattern of f_i is $(121234)^{|d_i|}$ if $i^\# = 1$ and $(34)^{|d_i|}$ if $i^\# = 2$.*

Proof Since C_1 intersects A in six points then, if $i^\# = 1$, it must be that $f_i^{-1}(A) \cap C_i$ consists of $6|d_i|$ points because f is in reduced form. Moreover, by starting at a suitable place in C_i and choosing the correct direction, the points of $f_i^{-1}(A) \cap C_i$ that we encounter will be first mapped to A_1, then to A_2, next to A_1 again and so on to produce the pattern $(121234)^{|d_i|}$ (see Figure 1). On the other hand, C_2 intersects A only at one point each of A_3 and A_4. Therefore, if $i^\# = 2$ then, choosing the correct starting point and direction, the pattern in this case will be $(34)^{|d_i|}$. □

Theorem 6 *Let $f\colon (X, \partial X) \to (X, \partial X)$ be a map in reduced form such that f_1 is essential, then every proper arc in $f^{-1}(A)$ is one of the following types: $[1,2], [1,3], [2,3]$ or $\{1,0\}$.*

Proof We first show that there are no arcs of type $[3,4]$ in $f^{-1}(A)$. Suppose that there are and let K by the "innermost" such arc. That is, letting L be the component of $X - K$ that contains C_2, then no other arc of that type is in L. Arcs that are components of $f^{-1}(A)$ that lie in L must be of type $[2,3]$. Let $f(K) = A_j$. If there were no arcs of type $[2,3]$ in L, the pattern of f_1 would contain jj, contrary to the Lemma 7. Thus there must be type $[2,3]$ arcs in L and these are the only components of $f^{-1}(A)$ that intersect C_2. Therefore the number of type $[2,3]$ arcs is either $6|d_2|$ or $2|d_2|$, depending on whether C_2 is mapped to C_1 or to C_2. But that is an *even* number in either case whereas, by Lemma 7, the pattern of f_1 is either $(121234)^{d_1}$ or $(34)^{|d_1|}$. A contradiction occurs because, in either possible pattern for f_1, the number of integers between any repetition of an integer j is *odd*. Thus there are no arcs of type $[3,4]$. Basically the same argument demonstrates that arcs of $\{0,1\}$ cannot be components of $f^{-1}(A)$, with arcs of type $[1,2]$ taking the role of the arcs of type $[2,3]$ in the previous argument.

We next eliminate arcs of $\{2,0\}$ as possible components of $f^{-1}(A)$. Suppose there are arcs of this type and let K denote the "innermost" such component.

That is, there is an arc α in C_2 connecting the endpoints of K, and with its interior in the component of $X - K$ that does not contain C_1, such that no other arc of type $\{0, 2\}$ intersects α. Let $f(K) = A_j$. If there were no other component of $f^{-1}(A)$ with endpoints in the interior of α, then the pattern of f_2 would contain jj, contrary to Lemma 7. However, a component of $f^{-1}(A)$ with endpoints in α would be of type $\{1, 0\}$ and the innermost of these would, in the same way, produce a pattern for f_2 contrary to Lemma 7. Thus there are no arcs of type $\{2, 0\}$ that are components of $f^{-1}(A)$.

The proof that there are no components of $f^{-1}(A)$ of type $\{2, 1\}$ is similar to the argument for type $\{2, 0\}$. This time let K be the "outermost" component of type $\{2, 1\}$. This means that there is an arc β in C_2 connecting the endpoints of K, and with its interior in the component of $X - K$ that does not contain C_1, such that no other arc of type $\{2, 1\}$ has its endpoints in β. If there were no components of $f^{-1}(A)$ intersecting the interior of β, the pattern of f_2 would again contain jj where $f(K) = A_j$, contrary to Lemma 7. If there were such components, however, they would be of type $\{1, 0\}$ and the "outermost" of these would lead to the same contradiction. Thus there are no components of $f^{-1}(A)$ that are arcs of type $\{2, 1\}$. By Theorem 5 and Lemma 5, the four types listed in the statement of the theorem are the only possibilities remaining (see Figure 4). □

