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An invariant of link cobordisms from Khovanov homology

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Abstract In [10], Mikhail Khovanov constructed a homology theory for oriented links, whose graded Euler characteristic is the Jones polynomial. He also explained how every link cobordism between two links induces a homomorphism between their homology groups, and he conjectured the invariance (up to sign) of this homomorphism under ambient isotopy of the link cobordism. In this paper we prove this conjecture, after having made a necessary improvement on its statement. We also introduce polynomial Lefschetz numbers of cobordisms from a link to itself such that the Lefschetz polynomial of the trivial cobordism is the Jones polynomial. These polynomials can be computed on the chain level.

AMS Classification 57Q45; 57M25

Keywords Khovanov homology, link cobordism, Jones polynomial

1 Introduction

In [10], M Khovanov associated to any diagram D of an oriented link a chain complex $C(D)$ of abelian groups, whose Euler characteristic is the Jones polynomial [7]. He proved that for any two diagrams of the same link the corresponding complexes are chain equivalent. Hence, the homology groups $\mathcal{H}(D)$ are link invariants up to isomorphism. For a definition of the chain complex, see Definitions 1 through 4, Section 2.1 below. See also Bar-Natan [2] for a treatment of Khovanov homology.

One of the motivations of Khovanov's work was the hope of finding a lift of the Penrose–Kauffman spin networks calculus to a calculus of surfaces in the 4–sphere. To finish this program he suggested the following TQFT construction of an invariant of link cobordisms.

Any link cobordism can be described as a one-parameter family $D_t, t \in [0,1]$ of planar diagrams, called a *movie*. The D_t are link diagrams, except at finitely many t–values where the topology changes: the diagram undergoes a *local move*, which is either a Reidemeister move or a Morse modification. Away from these values the diagram experiences a planar isotopy as t varies. Khovanov explained how local moves induce chain maps between complexes, hence homomorphisms between homology groups. The same is true for planar isotopies. Hence, the composition of these chain maps defines a homomorphism between the homology groups of the diagrams of the boundary links.

Khovanov conjectured that, up to multiplication by -1, this homomorphism is invariant under ambient isotopy of the link cobordism. (For the exact formulation see Section 4.)

In this paper, we show that this conjecture is not properly stated, by giving simple counterexamples (Theorem 1, Section 4.1). In fact, there are very simple examples for multi-component links, but we also give an example in the case of knots. We then show that this can be remedied by considering only ambient isotopies which leave the links in the boundary setwise fixed. If the conjecture is modified in this way, it is indeed true. This is the main result of the paper.

Theorem (Theorem 2, Section 4.2) *For oriented links L_0 and L_1, presented by diagrams D_0 and D_1, an oriented link cobordism Σ from L_0 to L_1, defines a homomorphism $\mathcal{H}(D_0) \to \mathcal{H}(D_1)$, invariant up to multiplication by -1 under ambient isotopy of Σ leaving $\partial\Sigma$ setwise fixed. Moreover, this invariant is non-trivial.*

The proof of Khovanov's conjecture implies the existence of a family of derived invariants of link cobordisms with the same source and target, which are analogous to the classical Lefschetz numbers of endomorphisms of manifolds. We call these *Lefschetz polynomials of link endocobordisms* and like their classical analogues they are computable on the chain level. The Jones polynomial appears as the Lefschetz polynomial of the identity cobordism (Section 6).

A knotted closed surface is a link cobordism between empty links. The grading properties of the theory force the invariant to be zero on such a surface unless its Euler characteristic is zero (see Section 3.5).

To get a theory which does not immediately rule out knotted spheres, Khovanov actually stated his conjecture in a more general setting, defining the chain complex as a module over the polynomial ring $\mathbb{Z}[c]$, with a differential also depending on c. The integer theory is obtained by setting $c = 0$. Surprisingly, it turns out that the conjectured invariant with polynomial coefficients does not exist. A proof of this fact is given in [9].

The following is an outline of the paper. Section 2 contains an elementary description of Khovanov homology, suggested by Oleg Viro [13]. In Section 3 we introduce link cobordisms and their diagrams, and explain how they induce maps between homology groups. Section 4 contains the counterexamples previously mentioned, and Section 5 contains the main result of the paper, the proof of Khovanov's conjecture. Finally, the last section introduces the Lefschetz polynomials.

2 Khovanov homology

In this section we give an elementary description of Khovanov's homology theory.

2.1 Chain groups

Let D be an oriented link diagram and a one of its crossings. A *marker at* a is a choice of a pair of vertical angles at a. In pictures this choice is indicated by a short bar connecting the chosen angles (see Figure 1).

If the bar is in the region swept out by the overcrossing line as it is turned counterclockwise toward the undercrossing line, the marker is called positive. Otherwise, it is negative.

To a distribution s of markers over the crossings of D corresponds an embedded collection $C_1, ..., C_{r_s}$ of circles in the plane, called the *resolution of* D *according to* s. It is obtained by replacing a small neighbourhood of each crossing point by a pair of parallel arcs in the regions not specified by the marker (see Figure 1). A *signed resolution* is a resolution together with a choice of $+/-$–sign for each circle C_i. A *state of* D is a distribution of markers at the crossing points, together with a signed resolution of it. The signed resolution of a state S will be denoted by $\mathrm{res}(S)$ (see Figure 2).

Let S be a state. Denote by $\sigma(S)$ the sum of the signs of the markers in S and by $\tau(S)$ the sum of the signs in its signed resolution $\mathrm{res}(S)$. Furthermore, let $w(D)$ denote the writhe number of the diagram D. The following functions will be the grading parameters of Khovanov's chain complex.

$$i(S) = \frac{w(D) - \sigma(S)}{2}$$

$$j(S) = -\frac{\sigma(S) + 2\tau(S) - 3w(D)}{2}$$

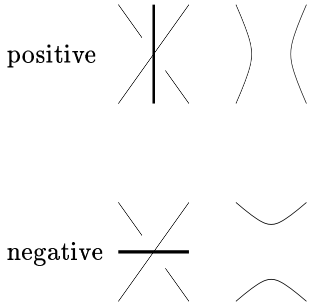

Figure 1: Resolution of a state according to markers

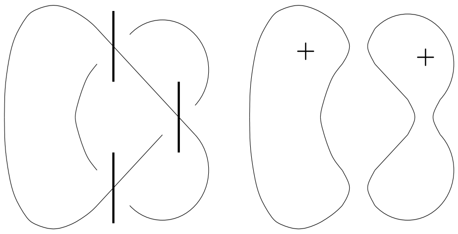

Figure 2: A state of a diagram of the unknot: here $i(S)=0$, $j(S)=-3$. Recall that the writhe of a knot (but not of a link) is independent of the choice of orientation.

Remark The definition of a state of a diagram is a refinement of L Kauffman's definition, which was used in [11] to construct the Jones polynomial V_L of a link L. The refinement consists of the signs associated to the components of the resolution. In [11] a state is only a distribution of markers together with the corresponding resolution. To understand this refinement, consider the equalities below.

$$\begin{aligned}
V_L &= \sum_{\text{Kauffman states s of D}} A^{\sigma(s)}(-A)^{-3w(D)}(-A^2-A^{-2})^{r_s} \\
&= \sum_{\text{“refined” states S of D}} (-1)^{w(D)}A^{\sigma(S)-3w(D)}(-A^{-2})^{-\tau(S)} \\
&= \sum_{\text{“refined” states S of D}} (-1)^{\frac{w(D)-\sigma(S)}{2}}q^{-\frac{\sigma(S)+2\tau(S)-3w(D)}{2}} \\
&= \sum_{\text{“refined” states S of D}} (-1)^{i(S)}q^{j(S)}
\end{aligned}$$

The first equality is Kauffman's definition. Each term in the sum corresponds to an “unrefined” state s. Each term is polynomial and contains a factor $(-A^2-A^{-2})^{r_s}$. If we associate the term $-A$ with positive circles and the term

$-A^{-2}$ with negative circles, then the refinement identifies each monomial in the binomial expansion of this polynomial term with a state in our sense. Letting $q = -A^{-2}$, we get $V_L(q)$ as a sum over the refined states.

Definition 1 Let L be a subset of the set I of crossings of D. Let $C_L^{i,j}(D)$ be the free abelian group on the set of states S which have $i(S) = i$, $j(S) = j$ and L as the set of crossings with negative markers.

Remark Observe that the cardinality $|L| = n_i$ of L is uniquely determined by i.

For any finite set S, let FS be the free abelian group generated by S. For bijections $f, g : \{1, ...|S|\} \to S$, let $p(f,g) \in \{0,1\}$ be the parity of the permutation $f^{-1}g$ of $\{1, ...|S|\}$. Let $Enum(S)$ be the set of all such bijections f, g.

Definition 2 For S as above, we define

$$E(S) = FEnum(S)/((-1)^{p(f,g)}f - g).$$

Khovanov's chain groups are defined as follows.

Definition 3 The (i,j)-th chain group of the chain complex is

$$C^{i,j}(D) = \bigoplus_{L \subset I, |L| = n_i} C_L^{i,j}(D) \otimes E(L).$$

2.2 The differential

To define the differential, note that replacing a positive marker in a state S with a negative one changes the resolution in one of two possible ways; the number of circles either increases or decreases by one.

Definition 4 Let S belong to $C_L^{i,j}(D)$. The differential of $S \otimes [x]$ is the sum

$$d(S \otimes [x]) = \sum_T T \otimes [xa],$$

where the T:s run over all states in $C^{i+1,j}(D)$ which satisfy the restrictions in the itemized list below, determined by S. Here, x is an ordered sequence of crossings and $a = a(T)$ a special crossing explained below. Square brackets around a sequence of crossings denote its equivalence class in $E(L)$.

- T has the same markers as S except at one crossing point a. At this point T has a negative marker, whereas S has a positive one.
- This restriction on markers implies that the resolutions of T and S coincide outside a small disc neighbourhood of a. We require the signed resolution $\mathrm{res}(T)$ to coincide with $\mathrm{res}(S)$ on the circles which do not intersect this disc. The disc is intersected by either one or two components of $\mathrm{res}(S)$. In the former case $\mathrm{res}(T)$ intersects the disc with two of its components; in the latter case, one. We require the total sum of signs of the circles of T to be greater by one than that of S (see Figure 3 and the remark below).

It is straightforward to check that the map d thus defined is actually a differential (i.e. that $d^2 = 0$) and that the bidegree of d is $(1, 0)$.

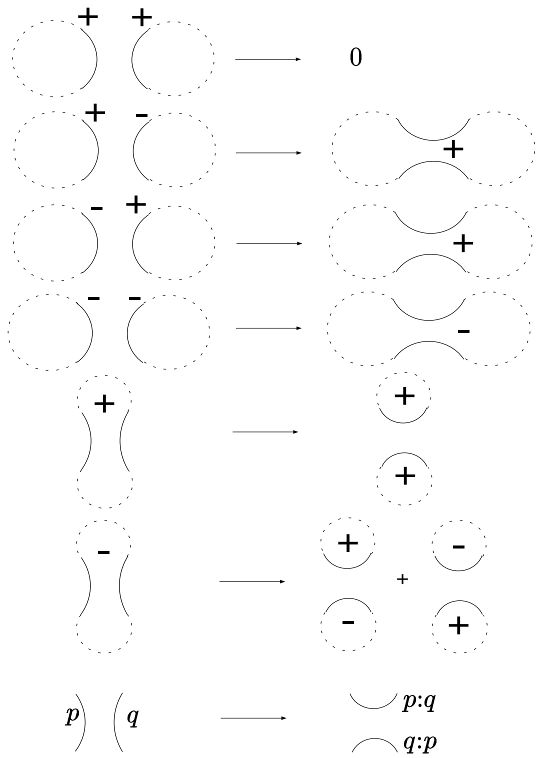

Figure 3: A Frobenius calculus of signed circles

Remark When both arcs passing the crossing a belong to the *same negative* component of $\mathrm{res}(S)$, there are two ways to distribute signs to $\mathrm{res}(T)$ in accordance with the restrictions. This explains why (and in what sense) the sixth row of Figure 3 is a sum. When there are *two positive components* of S passing a, the resolution of T cannot be supplied with signs consistent with the restrictions. Hence the zero in the first row.

