ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 4 (2004) 1125–1144
Published: 25 November 2004

Transverse contact structures on Seifert 3–manifolds

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Abstract We characterize the oriented Seifert–fibered three–manifolds which admit positive, transverse contact structures.

AMS Classification 57R17

Keywords Transverse contact structures, Seifert three–manifolds

1 Introduction and statement of results

Foliations and contact structures are arguably at opposite ends of the spectrum of the possible 2-plane fields ξ on a 3-manifold. While foliations are integrable plane fields, contact structures are totally non-integrable. If the planes in the distribution ξ are given as kernels of a one form α, then they form a foliation if $\alpha \wedge d\alpha = 0$, while they form a contact structure if everywhere pointwise $\alpha \wedge d\alpha \neq 0$. Eliashberg and Thurston [4] introduced the notion of confoliation, which generilizes the notion of foliation as well as the notion of contact structure. They also showed that a taut foliation can be C^0–approximated by a fillable contact structure.

The problem of existence of foliations transverse to the S^1–action on a Seifert fibered 3-manifold M was studied, and almost completely solved, in the 1980's by Eisenbud, Hirsch, Jankins and Neumann who translated, via holonomy, the original problem into a problem about homeomorphisms of the circle [1, 9, 10]. The problem was settled by Naimi [14].

The existence of transverse contact structures is clearly a related question. A foliation transverse to the S^1–action on a Seifert 3–manifold is taut since any orbit provides a closed loop transverse to the leaves of the foliation. If a transverse foliation $\mathcal{F}$ exists and the underlying manifold is not $S^1 \times S^2$ then, according to Eliashberg and Thurston [4], $\mathcal{F}$ can be approximated by a fillable contact structure ξ. The approximation is still transverse to the circle action on M. However, often M supports a transverse tight contact structure even when there are no transverse taut foliations.

Let $p\colon Y \to \Sigma_g$ be an oriented three–dimensional circle bundle with Euler class $e(Y)$ and base of genus g. Here $g \in \mathbb{Z}$ is defined so that

$$\chi(\Sigma_g) = \begin{cases} 2-2g & \text{if } \Sigma_g \text{ is orientable,} \\ 2+g & \text{if } \Sigma_g \text{ is non–orientable,} \end{cases}$$

where $\chi(\Sigma_g)$ is the Euler characteristic of Σ_g. Note that, with this convention, Σ_g is orientable for $g \geq 0$ and non–orientable for $g < 0$. Milnor and Wood studied the case of foliations transverse to the fibers of p. According to [12, 20], Y carries a transverse foliation if and only if one of the following holds:

- $\chi(\Sigma_g) \leq 0$ and $|e(Y)| \leq -\chi(\Sigma_g)$,
- $\chi(\Sigma_g) \geq 0$ and $e(Y) = 0$.

More recently, Giroux [6] and Sato and Tsuboi [19] proved that Y admits a positive, transverse contact structure if and only if one of the following holds:

- $\chi(\Sigma_g) \leq 0$ and $e(Y) \leq -\chi(\Sigma_g)$,
- $\chi(\Sigma_g) > 0$ and $e(Y) < 0$.

Let $p\colon M \to \Sigma_g$ be an oriented three–dimensional Seifert fibration with base of genus g and normalized (in the sense of [16]) Seifert invariants

$$\{b, g; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}.$$

Define

$$e(M) := -b - \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i}, \quad e_0(M) := -b - r, \quad \text{and} \quad \Gamma(M) := (\gamma_1, \ldots, \gamma_r),$$

where

$$\gamma_i := 1 - \frac{\beta_i}{\alpha_i}, \quad i = 1, \ldots, r.$$

Recall that the normalized Seifert invariants of $-M$ are

$$\{-b-r, g; (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_r, \alpha_r - \beta_r)\}.$$

Therefore,

$$e(-M) = -e(M), \quad e_0(-M) = -e_0(M) - r = b, \quad \Gamma(-M) = (1-\gamma_1, \ldots, 1-\gamma_r).$$

Definition 1.1 We say that $\Gamma = (\gamma_1, \ldots, \gamma_r) \in (\mathbb{Q} \cap (0,1))^r$ is *realizable* if $r \geq 3$ and there exist a permutation $\sigma \in \mathcal{S}_r$ and coprime integers $m > a > 0$ such that:

$$\gamma_{\sigma(1)} < \frac{a}{m}, \quad \gamma_{\sigma(2)} < \frac{m-a}{m}, \quad \gamma_{\sigma(3)}, \ldots, \gamma_{\sigma(r)} < \frac{1}{m}.$$

The main results of Eisenbud, Hirsch, Jankins, Neumann and Naimi on the existence of transverse foliations can be summarized in the following statement.

Theorem 1.2 [1, 10, 14] *Let $p\colon M \to \Sigma_g$ be an oriented three–dimensional Seifert fibration as above. Then, M carries a smooth foliation transverse to the fibration if and only if one of the following holds:*

(a) $e_0(M) \leq -\chi(\Sigma_g)$ *and* $e_0(-M) \leq -\chi(\Sigma_g)$
(b) $g = 0$ *and* $e(M) = e(-M) = 0$
(c) $g = 0$, $e_0(M) = -1$ *and* $\Gamma(M)$ *is realizable*
(d) $g = 0$, $e_0(-M) = -1$ *and* $\Gamma(-M)$ *is realizable*

Warning The reader should be aware of a difference in conventions: the quantity denoted by 'b' in [10] corresponds to $-b$ in our present notation.