2 Boundary inessential maps

Definition Represent the points of the unit circle S^1 by $e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Let R be an annulus embedded in X and α an arc in R such that there is a homeomorphism $\psi\colon [0,1] \times S^1 \to R$ with $\psi([0,1] \times \{e^{i0} = 1\}) = \alpha$. Given a map $g\colon \alpha \to X$, the *annulus extension* of g is the map $g\colon R \to X$ defined by $g(\psi(t, e^{i\theta})) = g(\psi(t, 1))$ for all $0 \leq t \leq 1$ and $0 \leq \theta \leq 2\pi$.

Theorem 7 *If $f\colon (X, \partial X) \to (X, \partial X)$ is boundary inessential, then $MF_\partial[f] = N_\partial(f)$.*

Proof By Lemma 3, we may assume that f is in reduced form. Therefore the restriction of f to each component C_j is a constant map to one of the points x_1 or x_2; we write $f(C_j) = x_{j\#}$. Consequently, the components of $f^{-1}(A)$ must be simple closed curves that, by Theorem 4, we may assume are of type $(0, 1)$. See Figure 5. We denote the components by $K_1, \ldots, K_m$ where

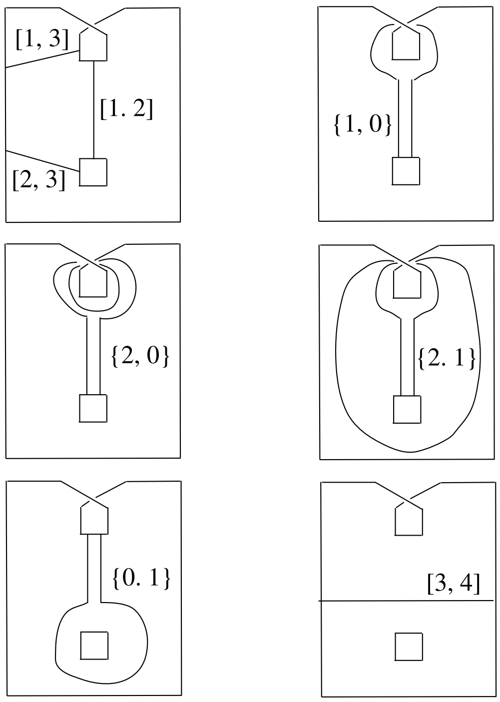

Figure 4

K_{i+1} is the component of $X - K_i$ containing C_2 and K_i and K_{i+1} bound an annulus R_i that contains no other component of $f^{-1}(A)$. We will describe a map $g\colon (X, \partial X) \to (X, \partial X)$ which is boundary-preserving homotopic to f, is also constant on each component of ∂X, and that has a single fixed point in the interior of X if $Im_\partial(f) = 2$ and $f(C_1) = C_2$. In that case, g has no fixed points on ∂X and, since its Lefschetz number is nonzero, we have $MF_\partial[f] = N_\partial(f)$. Otherwise, g will have no fixed points in the interior of X so obviously $MF_\partial[f] = N_\partial(f)$.

Let $[x_1, x_2]$ denote the line segment and let h_1 and h_2 be the simple closed curves shown in Figure 5.

We define a map g' that sends all of X to $G = [x_1, x_2] \cup h_1 \cup h_2$. Let R be the annulus in X bounded by the simple closed curves h_2 and C_2. We map the closure of $X - R$ to the line segment $[x_0, x_{1\#}]$ by sending h_2 to x_0 and extending. We will obtain g' on R by defining a map on the line segment $[x_0, x_2]$ and then extending to all of R by using annulus extensions. The behavior of the intersection of $[x_0, x_2]$ with each R_i will determine how g' is defined on R_i. We must send x_2 to $x_{2\#} = f(C_2)$, we let $g'(x_0') = x_0$ and we extend linearly to the line segment $[x_2, x'_0]$. Now each annulus R_i is mapped by f either to D or to one of the handles H_j and, for adjacent annuli R_i and R_{i+1}, one must go to D and the other to a handle. For each R_i mapped to D, choose a point x_i in the interior of the line segment $R_i \cap [x_0, x'_0]$. For each such i, the map g' sends the line segment $[x_i, x_{i+2}]$ once around h_1 or once around h_2, with x_i and x_{i+2} sent to x_0, depending on whether f maps R_{i+1} to H_1 or to H_2. Annulus extension on each R_i completes the definition of g'.

