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Character varieties of mutative 3–manifolds

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Abstract We describe a birational map between subvarieties in the character varieties of mutative 3–manifolds. By studying the birational map, one can decide in certain circumstances whether a mutation surface is detected by an ideal point of the character variety.

AMS Classification 57M27; 14E07

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1 Introduction

Let M be an irreducible 3–manifold. A *mutation surface* (S,τ) is a properly embedded incompressible, ∂–incompressible surface S, which is not boundary parallel, and an orientation preserving involution τ of S. The manifold obtained by cutting M open along S and regluing via τ is a (S,τ)*–mutant* of M, and denoted by M^τ. Two manifolds are *mutative* if they are related by a finite sequence of mutations. Mutative manifolds share many geometric and topological properties. Work of Ruberman [5] and Cooper and Long [2] shows that a relationship between the character varieties of mutants exists, and we now formalise this for both $SL_2(\mathbb{C})$– and $PSL_2(\mathbb{C})$–character varieties.

We construct a birational map between certain subvarieties of the character varieties of M and M^τ, which shows that in many cases the character varieties are birationally equivalent. A subvariety $\mathfrak{X}_\tau(M)$ in the $SL_2(\mathbb{C})$–character variety $\mathfrak{X}(M)$ will be defined, and the birational equivalence is defined for subvarieties of $\mathfrak{X}_\tau(M)$ which contain a dense set of *mutable* characters. All these notions descend to the $PSL_2(\mathbb{C})$–character variety, and the objects are denoted by placing a bar over the previous notation.

Proposition 1 *Let (S,τ) be a mutation surface in an irreducible 3–manifold M, and let C be an irreducible component of $\overline{\mathfrak{X}}_\tau(M)$ which contains the character of a representation whose restriction to $\mathrm{im}(\pi_1(S)\to\pi_1(M))$ is irreducible and has trivial centraliser.*

Then C is birationally equivalent to an irreducible component of $\overline{\mathfrak{X}}_\tau(M^\tau)$.

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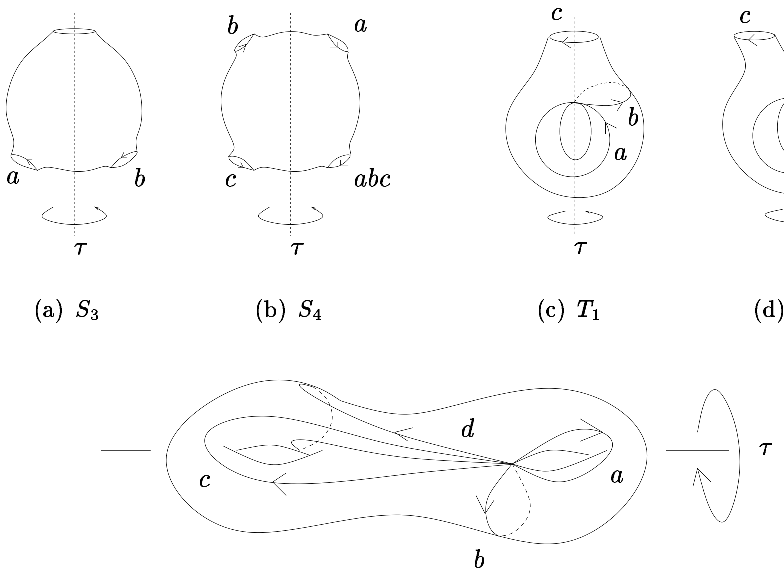

(a) S_3 (b) S_4 (c) T_1 (d) T_2

(e) G_2

Figure 1: Ruberman's symmetric surfaces: a thrice-punctured sphere S_3, a four-punctured sphere S_4, a once-punctured torus T_1, a twice-punctured torus T_2 and a genus-two surface G_2 along with the shown involutions.

