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Enrichment over iterated monoidal categories

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Abstract Joyal and Street note in their paper on braided monoidal categories [9] that the 2–category $\mathcal{V}$–Cat of categories enriched over a braided monoidal category $\mathcal{V}$ is not itself braided in any way that is based upon the braiding of $\mathcal{V}$. The exception that they mention is the case in which $\mathcal{V}$ is symmetric, which leads to $\mathcal{V}$–Cat being symmetric as well. The symmetry in $\mathcal{V}$–Cat is based upon the symmetry of $\mathcal{V}$. The motivation behind this paper is in part to describe how these facts relating $\mathcal{V}$ and $\mathcal{V}$–Cat are in turn related to a categorical analogue of topological delooping. To do so I need to pass to a more general setting than braided and symmetric categories — in fact the k–fold monoidal categories of Balteanu et al in [2]. It seems that the analogy of loop spaces is a good guide for how to define the concept of enrichment over various types of monoidal objects, including k–fold monoidal categories and their higher dimensional counterparts. The main result is that for $\mathcal{V}$ a k–fold monoidal category, $\mathcal{V}$–Cat becomes a $(k-1)$–fold monoidal 2–category in a canonical way. In the next paper I indicate how this process may be iterated by enriching over $\mathcal{V}$–Cat, along the way defining the 3–category of categories enriched over $\mathcal{V}$–Cat. In future work I plan to make precise the n–dimensional case and to show how the group completion of the nerve of $\mathcal{V}$ is related to the loop space of the group completion of the nerve of $\mathcal{V}$–Cat.

This paper is an abridged version of [8].

AMS Classification 18D10; 18D20

Keywords Loop spaces, enriched categories, n–categories, iterated monoidal categories

1 Introduction

A major goal of higher dimensional category theory is to discover ways of exploiting the connections between homotopy coherence and categorical coherence. Stasheff [15] and Mac Lane [13] showed that monoidal categories are precisely analogous to 1–fold loop spaces. There is a similar connection between symmetric monoidal categories and infinite loop spaces. The first step in filling in the gap between 1 and infinity was made in [6] where it is shown that the group completion of the nerve of a braided monoidal category is a 2–fold loop space. In [2] the authors finished this process by, in their words, "pursuing an analogy to the tautology that an n–fold loop space is a loop space in the category of $(n-1)$–fold loop spaces." The first thing they focus on is the fact that a braided category is a special case of a carefully defined 2–fold monoidal category. Based on their observation of the correspondence between loop spaces and monoidal categories, they iteratively define the notion of n–fold monoidal category as a monoid in the category of $(n-1)$–fold monoidal categories. In their view "monoidal" functors should be defined in a more "lax" way than is usual in order to avoid strict commutativity of 2–fold and higher monoidal categories. In [2] a symmetric category is seen as a category that is n–fold monoidal for all n.

The main result in [2] is that their definition of iterated monoidal categories exactly corresponds to n–fold loop spaces for all n. They show that the group completion of the nerve of an n–fold monoidal category is an n–fold loop space. Then they describe an operad in the category of small categories which parameterizes the algebraic structure of an iterated monoidal category. They show that the nerve of this categorical operad is a topological operad which is equivalent to the little n–cubes operad. This latter operad, as shown in [3] and [14], characterizes the notion of n–fold loop space. Thus the main result in [2] is a categorical characterization of n–fold loop spaces.

The present paper pursues the hints of a categorical delooping that are suggested by the facts that for a symmetric category, the 2–category of categories enriched over it is again symmetric, while for a braided category the 2–category of categories enriched over it is merely monoidal. Section 2 reviews enrichment. Section 3 goes over the recursive definition of the k–fold monoidal categories of [2], altered here to include a coherent associator. The immediate question is whether the delooping phenomenon happens in general for these k–fold monoidal categories. The answer is yes, once enriching over a k–fold monoidal category is carefully defined in Section 4, where we see that all the information included in the axioms for the k–fold category is exhausted in the process. The definition also provides for iterated delooping as is previewed in Section 5.

I have organized the paper so that sections can largely stand alone, so please skip them when able, and forgive redundancy when it occurs. Thanks to my advisor, Frank Quinn, for inspirational suggestions. Thanks to XY-pic for the diagrams. Thanks especially to the authors of [2] for making their source available–

I learned and borrowed from their use of LaTeX as well as from their insights into the subject matter.

2 Review of categories enriched over a monoidal category

In this section I briefly review the definition of a category enriched over a monoidal category $\mathcal{V}$. Enriched functors and enriched natural transformations make the collection of enriched categories into a 2-category $\mathcal{V}$-Cat. This section is not meant to be complete. It is included due to, and its contents determined by, how often the definitions herein are referred to and followed as models in the rest of the paper. The definitions and proofs can be found in more or less detail in [10] and [5] and of course in [12].

Definition 1 For our purposes a *monoidal category* is a category $\mathcal{V}$ together with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an object I such that:

(1) $\otimes$ is associative up to the coherent natural transformations α. The coherence axiom is given by the commuting pentagon:

$$\begin{tikzcd}
& ((U \otimes V) \otimes W) \otimes X \arrow[rr, "\alpha_{UVW} \otimes 1_X"] \arrow[dl, "\alpha_{(U\otimes V)WX}"'] & & (U \otimes (V \otimes W)) \otimes X \arrow[dr, "\alpha_{U(V\otimes W)X}"] & \\
(U \otimes V) \otimes (W \otimes X) \arrow[drr, "\alpha_{UV(W\otimes X)}"'] & & & & U \otimes ((V \otimes W) \otimes X) \arrow[dll, "1_U \otimes \alpha_{VWX}"] \\
& & U \otimes (V \otimes (W \otimes X)) & &
\end{tikzcd}$$

(2) I is a strict 2-sided unit for $\otimes$.

Definition 2 A (small) $\mathcal{V}$ *–Category* $\mathcal{A}$ is a set $|\mathcal{A}|$ of *objects*, a *hom-object* $\mathcal{A}(A, B) \in |\mathcal{V}|$ for each pair of objects of $\mathcal{A}$, a family of *composition morphisms* $M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple of objects, and an *identity element* $j_A : I \to \mathcal{A}(A, A)$ for each object. The composition morphisms are subject to the associativity axiom which states that the following pentagon commutes

$$\begin{tikzcd}
& (\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) \arrow[rr, "\alpha"] \arrow[dl, "M\otimes 1"'] & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \arrow[dr, "1\otimes M"] & \\
\mathcal{A}(B,D) \otimes \mathcal{A}(A,B) \arrow[drr, "M"'] & & & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[dll, "M"] \\
& & \mathcal{A}(A,D)) & &
\end{tikzcd}$$

and to the unit axioms which state that both the triangles in the following diagram commute.

$$\begin{tikzcd}
I \otimes \mathcal{A}(A,B) \arrow[dr, "="] \arrow[dd, "j_B \otimes 1"'] & & \mathcal{A}(A,B) \otimes I \arrow[dl, "="'] \arrow[dd, "1 \otimes j_A"] \\
& \mathcal{A}(A,B) & \\
\mathcal{A}(B,B) \otimes \mathcal{A}(A,B) \arrow[ur, "M_{ABB}"] & & \mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \arrow[ul, "M_{AAB}"']
\end{tikzcd}$$

In general a $\mathcal{V}$–category is directly analogous to an (ordinary) category enriched over **Set**. If $\mathcal{V} = \mathbf{Set}$ then these diagrams are the usual category axioms.

Definition 3 For $\mathcal{V}$–categories $\mathcal{A}$ and $\mathcal{B}$, a *$\mathcal{V}$–functor* $T : \mathcal{A} \to \mathcal{B}$ is a function $T : |\mathcal{A}| \to |\mathcal{B}|$ and a family of morphisms $T_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(TA,TB)$ in $\mathcal{V}$ indexed by pairs $A, B \in |\mathcal{A}|$. The usual rules for a functor that state $T(f \circ g) = Tf \circ Tg$ and $T1_A = 1_{TA}$ become in the enriched setting, respectively, the commuting diagrams

$$\begin{tikzcd}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \arrow[r, "M"] \arrow[d, "T \otimes T"] & \mathcal{A}(A,C) \arrow[d, "T"] \\
\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) \arrow[r, "M"] & \mathcal{B}(TA,TC)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathcal{A}(A,A) \arrow[dd, "T_{AA}"] \\
I \arrow[ur, "j_A"] \arrow[dr, "j_{TA}"'] & \\
& \mathcal{B}(TA,TA).
\end{tikzcd}$$

$\mathcal{V}$–functors can be composed to form a category called $\mathcal{V}$–Cat. This category is actually enriched over **Cat**, the category of (small) categories with cartesian product.