The main result of this paper is the following theorem, giving necessary and sufficient conditions for the existence of contact structures transverse to the fibers of a Seifert fibration:

Theorem 1.3 *Let $p\colon M \to \Sigma_g$ be an oriented three–dimensional Seifert fibration as above. Then, M carries a positive contact structure transverse to the fibers of p if and only if one of the following holds:*

(a) $e_0(M) \leq -\chi(\Sigma_g)$
(b) $g = 0$, $r \leq 2$ *and* $e(M) < 0$
(c) $g = 0$, $e_0(M) = -1$ *and* $\Gamma(M)$ *is realizable.*

In view of the Eliashberg–Thurston's approximation theorem, one might wonder why Condition (b) from Theorem 1.2 does not appear explicitely in the statement of Theorem 1.3. The reason is that $e(M) = e(-M) = 0$ implies $e_0(M) \leq -1$ and $e_0(-M) \leq -1$. If both inequalities are strict, case (a) of Theorem 1.3 holds for both M and $-M$. If both inequalities are equalities, then $r = 2$ and it is easy to check that $M = S^1 \times S^2$, therefore the Eliashberg–Thurston's perturbation theorem cannot be applied. If $e_0(M) = -1$ and $e_0(-M) \leq -2$, (a) of Theorem 1.3 holds for $-M$ and, since $r \geq 3$, it follows from [10, 14] that (c) of Theorem 1.3 holds for M. Similarly, if $e_0(M) = -2$ and $e_0(-M) \leq -1$ then (a) of Theorem 1.3 holds for M and (c) of Theorem 1.3 for $-M$.

It should be mentioned that Honda [7] uses monodromy arguments (parallel to the ones used in the foliation case) to study the existence of transverse contact structures, obtaining results similar to ours.

Our approach is completely different. We show that the conditions of Theorem 1.3 are imposed on a Seifert 3–manifold M by the existence of a transverse contact structure ξ because of a certain property of a specific symplectic filling X_M of (M,ξ). The property is simply the fact that surfaces embedded in X_M and any of its blowdowns satisfy the adjunction inequalities.

To prove Theorem 1.3 when Σ_g is orientable, we look at M as the boundary of an equivariant plumbing X_M built according to a weighted star Γ of Figure 1 as described in [16]. McCarthy and Wolfson [13] showed that X_M carries a symplectic structure such that the orbits of the S^1–action on M are tangent to the kernel of the restriction of the symplectic form. For any transverse contact structure ξ on M, this makes X_M a symplectic filling of (M,ξ). To prove that the conditions of Theorem 1.3 are necessary, we impose that the adjunction inequalities be satisfied by surfaces embedded in X_M and, if the central node of Γ is a (-1)-sphere, into successive blow-downs of X_M.

It turns out that the most difficult case to analyze is case (c) of Theorem 1.3, involving the "realizability" condition. It is very interesting to see it appear from this new angle. In fact, one may think of the proof of Theorem 1.3 as a way of checking whether a given $\Gamma(M)=(\gamma_1,\dots,\gamma_r)$ is realizable. In short, $\Gamma(M)$ is realizable if and only if in the successive blow-downs of the plumbed manifold X_M there are no obvious surfaces violating the adjunction inequality.

The case when Σ_g is non–orientable is reduced to the orientable case by pulling back the fibration $M\to\Sigma_g$ to the orientable double cover of Σ_g.

To prove the existence of a transverse contact structure under the assumptions of Theorem 1.3, we either use the results on foliations and the Eliashberg-Thurston approximation result, or we construct the contact structure directly.

The paper is organized as follows. In Section 2 we describe in detail the symplectic filling X_M, and we collect its necessary properties. In Section 3 we characterize those Seifert 3–manifolds which admit S^1–invariant transverse contact structures, and in Section 4 we prove Theorem 1.3.

2 Symplectic fillings

Let $p\colon M\to\Sigma_g$ be a Seifert 3–manifold with orientable base and Seifert invariants

$$\{b,g;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r)\}.$$

Suppose that
$$\frac{\alpha_i}{\alpha_i - \beta_i} = [b_{i,1}, \dots, b_{i,s_i}], \quad i = 1, \dots, r,$$
where
$$[b_1, \dots, b_k] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}, \quad b_1, \dots, b_k \geq 2$$
Note that, since we assume $0 < \beta_i < \alpha_i$, there is a unique continued expansion with all $b_{i,j} \geq 2$.

By [16, Chapter 2], M is isomorphic (as a 3–manifold with S^1–action) to the boundary of a 4–manifold with S^1–action X_M obtained by equivariant plumbing according to the graph Γ of Figure 1, where the central vertex represents a disk bundle with Euler number $e_0(M)$ over a surface of genus g, and all the other vertices represent disk bundles over the sphere with Euler numbers $-b_{i,j}$.

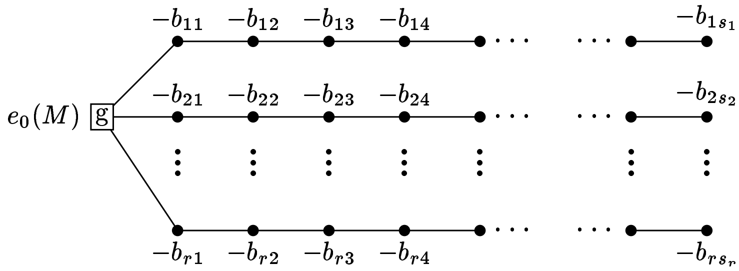

Figure 1: The plumbing X_M

Theorem 2.1 ([13]) *The S^1–manifold X_M carries a symplectic form ω such that every orbit of the S^1–action on ∂X_M is tangent to the kernel of $\omega|_{\partial X_M}$.* $\square$

Recall that a *symplectic filling* of a closed contact 3–manifold (M, ξ) is a symplectic 4–manifold (X, ω) such that (i) X is oriented by $\omega \wedge \omega$, (ii) $\partial X = M$ as oriented manifolds, and (iii) $\omega|_\xi \neq 0$ at every point of M.

