ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 4 (2004) 1111–1123
Published: 25 November 2004

Alexander polynomial, finite type invariants and volume of hyperbolic knots

Efstratia Kalfagianni

Abstract We show that given $n > 0$, there exists a hyperbolic knot K with trivial Alexander polynomial, trivial finite type invariants of order $\leq n$, and such that the volume of the complement of K is larger than n. This contrasts with the known statement that the volume of the complement of a hyperbolic alternating knot is bounded above by a linear function of the coefficients of the Alexander polynomial of the knot. As a corollary to our main result we obtain that, for every $m > 0$, there exists a sequence of hyperbolic knots with trivial finite type invariants of order $\leq m$ but arbitrarily large volume. We discuss how our results fit within the framework of relations between the finite type invariants and the volume of hyperbolic knots, predicted by Kashaev's hyperbolic volume conjecture.

AMS Classification 57M25; 57M27, 57N16

Keywords Alexander polynomial, finite type invariants, hyperbolic knot, hyperbolic Dehn filling, volume.

1 Introduction

Let $c(K)$ denote the crossing number and let $\Delta_K(t) := \sum_{i=0}^{k} c_i t^i$ denote the Alexander polynomial of a knot K. If K is hyperbolic, let $\mathrm{vol}(S^3 \setminus K)$ denote the volume of its complement. The determinant of K is the quantity $\det(K) := |\Delta_K(-1)|$. Thus, in general, we have

$$\det(K) \leq ||\Delta_K(t)|| := \sum_{i=0}^{k} |c_i|. \tag{1}$$

It is well know that the degree of the Alexander polynomial of an alternating knot equals twice the genus of the knot. On the other hand one can easily construct knots with arbitrarily large genus and trivial Alexander polynomial by taking connected sums of untwisted Whitehead doubles. In this note, we construct hyperbolic knots of arbitrarily large genus and volume that have trivial Alexander polynomial. To put our results into the appropriate context, let us first recall what is known about the relation between volume and Alexander polynomial of alternating knots. In general, it is known (see [6] and references therein) that there exists a universal constant $C > 0$ such that if K is a hyperbolic knot with crossing number $c(K)$, then

$$\mathrm{vol}(S^3 \setminus K) \leq Cc(K). \tag{2}$$

If in addition K is alternating then one also has ([5])

$$c(K) \leq \det(K) \;\text{ and }\; \det(K) = ||\Delta_K(t)||. \tag{3}$$

By a result of Menasco [13], prime, alternating knots are either torus knots or hyperbolic. Combining this with (1)-(3), we derive that there is a universal constant $C > 0$ such that we have

$$\mathrm{vol}(S^3 \setminus K) \leq C\det(K) \;\text{ and }\; \mathrm{vol}(S^3 \setminus K) \leq C||\Delta_K(t)||, \tag{4}$$

for every prime, alternating non-torus knot K. (In fact, in [17], A. Stoimenow showed that the quantity $\det(K)$ in (4) can be replaced by $\log(\det(K))$.) Thus, the Alexander polynomial of prime alternating knots, dominates two of the most important geometric knot invariants; namely the volume and the genus. In contrast to that, the main result in this paper implies the following.

Corollary 1.1 *Given $n \in \mathbf{N}$, there exists a hyperbolic knot K with trivial Alexander polynomial for which we have* $\mathrm{vol}(S^3 \setminus K) > n$ *and* $\mathrm{genus}(K) > \frac{n}{6}$.

The development of quantum topology led to many new knot invariants that generalized the Alexander polynomial. These are the Jones type polynomial invariants (quantum invariants) and their perturbative counterparts known as finite type invariants (or Vassiliev invariants). An open conjecture relating these invariants to the geometry of the knot complement is the hyperbolic volume conjecture of R. Kashaev [12]. Kashaev's conjecture, as rephrased after the work of H. Murakami and J. Murakami [14], asserts that the volume of the complement of a hyperbolic knot K is equal to a certain limit of values of the colored Jones polynomials of K. A convenient way to organize these polynomials is via the so-called colored Jones function. This is a 2-variable formal power series