Definition 4 For $\mathcal{V}$–functors $T, S : \mathcal{A} \to \mathcal{B}$ a *$\mathcal{V}$–natural transformation* $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$ is an $|\mathcal{A}|$–indexed family of morphisms $\alpha_A : I \to \mathcal{B}(TA, SA)$ satisfying the $\mathcal{V}$–naturality condition expressed by the commutativity of the following hexagonal diagram:

$$
\begin{tikzcd}[column sep=small]
 & I \otimes \mathcal{A}(A,B) \arrow[r, "\alpha_B \otimes T_{AB}"] & \mathcal{B}(TB,SB) \otimes \mathcal{B}(TA,TB) \arrow[dr, "M"] & \\
\mathcal{A}(A,B) \arrow[ur, "="] \arrow[dr, "="'] & & & \mathcal{B}(TA,SB) \\
 & \mathcal{A}(A,B) \otimes I \arrow[r, "S_{AB} \otimes \alpha_A"'] & \mathcal{B}(SA,SB) \otimes \mathcal{B}(TA,SA) \arrow[ur, "M"'] &
\end{tikzcd}
$$

For two $\mathcal{V}$–functors T, S to be equal is to say $TA = SA$ for all A and for the $\mathcal{V}$–natural isomorphism α between them to have components $\alpha_A = j_{TA}$. This latter implies equality of the hom–object morphisms: $T_{AB} = S_{AB}$ for all pairs of objects. The implication is seen by combining the second diagram in Definition 2 with all the diagrams in Definitions 3 and 4.

We want to check that $\mathcal{V}$–natural transformations can be composed so that $\mathcal{V}$–categories, $\mathcal{V}$–functors and $\mathcal{V}$–natural transformations form a 2–category. First the vertical composite of $\mathcal{V}$–natural transformations corresponding to the picture

$$
\begin{tikzcd}[column sep=large]
\mathcal{A} \arrow[r, bend left=60, "T"{name=T}] \arrow[r, "S"{name=S}] \arrow[r, bend right=60, "R"'{name=R}] & \mathcal{B}
\arrow[Rightarrow, from=T, to=S, "\alpha"]
\arrow[Rightarrow, from=S, to=R, "\beta"]
\end{tikzcd}
$$

has components given by $(\beta \circ \alpha)_A =$

$$
\begin{tikzcd}
I \cong I \otimes I \arrow[d, "\beta_A \otimes \alpha_A"'] \\
\mathcal{B}(SA, RA) \otimes \mathcal{B}(TA, SA) \arrow[d, "M"'] \\
\mathcal{B}(TA, RA).
\end{tikzcd}
$$

The reader should check that this composition produces a valid $\mathcal{V}$–natural transformation. Associativity of composition also follows from the pentagonal axioms. The identity 2-cells are the identity $\mathcal{V}$-natural transformations $\mathbf{1}_Q : Q \to Q : \mathcal{B} \to \mathcal{C}$. These are formed from the unit morphisms in $\mathcal{V}$: $(\mathbf{1}_Q)_B = j_{QB}$.

In order to define composition of all allowable pasting diagrams in the 2-category, we need to define the composition described by left and right whiskering diagrams. The first picture shows a 1-cell ($\mathcal{V}$–functor) following a 2-cell

($\mathcal{V}$–natural transformation). These are composed to form a new 2-cell as follows

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "T"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, "Q"] & \mathcal{C} \end{tikzcd}$$ is composed to become $$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "QT"] \arrow[r, bend right, "QS"'] \arrow[r, phantom, "\Downarrow Q\alpha"] & \mathcal{C} \end{tikzcd}$$

where QT and QS are given by the usual compositions of their set functions and morphisms in $\mathcal{V}$, and $Q\alpha$ has components given by $(Q\alpha)_A =$

$$\begin{tikzcd} I \arrow[d, "\alpha_A"'] \\ \mathcal{B}(TA, SA) \arrow[d, "Q_{TA,SA}"'] \\ \mathcal{C}(QTA, QSA). \end{tikzcd}$$

The second picture shows a 2-cell following a 1-cell. These are composed as follows

$$\begin{tikzcd} \mathcal{D} \arrow[r, "P"] & \mathcal{A} \arrow[r, bend left, "T"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$$ is composed to become $$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "TP"] \arrow[r, bend right, "SP"'] \arrow[r, phantom, "\Downarrow \alpha P"] & \mathcal{B} \end{tikzcd}$$

where αP has components given by $(\alpha P)_D = \alpha_{PD}$. What we have developed here are the partial functors of the composition morphism implicit in enriching over **Cat**. The partial functors can be combined to make the functor of two variables as shown in [5].

Having ascertained that we have a 2–category we review the morphisms between two such things. A *2–functor* $F : U \to V$ sends objects to objects, 1–cells to 1–cells, and 2–cells to 2–cells and preserves all the categorical structure. A *2–natural transformation* $\theta : F \to G : U \to V$ is a function that sends each object $A \in U$ to a 1–cell $\theta_A : FA \to GA$ in V in such a way that for each 2–cell in U the compositions of the following diagrams are equal in V.

$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \arrow[r, "\theta_B"] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\theta_A"] & GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GB \end{tikzcd}$$

3 k-fold monoidal categories

In this section I closely follow the authors of [2] in defining a notion of iterated monoidal category. For those readers familiar with that source, note that I vary from their definition only by including associativity up to coherent natural isomorphisms. This includes changing the basic picture from monoids to something that is a monoid only up to a monoidal natural transformation. We (and in this section "we" is not merely imperial, since so much is directly from [2]) start by defining a slightly nonstandard variant of monoidal functor. It is usually required in a definition of monoidal functor that η be an isomorphism. The authors of [2] note that it is crucial not to make this requirement.

Definition 5 A *monoidal functor* $(F, \eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(I) = I$ together with a natural transformation

$$\eta_{AB} : F(A) \otimes F(B) \to F(A \otimes B),$$

which satisfies the following conditions.

(1) Internal Associativity: The following diagram commutes.

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "\eta_{AB} \otimes 1_{F(C)}"] \arrow[d, "\alpha"] & F(A \otimes B) \otimes F(C) \arrow[d, "\eta_{(A\otimes B)C}"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[d, "1_{F(A)} \otimes \eta_{BC}"] & F((A \otimes B) \otimes C) \arrow[d, "F\alpha"] \\
F(A) \otimes F(B \otimes C) \arrow[r, "\eta_{A(B\otimes C)}"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

(2) Internal Unit Conditions: $\eta_{AI} = \eta_{IA} = 1_{F(A)}$.

Given two monoidal functors $(F, \eta) : \mathcal{C} \to \mathcal{D}$ and $(G, \zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF, \xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A) \otimes GF(B) \xrightarrow{\zeta_{F(A)F(B)}} G\big(F(A) \otimes F(B)\big) \xrightarrow{G(\eta_{AB})} GF(A \otimes B).$$

It is easy to verify that ξ satisfies the internal associativity condition above by subdividing the necessary commuting diagram into two regions that commute by the axioms for η and ζ respectively and two that commute due to their naturality. **MonCat** is the monoidal category of monoidal categories and monoidal functors, with the usual Cartesian product as in **Cat**.

A *monoidal natural transformation* $\theta : (F, \eta) \to (G, \zeta) : \mathcal{D} \to \mathcal{E}$ is a natural transformation $\theta : F \to G$ between the underlying ordinary functors of F and

G such that the following diagram commutes

$$\begin{tikzcd} F(A)\otimes F(B) \arrow[r, "\eta"] \arrow[d, "\theta_A\otimes\theta_B"] & F(A\otimes B) \arrow[d, "\theta_{A\otimes B}"] \\ G(A)\otimes G(B) \arrow[r, "\zeta"] & G(A\otimes B) \end{tikzcd}$$

Definition 6 For our purposes a 2*-fold monoidal category* is a tensor object, or pseudomonoid, in **MonCat**. This means that we are given a monoidal category $(\mathcal{V}, \otimes_1, \alpha^1, I)$ and a monoidal functor $(\otimes_2, \eta) : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ which satisfies:

(1) External Associativity: the following diagram describes a monoidal natural transformation α^2 in **MonCat**.

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V}\times\mathcal{V} \arrow[r, "(\otimes_2,\eta)\times 1_{\mathcal{V}}"] \arrow[d, "1_{\mathcal{V}}\times(\otimes_2,\eta)"'] & \mathcal{V}\times\mathcal{V} \arrow[d, "(\otimes_2,\eta)"] \arrow[dl, Rightarrow, "\alpha^2"] \\ \mathcal{V}\times\mathcal{V} \arrow[r, "(\otimes_2,\eta)"'] & \mathcal{V} \end{tikzcd}$$

(2) External Unit Conditions: the following diagram commutes in **MonCat**.

$$\begin{tikzcd} \mathcal{V}\times I \arrow[r, "\subseteq"] \arrow[dr, "\cong"] & \mathcal{V}\times\mathcal{V} \arrow[d, "(\otimes_2,\eta)"] & I\times\mathcal{V} \arrow[l, "\supseteq"'] \arrow[dl, "\cong"] \\ & \mathcal{V} & \end{tikzcd}$$

(3) Coherence: The underlying natural transformation α^2 satisfies the usual coherence pentagon.