Corollary 2.2 *Let $p\colon M \to \Sigma_g$ be an oriented Seifert fibered 3–manifold with orientable base Σ_g. Let ξ be a a positive contact structure on M transverse to the fibration. Then, (X_M, ω) from Theorem 2.1 is a symplectic filling of (M, ξ). Moreover, ξ is universally tight.*

Proof The first part follows immediately from the definitions together with Theorem 2.1. For the second part, recall that the fundamental group of a Seifert 3–manifold is residually finite [8, pp. 176–177]. Therefore, in order to prove that the contact structure is universally tight it is enough to show that the pull–back of the contact structure to every finite cover is tight. This follows from the first part of the statement, because every finite cover is still Seifert fibered, the pulled–back structure is transverse, and fillable contact structures are tight [2]. □

Corollary 2.3 *Let $p\colon M \to \Sigma_g$ be an oriented Seifert fibered 3–manifold with orientable base Σ_g. Suppose that M carries a positive contact structure transverse to the fibration. Let $S \subset X_M$ be an oriented surface of genus $g(S)$ smoothly embedded in the 4–manfild X_M given by Figure 1. Then,*

$$\begin{cases} S\cdot S \leq -1 & \text{if} \quad g(S) = 0, \\ S\cdot S \leq -\chi(S) & \text{if} \quad g(S) > 0. \end{cases}$$

Proof Let ξ be a a positive contact structure on M transverse to the fibration. By Corollary 2.2, (M,ξ) has a symplectic filling of the form (X_M,ω). By [3], (X_M,ω) can be compactified to a closed, symplectic 4–manifold $\widehat{X}_M$. Moreover, up to adding a suitable Stein cobordism to X_M (using e.g. [5, Theorem 2.5]) it can be arranged that $b_2^+(\widehat{X}_M) > 1$. Since the stated inequalities are satisfied by surfaces in $\widehat{X}_M$ [17], the conclusion follows. □

3 S^1–invariant structures

This section consists of the following proposition, which characterizes the oriented Seifert 3–manifolds carrying positive, S^1–invariant transverse contact structures.

Proposition 3.1 *Let $p\colon M \to \Sigma_g$ be an oriented Seifert 3–manifold. Then, M carries a positive, S^1–invariant transverse contact structure if and only if $e(M) < 0$.*

Proof Suppose that M has Seifert invariants

$$\{b, g; (\alpha_1,\beta_1), \dots, (\alpha_r,\beta_r)\}.$$

Let $a = \alpha_1\cdots\alpha_r$. The cyclic group $\mathbb{Z}/a\mathbb{Z}$ acts on M when it is regarded as a subgroup of S^1. The quotient $M' = M/\mathbb{Z}/a\mathbb{Z}$ is a genuine S^1–bundle over the same base Σ_g. Moreover, by [15, Theorem 1.2],

$$e(M') = ae(M). \tag{3.1}$$

Thus, if $e(M) < 0$ then $e(M') < 0$ and by [6, 19] M' carries an S^1–invariant contact structure ξ transverse to the fibers. The pull–back of ξ to M is also S^1–invariant and tranverse, therefore we have proved the first half of the statement.

Now let ξ be an S^1–invariant transverse contact structure on M. We claim that there exists an induced S^1–invariant transverse contact structure $\overline{\xi}$ on M'. By [6, 19] and Equation (3.1), proving the claim clearly suffices to finish the proof.

To prove the claim, we first argue locally around a singular fiber. Recall that a neighborhood of a singular fiber F is of the form

$$(D^2 \times S^1)/\mathbb{Z}/p\mathbb{Z},$$

where the generator $g \in \mathbb{Z}/p\mathbb{Z}$ acts by $(x, t) \mapsto (xg, tg^q)$ for some $0 < q < p$ coprime with p. The contact structure ξ lifts to a contact structure $\tilde{\xi}$ on $D \times S^1$ which is $\mathbb{Z}/p\mathbb{Z}$–invariant and S^1–invariant under the standard S^1–action on the second factor. Since ξ is transverse to the singular fiber F, after a suitable change of trivialization of the neighborhood of F $\tilde{\xi}$ is given as the zero set of a 1–form $\alpha + d\theta$, where $d\theta$ is the standard angular 1–form on S^1, $\alpha \in \Omega^1(D^2)$ is S^1–invariant and $d\alpha \in \Omega^2(D^2)$ is a volume form.

Since the $\mathbb{Z}/p\mathbb{Z}$–action and the $\mathbb{Z}/a\mathbb{Z}$–action commute, the quotient map from $D^2 \times S^1$ onto a neighorhood of the image $F' \subset M'$ of F can be factored as follows:

$$D^2 \times S^1 \xrightarrow{f} (D^2 \times S^1)/\mathbb{Z}/a\mathbb{Z} \xrightarrow{g} \left((D^2 \times S^1)/\mathbb{Z}/a\mathbb{Z}\right)/\mathbb{Z}/p\mathbb{Z}.$$

Here $\mathbb{Z}/a\mathbb{Z}$ acts freely on the second factor as a subgroup of S^1, and after the identification $(D^2 \times S^1)/\mathbb{Z}/a\mathbb{Z} \cong D^2 \times S^1$, $\mathbb{Z}/p\mathbb{Z}$ acts by rotations on the first factor. Since α and $d\theta$ are S^1–invariant, $\tilde{\xi}$, and therefore ξ, descends to a smooth contact structure $\overline{\xi}$ on a neighborhood of F'. Since ξ also descends to a transverse S^1–invariant contact structure when restricted to the complement of the the singular fibers, the claim is proved. $\square$

4 The proof of Theorem 1.3

4.1 Sufficiency of the conditions

The case $\chi(\Sigma_g) \leq 0$

Suppose that the inequality stated in case (a) of Theorem 1.3 holds. Let us first assume that $e_0(-M) \leq -\chi(\Sigma_g)$. Then, it follows from case (a) of Theorem 1.2 that M carries a smooth foliation $\mathcal{F}$ transverse to the Seifert fibration. Moreover, $M \neq S^1 \times S^2$ because $\chi(\Sigma_g) \leq 0$ implies that the fundamental group of Σ_g is non–trivial, and this is incompatible with $M = S^1 \times S^2$. Therefore, by [4] there exist positive contact structures arbitrarily C^0–near $\mathcal{F}$, hence transverse to the fibration.