$$J_K(h, d) = \sum_{i \geq 0, j \leq i} a_{ij} h^i d^j \;\in \mathbf{Q}[h, d],$$

such that a_{ij} is a Vassiliev invariant of order i for K. By the Melvin-Morton-Rozansky conjecture, the first proof of which was given in [3], the diagonal subseries $D_K(h) := \sum_i a_{ii} h^i$ is essentially equivalent to the Alexander polynomial of K. The volume conjecture implies that the colored Jones function of a hyperbolic knot K determines the volume of the complement of K. In particular, it implies that if there exist two hyperbolic knots that have *all* of their finite type invariants the same then their complements should have equal volumes. In this setting, Corollary 1.1 says that the part $D_K(h)$ of $J_K(h,d)$ is far from controlling the volume of hyperbolic knots. The following theorem, which is the main result of the paper, provides a stronger assertion. It implies that there can be no universal value $n \in \mathbf{N}$ such that, for every hyperbolic knot K, the finite type invariants of orders $\leq n$ together with $D_K(h)$ determine the volume of the complement of K.

Theorem 1.2 *For every $n \in \mathbf{N}$, there exists a hyperbolic knot K such that:*

(a) *We have* $\Delta_K(t) = 1$.

(b) *All the finite type invariants of orders $< 2n-1$ vanish for K.*

(c) *We have* $\mathrm{vol}(S^3 \setminus K) > n$.

(d) *We have* $\mathrm{genus}(K) > \frac{n}{6}$.

Theorem 1.2 has the following corollary which implies that for every $m \in \mathbf{N}$, there are hyperbolic knots with arbitrarily large volume but trivial "m- truncated" colored Jones function. The proof of the corollary is spelled out at the end of Section 3.

Corollary 1.3 *For every $n, m \in \mathbf{N}$, there exists a hyperbolic knot K with trivial finite type invariants of orders $\leq m$ and such that* $\mathrm{vol}(S^3 \setminus K) > n$.

The proof of Theorem 1.2 combines results of [2] and [10] with techniques from hyperbolic geometry [18]. Given $n \in \mathbf{N}$, [2] provides a method for constructing knots with trivial Alexander polynomial and trivial finite type invariants of orders $< 2n-1$. To obtain hyperbolicity and appropriate volume growth we combine a result from [10] with the work of Thurston and a result of Adams [1]. The results of [2], [10] needed in this paper are summarized in Section 2 where we also prove some auxiliary lemmas. Section 3 is devoted to the proof of Theorem 1.2.

Acknowledgment The author thanks Oliver Dasbach and Xiao-Song Lin for useful discussions and correspondence. She also thanks Alex Stoimenow for his interest in this work and Dan Silver for helpful comments and for informing her about his work [16]. In addition, she acknowledges the support of the NSF through grants DMS-0104000 and DMS-0306995 and of the Institute for Advanced Study through a research grant.

2 Preliminaries

In this section we recall results and terminology that will be used in the subsequent sections and we prove some auxiliary lemmas needed for the proof of the main results. A crossing disc for a knot K is a disc D that intersects K only in its interior exactly twice with zero algebraic intersection number; the boundary ∂D is called a crossing circle. Let $q \in \mathbf{Z}$. A knot K' obtained from K by twisting q-times along D is said to be obtained by a generalized crossing change of order $|q|$. Clearly, if $q = 0$ we have $K = K'$ and if $|q| = 1$ then K, K' differ by an ordinary crossing change. Note that K' can also be viewed as the result from K under $\frac{1}{q}$-surgery of S^3 along ∂D.

Definition 2.1 [10] *We will say that K is n-adjacent to the unknot, for some $n \in \mathbf{N}$, if K admits an embedding containing n generalized crossings such that changing any $0 < m \leq n$ of them yields an embedding of the unknot. A collection of crossing circles corresponding to these crossings is called an n-trivializer.*

Let V be a solid torus in S^3 and suppose that a knot K is embedded in V so that it is geometrically essential. We will use the term "K is n-adjacent to the unknot in V" to mean the following: There exists an embedding of K in V that contains n generalized crossings such that changing any $0 < m \leq n$ of them transforms K into a knot that bounds an embedded disc in intV.