Explicitly this means that we are given a second associative binary operation $\otimes_2 : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$, for which I is also a two-sided unit. We are also given a natural transformation

$$\eta_{ABCD} : (A\otimes_2 B)\otimes_1 (C\otimes_2 D)\to (A\otimes_1 C)\otimes_2 (B\otimes_1 D).$$

The internal unit conditions give $\eta_{ABII} = \eta_{IIAB} = 1_{A\otimes_2 B}$, while the external unit conditions give $\eta_{AIBI} = \eta_{IAIB} = 1_{A\otimes_1 B}$. The internal associativity condition gives the commutative diagram:

$$\begin{tikzcd}
((U \otimes_2 V) \otimes_1 (W \otimes_2 X)) \otimes_1 (Y \otimes_2 Z) \arrow[r, "\eta_{UVWX} \otimes_1 1_{Y \otimes_2 Z}"] \arrow[d, "\alpha^1"] & \big((U \otimes_1 W) \otimes_2 (V \otimes_1 X)\big) \otimes_1 (Y \otimes_2 Z) \arrow[d, "\eta_{(U \otimes_1 W)(V \otimes_1 X)YZ}"] \\
(U \otimes_2 V) \otimes_1 ((W \otimes_2 X) \otimes_1 (Y \otimes_2 Z)) \arrow[d, "1_{U \otimes_2 V} \otimes_1 \eta_{WXYZ}"] & ((U \otimes_1 W) \otimes_1 Y) \otimes_2 ((V \otimes_1 X) \otimes_1 Z) \arrow[d, "\alpha^1 \otimes_2 \alpha^1"] \\
(U \otimes_2 V) \otimes_1 \big((W \otimes_1 Y) \otimes_2 (X \otimes_1 Z)\big) \arrow[r, "\eta_{UV(W \otimes_1 Y)(X \otimes_1 Z)}"] & (U \otimes_1 (W \otimes_1 Y)) \otimes_2 (V \otimes_1 (X \otimes_1 Z))
\end{tikzcd}$$

The external associativity condition gives the commutative diagram:

$$\begin{tikzcd}
((U \otimes_2 V) \otimes_2 W) \otimes_1 ((X \otimes_2 Y) \otimes_2 Z) \arrow[r, "\eta_{(U \otimes_2 V)W(X \otimes_2 Y)Z}"] \arrow[d, "\alpha^2 \otimes_1 \alpha^2"] & \big((U \otimes_2 V) \otimes_1 (X \otimes_2 Y)\big) \otimes_2 (W \otimes_1 Z) \arrow[d, "\eta_{UVXY} \otimes_2 1_{W \otimes_1 Z}"] \\
(U \otimes_2 (V \otimes_2 W)) \otimes_1 (X \otimes_2 (Y \otimes_2 Z)) \arrow[d, "\eta_{U(V \otimes_2 W)X(Y \otimes_2 Z)}"] & ((U \otimes_1 X) \otimes_2 (V \otimes_1 Y)) \otimes_2 (W \otimes_1 Z) \arrow[d, "\alpha^2"] \\
(U \otimes_1 X) \otimes_2 \big((V \otimes_2 W) \otimes_1 (Y \otimes_2 Z)\big) \arrow[r, "1_{U \otimes_1 X} \otimes_2 \eta_{VWYZ}"] & (U \otimes_1 X) \otimes_2 ((V \otimes_1 Y) \otimes_2 (W \otimes_1 Z))
\end{tikzcd}$$

The authors of [2] remark that we have natural transformations

$$\eta_{AIIB} : A \otimes_1 B \to A \otimes_2 B \qquad \text{and} \qquad \eta_{IABI} : A \otimes_1 B \to B \otimes_2 A.$$

If they had insisted a 2-fold monoidal category be a tensor object in the category of monoidal categories and *strictly monoidal* functors, this would be equivalent to requiring that $\eta = 1$. In view of the above, they note that this would imply $A \otimes_1 B = A \otimes_2 B = B \otimes_1 A$ and similarly for morphisms.

Joyal and Street [9] considered a similar concept to Balteanu, Fiedorowicz, Schwänzl and Vogt's idea of 2–fold monoidal category. The former pair required the natural transformation η_{ABCD} to be an isomorphism and showed that the resulting category is naturally equivalent to a braided monoidal category. As explained in [2], given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\otimes_2$, and defining the commutativity isomorphism for the remaining operation $\otimes_1$ to be the composite

$$A \otimes_1 B \xrightarrow{\eta_{IABI}} B \otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B \otimes_1 A.$$

Just as in [2] we now define a 2–fold monoidal functor between 2–fold monoidal categories $F : \mathcal{V} \to \mathcal{D}$. It is a functor together with two natural transformations:

$$\lambda^1_{AB} : F(A) \otimes_1 F(B) \to F(A \otimes_1 B)$$
$$\lambda^2_{AB} : F(A) \otimes_2 F(B) \to F(A \otimes_2 B)$$

satisfying the same associativity and unit conditions as in the case of monoidal functors. In addition we require that the following hexagonal interchange diagram commutes.

$$\begin{tikzcd}
(F(A)\otimes_2 F(B))\otimes_1 (F(C)\otimes_2 F(D)) \arrow[r, "\eta_{F(A)F(B)F(C)F(D)}"] \arrow[d, "\lambda^2_{AB}\otimes_1\lambda^2_{CD}"] & (F(A)\otimes_1 F(C))\otimes_2 (F(B)\otimes_1 F(D)) \arrow[d, "\lambda^1_{AC}\otimes_2\lambda^1_{BD}"] \\
F(A\otimes_2 B)\otimes_1 F(C\otimes_2 D) \arrow[d, "\lambda^1_{(A\otimes_2 B)(C\otimes_2 D)}"] & F(A\otimes_1 C)\otimes_2 F(B\otimes_1 D) \arrow[d, "\lambda^2_{(A\otimes_1 C)(B\otimes_1 D)}"] \\
F((A\otimes_2 B)\otimes_1 (C\otimes_2 D)) \arrow[r, "F(\eta_{ABCD})"] & F((A\otimes_1 C)\otimes_2 (B\otimes_1 D))
\end{tikzcd}$$

We can now define the category $\mathbf{2-MonCat}$ of 2-fold monoidal categories and 2-fold monoidal functors, and then define a 3-fold monoidal category as a tensor object in $\mathbf{2-MonCat}$. From this point on, the iteration of this idea is straightforward and, paralleling the authors of [2], we arrive at the following definitions.

Definition 7 An *n-fold monoidal category* is a category $\mathcal{V}$ with the following structure.

(1) There are n distinct multiplications

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$$

for each of which the associativity pentagon commutes:

$$\begin{tikzcd}
 & ((U\otimes_i V)\otimes_i W)\otimes_i X \arrow[r, "\alpha^i_{UVW}\otimes_i 1_X"] \arrow[dl, "\alpha^i_{(U\otimes_i V)WX}"'] & (U\otimes_i (V\otimes_i W))\otimes_i X \arrow[dr, "\alpha^i_{U(V\otimes_i W)X}"] & \\
(U\otimes_i V)\otimes_i (W\otimes_i X) \arrow[drr, "\alpha^i_{UV(W\otimes_i X)}"] & & & U\otimes_i ((V\otimes_i W)\otimes_i X) \arrow[dl, "1_U\otimes_i\alpha^i_{VWX}"'] \\
 & & U\otimes_i (V\otimes_i (W\otimes_i X)) &
\end{tikzcd}$$

$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.

(2) For each pair (i,j) such that $1\le i<j\le n$ there is a natural transformation

$$\eta^{ij}_{ABCD} : (A\otimes_j B)\otimes_i (C\otimes_j D)\to (A\otimes_i C)\otimes_j (B\otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A\otimes_j B}$

(b) External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A\otimes_i B}$

(c) Internal associativity condition: The following diagram commutes.

$$\begin{tikzcd}
((U\otimes_j V)\otimes_i (W\otimes_j X))\otimes_i (Y\otimes_j Z) \arrow[r, "\eta^{ij}_{UVWX}\otimes_i 1_{Y\otimes_j Z}"] \arrow[d, "\alpha^i"] & \big((U\otimes_i W)\otimes_j (V\otimes_i X)\big)\otimes_i (Y\otimes_j Z) \arrow[d, "\eta^{ij}_{(U\otimes_i W)(V\otimes_i X)YZ}"] \\
(U\otimes_j V)\otimes_i ((W\otimes_j X)\otimes_i (Y\otimes_j Z)) \arrow[d, "1_{U\otimes_j V}\otimes_i\eta^{ij}_{WXYZ}"] & ((U\otimes_i W)\otimes_i Y)\otimes_j ((V\otimes_i X)\otimes_i Z) \arrow[d, "\alpha^i\otimes_j\alpha^i"] \\
(U\otimes_j V)\otimes_i \big((W\otimes_i Y)\otimes_j (X\otimes_i Z)\big) \arrow[r, "\eta^{ij}_{UV(W\otimes_i Y)(X\otimes_i Z)}"] & (U\otimes_i (W\otimes_i Y))\otimes_j (V\otimes_i (X\otimes_i Z))
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes.