If $e_0(-M) > -\chi(\Sigma_g)$, take N to be the Seifert fibered 3–manifold with Seifert invariants

$$(-\chi(\Sigma_g), g; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)).$$

Since

$$e_0(N) = \chi(\Sigma_g) - r \leq -\chi(\Sigma_g),$$

N carries a positive contact structure transverse to the fibration. Moreover, M can be obtained from N by a smooth surgery along a regular fiber $f \subset N$ with coefficient

$$\frac{1}{e_0(-M) + \chi(\Sigma_g)}$$

with respect to the framing induced by the fibration. It is not difficult to check(see [6, Lemma 1.3]) that transverse contact structures extend from the complement of a regular neighborhood of f in N to transverse contact structures on M.

The case $\chi(\Sigma_g) > 0$, i.e. $g \in \{-1, 0\}$

If case (a) of Theorem 1.3 holds, then

$$e(M) = e_0(M) + \sum_{i=1}^{r}(1 - \frac{\beta_i}{\alpha_i}) \leq -\chi(\Sigma_g) + \sum_{i=1}^{r}(1 - \frac{\beta_i}{\alpha_i}). \tag{4.1}$$

If $e(M) < 0$, then M carries a transverse contact structure by Proposition 3.1. If $e(M) \geq 0$, then Equation (4.1) implies $r \geq 2$ if $g = -1$, and $r \geq 3$ if $g = 0$, therefore $M \neq S^1 \times S^2$. Also, since $e(-M) = -e(M) \leq 0$,

$$e_0(-M) = e(-M) - \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} < 0,$$

therefore $e_0(-M) \leq -1$. If (i) $g = -1$ or (ii) $g = 0$ and $e_0(-M) \leq -2$, then by Theorem 1.2(a) together with the Eliashberg and Thurston's theorem we are done. If $g = 0$ and $e_0(-M) = -1$ then, since $e(-M) \leq 0$, by [9, Theorem 1] (see also [10, Theorem 3.1]) and [10, Theorem 3], $-M$, and therefore M, supports a smooth foliation $\mathcal{F}$ transverse to the fibration and we conclude as before.

If case (b) of Theorem 1.3 holds, then the conclusion follows by Proposition 3.1.

If (c) holds, then we get a transverse contact structure via Theorem 1.2(c) and [4].

4.2 Necessity of the conditions

The case $\chi(\Sigma_g) \leq 0$

If $g > 0$, observe that the smooth 4–manifold X_M of Section 2 contains a smooth surface Σ_g of genus g and self–intersection $e_0(M)$. If M carries a positive, transverse contact structure, then by Corollary 2.3 $e_0(M) \leq -\chi(\Sigma_g)$. Therefore case (a) of Theorem 1.3 holds.

If $g < -1$, let $\widehat{\Sigma}_g \to \Sigma_g$ be the orientable double cover of Σ_g, and let $\widehat{M} \to \widehat{\Sigma}_g$ be the pull–back of the fibration $M \to \Sigma_g$. According to [15, Theorem 1.2], $e_0(\widehat{M}) = 2e_0(M)$. If M carries a positive, transverse contact structure then so does $\widehat{M}$. Therefore, since $\widehat{\Sigma}_g$ has positive genus we have

$$e_0(\widehat{M}) \leq -\chi(\widehat{\Sigma}_g) = -2\chi(\Sigma_g),$$

hence case (a) of Theorem 1.3 holds.

The case $\chi(\Sigma_g) > 0$

Suppose first that $g = 0$. Then, Corollary 2.3 implies $e_0(M) \leq -1$. Hence, either case (a) of Theorem 1.3 holds, or $e_0(M) = -1$. Let us assume the latter.

If $r = 0$ then $e(M) = e_0(M) = -1$, therefore case (b) of Theorem 1.3 holds. If $r = 1$ or $r = 2$, we blow down (-1)–spheres in X_M as far as possible. The adjuction inequalities imply that no non–negative sphere appears, therefore X_M is negative definite and, since $e(M)$ is an eigenvalue of Q_X [15, Theorem 5.2], $e(M) < 0$. Thus, again case (b) of Theorem 1.3 holds.

When $g = 0$, we are left to consider the subcase $g = 0$ and $r \geq 3$. Before tackling this subcase we deal with the case $g = -1$.

If $g = -1$, let $\widehat{M} \to \widehat{\Sigma}_g$ be the pull–back of $M \to \Sigma_g$ under the orientable double cover of Σ_g. By [15, Theorem 1.2] we have

$$e_0(\widehat{M}) = 2e_0(M) \neq -1.$$

Thus, since $\widehat{\Sigma}_g = S^2$ and $\widehat{M}$ carries a positive, transverse contact structure, we have $e_0(\widehat{M}) \leq -2$. Therefore $e_0(M) \leq -1$, and case (a) of Theorem 1.3 holds for M.

The subcase $g = 0$, $e_0(M) = -1$, $r \geq 3$

This is the last and hardest subcase. Before delving into the proof, we need some preparation.

Algebraic preliminaries

Let

$$\rho \in \mathbb{Q}_{>1} := \{q \in \mathbb{Q} \mid q > 1\}.$$

Then, there is a unique way of writing ρ as a continued fraction

$$\rho = [a_1, \ldots, a_h], \quad a_i \geq 2,\ i = 1, \ldots, h.$$

It is well–known that if $\frac{p}{q} = [a_1, \ldots, a_h]$, with p, q coprime, then $[a_h, \ldots, a_1]$ is of the form $\frac{p}{q'}$ for some q' coprime with p.