When signs $+$, $-$ are used in figures, two arcs which belong to the same component are both labelled with one common, single sign, as in the first six rows of Figure 3.

However, we will also use the mnemonic notation on the last row of Figure 3. It summarizes the other rows in the following way. The labels p, q, $p{:}q$, $q{:}p$ represent the signs. Thus, in this notation there are always four labels on four arcs, regardless of whether the arcs are connected or not outside the figure. Furthermore, the labels $p{:}q$, $q{:}p$ on arcs in a figure may mean that the state appears with coefficient zero (if the arcs labelled p and q are in different and positive components, as in the first row), or that the sum of two states appears, with opposite signs on the two arcs (if the arcs labelled p and q are in the same, negative, component).

Remark One may regard the signs $+$ and $-$ as linear generators of a commutative Frobenius algebra A, with multiplication and comultiplication defined by the calculus in the previous remark. It is a theorem (see e.g. [1]) that commutative Frobenius algebras are in one-to-one correspondence with $(1+1)-$dimensional topological quantum field theories.

Indeed, this is Khovanov's starting point in [10]. To each collection of k circles in the plane his TQFT associates the tensor product of k copies of A. Any compact oriented surface with boundary the disjoint union of two such collections (with cardinalities k,n), induces a map $A^{\otimes k} \to A^{\otimes n}$, via its decomposition into discs and "pairs of pants", and the following descriptions.

To any pair of pants directed from the feet upwards corresponds the product in the algebra, $A \otimes A \xrightarrow{m} A$. To a pair of pants directed from the waist downwards corresponds the coproduct $A \xrightarrow{\Delta} A \otimes A$. A disc induces either a counit $A \xrightarrow{e} \mathbb{Z}$ or a unit $\mathbb{Z} \xrightarrow{i} A$, depending on whether it is regarded as directed from the empty set to the circle, or conversely. It follows from the axioms in the Frobenius algebra that the induced map is independent of the decomposition.

The values of multiplication and comultiplication on generators are given by the rules in Figure 3 in the following sense:

$$\begin{aligned} m(+\otimes +) &= 0 \\ m(+\otimes -) &= m(-\otimes +) = + \\ m(-\otimes -) &= - \\ \Delta(+) &= +\otimes + \\ \Delta(-) &= (+\otimes -) + (-\otimes +) \end{aligned}$$

The maps for discs are given by:

$$\begin{aligned} i(1) &= - \\ e(+) &= 1 \\ e(-) &= 0 \end{aligned}$$

2.3 Khovanov homology

As mentioned above, the differential d has bidegree $(1,0)$. Hence, for each j, $C^{i,j}(D)$ is an ordinary chain complex graded by i. The homology $\mathcal{H}^{i,j}(D)$ of this complex has an Euler characteristic which can be computed as the alternating sum of the ranks of the chain groups. Summing over j with coefficients q^j we obtain

$$\sum_{i,j}(-1)^i q^j \operatorname{rk}\mathcal{H}^{i,j}(D) = \sum_{i,j}(-1)^i q^j \operatorname{rk} C^{i,j}(D).$$

This is known as *the graded Euler characteristic* of the complex $C^{i,j}(D)$ (or of $\mathcal{H}^{i,j}(D)$). From the definition of i and j, it follows that the Jones polynomial is the graded Euler characteristic of Khovanov's chain complex.

3 Link cobordisms

3.1 Link cobordisms and their diagrams

Definition 5 Let $I = [0,1]$. An *(oriented) link cobordism* between two links $L_0 \subset \mathbb{R}^3 = \mathbb{R} \times \{0\}$ and $L_1 \subset \mathbb{R} = \mathbb{R}^3 \times \{1\}$ is a smooth, compact, oriented surface neatly embedded in $\mathbb{R}^3 \times I$ with boundary $\partial\Sigma = \Sigma \cap (\mathbb{R}^3 \times \partial I) = L_0 \cup L_1$. Close to the boundary, Σ is orthogonal to $\mathbb{R}^3 \times \partial I$.

Remark We will refer to L_0 and L_1 as the *source* and *target* of the cobordism Σ.

The orientation of Σ induces an orientation on all the intersections L_t of Σ with the constant time hyperplanes $\mathbb{R}^3 \times \{t\}$. A tangent vector v to a link component is in the positive direction if (v, w) gives the orientation of Σ whenever w is a vector tangent to Σ in the direction of increasing time.

Definition 6 A link cobordism Σ is *generic* if time t is a Morse function on Σ with distinct critical values.

A generic link cobordism intersects hyperplanes of constant $t \in [0,1]$ in embedded links except for a finite set of values t, for which the intersection L_t either has a single transversal double point or is the disjoint union of a link and an isolated point.

A generic link cobordism can be represented by a three-dimensional *surface diagram*, directly analogous to the two-dimensional diagrams of classical links (or tangles). Such a diagram is the image of the link cobordism under a projection to $\mathbb{R}^2 \times [0,1]$ which preserves the t–variable. The projection is required to be generic in the sense that the only singular points in the interior of the surface diagram are double points, Whitney umbrella points and triple points. At a double point, the diagram looks like the transversal intersection of two planes. Whitney umbrellas and triple points occur as the (isolated) boundary points of the double point set in the interior of $\mathbb{R}^2 \times [0,1]$.

A full set of Reidemeister-type moves for surface diagrams was given by D Roseman [12] and are called *Roseman moves*. These moves (plus ambient isotopy of the diagram) are sufficient to transform any two surface diagrams of ambient isotopic surfaces into each other. We will use another way of presenting a link cobordism.

Definition 7 A *movie* of a generic link cobordism with a given surface diagram as above, is the intersection of the diagram with planes $\mathbb{R}^2 \times \{t\}$, regarded as a function of time t. The intersection for a fixed t is called a *still*. The (finitely many) t for which the intersection is not a link diagram are called *critical levels*.

Observe that the restriction of the movie to a small interval of time around a critical level shows a link diagram which undergoes either a Reidemeister move or an oriented Morse modification. The Reidemeister moves occur at (interior) levels where the double point set has a boundary point or a local maximum or minimum. The Morse modifications occur at smooth points of the surface diagram which are minima, maxima or saddle points of the time function. Between two critical levels the diagram undergoes a planar isotopy.

Definition 8 Reidemeister moves and Morse modifications will be called *local moves*. Each such move is localized to a small disc, called a *changing disc*.

When a movie is studied, not all the stills of it are considered, but only as many as are necessary for a picture of the surface; one still between each pair of consecutive critical points and stills of the source and target diagrams (and sometimes a few additional ones, for clarity).

3.2 Movie moves

Carter and Saito [4] have found the Reidemeister-type moves for movies. These include movie versions of the Roseman moves mentioned above, but also additional moves to handle the additional structure of a time function on the diagram. The additional moves do not change the local topology of the surface diagram, but the Roseman moves do. We give pictures of the *movie moves* in Figures 4, 5 and 6. Each move has two sides, which will be called the left and right side according to their position in these figures. Any two movies of ambient isotopic link cobordisms can be related by a sequence of movie moves and interchanges of distant critical points.

Remark There are in fact more movie moves than are displayed in the figures. First, each move can be read in two directions, upwards or downwards. Second, each move comes in several versions, with varying crossing information. Each move has a mirror image, obtained by changing all the crossings. Moves 6 and 15 also have additional versions depending on the relative positions of the ingoing strands. The complicated Move 7 comes in several versions, but as we explain in the proof of the main theorem (Section 5), they are all equivalent modulo other moves. For details on the movie moves, see for example [6], where an even more refined set is described, which is not needed for our purposes. Finally, let us remark that it will not be necessary to consider any orientations on the strings.

3.3 Local moves

Khovanov associated to each local move on a link diagram a homomorphism between the homology groups of its source and target diagrams. In this section we describe how this is done.

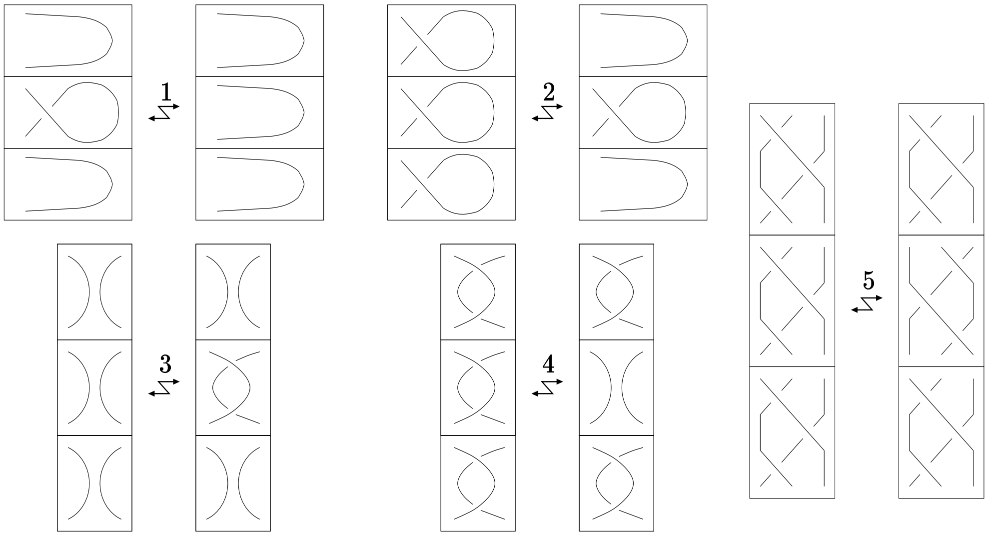

Figure 4: Movie moves (Roseman moves)

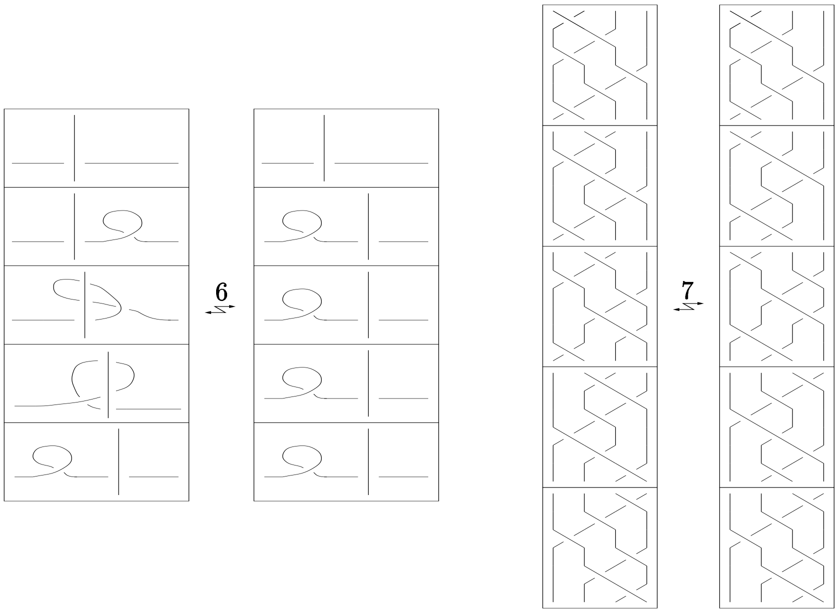

Figure 5: Movie moves (Roseman moves)