$$\begin{tikzcd}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) \arrow[r, "\eta^{ij}_{(U \otimes_j V)W(X \otimes_j Y)Z}"] \arrow[d, "\alpha^j \otimes_i \alpha^j"] & ((U \otimes_j V) \otimes_i (X \otimes_j Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}"] \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) \arrow[d, "\eta^{ij}_{U(V \otimes_j W)X(Y \otimes_j Z)}"] & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\alpha^j"] \\
(U \otimes_i X) \otimes_j ((V \otimes_j W) \otimes_i (Y \otimes_j Z)) \arrow[r, "1_{U \otimes_i X} \otimes_j \eta^{ij}_{VWYZ}"] & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{tikzcd}$$

(e) Finally it is required for each triple (i, j, k) satisfying $1 \leq i < j < k \leq n$ that the giant hexagonal interchange diagram commutes.

$$\begin{tikzcd}
 & ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}"'] \arrow[dr, "\eta^{ij}_{(A \otimes_k A')(B \otimes_k B')(C \otimes_k C')(D \otimes_k D')}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\eta^{ik}_{(A \otimes_j B)(A' \otimes_j B')(C \otimes_j D)(C' \otimes_j D')}"'] & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\eta^{jk}_{(A \otimes_i C)(A' \otimes_i C')(B \otimes_i D)(B' \otimes_i D')}"] \\
 & ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}$$

Definition 8 An *n–fold monoidal functor* $(F, \lambda^1, \ldots, \lambda^n) : \mathcal{C} \to \mathcal{D}$ between n–fold monoidal categories consists of a functor F such that $F(I) = I$ together with natural transformations

$$\lambda^i_{AB} : F(A) \otimes_i F(B) \to F(A \otimes_i B) \quad i = 1, 2, \ldots, n$$

satisfying the same associativity and unit conditions as monoidal functors. In addition the following hexagonal interchange diagram commutes.

$$\begin{tikzcd}
(F(A) \otimes_j F(B)) \otimes_i (F(C) \otimes_j F(D)) \arrow[r, "\eta^{ij}_{F(A)F(B)F(C)F(D)}"] \arrow[d, "\lambda^j_{AB} \otimes_i \lambda^j_{CD}"] & (F(A) \otimes_i F(C)) \otimes_j (F(B) \otimes_i F(D)) \arrow[d, "\lambda^i_{AC} \otimes_j \lambda^i_{BD}"] \\
F(A \otimes_j B) \otimes_i F(C \otimes_j D) \arrow[d, "\lambda^i_{(A \otimes_j B)(C \otimes_j D)}"] & F(A \otimes_i C) \otimes_j F(B \otimes_i D) \arrow[d, "\lambda^j_{(A \otimes_i C)(B \otimes_i D)}"] \\
F((A \otimes_j B) \otimes_i (C \otimes_j D)) \arrow[r, "F(\eta^{ij}_{ABCD})"] & F((A \otimes_i C) \otimes_j (B \otimes_i D))
\end{tikzcd}$$

Composition of n-fold monoidal functors is defined as for monoidal functors.

The authors of [2] point out that it is necessary to check that an $(n+1)$–fold monoidal category is the same thing as a tensor object in $\mathbf{n-MonCat}$, the category of n–fold monoidal categories and functors. Also as noticed in [2], the hexagonal interchange diagrams for the $(n+1)$–st monoidal operation regarded as an n–fold monoidal functor are what give rise to the giant hexagonal diagrams involving $\otimes_i$, $\otimes_j$ and $\otimes_{n+1}$.

The authors of [2] note that a symmetric monoidal category is n-fold monoidal for all n. Just let

$$\otimes_1 = \otimes_2 = \cdots = \otimes_n = \otimes$$

and define (associators added by myself)

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$.

4 Categories enriched over a k–fold monoidal category

Theorem 1 *For $\mathcal{V}$ a k–fold monoidal category $\mathcal{V}$–Cat is a $(k-1)$–fold monoidal 2-category.*

Example 1

We begin by describing the $k = 2$ case. $\mathcal{V}$ is 2–fold monoidal with products $\otimes_1, \otimes_2$. $\mathcal{V}$–categories (which are the objects of $\mathcal{V}$–Cat) are defined as being enriched over $(\mathcal{V}, \otimes_1, \alpha^1, I)$. Here $\otimes_1$ plays the role of the product given by $\otimes$ in the axioms of section 1. We need to show that $\mathcal{V}$–Cat has a product.

The unit object in $\mathcal{V}$–Cat is the enriched category $\mathcal{I}$ where $|\mathcal{I}| = \{0\}$ and $\mathcal{I}(0,0) = I$. Of course $M_{000} = 1 = j_0$. The objects of the tensor $\mathcal{A} \otimes^{(1)}_1 \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are simply pairs of objects, that is, elements of $|\mathcal{A}| \times |\mathcal{B}|$. The hom–objects in $\mathcal{V}$ are given by $(\mathcal{A} \otimes^{(1)}_1 \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_2 \mathcal{B}(B,B')$. The composition morphisms that make $\mathcal{A} \otimes^{(1)}_1 \mathcal{B}$ into a $\mathcal{V}$–category are immediately apparent as generalizations of the braided case. Recall that we are describing $\mathcal{A} \otimes^{(1)}_1 \mathcal{B}$ as a category enriched over $\mathcal{V}$ with product $\otimes_1$. Thus

$$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A} \otimes^{(1)}_1 \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes^{(1)}_1 \mathcal{B})((A,B),(A',B'))$$
$$\to (\mathcal{A} \otimes^{(1)}_1 \mathcal{B})((A,B),(A'',B''))$$

is given by:

$$
\begin{array}{c}
(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A',B')) \\
\Big\Vert \\
(\mathcal{A}(A',A'') \otimes_2 \mathcal{B}(B',B'')) \otimes_1 (\mathcal{A}(A,A') \otimes_2 \mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle \eta^{1,2}} \\
(\mathcal{A}(A',A'') \otimes_1 \mathcal{A}(A,A')) \otimes_2 (\mathcal{B}(B',B'') \otimes_1 \mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle M_{AA'A''} \otimes_2 M_{BB'B''}} \\
(\mathcal{A}(A,A'') \otimes_2 \mathcal{B}(B,B'')) \\
\Big\Vert \\
(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A'',B''))
\end{array}
$$

Example 2

Next we describe the $k = 3$ case. $\mathcal{V}$ is 3–fold monoidal with products $\otimes_1, \otimes_2$ and $\otimes_3$. $\mathcal{V}$–categories are defined as being enriched over $(\mathcal{V}, \otimes_1, \alpha^1, I)$. Now $\mathcal{V}$–Cat has two products. The objects of both possible tensors $\mathcal{A} \otimes_1^{(1)} \mathcal{B}$ and $\mathcal{A} \otimes_2^{(1)} \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are elements in $|\mathcal{A}| \times |\mathcal{B}|$. The hom–objects in $\mathcal{V}$ are given by

$$(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_2 \mathcal{B}(B,B')$$

just as in the previous case, and by

$$(\mathcal{A} \otimes_2^{(1)} \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_3 \mathcal{B}(B,B').$$

The composition that makes $(\mathcal{A} \otimes_2^{(1)} \mathcal{B})$ into a $\mathcal{V}$–category is analogous to that for $(\mathcal{A} \otimes_1^{(1)} \mathcal{B})$ but uses $\eta^{1,3}$ as its middle exchange morphism.

Now we need an interchange 2–natural transformation $\eta^{(1)1,2}$ for $\mathcal{V}$–Cat. The family of morphisms $\eta^{(1)1,2}_{\mathcal{ABCD}}$ that make up a 2–natural transformation between the 2–functors $\times^4\mathcal{V}$–Cat $:\to \mathcal{V}$–Cat in question is a family of enriched functors. Their action on objects is to send

$$((A,B),(C,D)) \in \left|(\mathcal{A} \otimes_2^{(1)} \mathcal{B}) \otimes_1^{(1)} (\mathcal{C} \otimes_2^{(1)} \mathcal{D})\right|$$

$$\text{to } ((A,C),(B,D)) \in \left|(\mathcal{A} \otimes_1^{(1)} \mathcal{C}) \otimes_2^{(1)} (\mathcal{B} \otimes_1^{(1)} \mathcal{D})\right|.$$

The correct construction of the family of hom–object morphisms in $\mathcal{V}$–Cat for each of these functors is also clear. Noting that

$$\begin{aligned}
&[(\mathcal{A} \otimes_2^{(1)} \mathcal{B}) \otimes_1^{(1)} (\mathcal{C} \otimes_2^{(1)} \mathcal{D})](((A,B),(C,D)),((A',B'),(C',D'))) \\
&\quad = (\mathcal{A} \otimes_2^{(1)} \mathcal{B})((A,B),(A',B')) \otimes_2 (\mathcal{C} \otimes_2^{(1)} \mathcal{D})((C,D),(C',D')) \\
&\qquad = (\mathcal{A}(A,A') \otimes_3 \mathcal{B}(B,B')) \otimes_2 (\mathcal{C}(C,C') \otimes_3 \mathcal{D}(D,D'))
\end{aligned}$$

and similarly

$$\begin{aligned}
&[(\mathcal{A} \otimes_1^{(1)} \mathcal{C}) \otimes_2^{(1)} (\mathcal{B} \otimes_1^{(1)} \mathcal{D})](((A,C),(B,D)),((A',C'),(B',D'))) \\
&\qquad = (\mathcal{A}(A,A') \otimes_2 \mathcal{C}(C,C')) \otimes_3 (\mathcal{B}(B,B') \otimes_2 \mathcal{D}(D,D'))
\end{aligned}$$

we make the obvious identification, where by obvious I mean based upon the corresponding structure in $\mathcal{V}$. For a detailed discussion of this construction for the case of braided $\mathcal{V}$ see [8]. Here "based upon" is more freely interpreted as allowing a shift in index. Thus we write:

$$\eta^{(1)1,2}_{\mathcal{ABCD}_{(ABCD)(A'B'C'D')}} = \eta^{2,3}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')\mathcal{D}(D,D')}$$

Much needs to be verified. Existence and coherence of required natural transformations, satisfaction of enriched axioms and of k–fold monoidal axioms all must be checked. These will be dealt with next.