The set of finite sequences of integers has a natural linear order $\preccurlyeq$ given by the following definition.

Definition 4.1 Declare

$$(a_1, \ldots, a_h) \preccurlyeq (b_1, \ldots, b_k)$$

if and only if there exists an index $1 \leq j \leq \min(h, k)$ such that

- $a_i = b_i$ for $i = 1, \ldots, j-1$, and
- either (i) $a_j < b_j$ or (ii) $j = k \leq h$ and $a_k = b_k$.

For example, we have

$$(2,2,2) \preccurlyeq (2,2,3) \preccurlyeq (2,2)$$

Lemma 4.2 *Let $\rho, \sigma \in \mathbb{Q}_{>1}$, with $\rho = [a_1, \ldots, a_h]$ and $\sigma = [b_1, \ldots, b_k]$. Then,*

$$\rho \leq \sigma \quad \Longleftrightarrow \quad (a_1, \ldots, a_h) \preccurlyeq (b_1, \ldots, b_k).$$

Proof Suppose that $(a_1,\dots,a_h) \preccurlyeq (b_1,\dots,b_k)$. Observe that the rational expression $[a_1,\dots,a_h]$ makes sense if $a_1,\dots,a_h \in \mathbb{Q}_{>1}$. Let $1 \le j \le \min(h,k)$ be such that $a_i = b_i$ for $1 \le i \le j-1$ and either (i) $a_j < b_j$ or (ii) $j = k \le h$ and $a_k = b_k$. If (i) holds then

$$[a_j,\dots,a_h] \le a_j \le b_j - 1 < [b_j,\dots,b_k],$$

therefore we have

$$\begin{aligned}\rho &= [a_1,\dots,a_{j-1},[a_j,\dots,a_h]] = \\ &= [b_1,\dots,b_{j-1},[a_j,\dots,a_h]] < \sigma = [b_1,\dots,b_{j-1},[b_j,\dots,b_k]].\end{aligned}$$

If (ii) holds then, since $[b_k,a_{k+1},\dots,a_h] \le b_k$,

$$\begin{aligned}\rho &= [a_1,\dots,a_k,a_{k+1},\dots,a_h] = \\ &= [b_1,\dots,b_{k-1},[b_k,a_{k+1},\dots,a_h]] \le \sigma = [b_1,\dots,b_k].\end{aligned}$$

Conversely, suppose $\rho \le \sigma$. If $(a_1,\dots,a_h) \not\preccurlyeq (b_1,\dots,b_k)$ then $(a_1,\dots,a_h) \neq (b_1,\dots,b_k)$, i.e. $\rho \neq \sigma$ and $(b_1,\dots,b_k) \preccurlyeq (a_1,\dots,a_h)$, which implies $\sigma < \rho$. Therefore we must have $(a_1,\dots,a_h) \preccurlyeq (b_1,\dots,b_k)$. □

Consider the involution $\mathbb{Q}_{>1} \to \mathbb{Q}_{>1}$ which maps $\rho \in \mathbb{Q}_{>1}$ to the only solution $\rho' \in \mathbb{Q}_{>1}$ of the equation:

$$\frac{1}{\rho} + \frac{1}{\rho'} = 1.$$

Observe that $2 = 2'$, and $\rho_1 \le \rho_2$ if and only if $\rho_2' \le \rho_1'$. Therefore, the involution maps the interval $(1,2)$ bijectively onto $(2,\infty)$ reversing the standard linear order.

Given

$$\rho = [a_1,\dots,a_h] \in \mathbb{Q}_{>1},$$

the *Riemenschneider's point diagram* [18] $D(\rho,\rho')$ says how to compute the coefficients in the expansion

$$\rho' = [b_1,\dots,b_k].$$

The diagram $D(\rho,\rho')$ consists of h rows of dots, with the i–th row consisting of $a_i - 1$ dots and whose first dot lies under the last dot of the $(i-1)$–st row. Then, b_j is given by the number of elements in the j–th column of $D(\rho,\rho')$ increased by one. For example, if $\rho = [3,4,3]$ and $\rho' = [2,3,2,3,2]$, the diagram is given by Figure 2.

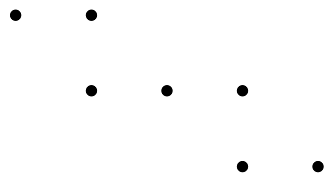

Figure 2: The point diagram $D([3,4,3],[2,3,2,3,2])$

Setting up the stage for the proof

Recall that we are assuming $g=0$, $e_0(M)=-1$ and $r\geq 3$. We need to show that if M carries a positive, transverse contact structure then $\Gamma(M)=(\gamma_1,\dots,\gamma_r)$ is realizable. Without loss of generality, we may assume

$$\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_r.$$

Define

$$\delta_i:=\frac{1}{\gamma_i},\ i=1,\dots,r.$$

The realizability of $\Gamma(M)$ is equivalent to the existence of coprime integers a,m such that $1\leq a<m$ and

$$\delta_1>\frac{m}{a},\quad \delta_2>\frac{m}{m-a},\quad \delta_3,\dots,\delta_r>m. \tag{4.2}$$

If $\delta_1>2$ then Inequalities (4.2) are satisfied for $m=2$, $a=1$, and $\Gamma(M)$ is realizable. Therefore we may assume $\delta_1\leq 2$. Under this assumption, we can write the continued fraction expansion of δ_1 as:

$$\delta_1=[\overbrace{2,\dots,2}^{n_1+1},n_2,\overbrace{2,\dots,2}^{n_3},n_4,\dots,n_{2p},\overbrace{2,\dots,2}^{n_{2p+1}}] \tag{4.3}$$

for some

$$n_1,n_3,\dots,n_{2p+1}\geq 0,\quad n_2,n_4,\dots,n_{2p}\geq 3.$$

Next, we claim that we may also assume $\delta_2>2$. In fact, if $\delta_2\leq 2$ then

$$\delta_2=[2,\dots],$$

and inspecting Figure 1 we see that X_M must contain a configuration of spheres dual to the graph of Figure 3. By blowing down (-1)–spheres, this immediately implies that X_M contains an embedded sphere of square zero, which is impossible by Corollary 2.3.