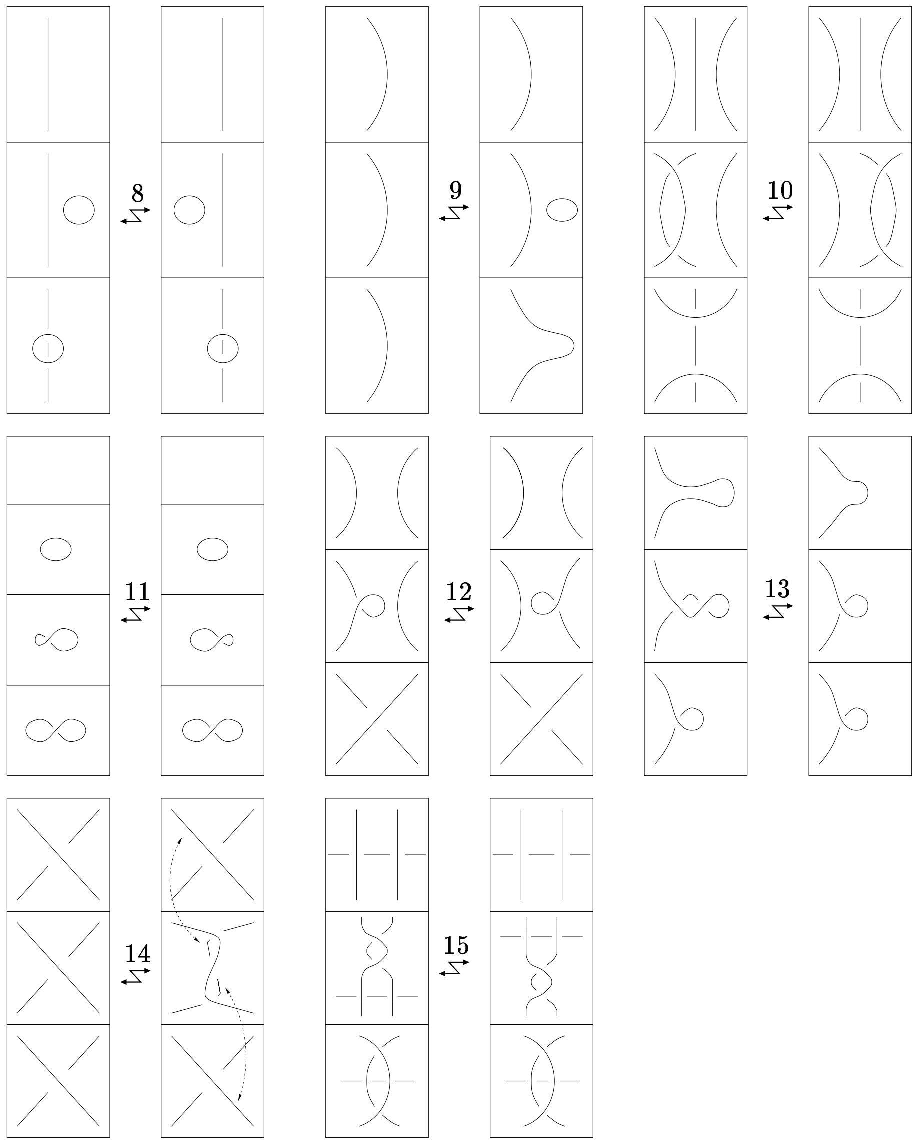

Figure 6: Movie moves (additional moves due to Carter/Saito)

3.3.1 Reidemeister moves

Let D and D' be link diagrams which differ by a single Reidemeister move of first or second type and assume that D' is the one with more crossings. Khovanov proved that the chain complex $C(D')$ splits as $C' \oplus C'_{contr}$, where there is an isomorphism $\psi : C' \xrightarrow{\cong} C(D)$ and C'_{contr} is chain contractible.

Thus, the composition Ψ of this isomorphism with the projection onto the first summand

$$C(D') = C' \oplus C'_{contr} \xrightarrow{pr_1} C' \xrightarrow{\psi} C(D)$$

is a chain equivalence with chain inverse Φ given by composition of the inclusion i_1 with ψ^{-1}.

If D differs from D' by a move of the third type, then $C(D)$ and $C(D')$ both split as above, $C(D) = C \oplus C_{contr}$ and $C(D') = C' \oplus C'_{contr}$, where there is an isomorphism $\psi : C \xrightarrow{\cong} C'$ and C_{contr} and C'_{contr} are chain contractible. Thus these chain complexes are chain equivalent as well via the map Ψ given by the composition

$$C(D') = C' \oplus C'_{contr} \xrightarrow{pr_1} C' \xrightarrow{\psi} C \xrightarrow{i_1} C \oplus C_{contr} = C(D).$$

Generators of the splitting factors, together with the isomorphisms ψ are given in Figures 7 through 10.

The figures should be interpreted in the following way. In the left columns the intersections of the generators with the changing disc are displayed. The image of each generator under ψ is displayed in the right column. Any state appearing as a term in the right column differs from the corresponding state(s) in the left column only on markers and circles intersecting the changing disc.

An explanation is probably needed for Figure 10. In the first row, the second term on the left hand side has no letter assigned to its upper left arc. It is supposed to have the same sign as in the first term, that is r, unless it is connected to one of the other arcs in the picture. The same applies to the right hand side.

In the second row, no marker is displayed at crossings a,b on any side. We mean that any markers are allowed at those crossings on the left and that the markers are the same on the right at the corresponding crossings. It follows that the resolutions are isotopic, and it is understood that the signs are preserved, too.

Generators of the contractible summands are given in Figures 11 through 14.

3.3.2 Morse moves

Khovanov also introduced a chain map of bidegree $(0,1)$ between the complexes of two diagrams which differ by a single Morse modification corresponding to a maximum or minimum, and a chain map of bidegree $(0,-1)$ if they differ by a saddle-point modification. These maps are described by Figure 15.

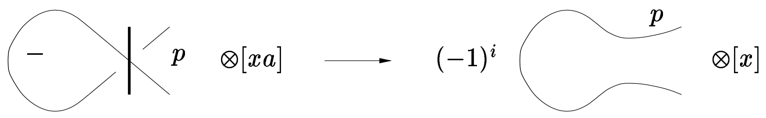

Figure 7: The isomorphism ψ for the first Reidemeister move, negative twist: the expressions on the left are generators for C'. The crossing in the twist is called a.

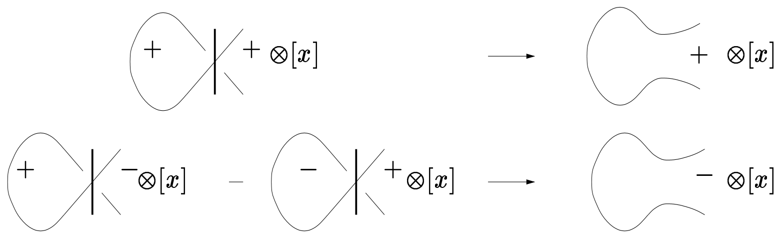

Figure 8: The isomorphism ψ for the first Reidemeister move, positive twist: the expressions on the left are generators for C'.

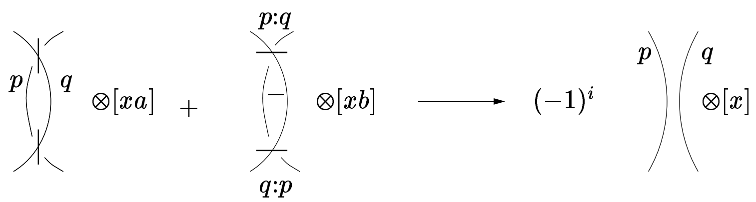

Figure 9: The isomorphism ψ for the second Reidemeister move: the expressions on the left are generators of C'. The upper crossing is called a, the lower b.

3.3.3 Chain maps in the basis of states

The states form a canonical basis (up to reversing signs) for Khovanov's chain complex. In the following proposition we give an explicit description of the value of the chain maps on the states. Recall the chain equivalences Ψ and their homotopy inverses Φ from Section 3.3.1.

Proposition 3.1 *The values of the chain maps Ψ and Φ on states are as shown in Figure 16 and 17 for the first two Reidemeister moves and in Figure 18 for the third Reidemeister move. The Morse move maps are as shown in Figure 15.*

Proof For Morse moves there is nothing to prove. We rewrite the chain equivalences Ψ and Φ in the basis of states. We give a sample calculation here and

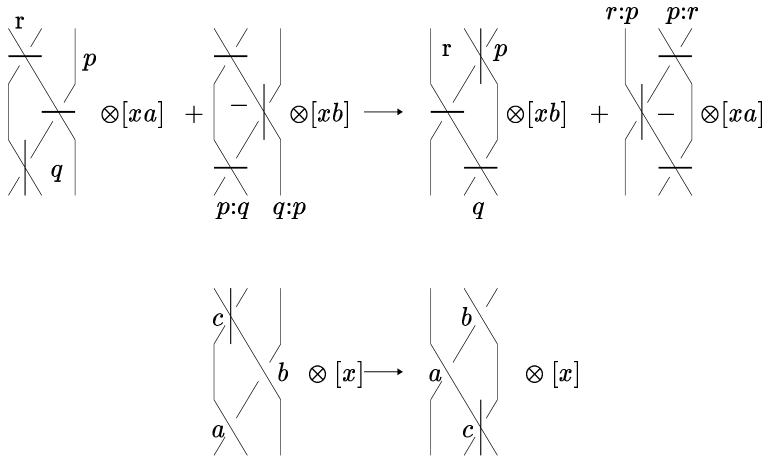

Figure 10: The isomorphism ψ for the third Reidemeister move: expressions on the left and right are generators for C and C' respectively.

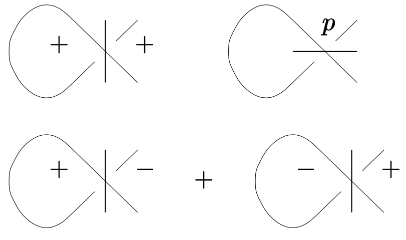

Figure 11: States with these configurations in the changing disc generate the contractible subcomplex C_{contr} in the case of a negative twist.

Figure 12: States with these configurations in the changing disc generate the contractible subcomplex C_{contr} in the case of a positive twist.

refer to [8] for more details.

Let us explain the second row in Figure 17. We call the upper crossing a and the lower crossing b.

Let $S_{+-} \otimes [xb]$ be the state on the left of this row. We use the index "$+-$" to stand for "positive marker at a and negative at b". Then there is a unique

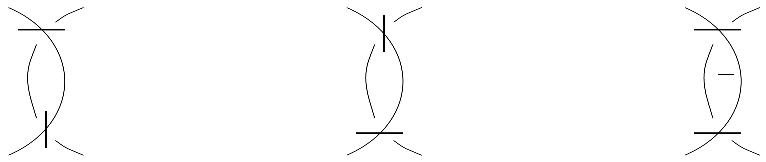

Figure 13: Generators for the contractible subcomplex C_{contr}, for the second Reidemeister move

Figure 14: Generators for the contractible subcomplexes C_{contr} and C'_{contr}, for the third Reidemeister move: note that they coincide after a rotation by π.

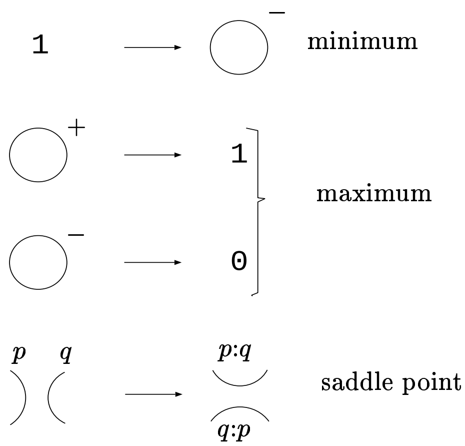

Figure 15: The effect of Morse modifications on states

state $S_{++} \otimes [x]$, such that S_{+-} and S_{++} coincide outside the changing disc. Note that $S_{++} \otimes [x]$ lies in C_{contr} (see Figure 13). Now apply the differential to this state.

$$d(S_{++} \otimes [x]) = S_{+-} \otimes [xb] + S_{-+} \otimes [xa] + (S_{+-,-} \otimes [xb]) + \sum_t S^t_{++} \otimes [xt]$$

Here, t ranges over those crossings outside the changing disc where S_{++} has a positive marker. $S_{+-,-}$ has positive marker at a, negative marker at b and a negative sign on the small circle enclosed in the changing disc. It appears only if the sign of the bottom arc in S_{+-} is negative.