Recall that if K is a non-trivial satellite with *companion knot* $\hat{C}$ and *model knot* $\hat{K}$ then:

i) $\hat{C}$ is non-trivial;

ii) $\hat{K}$ is geometrically essential in a standardly embedded solid torus $V_1 \subset S^3$ but not isotopic to the core of V_1; and

iii) there is a homeomorphism $h : V_1 \longrightarrow V := h(V_1)$, such that $h(\hat{K}) = K$ and $\hat{C}$ is the core of V.

The following theorem will be used to ensure that for every $n > 0$ there is a hyperbolic knot that is n-adjacent to the unknot.

Theorem 2.2 ([10], Theorem 1.3) *Let K be a non-trivial satellite knot and let V be any companion solid torus of K. If K is n-adjacent to the unknot, for some $n > 0$, then it is n-adjacent to the unknot in V. Furthermore, any model knot of K is n-adjacent to the unknot in the standard solid torus V_1.*

The next result compiles the statements of Theorems 4.5, 5.2 and 5.6 of [2].

Theorem 2.3 [2] *Suppose that K is a knot that is n-adjacent to the unknot for some $n > 2$. Then, the following are true:*

(a) *All finite type invariants of orders less than $2n-1$ vanish for K.*

(b) *We have $\Delta_K(t) = 1$.*

Given a knot K, a collection of crossing circles $K_1, \ldots, K_n$ and an n-tuple of integers $\mathbf{r} := (r_1 \ldots, r_n)$, let $K(\mathbf{r})$ denote the knot obtained from K by performing a generalized crossing of order $|r_i|$ along K_i, for all $1 \leq i \leq n$.

Lemma 2.4 *Suppose that K is n-adjacent to the unknot, for some $n > 1$, and that the crossing circles $K_1, \ldots, K_n$ constitute an n-trivializer for K. Then, for every $\mathbf{r}$ as above, $K(\mathbf{r})$ is also n-adjacent to the unknot.*

Proof It follows immediately from Theorem 4.4 and Remark 5.5 of [2]. □

For a knot K that is n-adjacent to the unknot and an n-trivializer $L_n := K_1 \cup \ldots \cup K_n$ let $\eta(L_n \cup K)$ denote a tubular neighborhood of $L_n \cup K$. We close this section with two technical lemmas that we need in the next section.

Lemma 2.5 *Suppose that K is a non-trivial knot that is n-adjacent to the unknot, for some $n > 0$. Then for every n-trivializer L_n, the 3-manifold $M(K, L_n) := S^3 \setminus \eta(L_n \cup K)$ is irreducible and ∂ -incompressible.*

Proof Let $D_1, \ldots, D_n$ be crossing discs bounded by $K_1, \ldots, K_n$, respectively. Assume, on the contrary, that $M(K, L_n)$ contains an essential 2-sphere Σ. Assume that Σ has been isotoped so that the intersection $I := \Sigma \cap (\cup_{i=1}^n D_i)$ is minimal. We must have $I \neq \emptyset$ since otherwise Σ would bound a 3-ball in $M(K, L_n)$. Let $c \in (\Sigma \cap D_i)$ denote a component of I that is innermost on Σ; it bounds a disc $E \subset \Sigma$ whose interior is disjoint from $\cup_{i=1}^n D_i$. Since Σ is separating in $M(K, L_n)$, E can't contain just one point of $K \cap D_i$ and, by the minimality assumption, E can't be disjoint from K. Hence E contains both points of $K \cap D_i$ and so $c = \partial E$ is parallel to ∂D_i in $D_i \setminus K$. It follows that K_i bounds an embedded disc in the complement of K. But then no twist along K_i can unknot K, contrary to our assumption that K_i is part of an n-trivializer of K. This contradiction finishes the irreducibility claim. To finish the proof of the lemma, suppose that $M(K, L_n)$ is ∂ -compressible. Then, a component T of $\partial M(K, L_n)$ admits a compressing disc E'. Since K is non-trivial, we must have $T = \partial \eta(K_i)$, for some $1 \leq i \leq n$. But then, ∂E must be a longitude of $\eta(K_i)$. Thus K_i bounds a disc in the complement of K; a contradiction. □