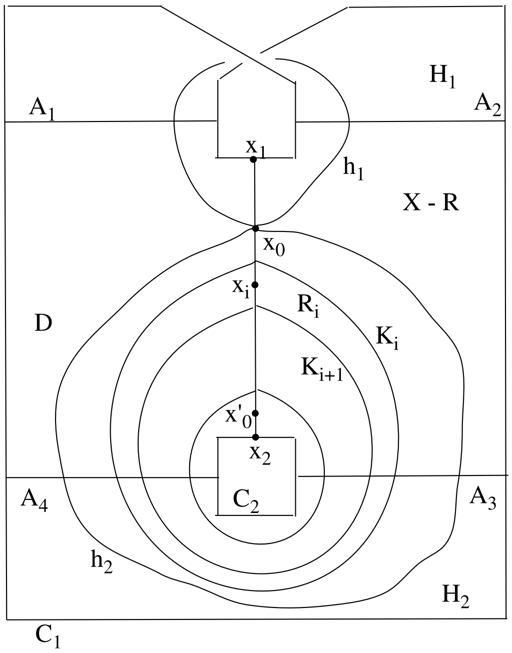

Figure 5

If $f(C_1) = x_1$, then g' is the identity on the line segment $[x_0, x_1]$ but there are no other fixed points on the interior of X. Let $G' = [x_1, x_2] \cup h'_1 \cup h'_2$ where h'_i is isotopic to h_i and intersects $[x_1, x_2]$ only at x_1. Let $\rho\colon G \to G'$ map $[x_1, x_2]$ to itself, taking $[x_1, x_0]$ to x_1 and extending linearly over $[x_0, x_2]$, and sending each h_i to h'_i by a homeomorphism. Note that $\rho g'$ is homotopic to f as a map of the pair $(X, \partial X)$. If $f(C_1) = x_1$ and $f(C_2) = x_2$, then x_1 and x_2 are the only fixed points of $g = \rho g'$. If $f(C_1) = f(C_2) = x_1$, then $g = \rho g'$ has a fixed point only at x_1. Now suppose $f(C_1) = f(C_2) = x_2$ then, by Lemma 6, the map $g = \phi \rho g' \phi$ is homotopic to f and its only fixed point is x_2. Finally, if $f(C_1) = x_2$ and $f(C_2) = x_1$, then we can set $g = g'$ because g' will have the required property, namely, one fixed point in the interior of X, at x_0. □

3 Boundary essential maps

Theorem 8 *If $f\colon (X,\partial X) \to (X,\partial X)$ is a boundary essential map such that $Im_\partial(f) = 2$, then $N_\partial(f) = MF_\partial[f]$.*