The left hand side of this equation is in C_{contr} since S_{++} is. So is the third term and the big sum on the right hand side. Modulo the contractible subcomplex C_{contr}, we are left with

$$0 = S_{+-} \otimes [xb] + S_{-+} \otimes [xa].$$

By the definition of the differential the signs of the circles of S_{-+} depend on those of S_{+-} as determined by the "Frobenius calculus of signed circles" in Figure 3. Applying the isomorphism ψ to this expression it follows that the value on S is as displayed in Figure 17. □

3.4 Yet another move

Below we use the notation Ω_k for the k-th Reidemeister move. We will need one additional move whose induced map is not yet explicitly described; the mirror image $\overline{\Omega}_3$ of Ω_3. The chain equivalence corresponding to this move was not mentioned by Khovanov in [10] since it was not needed for his purposes there. It is well-known that it can be expressed in the other moves. We will fix one such expression to ensure that we have a unique induced homomorphism. To this end, consider Figure 19. There $\overline{\Omega}_3$ is expressed as the composition of an Ω_2-move, the original Ω_3-move and a second Ω_2-move. We have a choice as to where to make the first Ω_2. We choose, as shown in the picture, the wedge formed by the upper and middle strands. The composition induces a well-defined isomorphism on homology, in view of the previous sections.

3.5 Definition of the invariant

Let (Σ, L_0, L_1) be a link cobordism and let a movie presentation of Σ be given. Close to a critical level, the movie realizes a single Reidemeister move or oriented Morse modification. It follows that each critical level induces a chain map between the complexes of the diagrams just before and after this level. Between

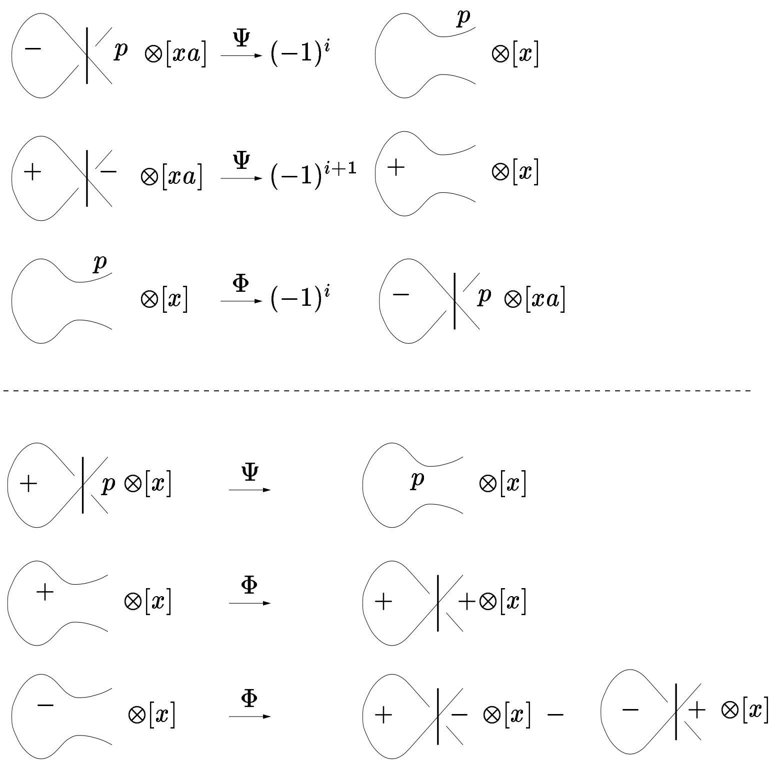

Figure 16: The effect of the first Reidemeister moves on states: the negative twist is above the dashed line, the positive twist below. States with any other local configuration map to zero.

two consecutive critical levels the diagram undergoes a planar isotopy. There is an obvious canonical isomorphism associated to this planar isotopy, given by tracing each state through it.

Composing all these maps, we obtain a chain homomorphism ϕ_Σ from $C^{i,j}(D_0)$ to $C^{i,j+\chi(\Sigma)}(D_1)$.

The reason for the grading of this map is plain. The Reidemeister moves are grading preserving, and the Morse modifications change the grading by $+1$ for extrema and by -1 for saddle points. That is, for each attachment of a 0– or 2–handle the grading increases by 1 and for each attachment of a 1–handle the grading decreases by 1. The sum of these changes is the Euler characteristic of Σ.

The chain map ϕ_Σ induces a map $\phi_{\Sigma*}$ on homology and the purported invariant

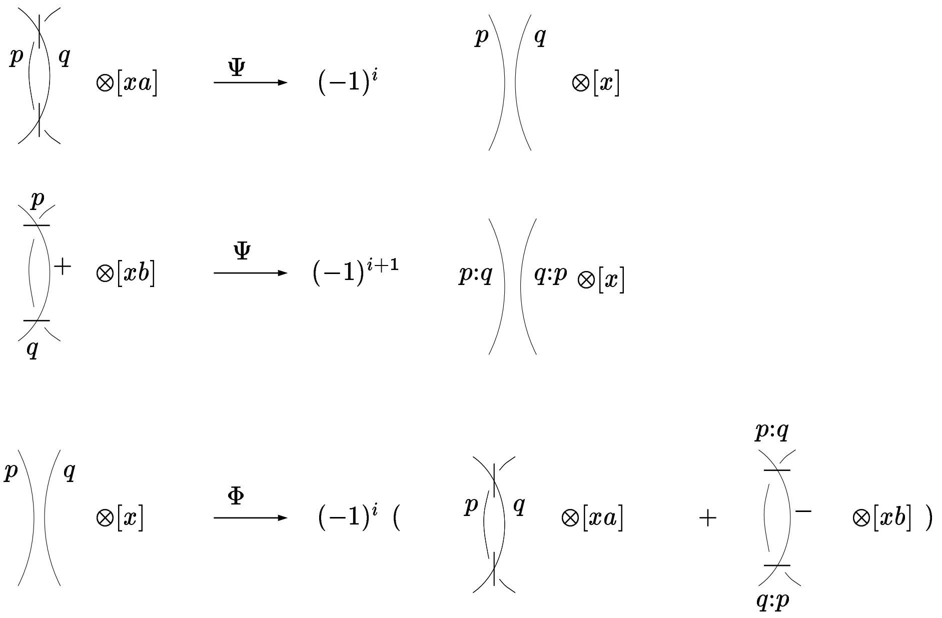

Figure 17: The effect of the second Reidemeister move on states: states with any other local configuration map to zero.

of link cobordisms is defined by

$$\Sigma \mapsto \pm\phi_{\Sigma *}.$$

The invariance is shown in Section 5.

Remark A closed knotted surface Σ in 4–space is a cobordism between empty links. The empty link has a single homology group $\mathbb{Z}$ in degree $(0,0)$, so in this case the invariant is a single natural number. If the Euler characteristic of Σ is non-zero this number must be zero, because of the above grading properties. Finally note that for an unknotted torus the number is 2.

4 Khovanov's conjecture

4.1 Original statement and examples

In [10], the following conjecture was made.

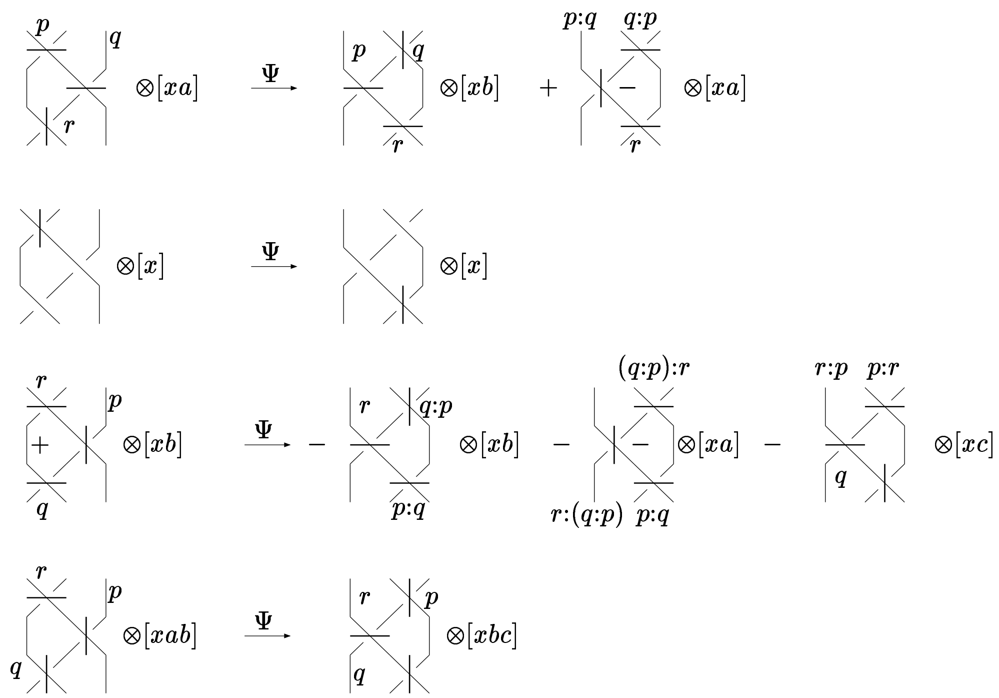

Figure 18: The effect of the third Reidemeister move on states: states with any other local configuration map to zero. The crossings are, from bottom to top, a,b,c on the left and c,a,b on the right.

Conjecture (Khovanov, [10], page 40) If two movies of a link cobordism Σ have the property that the corresponding source and target link diagrams of the two movies are isomorphic, the induced map on homology is the same up to an overall sign.

Any ambient isotopy of a link in 3–space gives rise to a link cobordism, which is ambient isotopic to a cylinder on the link, i.e. to the trivial cobordism. Furthermore, the effect of an ambient isotopy on a link diagram is a sequence of planar isotopies and Reidemeister moves. Hence, if taken literally, this conjecture implies that any sequence of planar isotopies and Reidemeister moves, which starts and finishes with the same link diagram either induces the identity map in all of the homology groups associated to this diagram, or induces multiplication by -1 in all of them. This is not true, even for very simple link diagrams, as the following example shows.

Remark It is obvious that the homology groups of the unnested unlink dia-

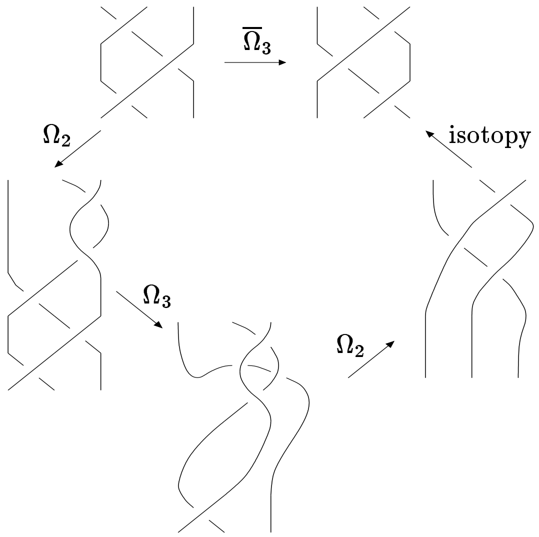

Figure 19: Another third Reidemeister move

gram 0_1^2 of two components without crossings coincide with the chain groups.

$$\begin{aligned} C^{0,-2} &= \mathcal{H}^{0,-2} \cong \mathbb{Z} \\ C^{0,0} &= \mathcal{H}^{0,0} \cong \mathbb{Z} \oplus \mathbb{Z} \\ C^{0,2} &= \mathcal{H}^{0,2} \cong \mathbb{Z} \end{aligned}$$

A planar isotopy which interchanges the two components also interchanges the two generators of $C^{0,0}$ (the states which have different signs on the two circles) and is neither plus or minus the identity. (This fact was first pointed out by Olof-Petter Östlund [14].) Clearly, similar things can be done with more and non-trivial components. The question whether non-trivial maps can be induced on the homology of a *knot* diagram is now natural. In Theorem 1 below we find an affirmative answer to that question.