Lemma 2.6 *Suppose that K is a knot that is embedded in a standard solid torus $V_1 \subset S^3$ so that it is geometrically essential in V_1. Suppose, moreover, that K is n-adjacent to the unknot in V_1, for some $n > 1$. Then, K is not the unknot.*

Proof Let L_n be an n-trivializer that exhibits K as n-adjacent to the unknot in V_1. Let $\mathcal{D}$ denote the disjoint union of a collection of crossing discs corresponding to the components of L_n. Suppose, on the contrary, that K is the unknot. By Lemma 4.1 and the discussion in Remark 5.5 of [2], K bounds an embedded disc Δ in the complement of L_n. Furthermore, $\Delta \cap \mathcal{D}$ is a collection $\mathcal{A}$ of n disjoint properly embedded arcs in Δ; one for each component of $\mathcal{D}$. Now making a crossing change on K supported on a component $K_i \subset L_n$ is the same as twisting Δ along the arc in $\mathcal{A}$ corresponding to K_i. Since $n > 1$, $\mathcal{A}$ has at least two components, say α_1, α_2, which cut Δ into three subdiscs. Since K is geometrically essential in V, but can be unknotted by twisting along either of α_1, α_2, exactly two of these subdiscs must intersect ∂V_1 in a longitude. Recall that α_1, α_2 correspond to crossings that exhibit K as 2-adjacent to the unknot in V_1. Thus K can be also be uknotted by performing the aforementioned twists simultaneously on $\alpha_1 \cup \alpha_2$. A straightforward checking will convince the reader that this is impossible. □

3 Knot adjacency and hyperbolic volume

3.1 Trivializers and hyperbolic Dehn filling

In this section we will prove the main results of this note. As we will show Theorem 1.2 follows from the following theorem and known hyperbolic volume techniques.

Theorem 3.1 *For every $n > 2$, there exists a knot $\hat{K}$ that is n-adjacent to the unknot and it admits an n-trivializer L_n such that the interior of $M_n := M(\hat{K}, L_n)$ admits a complete hyperbolic structure with finite volume. Furthermore, we have $\mathrm{vol}(M_n) > n\, v_3$, where v_3 (≈ 1.01494) is the volume of a regular hyperbolic ideal tetrahedron.*

The proof of Theorem 3.1 uses Thurston's uniformization theorem for Haken 3-manifolds [18] in combination with a result of Adams [1]. Having Theorem 3.1 at hand, Theorem 1.2 follows from Thurston's hyperbolic Dehn filling theorem.

The statements of these results of Thurston can also be found in [4]. By abusing the terminology, we will say M_n is hyperbolic to mean that the interior of M_n admits a complete hyperbolic structure with finite volume.

Next let us show how Theorem 1.2 follows from Theorem 3.1. For the convenience of the reader we repeat the statement of Theorem 1.2.

Theorem 3.2 *For every $n \in \mathbf{N}$, there exists a hyperbolic knot K such that:*

(a) *We have $\Delta_K(t) = 1$.*

(b) *All the finite type invariants of orders $< 2n-1$ vanish for K.*

(c) *We have $\mathrm{vol}(S^3 \setminus K) > n$.*

(d) *We have $\mathrm{genus}(K) > \frac{n}{6}$.*

Proof First assume that $n \leq 2$. Then, we have $2n-1 \leq 3$ and thus to satisfy part(c) we need a knot that has trivial finite type invariants of orders ≤ 2. It is known that the only finite type invariant of order ≤ 2 is essentially the second derivative of the Alexander polynomial at $t=1$. Thus, a knot with trivial Alexander polynomial has trivial finite type invariants of order ≤ 2. There are hyperbolic knots with trivial Alexander polynomial; the first such knots occur at thirteen crossings and all have volume > 2. Thus for $n \leq 2$ there are hyperbolic knots satisfying parts (a)-(c). But note that (d) is trivially satisfied.