Proof Suppose for now that $f(C_i) = C_i$ for $i = 1,2$. Throughout the proof, homotopies will be understood to be relative to ∂X. We will show that f maps X onto itself. Let $Z = \partial X \cup A_1 \cup A_3$. Suppose f is not onto, then we may assume there is a point of $X - Z$ that is not in the image of f. By retracting, there is a map homotopic to f whose image is Z. We use [3], to obtain a map $g\colon X \to Z$ such that, for $i = 1,2$, the set $L_i = g^{-1}(x_i)$ is a 1-dimensional subpolyhedron of X. There is a component α_i of L_i that is an essential properly embedded arc or an essential simple closed curve intersecting C_i in a single point or else connected to C_i by an arc meeting C_i in a point. If no such α_i existed, there would be a simple closed curve in the complement of L_i that is isotopic to C_i. But that would imply that $f_i\colon C_i \to C_i$ is inessential, contrary to the hypotheses. If α_1 and α_2 are both essential properly embedded arcs, then it follows from Lemma 5 that $\alpha_1 \cap \alpha_2 \neq \emptyset$ (see Figure 4). If one or both of the α_i is a simple closed curve intersecting ∂X in one point or connected to it by an arc and it were true that $\alpha_1 \cap \alpha_2 = \emptyset$, then there would be nearby properly embedded arcs with the same property, contrary to Lemma 5, so we conclude that $\alpha_1 \cap \alpha_2 \neq \emptyset$. But $L_1 \cap L_2 = \emptyset$ by definition so the contradiction proves that f maps onto X. Every map of X that is homotopic to f through boundary-preserving homotopies is boundary-essential and therefore onto, as we have shown, so the geometric degree of f is nonzero. Consequently, by a theorem of Edmonds (see [14], Theorem 1.1 for a corrected version of the result), f is homotopic to qp where $p\colon X \to Y$ is a pinch and $q\colon Y \to X$ is an n-fold branched cover. Triangulate X so that the branch points are vertices and let v be the number of vertices in the triangulation. Triangulate Y so that q is a simplicial map, then the number of vertices of this triangulation of Y is $nv - b$ for some $b \geq 0$ depending on the branching. Therefore, the Euler characteristic of Y is $\chi(Y) = n\chi(X) - b = -n - b < 0$. If p were a nontrivial pinch, then there would be a subsurface P of X, not a disc, such that ∂P is connected and $Y = X/P$. But that would imply $\chi(Y) = \chi(X) - \chi(P) + 1 = -\chi(P) \geq 0$ so the pinch must be trivial, that is, $Y = X$ and p is the identity map. But then we have $-1 = -n - b$ which implies that $b = 0$ and $n = 1$, that is, q is a homeomorphism. It follows by [7] that $N_\partial(q) = MF_\partial[q]$ and therefore, since f is homotopic to q, that $N_\partial(f) = MF_\partial[f]$ when $f(C_i) = C_i$ for $i = 1,2$. Now suppose $f(C_1) = C_2$ and $f(C_2) = C_1$. Let $\phi\colon X \to X$ be the homeomorphism that interchanges C_1 and C_2. The map ϕf has the property that $\phi f(C_i) = C_i$ for $i = 1, 2$ so, as we have just demonstrated, ϕf is homotopic to a homeomorphism q. But then f is homotopic to the homeomorphism ϕq so we may again apply [7] to complete the proof. □

Theorem 9 *If $f\colon (X, \partial X) \to (X, \partial X)$ is a boundary essential map such that $Im_\partial(f) = 1$, then $N_\partial(f) = MF_\partial[f]$.*

Proof We first consider the case that $f(\partial X) = C_2$. We claim that f, assumed to be in reduced form, is homotopic modulo the boundary to a map $g\colon X \to C_2$ which agrees with f on ∂X and that would establish $N_\partial(f) = MF_\partial[f]$ in this case. Since f is an essential map on C_2, there are components of $f^{-1}(A)$ that are arcs with at least one endpoint in C_2. No simple closed curve of type $(0, 1)$ can lie in the complement of such an arc; see Figure 4. Thus there are no simple closed curves in $f^{-1}(A)$ since, by Theorem 4, we need consider only those of type $(0, 1)$. Each arc K that is a component of $f^{-1}(A)$ is mapped by f in such a way that the endpoints of K are mapped to the same point of $C_2 \cap A$; call that point a_j. We let g agree with f on ∂X and define g on each component K of $f^{-1}(A)$ by defining $g(K) = a_j$ where $f(K) \subseteq A_j$ for $j = 3, 4$. Each component V of $X - (f^{-1}(A) \cup \partial X)$ is a disk bounded by components of $f^{-1}(A)$ and arcs in ∂X. Thus g has been defined to send the simple closed curve ∂V to C_2 and it is homotopic to f on ∂V where f extends to V. Since the inclusion of C_2 into X induces a monomorphism of the fundamental groups, the degree of g on ∂V is zero and therefore it extends to a map of V to C_2. In this way we define $g\colon X \to C_2$ as we claimed. If $f(\partial C) = C_1$, we can apply the first part of the proof to $\phi f \phi$ to obtain $g\colon X \to C_2$. By Lemma 6, the map $\phi g \phi\colon X \to C_1$ establishes that $N_\partial(f) = MF_\partial[f]$ in this case. □