Proof of Theorem 1 As in the examples, $\mathcal{V}$–Cat is made up of categories enriched over $(\mathcal{V}, \otimes_1, \alpha^1, I)$. Here we define products $\otimes_1^{(1)} \ldots \otimes_{k-1}^{(1)}$ in $\mathcal{V}$–Cat for $\mathcal{V}$ k–fold monoidal. We check that our products do make $\mathcal{A} \otimes_2^{(1)} \mathcal{B}$ into a $\mathcal{V}$–category. Then we check that $\mathcal{V}$–Cat has the required coherent 2–natural transformations of associativity and units. We then define interchange 2–natural transformations $\eta^{(1)i,j}$ and check that the interchange transformations are 2–natural and obey all the axioms required of them. It is informative to observe how these axioms are satisfied based upon the axioms that $\mathcal{V}$ itself satisfies. It is here that we should look carefully for the algebraic reflection of the topological functor Ω.

Again, the unit object in $\mathcal{V}$–Cat is the enriched category $\mathcal{I}$ where $|\mathcal{I}| = \{0\}$ and $\mathcal{I}(0,0) = I$. For $\mathcal{V}$ k–fold monoidal we define the ith product of $\mathcal{V}$–categories $\mathcal{A} \otimes_i^{(1)} \mathcal{B}$ to have objects $\in |\mathcal{A}| \times |\mathcal{B}|$ and to have hom–objects in $\mathcal{V}$ given by

$$(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_{i+1} \mathcal{B}(B,B').$$

Immediately we see that $\mathcal{V}$–Cat is $(k-1)$–fold monoidal by definition. The composition morphisms are

$$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B'))$$

$$\to (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A'',B''))$$

given by

$$\begin{array}{c}
(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) \\
\| \\
(\mathcal{A}(A',A'') \otimes_{i+1} \mathcal{B}(B',B'')) \otimes_1 (\mathcal{A}(A,A') \otimes_{i+1} \mathcal{B}(B,B')) \\
\downarrow{\scriptstyle \eta^{1,i+1}} \\
(\mathcal{A}(A',A'') \otimes_1 \mathcal{A}(A,A')) \otimes_{i+1} (\mathcal{B}(B',B") \otimes_1 \mathcal{B}(B,B')) \\
\downarrow{\scriptstyle M_{AA'A''} \otimes_2 M_{BB'B''}} \\
(\mathcal{A}(A,A'') \otimes_{i+1} \mathcal{B}(B,B'')) \\
\| \\
(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A'',B'')).
\end{array}$$

The identity element is given by $j_{(A,B)} =$

$$\begin{array}{c}
I = I \otimes_{i+1} I \\
\downarrow{\scriptstyle j_A \otimes_{i+1} j_B} \\
\mathcal{A}(A,A) \otimes_{i+1} \mathcal{B}(B,B) \\
\| \\
(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A,B)).
\end{array}$$

The product $\otimes_i^{(1)}$ of enriched functors is defined in the obvious way.

Here we first check that $\mathcal{A} \otimes_i^{(1)} \mathcal{B}$ is indeed properly enriched over $\mathcal{V}$. Our definition of M must obey the axioms for associativity and respect of the unit. For associativity the following diagram must commute, where the initial bullet represents

$$[(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A'',B''),(A''',B''')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A',B'),(A'',B''))]$$

$$\otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')).$$

$$
\begin{tikzcd}
& \bullet \arrow[rr, "\alpha"] \arrow[dl, "M\otimes 1"] & & \bullet \arrow[dr, "1\otimes M"] & \\
\bullet \arrow[drr, "M"] & & & & \bullet \arrow[dll, "M"] \\
& & \bullet & &
\end{tikzcd}
$$

In the expanded diagram given in Figure 1 let $X = \mathcal{A}(A, A')$, $X' = \mathcal{A}(A', A'')$, $X'' = \mathcal{A}(A'', A''')$, $Y = \mathcal{B}(B, B')$, $Y' = \mathcal{B}(B', B'')$ and $Y'' = \mathcal{B}(B'', B''')$. The exterior of the diagram is required to commute.

The lower pentagon in Figure 1 commutes since it is two copies of the associativity axiom–one for $\mathcal{A}$ and one for $\mathcal{B}$. The two diamonds commute by the naturality of η. The upper hexagon commutes by the internal associativity of η.

For the unit axioms we have the following compact diagram.

$$
\begin{tikzcd}
I \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) \arrow[dd, "j_{(A',B')}\otimes_1 1"'] \arrow[dr, "="] & & (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) \otimes_1 I \arrow[dd, "1\otimes_1 j_{(A,B)}"] \arrow[dl, "="'] \\
& (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) & \\
(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A',B'),(A',B')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) \arrow[ur, "M_{(A,B)(A',B')(A',B')}"] & & (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A,B)) \arrow[ul, "M_{(A,B)(A,B)(A',B')}"']
\end{tikzcd}
$$

I expand the left triangle, abbreviating $X = \mathcal{A}(A, A')$, $Y = \mathcal{A}(A', A')$, $Z = \mathcal{B}(B, B')$ and $W = \mathcal{B}(B', B')$. The exterior of the following must commute:

$$
\begin{tikzcd}
I \otimes_1 (X \otimes_{i+1} Z) \arrow[dd, "="] \arrow[dr, "="] & & \\
& (I \otimes_1 X) \otimes_{i+1} (I \otimes_1 Z) \arrow[dd, "(j_{A'}\otimes_1 1)\otimes_{i+1}(j_{B'}\otimes_1 1)"'] \arrow[dr, "="] & \\
(I \otimes_{i+1} I) \otimes_1 (X \otimes_{i+1} Z) \arrow[ur, "\eta^{1,i+1}_{IIXZ}"] \arrow[dd, "(j_{A'}\otimes_{i+1} j_{B'})\otimes_1(1\otimes_{i+1}1)"'] & & (X \otimes_{i+1} Z) \\
& (Y \otimes_1 X) \otimes_{i+1} (W \otimes_1 Z) \arrow[ur, "M\otimes_{i+1}M"] & \\
(Y \otimes_{i+1} W) \otimes_1 (X \otimes_{i+1} Z) \arrow[ur, "\eta^{1,i+1}_{YWXZ}"] & &
\end{tikzcd}
$$

$$\begin{tikzcd}[column sep=small]
 & {[(X'' \otimes_{i+1} Y'') \otimes_1 (X' \otimes_{i+1} Y')] \otimes_1 (X \otimes_{i+1} Y)} \arrow[rr, "\alpha^1"] \arrow[d, "\eta^{1,i+1} \otimes_1 1"'] & & {(X'' \otimes_{i+1} Y'') \otimes_1 [(X' \otimes_{i+1} Y') \otimes_1 (X \otimes_{i+1} Y)]} \arrow[d, "1 \otimes_1 \eta^{1,i+1}"] & \\
 & {[(X'' \otimes_1 X') \otimes_{i+1} (Y'' \otimes_1 Y')] \otimes_1 (X \otimes_{i+1} Y)} \arrow[d, "\eta^{1,i+1}"] \arrow[ddl, "(M \otimes_{i+1} M) \otimes_1 (1 \otimes_{i+1} 1)"'] & & {(X'' \otimes_{i+1} Y'') \otimes_1 [(X' \otimes_1 X) \otimes_{i+1} (Y' \otimes_1 Y)]} \arrow[d, "\eta^{1,i+1}"'] \arrow[ddr, "(1 \otimes_{i+1} 1) \otimes_1 (M \otimes_{i+1} M)"] & \\
 & {[(X'' \otimes_1 X') \otimes_1 X] \otimes_{i+1} [(Y'' \otimes_1 Y') \otimes_1 Y]} \arrow[rr, "\alpha^1 \otimes_{i+1} \alpha^1"] \arrow[dd, "(M \otimes_1 1) \otimes_{i+1} (M \otimes_1 1)"] & & {[X'' \otimes_1 (X' \otimes_1 X)] \otimes_{i+1} [Y'' \otimes_1 (Y' \otimes_1 Y)]} \arrow[dd, "(1 \otimes_1 M) \otimes_{i+1} (1 \otimes_1 M)"'] & \\
(\mathcal{A}(A', A''') \otimes_{i+1} \mathcal{B}(B', B''')) \otimes_1 (X \otimes_{i+1} Y) \arrow[dr, "\eta^{1,i+1}"] & & & & (X'' \otimes_{i+1} Y'') \otimes_1 (\mathcal{A}(A, A'') \otimes_{i+1} \mathcal{B}(B, B'')) \arrow[dl, "\eta^{1,i+1}"'] \\
 & (\mathcal{A}(A', A''') \otimes_1 X) \otimes_{i+1} (\mathcal{B}(B', B''') \otimes_1 Y) \arrow[dr, "M \otimes_{i+1} M"] & & (X'' \otimes_1 \mathcal{A}(A, A'')) \otimes_{i+1} (Y'' \otimes_1 \mathcal{B}(B, B'')) \arrow[dl, "M \otimes_{i+1} M"'] & \\
 & & \mathcal{A}(A, A''') \otimes_{i+1} \mathcal{B}(B, B''') & &
\end{tikzcd}$$

Figure 1

The parallelogram commutes by naturality of η, the rightmost triangle by the unit axioms of the individual $\mathcal{V}$–categories, and the top triangle by the internal unit condition for η. The right-hand triangle in the axiom is checked similarly.