Suppose now that $n > 2$. By Theorem 3.1, there exists a non-trivial knot $\hat{K}$, that is n-adjacent to the unknot, and an n-trivializer, say L_n, so that M_n is hyperbolic and $\mathrm{vol}(M_n) > n\, v_3$. Let $K_1, \ldots, K_n$ denote the components of L_n. Given an n-tuple of integers $\mathbf{r} := (r_1, \ldots, r_n)$, let $M_n(\mathbf{r})$ denote the 3-manifold obtained from M_n as follows: For $1 \leq i \leq n$, perform Dehn filling with slope $\frac{1}{r_i}$-surgery along $\partial\eta(K_i)$. Let $\hat{K}(\mathbf{r})$ denote the image of $\hat{K}$ in $M_n(\mathbf{r})$. Clearly, $M_n(\mathbf{r}) := S^3 \setminus \eta(\hat{K}(\mathbf{r}))$ and $\hat{K}(\mathbf{r})$ is obtained from $\hat{K}$ by generalized crossing changes. By Lemma 2.4, $\hat{K}(\mathbf{r})$ is n-adjacent to the unknot, for every n-tuple $\mathbf{r}$. On the other hand, by Thurston's hyperbolic Dehn filling theorem, if $r_i >> 0$ then $M_n(\mathbf{r})$ admits a complete hyperbolic structure of finite volume; thus $\hat{K}(\mathbf{r})$ is a hyperbolic knot. By the proof of Thurston's theorem, the hyperbolic metric on $M_n(\mathbf{r})$ can be chosen so that it is arbitrarily close to the metric of M_n, provided that the numbers r_i are all sufficiently large. Thus by choosing the r_i's large we may ensure that the volume of $M_n(\mathbf{r})$ is arbitrarily close to that of M_n. Since $\mathrm{vol}(M_n) > n\, v_3$, we can choose the r_i's so that we have $\mathrm{vol}(M_n(\mathbf{r})) > n$. But since $\mathrm{vol}(M_n(\mathbf{r})) = \mathrm{vol}(S^3 \setminus \hat{K}(\mathbf{r}))$, setting $K := \hat{K}(\mathbf{r})$ we are done. This finishes the proof of parts (a)-(c) of the theorem. For part (d), recall that by the main result of [9] (or by Theorem 1.3 of [11]), we have $6\mathrm{genus}(K) - 3 \geq n$. Thus, $\mathrm{genus}(K) \geq \frac{n+3}{6} > \frac{n}{6}$ as desired. □

3.2 Ensuring hyperbolicity

The rest of the section is devoted to the proof of Theorem 3.1. Our purpose is to show that given $n > 2$, we can find a non-trivial knot $\hat{K}$, that is exhibited n-adjacent to the unknot by an n-trivializer L_n such that the 3-manifold $M_n := S^3 \setminus \eta(\hat{K} \cup L_n)$ has the following properties:

(i) M_n is irreducible.

(ii) M_n is ∂-incompressible.

(iii) M_n is atoroidal; every incompressible embedded torus is parallel to a component of ∂M_n.

(iv) M_n is anannular; every incompressible properly embedded annulus can be isotoped on ∂M_n.

Having properties (i)-(iv) at hand, we can apply Thurston's uniformization theorem for Haken 3-manifolds to conclude that M_n is hyperbolic. Now ∂M_n is a collection of tori. Associated with each component of ∂M_n, there is a cusp in $\mathrm{int}M_n$ homeomorphic to $T^2 \times [1, \infty)$. A result of Adams [1] states that a complete hyperbolic 3-manifold with N cusps must have volume at least $N.\ v_3$. Since ∂M_n has $n+1$ components, $\mathrm{int}M_n$ has $n+1$ cusps. Thus $\mathrm{vol}(M_n) \geq (n+1)\ v_3 > n\ v_3$ and the desired conclusion follows.

By [2], for every $n \in \mathbf{N}$, there exist many non-trivial knots that are n-adjacent to the unknot. Fix $n > 2$. Let $\bar{K}$ be any non-trivial knot that is n-adjacent to the unknot and let L_n be any n-trivializer corresponding to $\bar{K}$. By lemma 2.5, properties (i)-(ii) hold for $M_n := S^3 \setminus \eta(\bar{K} \cup L_n)$. Next we set off to show that we can choose $\bar{K}$ and L_n so that M_n satisfies properties (iii)-(iv) as well. We need the following lemma that will be used to control essential tori in the complement of n-trivializer's.