Assuming $\delta_2>2$, we have

$$\delta_2=[m_1,\overbrace{2,\dots,2}^{m_2},m_3,\dots,m_{2q-1},\overbrace{2,\dots,2}^{m_{2q}}], \tag{4.4}$$

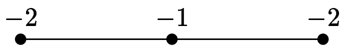

Figure 3: Configuration of spheres in X_M when $\delta_2 \leq 2$

for some

$$m_1, m_3, \ldots, m_{2q-1} \geq 3, \quad m_2, m_4, \ldots, m_{2q} \geq 0.$$

Since $\delta_3 \geq \delta_2$, we also have

$$\delta_3 = [d, \ldots], \quad \text{with} \quad d \geq 3.$$

Blowing down and adjunction inequalities

Since the central sphere in Figure 1 is a (-1)–sphere, we can blow it down. This shows that the 4–manifold X_M of Section 2 contains $X^{(0)} \# \overline{\mathbb{CP}}^2$, where $X^{(0)}$ is the plumbing associated to the graph of Figure 4. Blowing down $X^{(0)}$

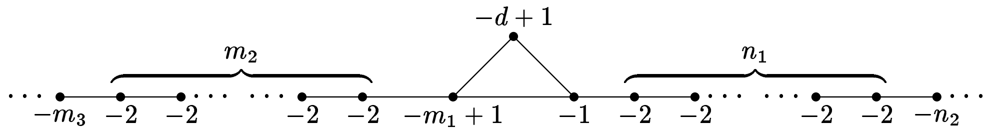

Figure 4: The manifold $X^{(0)}$

$n_1 + 1$ times, we obtain Figure 5, representing a four–manifold $X^{(1)}$ such that

$$X^{(0)} \cong X^{(1)} \# (n_1 + 1)\overline{\mathbb{CP}}^2.$$

Notice that each vertex in Figure 5 corresponds to an embedded sphere. In this picture and the following ones, a numerical weight of type "(n)" on the edge between two vertices of the graph denotes the intersection number between suitably chosen homology classes corresponding to the vertices.

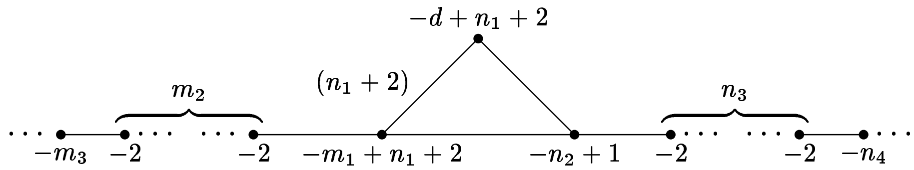

Figure 5: The manifold $X^{(1)}$

By Corollary 2.3, X_M contains no spheres of non–negative self–intersection. Therefore, since $X^{(1)}$ can be embedded in X_M, we have

$$-m_1 + n_1 + 2 \leq -1.$$

If $-m_1+n_1+2 = -1$ we blow down another m_2+1 times obtaining a 4–manifold $X^{(2)}$ such that

$$X^{(1)} \cong X^{(2)} \# (m_2+1)\overline{\mathbb{CP}}^2,$$

with $X^{(2)}$ given by Figure 6. In Figure 6 the vertices on the horizontal line correspond to embedded spheres and the top vertex can be represented by an immersed sphere with $(m_2+1)\binom{n_1+2}{2}$ positive, transverse self–intersections. Observe that, by smoothing out the self–intersections, we can represent the same homology class by an embedded surface of genus

$$g_2 = (m_2+1)\binom{n_1+2}{2}.$$

Moreover, by Corollary 2.3 we have

$$-n_2 + m_2 + 2 \leq -1.$$

Continuing in the same fashion, this process gives rise to a sequence of mani-

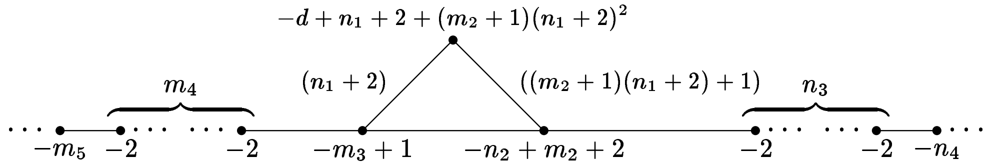

Figure 6: The manifold $X^{(2)}$

folds

$$X^{(0)} \to X^{(1)} \to X^{(2)} \to \cdots \to X^{(i)} \to \cdots$$

such that each $X^{(i)}$ (if defined) is given by Figure 7 for even i and by Figure 8 for odd i. By Corollary 2.3, we have

$$(-1)^i(m_i - n_i) + 2 \leq -1$$

for each i such that $X^{(i)}$ is defined. Moreover, if $X^{(i)}$ is defined, then in order for $X^{(i+1)}$ to be defined as well we need to have

$$(-1)^i(m_i - n_i) + 2 = -1. \tag{4.5}$$

If Equation (4.5) holds, then

$$X^{(i)} \cong \begin{cases} X^{(i+1)} \# (n_{i+1}+1)\overline{\mathbb{CP}}^2 & \text{if } i \text{ is even,} \\ X^{(i+1)} \# (m_{i+1}+1)\overline{\mathbb{CP}}^2 & \text{if } i \text{ is odd.} \end{cases}$$