Definition 9 The subgroup of the automorphism group of $\mathcal{H}(D)$ consisting of isomorphisms induced from sequences of Reidemeister moves and planar isotopies from D to itself we call *the monodromy group* of D. The monodromy group is called *trivial* if it consists of only the identity map and its negative.

Remark The monodromy group of 0_1^2 above is non-trivial.

Theorem 1 *The monodromy group of $\mathcal{H}^{i,j}(D)$ is non-trivial when D is the diagram of the knot 8_{18} pictured in Figure 20.*

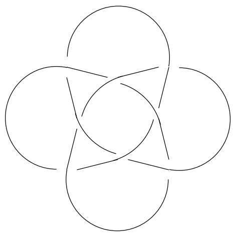

Figure 20: The knot diagram 8_{18}

Proof We will compute $C^{i,j}(D)$ for $j = -7$. Then the following formulas obviously hold.

$$w(D) = 0$$
$$\sigma(s) + 2\tau(S) = 14$$
$$-8 \le \sigma(s) \le 8$$

If all markers in S are positive then $\sigma(S) = 8$, i.e. $i = -4$. The resolution of S consists of five circles, four of which are pairwise unnested and enclosed by the fifth. By the second of the above equations $\tau(S) = 3$, so one of the circles has to be negative and the rest positive. There are five such states, $v_1, ..., v_5$, and they are shown in Figure 21.

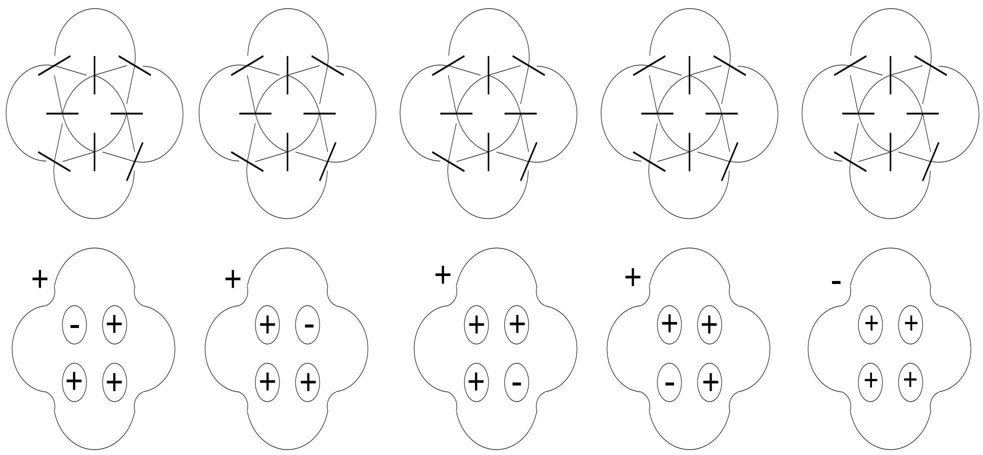

Figure 21: Generators $v_1, ..., v_5$ of $C^{-4,-7}(8_{18})$

If $i = -3$, then $\sigma = 6$ so $\tau = 4$. The resolution of any such state (i.e. with exactly one negative marker) consists of three unnested circles enclosed by a fourth. All circles are equipped with positive signs. There are eight states of this form, $u_1, ..., u_4$ and $w_1, ..., w_4$. They are shown in Figure 22.

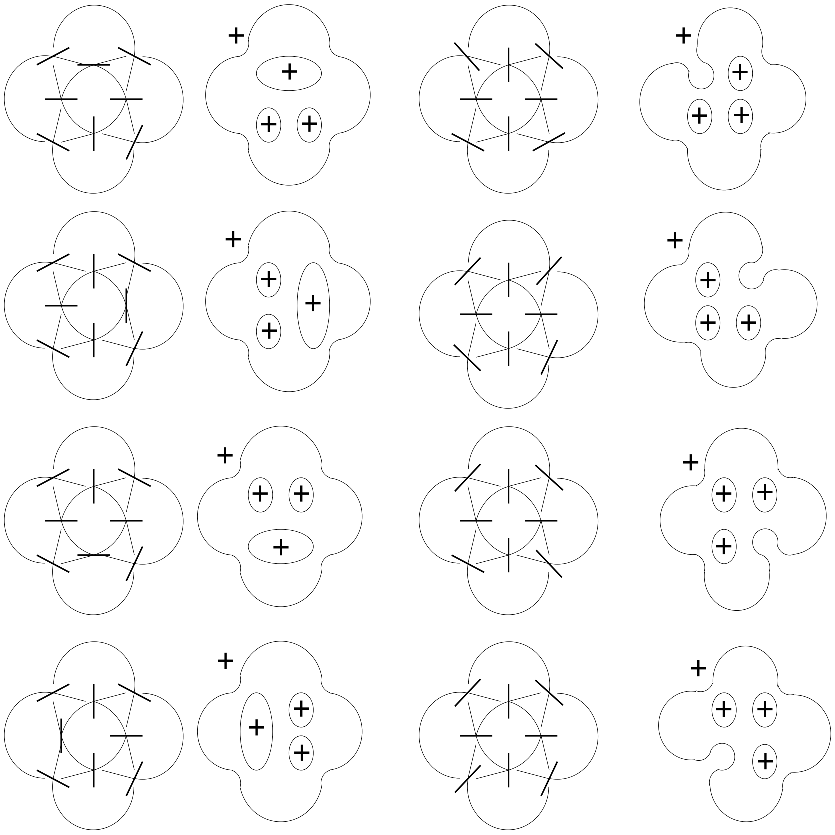

Figure 22: Generators $u_1, ..., u_4$ (left column) and $w_1, ..., w_4$ (right column) of $C^{-3,-7}(8_{18})$

The differential of any state S in $C^{-3,-7}(D)$ is zero since any change of a positive marker in S causes two positive circles to merge. (In fact, all groups $C^{i,-7}(D)$ are zero for i different from -3 or -4.) We have the following fragment of the chain complex:

$$\begin{tikzcd} 0 \arrow[r] & C^{-4,-7}(D) \arrow[r, "d"] \arrow[d, "\cong"] & C^{-3,-7}(D) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^5 \arrow[r, "A"] & \mathbb{Z}^8 \arrow[r] & 0 \end{tikzcd}$$

where the vertical isomorphisms are given by the choices of ordered bases $v_i \mapsto$

e_i and $w_i \mapsto e_i$, $u_i \mapsto e_{i+4}$, where $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{Z}^n$. The definition of d gives:

$$v_i \mapsto w_i + u_i + u_{i-1}, \quad 1 \leq i \leq 4$$
$$v_5 \mapsto \sum_i w_i$$

with the natural cyclic ordering of indices. We do not need to consider the $E(L)$-tensor factor, since for any state involved its singleton or empty set of negative markers is uniquely ordered. Thus, the matrix A is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A'$$

and A' is the Smith normal form of A. It follows that d is injective and that A and A' are presentation matrices of the isomorphism class of $\mathcal{H}^{-3,-7}(D)$. From A' we see that $\mathcal{H}^{-3,-7}(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

Let Σ be the planar isotopy which rigidly rotates D clockwise by an angle $\pi/2$ about the center. Denote by ϕ_Σ and $\phi_{\Sigma *}$ the maps induced on the chain complex and on homology, respectively.

Then it is clear that $\phi_\Sigma(w_i) = w_{i+1}$ and that $\phi_\Sigma(u_i) = u_{i+1}$ so that $\phi_{\Sigma *}([w_i]) = [w_{i+1}]$ and $\phi_{\Sigma *}([u_i]) = [u_{i+1}]$. Now, suppose the monodromy group were trivial. Then either $\phi_{\Sigma *}[w_i] = [w_i]$ and $\phi_{\Sigma *}[u_i] = [u_i]$ or $\phi_{\Sigma *}[w_i] = -[w_i]$ and $\phi_{\Sigma *}[u_i] = -[u_i]$. This gives two cases:

Case(+): Then for all i, $[w_i] = [w_{i+1}]$ and $[u_i] = [u_{i+1}]$. This immediately reduces the number of generators of $\mathcal{H}^{-3,-7}(D)$ to two and the relations to

$$w_1 + 2u_1 = 0$$
$$4w_1 = 0$$

which is a presentation of $\mathbb{Z}_8$. This contradicts the computationof $\mathcal{H}^{-3,-7}(D)$ above.

Case(−): In this case $[w_i] = -[w_{i+1}]$ and $[u_i] = -[u_{i+1}]$. This reduces the relations of $\mathcal{H}^{-3,-7}(D)$ to

$$w_i = 0, \quad 1 \leq i \leq 4$$
$$u_i = -u_{i+1}, \quad 1 \leq i \leq 4$$

which is a presentation of $\mathbb{Z}$, and we have another contradiction. This completes the proof. $\square$

Remark Let Σ be the cobordism of the planar isotopy in the proof above. With coefficients in $\mathbb{Q}$, $\mathcal{H}^{-3,-7}(D) \cong \mathbb{Q}^3$. Three independent cycles in the (ϕ_Σ–invariant) orthogonal complement of $dC^{-4,-7}(D)$ are

$$\begin{aligned} v_1 &= w_1 - w_3 - u_1 + u_2 \\ v_2 &= w_1 + w_2 - w_3 - w_4 - u_1 + u_3 \\ v_3 &= w_2 - w_4 - u_1 + u_4 \end{aligned}$$

and it is easy to see that in the basis given by the homology classes of these cycles the map $\phi_{\Sigma *}$ on $\mathcal{H}^{-3,-7}(D)$ is given by the following matrix.

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This map is of order four and its spectrum is $\{-1, i, -i\}$.

4.2 Recovering the conjecture

Because of the examples in the previous subsection it is necessary to reformulate the conjecture to have any hope of proving it. It turns out that it is enough to require that the isotopy is relative to the boundary.

Theorem 2 (Khovanov's conjecture) *For oriented links L_0 and L_1, presented by diagrams D_0 and D_1, an oriented link cobordism $\Sigma \subset \mathbb{R}^3 \times [0,1]$ from L_0 to L_1, defines homomorphisms $\mathcal{H}^{i,j}(D_0) \to \mathcal{H}^{i,j+\chi(\Sigma)}(D_1)$ invariant up to an overall multiplication by -1 under ambient isotopy of Σ leaving $\partial\Sigma$ setwise fixed. Moreover, this invariant is non-trivial.*

The proof of Theorem 2 occupies the next section. It uses the following result by Carter and Saito, previously alluded to in Section 3.2. It is given a slight reformulation to fit our purposes.

Theorem (Carter–Saito, [4]) *Two movies represent equivalent link cobordisms in the sense of Theorem 2 if and only if they can be related via a finite sequence of movie moves and interchanges of distant critical points of the time function t.*

5 Proof of Khovanov's conjecture

We split the proof into three lemmas.

Lemma 5.1 (Distant critical points) *The interchange of two distant critical points of the surface diagram does not change the induced map on homology.*

Proof Let D be a link diagram and let D' be obtained from D by two local moves in disjoint changing discs. There are two different orders in which these moves can be performed. They correspond to two different movies from D to D', inducing maps $\phi_l, \phi_r : C(D) \to C(D')$. Let S be a state of D.

If we temporarily forget the signs of the resolutions the moves are completely local and only affect S inside the changing discs.