Lemma 3.3 *Let L_n be an n-trivializer that shows a non-trivial knot $\bar{K}$ to be n-adjacent to the unknot. Suppose that $M_n := S^3 \setminus \eta(L_n \cup \bar{K})$ contains an essential torus T and let V denote the solid torus in S^3 bounded by T. Then, L_n can be isotoped in $\mathrm{int}V$ by an isotopy in the complement of $\bar{K}$. Furthermore, T remains essential in the complement of $\bar{K}$.*

Proof First we show that L_n can be isotoped in $\mathrm{int}V$, by an isotopy in the complement of $\bar{K}$. Suppose that a component $K_1 \subset L_n$ lies outside V and let D_1 be a crossing disc bounded by K_1. We can isotope D_1 so that every component of $D_1 \cap T$ is essential in $D_1 \setminus \bar{K}$. Thus, each component of $D_1 \cap T$ is either parallel to ∂D_1 on D_1, or it bounds a subdisc of D_1 that is intersected exactly once by $\bar{K}$. In the first case, K_1 can be isotoped to lie in int$V$. In the later case, each component of $D_1 \cap T$ bounds a subdisc $E \subset D_1$ that is intersected exactly once by $\bar{K}$. Since T is essential in the complement of $\bar{K}$, E must be a meridian disc of V. Thus $\bar{K}$ has wrapping number one in the follow-swallow torus. It follows that $\bar{K}$ is composite and that the crossing change corresponding to L_1 occurs within a factor of $\bar{K}$. But then such a crossing change cannot unknot $\bar{K}$ contradicting the fact that L_1 is part of an n-trivializer. Thus L_n can be isotoped to lie in int$V$ as desired. Now it is easy to see that L_n exhibits $\bar{K}$ as n-adjacent to the unknot in int$V$.

Next we show that T remains essential in the complement of $\bar{K}$.

Since the linking number of $\bar{K}$ and each component of L_n is zero, $\bar{K}$ bounds a Seifert surface in the complement of L_n. Let S be a Seifert surface of $\bar{K}$ that is of minimum genus (and thus of maximum Euler characteristic) among all such Seifert surfaces of $\bar{K}$. Since T, S are incompressible in M_n we can isotope them so that $S \cap T$ is a collection of parallel essential curves on T. Since twisting along any component of L_n unknots $\bar{K}$, the winding number $\bar{K}$ in V is zero. We conclude that $S \cap T$ is homologically trivial in T. Thus we may replace the components of $S \setminus T$ that lie outside V by annuli on T, and then isotope these annuli away from T in V, to obtained a Seifert surface S' for $\bar{K}$. Now S' lies in $M_n \cap V$. Since no component of $X := S \cap \overline{S^3 \setminus V}$ can be a disc, the Euler characteristic $\chi(X)$ is not positive. Thus $\chi(S') \geq \chi(S)$. Since S was chosen to be of maximum Euler characteristic for K in the complement of L_n, we have $\chi(S') \leq \chi(S)$. It follows that $\chi(S') = \chi(S)$ and thus genus(S') = genus(S). Hence S' is also a minimum genus Seifert surface for $\bar{K}$ in M_n. Let $\mathcal{D}$ be a collection of crossing discs; one for each component of L_n. For $D \in \mathcal{D}$, the components of $S' \cap D$ are closed curves that are parallel to ∂D on D and an arc properly embedded on S'. After an isotopy of S' we can arrange so that $S' \cap D$ is only an arc α properly embedded in S'. Thus, $S' \cap \mathcal{D}$ is a collection, say $\mathcal{A}$ of arcs properly embedded on S'. By Theorem 4.1 of [9], S' remains of minimum genus in the complement of $\bar{K}$. That is, genus$(\bar{K})$ = genus(S'). Thus S' is incompressible in the complement of $\bar{K}$. By assumption, twisting along all the components of L_n turns $\bar{K}$ into the unknot in V. This unknot can be isotoped to lie in a 3-ball $B \subset$ int$V$. Assume that $\bar{K}$ is inessential in V; then $\bar{K}$ can also be isotoped to lie in B. Since S' is incompressible, S' can also be isotoped to lie in B. But then, $\mathcal{A}$ and $\bar{K}$ will lie in B. Now since L_n can be isotoped so that it lies in a small neighborhood of $\mathcal{A}$ in a collar of S', it follows that $\bar{K} \cup L_n$ can be isotoped in B. But this contradicts the assumption that T is essential in M_n. $\square$