On a related note, we need to check that $\mathcal{I} \otimes_i^{(1)} \mathcal{A} = \mathcal{A}$ The object sets and hom–objects of the two categories in question are clearly equivalent. What needs to be checked is that the composition morphisms are the same. Note that the composition given by

$$\begin{tikzcd}
(\mathcal{I} \otimes_i^{(1)} \mathcal{A})((0,A'),(0,A'')) \otimes_1 (\mathcal{I} \otimes_i^{(1)} \mathcal{A})((0,A),(0,A')) \arrow[d, equal] \\
(I \otimes_{i+1} \mathcal{A}(A',A'')) \otimes_1 (I \otimes_{i+1} \mathcal{A}(A,A')) \arrow[d, "\eta^{1,i+1}_{I\mathcal{A}(A',A'')I\mathcal{A}(A,A')}"'] \\
(I \otimes_1 I) \otimes_{i+1} (\mathcal{A}(A',A'') \otimes_1 \mathcal{A}(A,A')) \arrow[d, "1\otimes_{i+t} M_{AA'A''}"'] \\
(I \otimes_{i+1} \mathcal{A}(A,A'')) \arrow[d, equal] \\
(\mathcal{I} \otimes_i^{(1)} \mathcal{A})((0,A),(0,A''))
\end{tikzcd}$$

is equivalent to simply $M_{AA'A''}$ by the external unit condition for η.

Associativity in $\mathcal{V}$–Cat must hold for each $\otimes_i^{(1)}$. The components of 2–natural isomorphism

$$\alpha^{(1)i}_{\mathcal{ABC}} : (\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C} \to \mathcal{A} \otimes_i^{(1)} (\mathcal{B} \otimes_i^{(1)} \mathcal{C})$$

are $\mathcal{V}$–functors that send ((A,B),C) to (A,(B,C)) and whose hom-components

$$\alpha^{(1)i}_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}} : [(\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C}](((A,B),C),((A',B'),C'))$$

$$\to [\mathcal{A} \otimes_i^{(1)} (\mathcal{B} \otimes_i^{(1)} \mathcal{C})]((A,(B,C)),(A',(B',C')))$$

are given by:

$$\alpha^{(1)i}_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}} = \alpha^{i+1}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')}$$

This guarantees that the 2–natural isomorphism $\alpha^{(1)i}$ is coherent. The commutativity of the pentagon for the objects is trivial, and the commutativity of the pentagon for the hom–object morphisms follows directly from the commutativity of the pentagon for α^{i+1}.

In order to be a functor the associator components must satisfy the commutativity of the diagrams in Definition 3.

(1)

$$
\begin{tikzcd}
\bullet \arrow[r, "M"] \arrow[d, "\alpha^{(1)i}\otimes\alpha^{(1)i}"'] & \bullet \arrow[d, "\alpha^{(1)i}"] \\
\bullet \arrow[r, "M"] & \bullet
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "\alpha^{(1)i}"] \\
I \arrow[ur, "j_{((A,B),C)}"] \arrow[dr, "j_{(A,(B,C))}"'] & \\
 & \bullet
\end{tikzcd}
$$

Expanding the first using the definitions just given we have that the initial position in the diagram is

$$[(\mathcal{A}\otimes_i^{(1)}\mathcal{B})\otimes_i^{(1)}\mathcal{C}](((A',B'),C'),((A'',B''),C''))\otimes_1[(\mathcal{A}\otimes_i^{(1)}\mathcal{B})\otimes_i^{(1)}\mathcal{C}](((A,B),C),((A',B'),C'))$$
$$= [(\mathcal{A}(A',A'')\otimes_{i+1}\mathcal{B}(B',B''))\otimes_{i+1}\mathcal{C}(C',C'')]\otimes_1[(\mathcal{A}(A,A')\otimes_{i+1}\mathcal{B}(B,B'))\otimes_{i+1}\mathcal{C}(C,C')]$$

We let $X = \mathcal{A}(A',A'')$, $Y = \mathcal{B}(B',B'')$, $Z = \mathcal{C}(C',C'')$, $X' = \mathcal{A}(A,A')$, $Y' = \mathcal{B}(B,B')$ and $Z' = \mathcal{C}(C,C')$. Then expanding the diagram, with an added interior arrow, we have:

$$
\begin{tikzcd}[column sep=tiny]
 & {[(X\otimes_{i+1}Y)\otimes_{i+1}Z]\otimes_1[(X'\otimes_{i+1}Y')\otimes_{i+1}Z']} \arrow[dl, "\alpha^{i+1}\otimes_1\alpha^{i+1}"'] \arrow[dr, "\eta^{1,i+1}_{(X\otimes_{i+1}Y)Z(X'\otimes_{i+1}Y')Z'}"] & \\
{[X\otimes_{i+1}(Y\otimes_{i+1}Z)]\otimes_1[X'\otimes_{i+1}(Y'\otimes_{i+1}Z')]} \arrow[d, "\eta^{1,i+1}_{X(Y\otimes_{i+1}Z)X'(Y'\otimes_{i+1}Z')}"] & & {[(X\otimes_{i+1}Y)\otimes_1(X'\otimes_{i+1}Y')]\otimes_{i+1}(Z\otimes_1 Z')} \arrow[d, "\eta^{1,i+1}_{XYX'Y'}\otimes_{i+1}1_{Z\otimes_1 Z'}"] \\
{(X\otimes_1 X')\otimes_{i+1}[(Y\otimes_{i+1}Z)\otimes_1(Y'\otimes_{i+1}Z')]} \arrow[d, "1_{X\otimes_1 X'}\otimes_{i+1}\eta^{1,i+1}_{YZY'Z'}"] & & {[(X\otimes_1 X')\otimes_{i+1}(Y\otimes_1 Y')]\otimes_{i+1}(Z\otimes_1 Z')} \arrow[d, "(M\otimes_{i+1}M)\otimes_{i+1}M"] \arrow[dll, "\alpha^{i+1}"'] \\
{(X\otimes_1 X')\otimes_{i+1}[(Y\otimes_1 Y')\otimes_{i+1}(Z\otimes_1 Z')]} \arrow[dr, "M\otimes_{i+1}(M\otimes_{i+1}M)"'] & & {(\mathcal{A}(A,A'')\otimes_{i+1}\mathcal{B}(B,B''))\otimes_{i+1}\mathcal{C}(C,C'')} \arrow[dl, "\alpha^{i+1}"] \\
 & {\mathcal{A}(A,A'')\otimes_{i+1}(\mathcal{B}(B,B'')\otimes_{i+1}\mathcal{C}(C,C''))} &
\end{tikzcd}
$$

The lower quadrilateral commutes by naturality of α, and the upper hexagon commutes by the external associativity of η.

The uppermost position in the expanded version of diagram number (2) is

$$\begin{aligned}&[(\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C}](((A,B),C),((A,B),C))\\ &= [(\mathcal{A}(A,A) \otimes_{i+1} \mathcal{B}(B,B)) \otimes_{i+1} \mathcal{C}(C,C)].\end{aligned}$$

The expanded diagram is easily seen to commute by the naturality of α.