4 Boundary semi-essential maps

Lemma 8 *If $f\colon (X, \partial X) \to (X, \partial X)$ is a boundary semi-essential map where $f(C_1) = C_1$ and f_1 is essential, then $N_\partial(f) = MF_\partial[f]$.*

Proof We assume f is in reduced form then, since f_2 is inessential, either $f(C_2) = x_1 \in T_1$ or $f(C_2) = x_2 \in T_2$ so $f^{-1}(A)$ does not intersect C_2. By Theorem 6, there are just four possible types of arcs that can be components of $f^{-1}(A)$ and only those of type $[1, 3]$ fail to intersect C_2. By Theorem 4, the only type of simple closed curve that could be a component of $f^{-1}(A)$ is $(0, 1)$. Referring to Figure 6, the complement of the line segment $[y_1, y_2]$ has two components that we call U and L, where $C_2 \subset L$. All the simple closed curves in $f^{-1}(A)$ are contained in the component of the complement of h_2 that contains C_2. We have chosen a point z in $U \cap C_1$ such that the line segment $[x_0, z]$ does not intersect $f^{-1}(A)$. Since all the points of $f^{-1}(A) \cap C_1$ lie in U, we may modify f so that it maps all of $L \cap C_1$ to z, without adding any fixed points to f_1. The points x_3 and x_4 on the line segment between x_0 and x_2 are chosen so that the segments $[x_0, x_3]$ and $[x_2, x_4]$ are disjoint from $f^{-1}(A)$. We will construct a map $g\colon (X, \partial X) \to (X, \partial X)$ that is homotopic to f as a map of pairs and identical to f on the boundary of X such that g has no fixed points on the interior of X. We obtain g on L by defining it on the line segment $[x_0, x_2]$ and then, using annulus extensions, we extend it to all of L. We introduce notation as follows. Letting $h = h_1 \cup h_2$ we set $a_i = h \cap A_i$. We number the simple closed curves of $f^{-1}(A)$ as $K_1, \ldots, K_m$ where K_{j+1} is in the component of $X - K_j$ that contains C_2 and we set $k_j = K_j \cap [x_0, x_2]$. We define $g\colon [x_0, x_2] \to [z, x_0] \cup [x_0, x_2] \cup h$ in the following way. Let $g(x_0) = z$ and extend to $[x_0, x_3]$ as a homeomorphism onto $[z, x_0]$. For $j = 1, \ldots, m$ we have $f(K_j) \subseteq A_{i(j)}$ for some $1 \leq i(j) \leq 4$ and we define $g(k_j) = a_{i(j)}$. We then define g on $[x_3, k_1]$ by sending it to the arc in h connecting x_0 to $a_{i(1)}$ and on each $[k_j, k_{j+1}]$ by mapping to an arc in h that connects $a_{i(j)}$ to $a_{i(j+1)}$ through the component of $X - A$ that contains $f(k_j, k_{j+1})$. Setting $g(x_4) = x_0$, we define g on $[k_m, x_4]$ by mapping onto the arc in h that connects $a_{i(m)}$ to x_0. Finally, g maps $[x_4, x_2]$ homeomorphically onto $[x_0, f(C_2)]$. We note that g will have a fixed point at x_2 in the case that $f(C_2) = x_2$ and no fixed points if $f(C_2) = x_1$. Choose two simple closed curves in L in the complement of the K_i intersecting $[x_0, x_2]$ in x_3 and x_4 respectively. If we delete these simple closed curves as well as all the K_i from L, then the closure of each component is an annulus containing a segment of $[x_0, x_2]$. Applying annulus extensions to each of these, we now have g defined on L.