From the Figures 15 through 18 describing the induced maps it is clear that the resolution $\mathrm{res}(T)$ of each term T of $\phi_l(S)$ or $\phi_r(S)$ can be obtained from $\mathrm{res}(S)$ by a sequence of applications of the signed circle calculus (recall Figure 3) inside the changing discs, together with the addition or removal of some circle enclosed in the changing disc. The latter changes are local and commute with the others. Hence without loss of generality we may forget about them. It is therefore sufficient to prove that two distant saddle point moves performed on the resolution commute.

This follows from the fact that the product m and coproduct Δ of the Frobenius algebra A are associative and coassociative respectively, and satisfy the relation

$$\Delta \circ m = (m \otimes id)(id \otimes \Delta)$$

In Figure 23 these relations are interpreted topologically. We immediately see that they express the commutativity of saddle point moves. This completes the proof. □

Lemma 5.2 (Carter–Saito moves) *For any Carter–Saito move M, the maps induced on homology by the two sides of M coincide up to an overall sign. The sign difference is given in Table 1.*

Remark The words "Downward" and "Upward" refer to the direction of the movies in Figures 4, 5 and 6. "As displayed" means as displayed in these figures. Observe that if the movies contain only Reidemeister moves, the induced maps are inverses, and it is enough to consider one time direction.

Proof We compute the maps on the chain level and analyze the effect on homology.

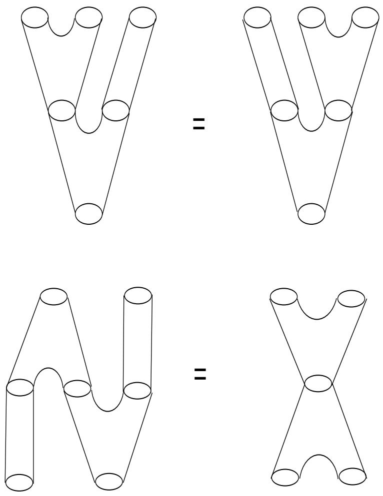

Figure 23: Associativity/coassociativity and an additional relation prove the commutativity of distant critical points.

Movie move no.	Downward time	Upward time
1,2,3,4,5	same sign	same sign
6, pos twist	different sign	different sign
6, neg twist	same sign	same sign
7	same sign	same sign
8	same sign	different sign
9	same sign	same sign
10	different sign	same sign
11, as displayed	different sign	same sign
11, mirror image	same sign	same sign
12, as displayed	different sign	same sign
12, mirror image	same sign	different sign
13, as displayed	same sign	same sign
13, mirror image	different sign	different sign
14, as displayed	different sign	different sign
14, mirror image	same sign	same sign
15	different sign	different sign

Table 1

Movie moves 1-5

The left hand side of each of these moves trivially induces the identity. The same holds for the right hand side, since it is the composition of a Reidemeister move and its inverse.

Movie move 6

Negative twist The right side is just the appearance of a negative twist. The induced map is given by Figure 16, top. The left hand side is described in Figure 25.

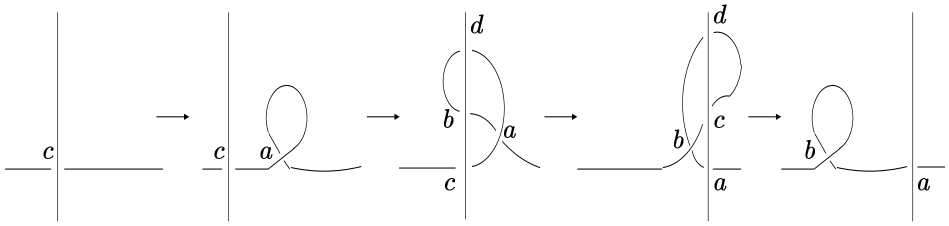

Figure 24: Enumeration of the crossings of the left side of Move 6

The two end results seem to coincide, but we still have to analyze the possible sign difference. That is, what happens to the $E(L)$ tensor factor. Again, on the right hand side this is given by Figure 16. As for the left hand side, pick an enumeration of the crossings as in Figure 24 and then consider Figure 25 again. The states in dashed boxes eventually map to zero. Therefore let us disregard them. Then the $E(L)$–factor transforms as below for the two types of states (negative respectively positive marker at the crossing).

$$[xc] \mapsto [xca] \mapsto [xcab] \mapsto [xcab] \mapsto [xab]$$

$$[x] \mapsto [xa] \mapsto [xab] \mapsto [xbc] \mapsto [xb]$$

We see that the signs are the same for both sides.

Positive twist In the positive twist case, the two induced chain maps do not coincide. However, their sum evaluated on a state is an element of the contractible subcomplex corresponding to the twist in the target diagram. The projection on the non-contractible part gives an isomorphism on homology. Tracing the signs as above shows that there is a difference in signs here. We omit details and note only that the Ω_3–move involved here is the "reflected" one $\overline{\Omega}_3$ from Section 3.4.

These remarks apply regardless of whether the horizontal strand is behind or in front of the vertical one.

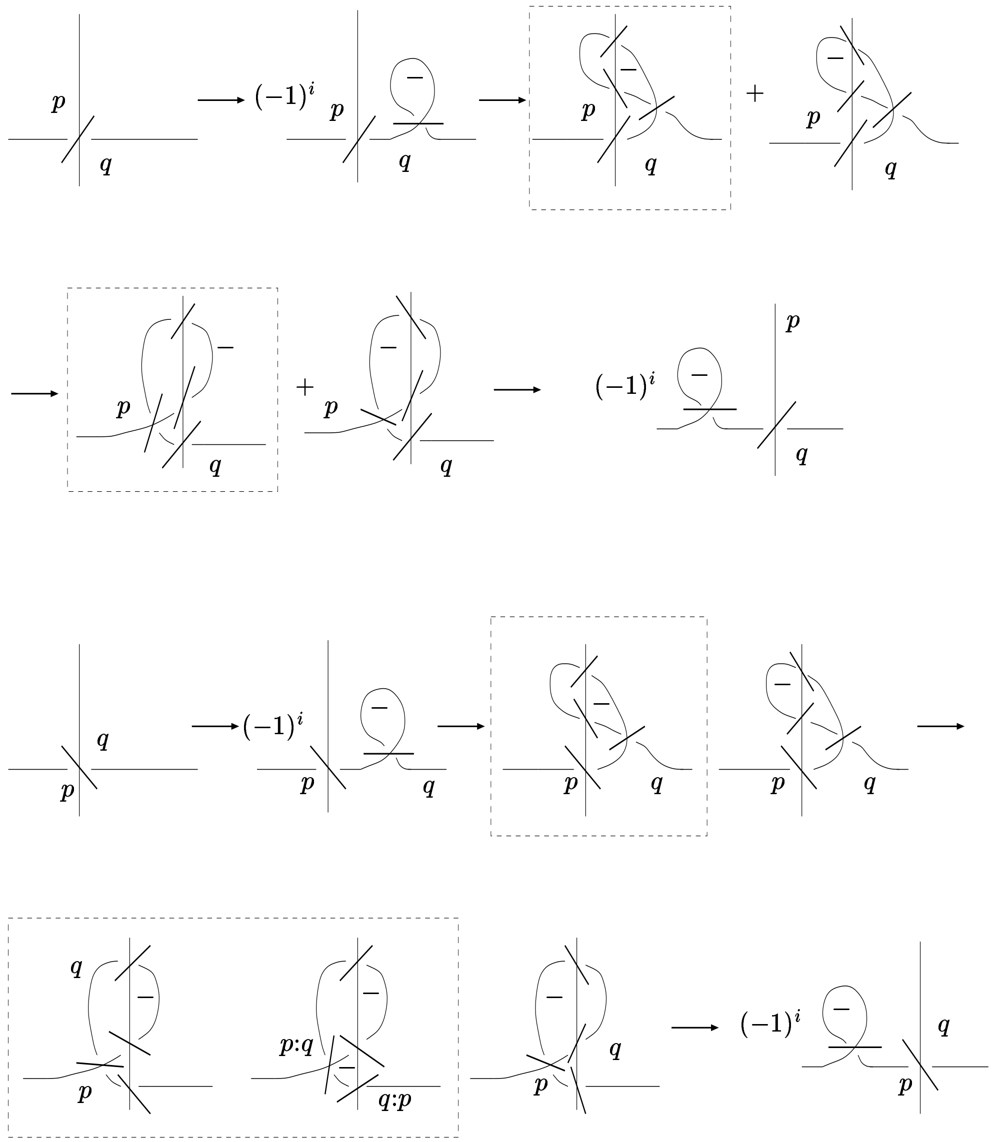

Figure 25: Move 6

Movie move 7

Reduction to one version of the move It is sufficient to consider a single version of this move, for the following reason. Let P be the three-dimensional configuration of four planes making up the left side of any quadruple point move. (A small 3–ball in which four sheets of the surface diagram form a tetrahedron.) Enumerate the four sheets in the order of increasing height with respect to the projection from four to three dimensions. Let P_0 be the standard configuration which has the $xy-$, $xz-$ and $yz-$planes as its first, second and third sheets, and with the fourth plane having normal $(1,1,1)$.

The first, second and third sheets of P can be isotoped to coincide with the first, second and third sheets of P_0 respectively. With this done, the fourth sheet of P can be made to sit as one of the eight planes $(\pm1,\pm1,\pm1)$. The moves which correspond to isotopies of the surface diagram are the moves 8–15 and the interchanges of distant critical points. The invariance under these moves is proved in other subsections (independently of the invariance under this move). Thus, this isotopy does not change the induced homomorphism (up to sign).

The quadruple point move corresponding to this new plane configuration can be replaced with a sequence of three movie moves consisting of one move of type 5 (adding two triple points in the diagram), one quadruple point move in a neighbouring octant, and another (inverse) move of type 5 (subtracting two triple points). (This is a four-dimensional counterpart of Figure 19, where one type of Ω_3–move was replaced by a sequence containing one Ω_2–move, the other Ω_3–move and an inverse Ω_2–move.) This result, and pictures explaining it in detail, can be found in [5], page 11. From the invariance under the moves of types 3 and 5 we may therefore assume that the move takes place in any octant we like, in particular with the standard configuration of sheets.

Finally, the result can be isotoped to the right hand side of the original move. Note that orientations never enter into the calculations. It follows that invariance under any version of move 7 is implied by the invariance under a single version. We choose one with its crossings as in Figure 26 below, and label the crossings as indicated there. The left movie is the upper one in this figure.

Computing the maps Consider the target diagram D' in Figure 26. The bottom right triangle, formed by the vertices d,e,f is the target of a third Reidemeister move and hence defines a splitting of the chain complex $C(D')$ as explained in Section 3.3, Figures 10 and 14. In the discussion that follows we will tacitly disregard all states in this subcomplex in the discussion that follows. Indeed, projecting out this subcomplex induces an isomorphism on homology and it is enough to consider the composition of right and left maps with the projection.

Similarily, the complex of the source diagram D splits according to the upper left triangle formed by the vertices d,e,f, where the first Ω_3–move in the

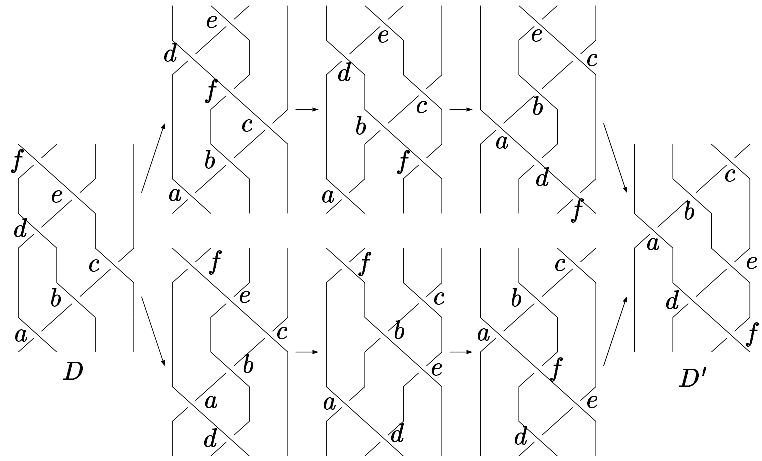

Figure 26: Enumeration of the crossings in Move 7

left movie takes place. We need to consider only the generators of the non-contractible factor of this splitting, since the inclusion of this factor induces an isomorphism on homology.