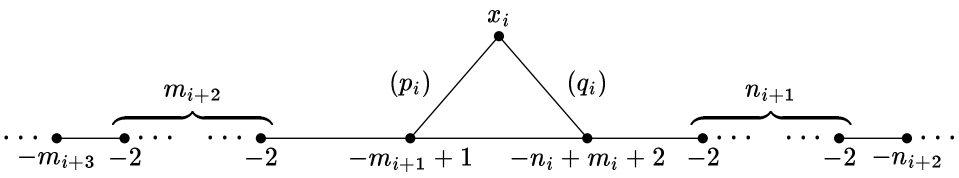

Figure 7: The manifold $X^{(i)}$ for even i

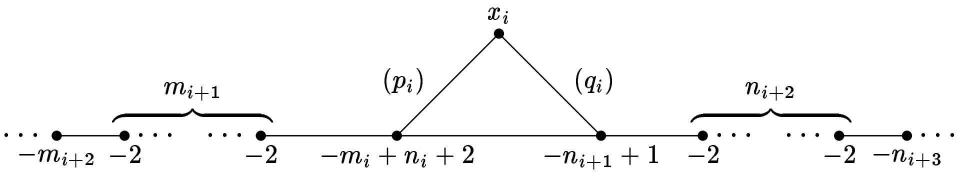

Figure 8: The manifold $X^{(i)}$ for odd i

As in the case of $X^{(2)}$, the homology class corresponding to the top vertex in each of Figures 7 and 8 can be represented in a natural way by an immersed sphere $S_i \subset X^{(i)}$. We denote by g_i the genus of the smooth surface Σ_i obtained by smoothing out the singularities of S_i. Then, the numbers p_i and q_i denote algebraic as well as geometric intersection numbers between Σ_i and two embedded spheres representing the homology classes which correspond to the vertices connected to the top. Moreover, it is easy to check that the following relations hold:

$$\begin{aligned}
x_{i+1} &= \begin{cases} x_i + (n_{i+1}+1)q_i^2 & \text{if } i \text{ is even} \\ x_i + (m_{i+1}+1)p_i^2 & \text{if } i \text{ is odd} \end{cases} \\
p_{i+1} &= \begin{cases} p_i + (n_{i+1}+1)q_i & \text{if } i \text{ is even} \\ p_i & \text{if } i \text{ is odd} \end{cases} \\
q_{i+1} &= \begin{cases} q_i & \text{if } i \text{ is even} \\ q_i + (m_{i+1}+1)p_i & \text{if } i \text{ is odd} \end{cases} \\
g_{i+1} &= \begin{cases} g_i + (n_{i+1}+1)\binom{q_i}{2} & \text{if } i \text{ is even} \\ g_i + (m_{i+1}+1)\binom{p_i}{2} & \text{if } i \text{ is odd} \end{cases}
\end{aligned} \tag{4.6}$$

Strictly speaking, Figures 7 and 8 should be taken literally only if

$$0 \le i < \min(2q, 2p+1). \tag{4.7}$$

(Recall that p and q were defined in Equations (4.3) and (4.4), respectively). When $i = \min(2q, 2p+1)$, the pictures should be suitably interpreted, because the diagram has only one horizontal leg. Similarly, Equations (4.6) shold be thought of as relations only when (4.7) holds, while for $i = \min(2q, 2p+1)$ they should be thought of as the definition of x_{i+1}, p_{i+1}, q_{i+1} and g_{i+1}.

Using (4.6) it is easy to check that the number

$$2g_i - 2 - x_i + p_i + q_i, \quad 0 \le i \le \min(2q+1, 2p+2),$$

is independent of i, and therefore it is always equal to its value for $i = 0$, i.e. $d-1$. On the other hand, the adjunction inequality implies

$$2g_i - 2 - x_i \ge 0,$$

hence

$$d > p_i + q_i, \quad 0 \le i \le \min(2q+1, 2p+2). \tag{4.8}$$

The end of the proof

By considering the Riemenschneider's point diagram $D(\delta_2, \delta_2')$, we see that

$$\delta_2' = [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3, \overbrace{2,\ldots,2}^{m_3-3}, m_4+3, \ldots, \overbrace{2,\ldots,2}^{m_{2q-1}-3}, m_{2q}+2].$$

Clearly, one of the following holds:

(1) There exists k such that $0 \le k \le \min(2q, 2p+1)$, Equation (4.5) holds for every $0 \le i < k$, and

$$(-1)^k(m_k - n_k) + 2 < -1.$$

(2) Equation (4.5) holds for every $0 \le i \le \min(2q, 2p+1)$.

We shall now treat separately the possible cases which can occur.

(1) holds and k is even

In this case we have

$$\begin{aligned}\delta_2' < [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3, \overbrace{2,\ldots,2}^{m_3-3}, \ldots, m_{k-2}+3, \overbrace{2,\ldots,2}^{m_{k-1}-3}, m_k+3] \le \\ \le [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \overbrace{2,\ldots,2}^{n_3}, \ldots, n_{k-2}, \overbrace{2,\ldots,2}^{n_{k-1}}, n_i-1] < \\ [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \overbrace{2,\ldots,2}^{n_3}, \ldots, n_{k-2}, \overbrace{2,\ldots,2}^{n_{k-1}}, n_i, \ldots] = \delta_1\end{aligned}$$

Lemma 4.3 *Let $0 \le k \le \min(2q+1, 2p+2)$, and suppose that Equation (4.5) holds for every $0 \le i < k$. Then, if k is even we have*

$$\frac{p_k+q_k}{p_k} = [m_k+3, \overbrace{2,\dots,2}^{m_{k-1}-3}, m_{k-2}+3, \dots, \overbrace{2,\dots,2}^{m_3-3}, m_2+3, \overbrace{2,\dots,2}^{m_1-2}],$$

while if k is odd then

$$\frac{p_k+q_k}{q_k} = [\overbrace{2,\dots,2}^{n_k+1}, n_{k-1}, \overbrace{2,\dots,2}^{n_{k-2}}, n_{k-3}, \dots, \overbrace{2,\dots,2}^{n_3}, n_2, \overbrace{2,\dots,2}^{n_1+1}].$$