Next we turn our attention to essential annuli in M_n. It is known that an atoroidal link complement that contains essential annuli admits a Seifert fibration over a sphere with at most three punctures. In particular such a link can have at most 3-components. Hence, the restriction $n > 2$ in our case implies that if M_n is atoroidal then it is anannular. For the convenience of the reader we give a direct argument that M_n is anannular in the next lemma.

Lemma 3.4 *Let L_n be an n-trivializer of a non-trivial knot $\bar{K}$. If $n > 2$ and M_n is atoroidal, then M_n is anannular.*

Proof Since $\bar{K}$ is non-trivial, by Lemma 2.5, M_n is irreducible. We will show that every incompressible, properly embedded annulus $(A, \partial A) \hookrightarrow (M_n, \partial M_n)$ can be isotoped to lie on ∂M_n.

First, suppose that A runs between two components of ∂M_n, say T_1, T_2. Let N denote a neighborhood of $A \cup T_1 \cup T_2$. Now ∂N has three tori components. Two of them are parallel to T_1, T_2, respectively. Let T denote the third one. Since $n > 2$, ∂M_n has at least four components. Thus T separates pairs of components of ∂M_n and it can't be boundary parallel. Hence, since M_n is atoroidal, T must be compressible. A compressing disc cuts T into a 2-sphere that separates pairs of components of ∂M_n and thus it is essential; a contradiction. Therefore, ∂A must lie on one component, say T_1, of ∂M_n. Let N denote a neighborhood on $A \cup T_1$. Now ∂N contains a torus T that separates T_1 from the rest of the components of ∂M_n. Suppose, for a moment, that T compresses. Then a compressing disc cuts T into a 2-sphere that separates T_1 from the rest of the components of ∂M_n. But such a sphere would be essential contradicting the irreducibility of M_n. Therefore we conclude that T must be parallel to a component of ∂M_n. But since T separates T_1 from at least three components of ∂M_n, it can only be parallel to T_1. Hence A is contained in a collar $T_1 \times I$ and it can be isotoped on T_1. This proves that A is inessential which finishes the proof of the lemma. □

3.3 Completing the proofs

We are now ready to give the proofs of the main results. We begin with the proof of Theorem 3.1.

Theorem 3.5 *For every $n > 2$, there exists a knot $\hat{K}$ that is n-adjacent to the unknot and it admits an n-trivializer L_n such that the interior of $M_n := M(\hat{K}, L_n)$ admits a complete hyperbolic structure with finite volume.*

Furthermore, we have $\mathrm{vol}(M_n) > n\,v_3$*, where* $v_3\,(\approx 1.01494)$ *is the volume of a regular hyperbolic ideal tetrahedron.*

Proof To complete the proof we need the following claim:

Claim For every $n > 2$, there is a non-trivial knot $\hat{K}$ that is not a satellite (i.e. its complement is atoroidal) and it is n-adjacent to the unknot.

Proof of Claim By [2] there is a non-trivial knot $\bar{K}$ that is n-adjacent to the unknot. Suppose $\bar{K}$ is a satellite knot. By applying Theorem 2.2 inductively we can find a pattern knot $\hat{K}$ of $\bar{K}$ such that i) $\hat{K}$ is not a non-trivial satellite; and ii) $\hat{K}$ is n-adjacent to the unknot in a standard solid torus V_1. By Lemma 2.6, $\hat{K}$ is not the unknot and the claim is proved.