The 2–naturality of $\alpha^{(1)}$ is essentially just the naturality of its components, but I think it ought to be expounded upon. Since the components of $\alpha^{(1)}$ are $\mathcal{V}$–functors the whisker diagrams for the definition of 2–naturality are defined by the whiskering in $\mathcal{V}$–Cat. Given an arbitrary 2–cell in $\times^3\mathcal{V}$–Cat, i.e. $(\beta,\gamma,\rho):(Q,R,S)\to(Q',R',S'):(\mathcal{A},\mathcal{B},\mathcal{C})\to(\mathcal{A}',\mathcal{B}',\mathcal{C}')$ the diagrams whose composition must be equal are:

$$\begin{tikzcd}[column sep=huge]
(\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C} \arrow[r, bend left, "(Q\otimes_i^{(1)}R)\otimes_i^{(1)}S"] \arrow[r, bend right, "(Q'\otimes_i^{(1)}R')\otimes_i^{(1)}S'"'] \arrow[r, phantom, "\Downarrow (\beta\otimes_i^{(1)}\gamma)\otimes_i^{(1)}\rho"] & (\mathcal{A}' \otimes_i^{(1)} \mathcal{B}') \otimes_i^{(1)} \mathcal{C}' \arrow[r, "\alpha^{(1)i}_{\mathcal{A}'\mathcal{B}'\mathcal{C}'}"] & \mathcal{A}' \otimes_i^{(1)} (\mathcal{B}' \otimes_i^{(1)} \mathcal{C}')
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}[column sep=huge]
(\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C} \arrow[r, "\alpha^{(1)i}_{\mathcal{A}\mathcal{B}\mathcal{C}}"] & \mathcal{A} \otimes_i^{(1)} (\mathcal{B} \otimes_i^{(1)} \mathcal{C}) \arrow[r, bend left, "Q\otimes_i^{(1)}(R\otimes_i^{(1)}S)"] \arrow[r, bend right, "Q'\otimes_i^{(1)}(R'\otimes_i^{(1)}S')"'] \arrow[r, phantom, "\Downarrow \beta\otimes_i^{(1)}(\gamma\otimes_i^{(1)}\rho)"] & \mathcal{A}' \otimes_i^{(1)} (\mathcal{B}' \otimes_i^{(1)} \mathcal{C}')
\end{tikzcd}$$

This is quickly seen to hold when we translate using the definitions of whiskering in $\mathcal{V}$–Cat, as follows. The $ABCD$ components of the new 2–cells are given by the exterior legs of the following diagram. They are equal by naturality of α^{i+1} and Mac Lane's coherence theorem.

$$\begin{tikzcd}
& I \arrow[dl, "="'] \arrow[dr, "="] & \\
(I \otimes_{i+1} I) \otimes_{i+1} I \arrow[rr, "\alpha^{i+1}"] \arrow[d, "(\beta_A\otimes_{i+1}\gamma_B\otimes_{i+1})\rho_C"] & & I \otimes_{i+1} (I \otimes_{i+1} I) \arrow[d, "\beta_A\otimes_{i+1}(\gamma_B\otimes_{i+1}\rho_C)"] \\
(\mathcal{A}'(QA,Q'A) \otimes_{i+1} \mathcal{B}'(RB,R'B)) \otimes_{i+1} \mathcal{C}'(SC,S'C) \arrow[rr, "\alpha^{i+1}"] & & \mathcal{A}'(QA,Q'A) \otimes_{i+1} (\mathcal{B}'(RB,R'B) \otimes_{i+1} \mathcal{C}'(SC,S'C))
\end{tikzcd}$$

Now we turn to consider the existence and behavior of interchange 2–natural transformations $\eta^{(1)ij}$ for $j \geq i+1$. As in the example, we define the component morphisms $\eta^{(1)i,j}_{\mathcal{ABCD}}$ that make a 2–natural transformation between 2–functors. Each component must be an enriched functor. Their action on objects is to send $((A,B),(C,D)) \in \left|(\mathcal{A} \otimes^{(1)}_j \mathcal{B}) \otimes^{(1)}_i (\mathcal{C} \otimes^{(1)}_j \mathcal{D})\right|$ to $((A,C),(B,D)) \in \left|(\mathcal{A} \otimes^{(1)}_i \mathcal{C}) \otimes^{(1)}_j (\mathcal{B} \otimes^{(1)}_i \mathcal{D})\right|$. The hom–object morphisms are given by:

$$\eta^{(1)i,j}_{\mathcal{ABCD}_{(ABCD)(A'B'C'D')}} = \eta^{i+1,j+1}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')\mathcal{D}(D,D')}$$

For this designation of $\eta^{(1)}$ to define a valid $\mathcal{V}$–functor, it must obey the axioms for compatibility with composition and units. We need commutativity of the following diagram, where the first bullet represents

$$[(\mathcal{A} \otimes^{(1)}_j \mathcal{B}) \otimes^{(1)}_i (\mathcal{C} \otimes^{(1)}_j \mathcal{D})](((A',B'),(C',D')),((A'',B''),(C'',D'')))$$

$$\otimes_1[(\mathcal{A} \otimes^{(1)}_j \mathcal{B}) \otimes^{(1)}_i (\mathcal{C} \otimes^{(1)}_j \mathcal{D})](((A,B),(C,D)),((A',B'),(C',D')))$$

and the last bullet represents

$$[(\mathcal{A} \otimes^{(1)}_i \mathcal{C}) \otimes^{(1)}_j (\mathcal{B} \otimes^{(1)}_i \mathcal{D})](((A,C),(B,D)),((A'',C''),(B'',D''))).$$

$$\begin{tikzcd}
\bullet \arrow[r, "M"] \arrow[d, "\eta^{(1)i,j} \otimes_1 \eta^{(1)i,j}"'] & \bullet \arrow[d, "\eta^{(1)i,j}"] \\
\bullet \arrow[r, "M"'] & \bullet
\end{tikzcd}$$

If we let $X = \mathcal{A}(A,A')$, $Y = \mathcal{B}(B,B')$, $Z = \mathcal{C}(C,C')$, $W = \mathcal{D}(D,D')$, $X' = \mathcal{A}(A',A'')$, $Y' = \mathcal{B}(B',B'')$, $Z' = \mathcal{C}(C',C'')$ and $W' = \mathcal{D}(D',D'')$ then the expanded diagram is given in Figure 2. The exterior must commute.

The lower quadrilateral in Figure 2 commutes by naturality of η and the upper hexagon commutes since it is an instance of the giant hexagonal interchange. As for $\alpha^{(1)}$, the compatibility with the unit of $\eta^{(1)i,j}$ follows directly from the naturality of $\eta^{i+1,j+1}$ and the fact that $j_{[(A,B),(C,D)]} = [(j_A \otimes_{j+1} j_B) \otimes_{i+1} (j_C \otimes_{j+1} j_D)]$.

Also the 2–naturality of $\eta^{(1)i,j}$ follows directly from the naturality of $\eta^{i+1,j+1}$ and the Mac Lane coherence theorem.

Since $\alpha^{(1)}$ and $\eta^{(1)}$ are both defined based upon α and η their $\mathcal{V}$–functor components satisfy all the axioms of the definition of a k–fold monoidal category. At this level of course it is actually a k–fold monoidal 2–category.

$$
\begin{tikzcd}[column sep=tiny]
& \left[(X' \otimes_{j+1} Y') \otimes_{i+1} (Z' \otimes_{j+1} W')\right] \otimes_1 \left[(X \otimes_{j+1} Y) \otimes_{i+1} (Z \otimes_{j+1} W)\right] \arrow[dl, "\eta^{i+1,j+1}_{X'Y'Z'W'} \otimes_1 \eta^{i+1,j+1}_{XYZW}"'] \arrow[dr, "\eta^{1,i+1}_{(X' \otimes_{j+1} Y')(Z' \otimes_{j+1} W')(X \otimes_{j+1} Y)(Z \otimes_{j+1} W)}"] & \\
\left[(X' \otimes_{i+1} Z') \otimes_{j+1} (Y' \otimes_{i+1} W')\right] \otimes_1 \left[(X \otimes_{i+1} Z) \otimes_{j+1} (Y \otimes_{i+1} W)\right] \arrow[d, "\eta^{1,j+1}_{(X' \otimes_{i+1} Z')(Y' \otimes_{i+1} W')(X \otimes_{i+1} Z)(Y \otimes_{i+1} W)}"'] & & \left[(X' \otimes_{j+1} Y') \otimes_1 (X \otimes_{j+1} Y)\right] \otimes_{i+1} \left[(Z' \otimes_{j+1} W') \otimes_1 (Z \otimes_{j+1} W)\right] \arrow[d, "\eta^{1,j+1}_{X'Y'XY} \otimes_{i+1} \eta^{1,j+1}_{Z'W'ZW}"] \\
\left[(X' \otimes_{i+1} Z') \otimes_1 (X \otimes_{i+1} Z)\right] \otimes_{j+1} \left[(Y' \otimes_{i+1} W') \otimes_1 (Y \otimes_{i+1} W)\right] \arrow[d, "\eta^{1,i+1}_{X'Z'XZ} \otimes_{j+1} \eta^{1,i+1}_{Y'W'YW}"'] & & \left[(X' \otimes_1 X) \otimes_{j+1} (Y' \otimes_1 Y)\right] \otimes_{i+1} \left[(Z' \otimes_1 Z) \otimes_{j+1} (W' \otimes_1 W)\right] \arrow[d, "{[M_{AA'A''} \otimes_{j+1} M_{BB'B''}] \otimes_{i+1} [M_{CC'C''} \otimes_{j+1} M_{DD'D''}]}"] \arrow[dll, "\eta^{i+1,j+1}_{(X' \otimes_1 X)(Y' \otimes_1 Y)(Z' \otimes_1 Z)(W' \otimes_1 W)}"'] \\
\left[(X' \otimes_1 X) \otimes_{i+1} (Z' \otimes_1 Z)\right] \otimes_{j+1} \left[(Y' \otimes_1 Y) \otimes_{i+1} (W' \otimes_1 W)\right] \arrow[dr, "{[M_{AA'A''} \otimes_{i+1} M_{CC'C''}] \otimes_{j+1} [M_{BB'B''} \otimes_{i+1} M_{DD'D''}]}"'] & & \left[\mathcal{A}(A, A'') \otimes_{j+1} \mathcal{B}(B, B'')\right] \otimes_{i+1} \left[\mathcal{C}(C, C'') \otimes_{j+1} \mathcal{D}(D, D'')\right] \arrow[dl, "\eta^{i+1,j+1}_{\mathcal{A}(A,A'')\mathcal{B}(B,B'')\mathcal{C}(C,C'')\mathcal{D}(D,D'')}"] \\
& \left[\mathcal{A}(A, A'') \otimes_{i+1} \mathcal{C}(C, C'')\right] \otimes_{j+1} \left[\mathcal{B}(B, B'') \otimes_{i+1} \mathcal{D}(D, D'')\right] &
\end{tikzcd}
$$