Proof Observe that Relations (4.6) imply

$$\frac{p_{i+1}}{q_{i+1}} = \frac{p_i}{q_i} + \begin{cases} (n_{i+1}+1) & \text{if } i \text{ is even,} \\ (m_{i+1}+1) & \text{if } i \text{ is odd,} \end{cases} \qquad 0 \le i < k. \tag{4.9}$$

Let us introduce the notation:

$$[n_1, \dots, n_k]^+ := n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}$$

Since Equation (4.5) holds for every $0 \le i < k$, by (4.9) we get, when k is even,

$$1 + \frac{q_k}{p_k} = [m_k+2, m_{k-1}-2, m_{k-2}+1, \dots, m_3-2, m_2+1, m_1-1]^+,$$

and when i is odd

$$1 + \frac{p_k}{q_k} = [n_k+2, n_{k-1}-2, n_{k-2}+1, \dots, n_3+1, n_2-1, n_1+2]^+.$$

The lemma follows from the following identities, which can be established by a straightforward induction:

$$[m_k+2, m_{k-1}-2, m_{k-2}+1, \dots, m_3-2, m_2+1, m_1-1]^+ =$$
$$[m_k+3, \overbrace{2,\dots,2}^{m_{k-1}-3}, m_{k-2}+3, \dots, \overbrace{2,\dots,2}^{m_3-3}, m_2+3, \overbrace{2,\dots,2}^{m_1-2}].$$

$$[n_{k+1}+2, n_k-2, n_{k-1}+1, \dots, n_3+1, n_2-1, n_1+2]^+ =$$
$$[\overbrace{2,\dots,2}^{n_{k+1}+1}, n_k, \overbrace{2,\dots,2}^{n_{k-1}}, n_{k-2}, \dots, \overbrace{2,\dots,2}^{n_3}, n_2, \overbrace{2,\dots,2}^{n_1+1}]. \qquad \square$$

By (4.8) and Lemma 4.3, δ_3 is bigger than the numerator of a fraction representing the number

$$\rho = [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3, \overbrace{2,\ldots,2}^{m_3-3}, \ldots, m_{k-2}+3, \overbrace{2,\ldots,2}^{m_{k-1}-3}, m_k+3].$$

If $\rho = \frac{m}{a}$ with a and m coprime, then

$$\delta_1 > \frac{m}{a}, \quad \delta_2 > \rho' = \frac{m}{m-a} \quad \text{and} \quad \delta_3 > m.$$

Therefore $\Gamma(M)$ is realizable.

(1) holds and k is odd

In this case we have

$$\begin{aligned}\delta_2' < [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3, \overbrace{2,\ldots,2}^{m_3-3}, \ldots, m_{k-3}+3, \overbrace{2,\ldots,2}^{m_{k-2}-3}, m_{k-1}+3, \overbrace{2,\ldots,2}^{m_k-3}] \leq \\ \leq [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \overbrace{2,\ldots,2}^{n_3}, \ldots, n_{k-3}, \overbrace{2,\ldots,2}^{n_{k-2}}, n_{k-1}, \overbrace{2,\ldots,2}^{n_k+1}] < \\ [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \overbrace{2,\ldots,2}^{n_3}, \ldots, n_{k-3}, \overbrace{2,\ldots,2}^{n_{k-2}}, n_{k-1}, \overbrace{2,\ldots,2}^{n_k}, \ldots] = \delta_1\end{aligned}$$

By (4.8) and Lemma 4.3, δ_3 is bigger than the numerator of

$$\frac{m}{a} = [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \overbrace{2,\ldots,2}^{n_3}, \ldots, n_{k-3}, \overbrace{2,\ldots,2}^{n_{k-2}}, n_{k-1}, \overbrace{2,\ldots,2}^{n_k+1}].$$

As in the previous case, it follows that $\Gamma(M)$ is realizable.

(2) holds and $2q < 2p+1$

We have

$$\begin{aligned}\delta_2' < [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3, \ldots, m_{2q}+3, \overbrace{2,\ldots,2}^{n_{2q+1}+1}] = \\ = [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \ldots, n_{2q}, \overbrace{2,\ldots,2}^{n_{2q+1}+1}] < \\ < [\overbrace{2,\ldots,2}^{n_1+1}, n_2, \ldots, n_{2q}, \overbrace{2,\ldots,2}^{n_{2q+1}}, \ldots] = \delta_1.\end{aligned}$$

By (4.8) and Lemma 4.3, δ_3 is bigger than the numerator of

$$[\overbrace{2,\ldots,2}^{n_1+1}, n_2, \ldots, n_{2q}, \overbrace{2,\ldots,2}^{n_{2q+1}+1}],$$

therefore $\Gamma(M)$ is realizable.

(2) holds and $2p+1<2q$

We have

$$\delta_2' < [\overbrace{2,\ldots,2}^{m_1-2}, m_2+3,\ldots,m_{2p}+3,\overbrace{2,\ldots,2}^{m_{2p+1}-3}, m_{2p+2}+3] =$$
$$= [\overbrace{2,\ldots,2}^{n_1+1}, n_2,\ldots,n_{2p},\overbrace{2,\ldots,2}^{n_{2p+1}}, m_{2p+2}+3] <$$
$$< [\overbrace{2,\ldots,2}^{n_1+1}, n_2,\ldots,n_{2p},\overbrace{2,\ldots,2}^{n_{2p+1}}] = \delta_1.$$

By (4.8) and Lemma 4.3, δ_3 is bigger than the numerator of

$$[\overbrace{2,\ldots,2}^{m_1-2}, m_2+3,\ldots,m_{2p}+3,\overbrace{2,\ldots,2}^{m_{2p+1}-3}, m_{2p+2}+3],$$

therefore $\Gamma(M)$ is realizable.

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Received: 21 June 2004