The map g is defined on U as follows. Each component of $f^{-1}(A_3 \cup A_4)$ has both its endpoints mapped a single point, either $C_1 \cap A_3$ or $C_1 \cap A_4$. Consider a component K of $f^{-1}(A_1 \cup A_2)$ that is adjacent to a component K' of $f^{-1}(A_3 \cup A_4)$. As the image of the two arcs of C_1 between K and K' must map to the same arc of $D \cap C_1$, then the endpoints of K must map to the same point, either $T_3 \cap A_1$ or $T_4 \cap A_2$. Next, for a component of $f^{-1}(A)$ adjacent to either K or K', its endpoints must also map to a single point by the same argument. Continuing in this manner, we conclude that both endpoints of any component K of $f^{-1}(A)$ are sent by f to the same point $a_K \in A \cap C_1$. Letting $g(K) = a_K$ for each component K of $f^{-1}(A)$ then, noting that $g([y_1, y_2]) = z$, we can extend to all of U so that $g(U) \subseteq C_1$ and $g^{-1}(A) = f^{-1}(A)$. Since

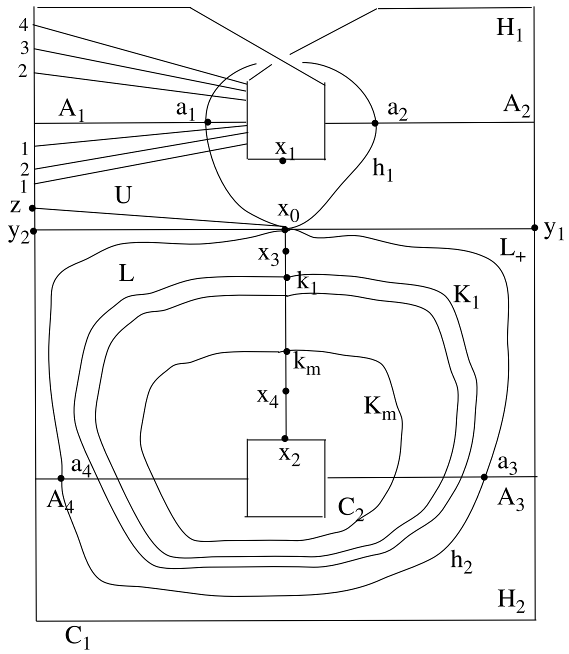

Figure 6

all the fixed points of g in U are therefore in C_1, we conclude that g has no fixed points except those on ∂X which are the fixed points of f on ∂X. This completes the definition of g on X and, since it has no fixed points in the interior of X, we have demonstrated that $N_\partial(f) = MF_\partial[f]$. □

Theorem 10 *If $f\colon (X, \partial X) \to (X, \partial X)$ is a boundary semi-essential map such that $Im_\partial(f) = 2$, then $N_\partial(f) = MF_\partial[f]$.*

Proof The proof is divided into four cases, depending on whether or not the boundary components are mapped to themselves and which of the two boundary maps is essential. The case in which each boundary component is mapped to itself by f, that f_1 is essential and f_2 is inessential is immediate from Lemma 8. Next we consider the case where still f_1 is essential and f_2 is inessential but now $f(C_1) = C_2$ and $f_2(C_2) = x_1$. Since f_2 is inessential, the components of $f^{-1}(A)$ are still the ones pictured in Figure 6. However, the arcs of type $[1, 3]$ would now be labelled $3434\ldots$ because only A_3 and A_4 intersect C_2. Clearly, both endpoints of each arc of type $[1, 3]$ must map to the same point. Thus we can define a map g such that $g(U) \subseteq C_2$ and $g([y_1, y_2]) = x_2$. Our definition of g on L will again consist of defining it on $[x_0, x_2]$ and extending to all of L by annulus extensions. First, g maps $[x_0, x_3]$ homeomorphically onto $[x_2, x_0]$ with $g(x_3) = x_0$, then g will have exactly one fixed point on $[x_0, x_3]$. Next g sends $[x_3, k_1]$ to the arc in h that connects x_0 to $f(k_1)$. As before, we continue in this manner so that g maps $[k_1, x_4]$ to h. Finally, $[x_4, x_2]$ is mapped homeomorphically onto the line segment $[x_0, x_1]$. Extending g to all of L by annulus extensions, we note that the map g has only a single fixed point on all of X. Since the Lefschetz number of f is nonzero, we conclude that $N_\partial(f) = MF_\partial[f] = 1$ in this case.