Figure 2

Notice that we have used all the axioms of a k–fold monoidal category. The external and internal unit conditions imply the unital nature of $\mathcal{V}$–Cat and the unit axioms for a product of $\mathcal{V}$–categories respectively. The external and internal associativities give us respectively the $\mathcal{V}$–functoriality of $\alpha^{(1)}$ and the associativity of the composition morphisms for products of $\mathcal{V}$–categories. This reflects the dual nature of the latter two axioms that was pointed out for the braided case in [8]. Finally the giant hexagon gives us precisely the $\mathcal{V}$–functoriality of $\eta^{(1)}$. Notice also that we have used in each case the instance of the axiom corresponding to $i = 1; j = 2 \dots k$. The remaining instances will be used as we iterate the categorical delooping. □

5 Further questions

For $\mathcal{V}$ k–fold monoidal we have demonstrated that $\mathcal{V}$–Cat is $(k-1)$–fold monoidal. By induction we have that this process continues, i.e. that $\mathcal{V}$–n–Cat $= \mathcal{V}$–$(n-1)$–Cat–Cat is $(k-n)$–fold monoidal for $k > n$. For example, let us expand our description of the next level: the fact that $\mathcal{V}$–2–Cat $= \mathcal{V}$–Cat–Cat is $(k-2)$–fold monoidal. Now we are considering enrichment over $\mathcal{V}$–Cat. All the constructions in the proof above are recursively repeated. The unit $\mathcal{V}$–2–category is denoted as $\boldsymbol{\mathcal{I}}$ where $|\boldsymbol{\mathcal{I}}| = \{\mathbf{0}\}$ and $\boldsymbol{\mathcal{I}}(\mathbf{0},\mathbf{0}) = \mathcal{I}$. Products of $\mathcal{V}$–2–categories are given by $\boldsymbol{\mathcal{U}} \otimes_i^{(2)} \boldsymbol{\mathcal{W}}$ for $i = 1 \dots k-2$. Objects are pairs of objects as usual, and that there are exactly $k-2$ products is seen when the definition of hom–objects is given. In $\mathcal{V}$–2–Cat,

$$[\boldsymbol{\mathcal{U}} \otimes_i^{(2)} \boldsymbol{\mathcal{W}}]((U,W),(U',W')) = \boldsymbol{\mathcal{U}}(U,U') \otimes_{i+1}^{(1)} \boldsymbol{\mathcal{W}}(W,W').$$

Thus we have that

$$\begin{aligned}
&[\boldsymbol{\mathcal{U}} \otimes_i^{(2)} \boldsymbol{\mathcal{W}}]((U,W),(U',W'))((f,f'),(g,g')) \\
&= [\boldsymbol{\mathcal{U}}(U,U') \otimes_{i+1}^{(1)} \boldsymbol{\mathcal{W}}(W,W')]((f,f'),(g,g')) \\
&= \boldsymbol{\mathcal{U}}(U,U')(f,g) \otimes_{i+2} \boldsymbol{\mathcal{W}}(W,W')(f',g').
\end{aligned}$$

The definitions of $\alpha^{(2)i}$ and $\eta^{(2)i,j}$ are just as in the lower case. For instance, $\alpha^{(2)i}$ will now be a 3–natural transformation, that is, a family of $\mathcal{V}$–2–functors

$$\alpha^{(2)i}_{\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{W}}} : (\boldsymbol{\mathcal{U}} \otimes_i^{(2)} \boldsymbol{\mathcal{V}}) \otimes_i^{(2)} \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{U}} \otimes_i^{(2)} (\boldsymbol{\mathcal{V}} \otimes_i^{(2)} \boldsymbol{\mathcal{W}}).$$

To each of these is associated a family of $\mathcal{V}$–functors

$$\alpha^{(2)i}_{\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{W}}_{(U,V,W)(U',V',W')}} = \alpha^{(1)i+1}_{\boldsymbol{\mathcal{U}}(U,U')\boldsymbol{\mathcal{V}}(V,V')\boldsymbol{\mathcal{W}}(W,W')}$$

to each of which is associated a family of hom–object morphisms:

$$\alpha^{(2)i}_{\mathcal{U}\mathcal{V}\mathcal{W}_{(U,V,W)(U',V',W')_{(f,g,h)(f',g',h')}}} = \alpha^{i+2}_{\mathcal{U}(U,U')(f,f')\mathcal{V}(V,V')(g,g')\mathcal{W}(W,W')(h,h')}$$

Verifications that these define a valid $(k-2)$–fold monoidal 3–category all follow just as in the lower dimensional case. The facts about the $\mathcal{V}$–functors are shown by using the original k–fold monoidal category axioms that involve i=2.

In the next paper [7] my aim is to show how enrichment increases categorical dimension as it decreases monoidalness. That paper also includes the definitions of $\mathcal{V}$–n–categories and of the morphisms of $\mathcal{V}$–n–Cat. In further work I want to relate enrichment more precisely to topological delooping as well as to other categorical constructions that have similar topological implications.

In [16] Street defines the nerve of a strict n–category. Recently Duskin in [4] has worked out the description of the nerve of a bicategory. This allows us to ask whether these nerves will prove to be the logical link to loop spaces for higher dimensional iterated monoidal categories.

Passing to the category of enriched categories basically reduces the number of products so that for $\mathcal{V}$ a k–fold monoidal n–category, $\mathcal{V}$–Cat becomes a $(k-1)$–fold monoidal $(n+1)$–category. This picture was anticipated by Baez and Dolan [1] in the context where the k–fold monoidal n–category is specifically a (weak) $(n+k)$–category with only one object, one 1–cell, etc. up to only one k–cell. Their version of categorical delooping simply consists of creating from a monoidal category $\mathcal{V}$ the one object bicategory that has its morphisms the objects of $\mathcal{V}$. Relating the two versions of delooping is important to an understanding of how categories model spaces.

References

[1] J. C. Baez and J. Dolan, Categorification, in "Higher Category Theory", eds. E. Getzler and M. Kapranov, Contemp. Math. 230 , American Mathematical Society, 1-36, (1998).

[2] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, Iterated Monoidal Categories, Adv. Math. 176 (2003), 277-349.

[3] J. M. Boardman and R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Mathematics, Vol. 347, Springer, 1973.

[4] J.W.Duskin, Simplicial Matrices and the Nerves of Weak n–Categories I: Nerves of Bicategories, Theory and Applications of Categories, 9, No. 10 (2002), 198-308.

[5] S. Eilenberg and G. M. Kelly, Closed Categories, Proc. Conf. on Categorical Algebra, Springer-Verlag (1965), 421-562.

[6] Z. Fiedorowicz, The symmetric bar construction, preprint.

[7] S. Forcey, Higher Dimensional Enrichment, `arXiv:math.CT/0306086`

[8] S. Forcey, Enrichment as Categorical Delooping I: Enrichment Over Iterated Monoidal Categories, `arXiv:math.CT/0304026`

[9] A. Joyal and R. Street, Braided tensor categories, Advances in Math. 102(1993), 20-78.

[10] G. M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Society Lecture Note Series 64, 1982.

[11] V. Lyubashenko, Category of A_∞–categories, Homology, Homotopy and Applications 5(1) (2003), 1-48.

[12] S. Mac Lane, Categories for the Working Mathematician 2nd. edition, Grad. Texts in Math. 5, 1998.

[13] S. Mac Lane, Categorical algebra, Bull. A. M. S. 71(1965), 40-106.

[14] J. P. May, The geometry of iterated loop spaces, Lecture Notes in Mathematics, Vol. 271, Springer, 1972

[15] J. D. Stasheff, Homotopy associativity of H-spaces I, Trans. A. M. S. 108(1963), 275-292.

[16] R. Street, The Algebra of Oriented Simplexes, J. Pure Appl. Algebra 49(1987), 283-335.

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