It is easily checked that a state as in Figure 27 maps to zero under the left hand as well as under the right hand map, regardless of the markers at a, b, c.

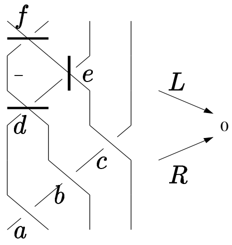

Figure 27: Move 7

It follows that we need only look at the states (let us call them A–states) which have positive markers at e, f and negative at d, and all states which have a negative marker at f (let us call these B–states).

It is also easy to verify that A–states with local configurations of markers as in Figure 28 all map to zero under both sides of the move.

A–states with negative markers at the three remaining crossings behave in the same way under both sides of the move (Figure 29).

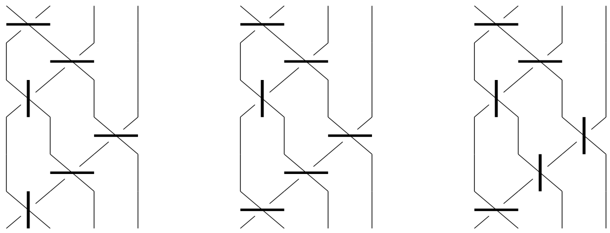

Figure 28: Move 7

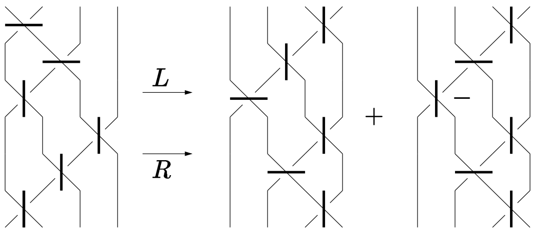

Figure 29: Move 7

We omit here most of the signs of the resolutions, but a closer examination [8] reveals that the same modifications of the signed resolutions take place on both sides, so that the final results indeed coincide.

It is tedious but straightforward to verify the results in Figures 30 and 31 below. Most of the information about the signed resolutions is again left out in these figures. Consider the expressions in the right column. They are written modulo the contractible subcomplex of the bottom right triangle. The resolutions in the left and right hand images are the same (that is, planar isotopic) for two states placed above each other. Indeed, they even have the same signed resolutions, as can be seen by going through the ways signs change under the two sides of the move.

The next thing to note is that the difference of two states enclosed in a dashed box is really in the contractible subcomplex. Such states differ only at the lower right triangle and as mentioned above they have the same signed resolutions. That is, they look like the first two terms on the right hand side of the equation in Figure 32. At first glance, it seems as though we should get the difference rather than the sum of these terms. However, a closer analysis reveals the correct sign hidden in the $E(L)$–factor. Since all terms except for these two are in C_{contr}, the statement follows.

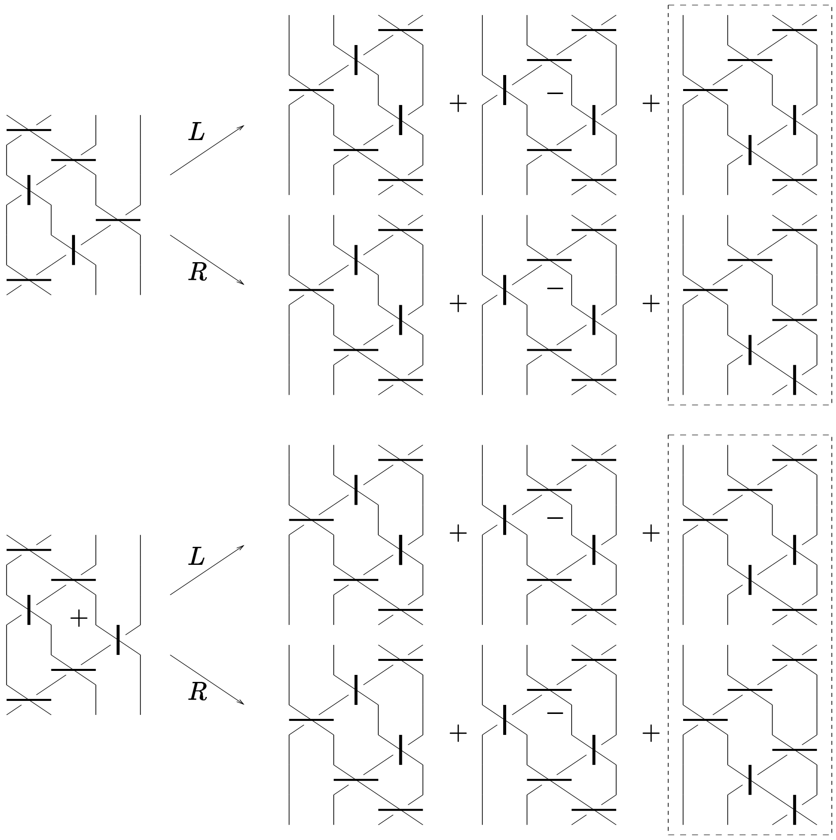

Figure 30: Move 7

Finally, consider again A–states with markers (pos, pos, neg) respectively (neg, pos, neg) at crossings (a, b, c), but this time with a small negative component (the circle *bced*) instead of a positive one. On these states both sides are zero as a short computation shows. This finishes the proof for A–states, except for the verification that the $E(L)$–factor really behaves as claimed. Using the enumeration of vertices above, this is a straightforward check.

As regards the B–states, i.e. states with a negative marker at f, these are more well-behaved. Indeed, the left and right images are the same for each of these states, so the maps are the same on the chain level, already. We omit details.

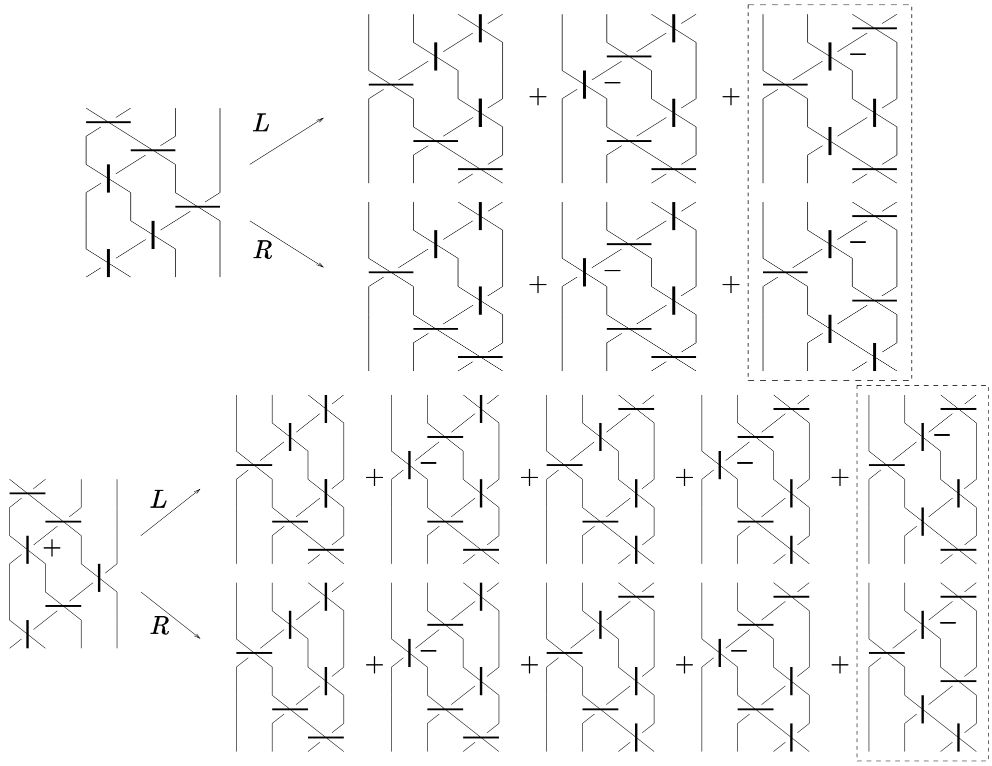

Figure 31: Move 7

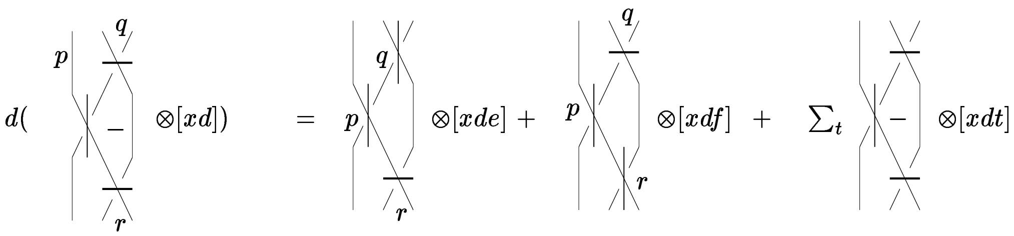

Figure 32: Rewriting dashed box differences

Movie move 8

In Figure 33, we give the computation for the move no. 8 with time flowing downwards. A new-born circle is always marked with a minus sign, and the Ω_2–move that follows eliminates the difference between the two sides.

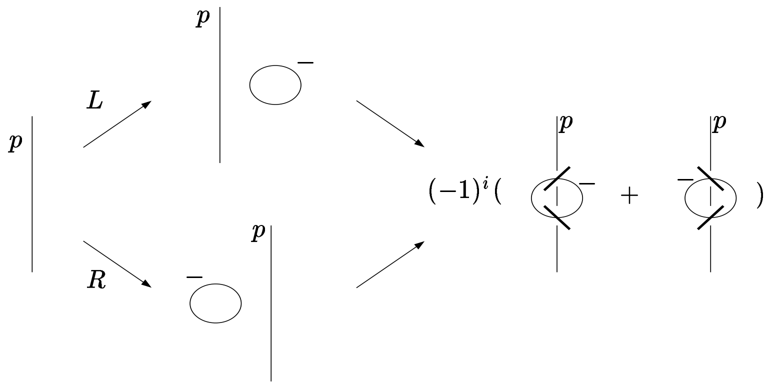

Figure 33: Move 8 in downward time

In upward time, we can start by splitting the chain complex into $C \oplus C_{contr}$ according to the Ω_2–move about to be performed on the left hand side. It is enough to check what happens to the generators of C (see Figure 9). The calculation is included in Figure 34.

No extra signs appear from the $E(L)$–factor, neither in upward nor in downward time. The version where the vertical strand is above the circle is almost identical.

Movie move 9

The computation here is very trivial on both sides of the move. Only maps associated to Morse modifications are used. We immediately see that the maps coincide up to sign. No markers are involved, so obviously the effect on the $E(L)$–factor is trivial. See Figure 35.

Movie move 10

In downward time the calculation in Figure 36 yields the result.

And in upward time, the one in Figure 37, in which one should note that the second level is written modulo C_{contr}. The states not shown all map to zero.

The action on the $E(L)$–factor is the same on both sides, and in both time directions. If the middle strand is behind the others, everything works similarily.