To continue with the proof of the theorem, fix $n > 2$ and let $\bar{K}$ be a knot as in the claim above. Also let L_n be an n-trivializer for $\hat{K}$. By Lemma 2.5, $M_n := S^3 \setminus \eta(L_n \cup \bar{K})$ is irreducible and ∂-incompressible. We claim that M_n is atoroidal. For, suppose that M_n contains an essential torus. By Lemma 3.3, T must remain essential in the complement of $\hat{K}$. But this is impossible since $\hat{K}$ was chosen to be non-satellite. Hence M_n is atoroidal and by Lemma 3.4 anannular. Now Thurston's uniformization theorem applies to conclude that M_n is hyperbolic. Since ∂M_n has $n+1$ components, the interior of M_n has $n+1$ cusps. By [1], $\mathrm{vol}(M_n) \geq (n+1)v_3$ and thus $\mathrm{vol}(M_n) > nv_3$ as desired. □

As we proved earlier, Theorem 1.2 follows from Theorem 3.1. Note that Corollary 1.1 follows immediately from Theorem 1.2. We will finish the section with the proof of Corollary 1.3.

Corollary 3.6 *For every* $n, m \in \mathbf{N}$*, there exists a hyperbolic knot* K *with trivial finite type invariants of orders* $\leq m$ *and such that* $\mathrm{vol}(S^3 \setminus K) > n$*.*

Proof Fix $m \in \mathbf{N}$. By Theorem 1.2, there is a hyperbolic knot K with $\mathrm{vol}(S^3 \setminus K) > m$ and with trivial finite type invariants of orders $\leq m$. Now this K works for all n with $n < m$. To obtain a knot corresponding to m, n for some $n > m$ just apply Theorem 1.2 for this n. □

4 Concluding discussion

4.1 Mahler measure of Alexander polynomials and volume

For a hyperbolic link L with an unknotted component K let $L(q)$ denote the link obtained from $L \setminus K$ by a twist of order q along K. Thurston's hyperbolic

Dehn filling theorem implies that as $q \longrightarrow \infty$, the volume of the complement of $L(q)$ converges to the volume of the complement of L. In [15], D. Silver and S. Williams showed that the Mahler measure of the Alexander polynomial of $L(q)$ exhibits an analogous behavior; it converges to the Mahler measure of the Alexander polynomial of L, as $q \longrightarrow \infty$. This analogy prompted the question of whether the Mahler measure of the Alexander polynomial of a hyperbolic knot is related to the volume of its complement [15]. In that spirit, Corollary 1.1 provides examples of non-alternating, hyperbolic knots with arbitrarily large volume whose Alexander polynomial has trivial Mahler measure. In [16], Silver, Stoimenow and Williams construct examples of alternating knots with these properties. However they also show that, for alternating knots, large Mahler measure of the Alexander polynomial implies large volume of the complement.

4.2 The Jones polynomial and volume

Corollary 1.1 shows that the Alexander polynomial of generic (non-alternating) hyperbolic knots is far from controlling the hyperbolic volume. The determinant of a knot can also be evaluated from its Jones polynomial. More specifically, if $J_K(t) := \sum_{i=0}^{r} a_i t^i$ denotes the Jones polynomial of an alternating knot K, then $\det(K) := |J_K(-1)| = \sum_{i=0}^{r} |a_i|$. Thus, the inequalities in (4) can also be expressed in terms of the Jones polynomial of K. In particular, the right hand side inequality becomes $\mathrm{vol}(S^3 \setminus K) \leq C(\sum_{i=0}^{r} |a_i|)$. A recent result O. Dasbach and X.-S. Lin [7], which preceded and inspired the results in this note, gives a striking relation between the Jones polynomial and the volume of alternating hyperbolic knots. More specifically, the *Volume-ish theorem* of [7] states that the volume of the complement of a hyperbolic alternating knot is bounded above and below by linear functions of absolute values of certain coefficients of the Jones polynomial of the knot. Furthermore, the authors record experimental data on the correlation between the coefficients of the Jones polynomial and the volume of knots up to 14 crossings. For generic knots, this data suggests a better correlation than the one between the coefficients of Alexander polynomial and volume. In the view of the experimental evidence of [7], and the examples constructed in this note, it becomes interesting to study the Jones polynomial of knots that are n-adjacent to the unknot, for $n \gg 0$.

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Department of Mathematics, Michigan State University, E. Lansing, MI 48824, USA
or
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

Email: `kalfagia@math.msu.edu` or `kalfagia@math.ias.edu`

URL: `http://www.math.msu.edu/~kalfagia`

Received: 22 September 2004