The remaining cases are (1) f_1 is an inessential map of C_1 to itself and f_2 is an essential map on C_2 and (2) f_1 is an inessential map from C_1 to C_2 and f_2 is an essential map from C_2 to C_1. Both cases follow by Lemma 6 from cases we have already established. □

The two results that follow complete the proof of Theorem 2 by analyzing the cases in which f is boundary semi-essential with $Im_\partial(f) = 1$.

Theorem 11 *If $f\colon (X, \partial X) \to (X, \partial X)$ is a boundary semi-essential map such that $f(\partial X) = C_1$ and f_1 is essential or if $f(\partial X) = C_2$ and f_2 is essential, then $N_\partial(f) = MF_\partial[f]$.*

Proof The case in which $f(\partial X) = C_1$ and f_1 is essential is included in Lemma 8 and the other case then follows by Lemma 6. □

Theorem 12 *If $f\colon (X, \partial X) \to (X, \partial X)$ is a boundary semi-essential map such that $f(\partial X) = C_2$ and f_1 is essential or if $f(\partial X) = C_1$ and f_2 is essential, then $MF_\partial[f] - N_\partial(f) \le 1$.*

Proof We will consider only the case that $f(\partial X) = C_2$ and f_1 is essential since the other case will follow from it by Lemma 6. In contrast to the two previous theorems, we cannot apply Lemma 8. However, the fact that f_1 is essential still allows us to use Theorem 6 to reduce the possible types of arcs among the components of $f^{-1}(A)$ to the four types listed there. Moreover, since f_2 is inessential, we may assume that no point of C_2 is mapped to A and therefore the only type of arc is $[1, 3]$. Theorem 4 thus assures us that we may still use Figure 6 for the present case. We will construct a map g that is boundary-preserving homotopic to f with the same number of fixed points on ∂X, that is $N(\bar{f})$ of them, and one fixed point in the interior of X. The construction is a modification of the proof of Lemma 8, as follows. We may homotope f on C_1 without changing the fixed points but mapping all of $C_1 \cap L$

to x_2. Define g: $[x_0, x_2] \to [x_0, x_2] \cup h$ by sending $[x_0, x_3]$ to x_0 and defining g on $[x_3, x_2]$ exactly as in the proof of Lemma 8. Extending g to L by annulus extensions gives us a map without any fixed points on the interior other than x_0.

In order to define g on U, observe that both endpoints of a component of $f^{-1}(A)$ in U are mapped to the same point of $C_2 \cap A$. Let V be the component of $U - f^{-1}(A)$ containing $[y_1, y_2]$, then g can map $U - V$ to C_2 in the same way it mapped U to C_1 in the proof of Lemma 8. The image under g of the boundary of V is contained in $[x_0, x_2] \cup T_2$. Since this set is contractible, we may extend g from ∂V to V and thus g is defined on all of X. There are no fixed points of g in U except x_0 and those on C_1. We conclude that g has a single fixed point on the interior of X. □

We conclude this paper with an observation about the difference between $MF_\partial[f]$ and $N_\partial(f)$ for f a boundary-preserving map of the punctured Möbius band X. An example in [11] shows that these are not always equal. In this paper we studied the question of equality by considering a classification of boundary-preserving maps based on their restrictions to the two boundary components of X. From Theorems 7 - 12 we can see that the only kind of map f: $(X, \partial X) \to (X, \partial X)$ for which $N_\partial(f)$ does not equal $MF_\partial[f]$ is one like the example from [11], which shows that Theorem 12 cannot be improved.

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Department of Mathematics, University of California
Los Angeles, CA 90095-1555, USA
and
Department of Mathematics and Computer Science, Loyola University
6363 St. Charles Avenue, New Orleans, LA 70118, USA

Email: `rfb@math.ucla.edu` and `kelly@loyno.edu`

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