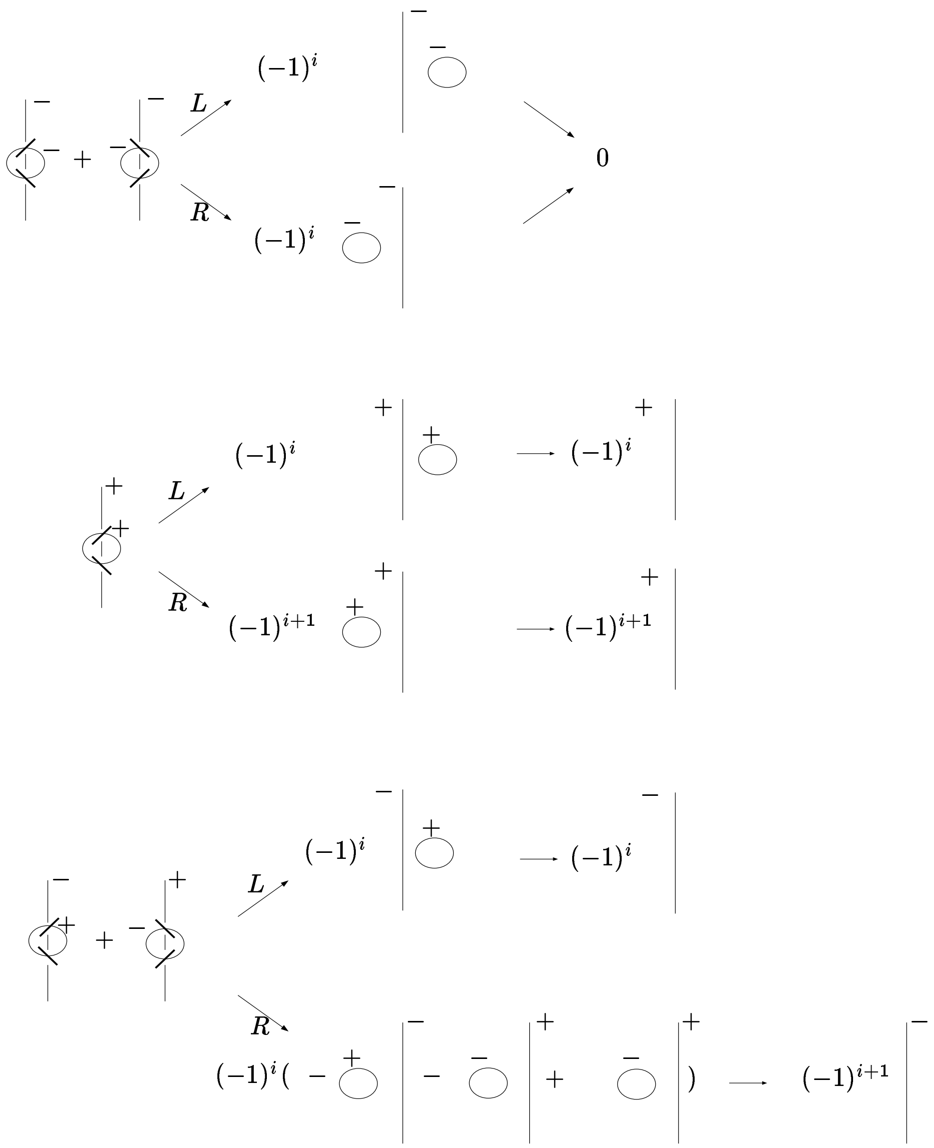

Figure 34: Move 8 in upward time

Movie moves 11 – 15

The calculations for the rest of the moves are similar to the above and straightforward (albeit sometimes tedious) given Proposition 3.1. We omit them, and

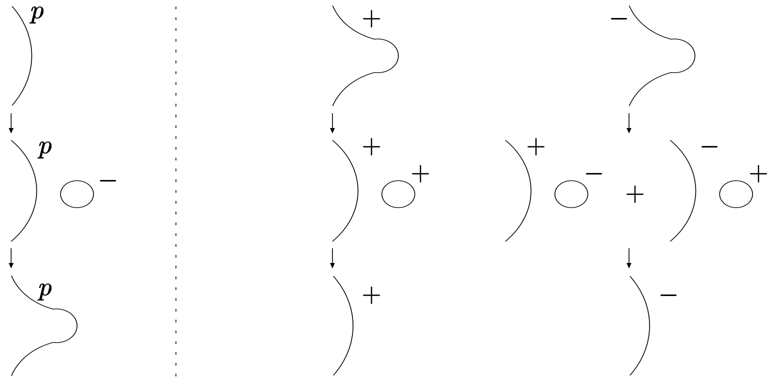

Figure 35: Move 9

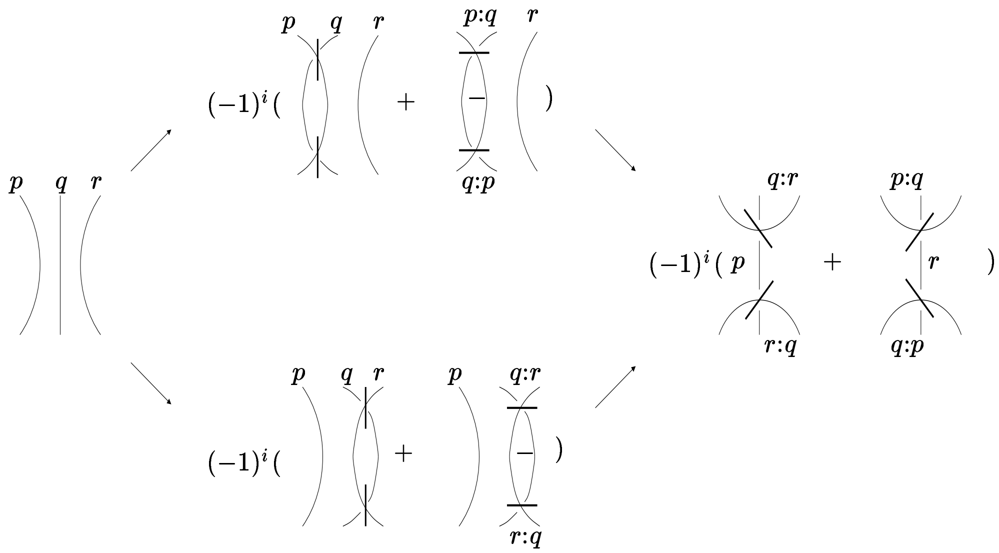

Figure 36: Move 10 in downward time

refer the interested reader to [8] for details. There the reader can also find more details for the previous cases, if (s)he so wishes. ☐

Lemma 5.3 *The invariant is non-trivial.*

Proof Let K be a knot with non-trivial Jones polynomial. (No non-trivial knots with trivial Jones polynomial are known at present.) For any knot K in

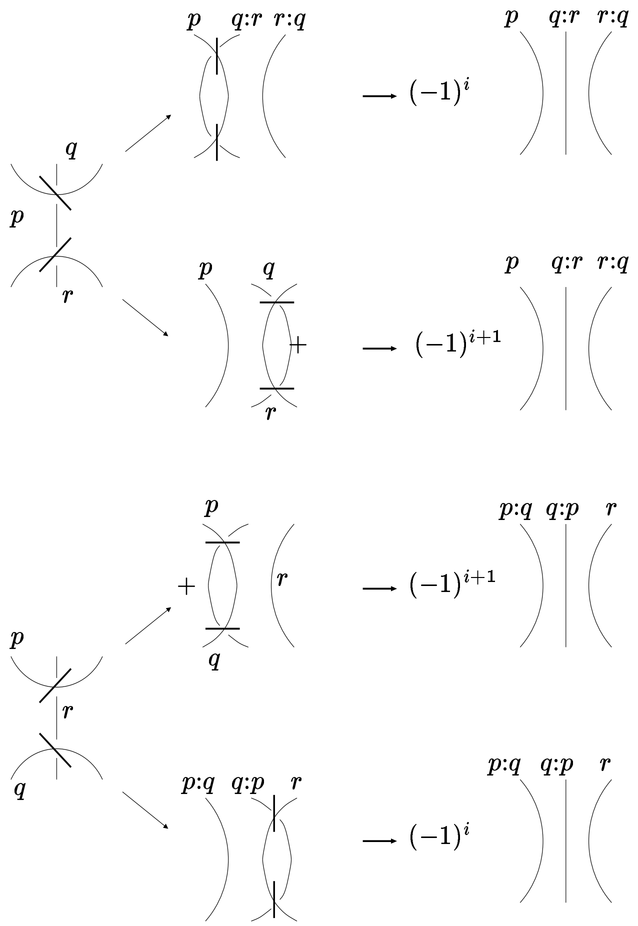

Figure 37: Move 10 in upward time: on the second level terms in C_{contr} are not drawn.

$\mathbb{R}$, it is well-known that the connected sum of K and its mirror image $\bar{K}$ is "slice", i.e. bounds an embedded disc in $\mathbb{R}^3 \times I$. Put one such disc at each end of $\mathbb{R}^3 \times I$ and connect the two discs with a vertical tube. This is a knotted cylinder and induces some map on homology. In general, this map has a big kernel, since it factors through the homology of the unknot in the space between the two discs. Therefore it is different from the identity cylinder on $K\#\bar{K}$ if the latter has non-trivial homology groups. This proves non-triviality, and concludes the whole proof. □

6 Lefschetz polynomials of link endocobordisms

The calculation of the induced homomorphism on homology induced by a link cobordism can be rather involved. However it is relatively easy to compute it on the level of chain groups, because of their explicit, geometric definition. It would therefore be interesting to find (possibly weaker) derived invariants of link cobordisms computable from the homomorphism induced on the chain group level.

In this section we make some remarks on special link cobordisms, namely those which have identical source and target L and the cobordism Σ a surface with zero Euler characteristic. We could call such objects (Σ, L) "links with endomorphism" or "link endocobordisms".

By Theorem 2, to every link endocobordism (Σ, L) and diagram D of L there is a well-defined (up to sign) endomorphism $\phi_* = \phi_{\Sigma *}$ on $\mathcal{H}(D)$. On $\mathcal{H}^{i,j}(D)$ we denote this map by $\phi_*^{i,j}$. Note that the bigrading is $(0,0)$. For fixed j this is an endomorphism of a chain complex, so it has a well-defined Lefschetz number $L_j(\Sigma)$ (also up to sign) which is the alternating sum of traces:

$$L_j(\Sigma) = \sum_i (-1)^i tr(\phi_*^{i,j} : \mathcal{H}^{i,j}(D) \otimes \mathbb{Q} \to \mathcal{H}^{i,j}(D) \otimes \mathbb{Q})$$

$L_j(\Sigma)$ is obviously invariant (up to sign) under ambient isotopy, as is the following alternating sum.

Definition 10 Summing over j with coefficients q^j gives us *the Lefschetz polynomial* of the link endocobordism:

$$L(\Sigma) = \sum_j L_j(\Sigma) q^j$$

Isotopies of L act in a natural way on endocobordisms (Σ, L) with diagram D. Namely, to each isotopy of L corresponds a sequence of Reidemeister moves on D, that is, a movie from D to another diagram D'. Conjugating Σ by the corresponding link cobordism, we get a new pair (Σ', L') with diagram D'. Since the Lefschetz polynomial is built from traces it is invariant under conjugation. In fact, even each L_j is invariant.

There is the well-known theorem that L_j can be computed directly on the chain level, without passing to homology, so that

$$L_j(\Sigma) = \sum_i (-1)^i tr(\phi^{i,j} : C^{i,j}(D) \otimes \mathbb{Q} \to C^{i,j}(D) \otimes \mathbb{Q}).$$

where $\phi^{i,j} = \phi_\Sigma^{i,j}$ is the chain map induced by Σ.

Remark It is easy to see that the map $\phi_\Sigma^{i,-7}$ of clockwise rotation in Section 4.1, only has the following non-zero matrices in the canonical bases of $C^{i,-7}(D)$.

$$\phi_\Sigma^{-4,-7} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\phi_\Sigma^{-3,-7} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

From this we immediately see that $L_{-7}(\Sigma) = (\pm)1$. (This can also be seen from the matrix of $\phi_{\Sigma*}$ in the remark at the end of Section 4.1.) By contrast, the identity map has this Lefschetz number equal to -3. Hence the induced maps cannot be the same.

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