Proposition 2 *Let (S,τ) be a* separating *mutation surface in an irreducible 3–manifold M, and let C be an irreducible component of $\mathfrak{X}_\tau(M)$ which contains the character of a representation whose restriction to $\operatorname{im}(\pi_1(S) \to \pi_1(M))$ is irreducible.*

Then C is birationally equivalent to an irreducible component of $\mathfrak{X}_\tau(M^\tau)$.

If M admits a complete hyperbolic structure of finite volume, then there is a discrete and faithful representation of $\pi_1(M)$ into $PSL_2(\mathbb{C})$, which lies on the *Dehn surgery component* $\overline{\mathfrak{X}}_0(M)$ of $\overline{\mathfrak{X}}(M)$, and lifts to a component $\mathfrak{X}_0(M)$ of $\mathfrak{X}(M)$. We now focus on the *symmetric surfaces* shown in Figure 1.

Corollary 3 (Hyperbolic knots) *Let $\mathfrak{k}$ be a hyperbolic knot and $\mathfrak{k}^\tau$ be a Conway mutant of $\mathfrak{k}$. Then $\overline{\mathfrak{X}}_0(\mathfrak{k})$ and $\overline{\mathfrak{X}}_0(\mathfrak{k}^\tau)$, as well as $\mathfrak{X}_0(\mathfrak{k})$ and $\mathfrak{X}_0(\mathfrak{k}^\tau)$, are birationally equivalent. Moreover, the associated factors of the A–polynomials are equal.*

It is noted in [9] that any Conway mutation of a knot can be realised by at most two mutations along genus two surfaces. The previous corollary is therefore a special case of the following:

Corollary 4 (Separating in hyperbolic) *Let (S,τ) be a separating symmetric surface in a finite volume hyperbolic 3–manifold M. If S is a twice–punctured torus or a genus two surface, then $\overline{\mathfrak{X}}_0(M)$ and $\overline{\mathfrak{X}}_0(M^\tau)$ as well as $\mathfrak{X}_0(M)$ and $\mathfrak{X}_0(M^\tau)$ are birationally equivalent.*

The restriction to the two surfaces in the above corollary is necessary in general. There are no separating incompressible and ∂–incompressible thrice punctured spheres and once–punctured tori in hyperbolic 3–manifolds, and it is easy to find examples of Conway mutation on links with the property that only a proper subvariety in each of the respective Dehn surgery components is contained in $\mathfrak{X}_\tau(M)$ and $\overline{\mathfrak{X}}_\tau(M)$. If the surface is non–separating, one can similarly find examples such that mutation along a twice–punctured torus or a thrice punctured sphere does not allow a general statement, which limits us to the following:

Corollary 5 (Non–separating in hyperbolic) *Let (S,τ) be a non–separating symmetric surface in a finite volume hyperbolic 3–manifold M. If S is a once–punctured torus or a genus two surface, then $\overline{\mathfrak{X}}_0(M)$ and $\overline{\mathfrak{X}}_0(M^\tau)$ are birationally equivalent.*

This corollary does not extend to $SL_2(\mathbf{C})$–Dehn surgery components in general: mutation of the figure eight knot complement along the fibre results in the associated sister manifold, and the smooth projective models of their $SL_2(\mathbf{C})$–Dehn surgery components are a torus and a sphere respectively.

The proofs of the above results are contained in Section 2. Some of the ideas in the proofs are useful in other settings; e.g. they produce examples of "holes in the eigenvalue variety" in [9]. The *extension lemma* (see Lemma 10) can be used to study the character variety of a 3–manifold by successively cutting along non–separating surfaces.

In certain cases, analysis of the points where the birational equivalence is not well–defined can be used to decide whether a mutation surface is detected by an ideal point of the character variety. It is still an open problem whether every essential surface is detected by an ideal point of the character variety. Necessary and sufficient conditions which have to be satisfied by a connected surface are given in Section 3, and the birational equivalence is used to show that symmetric surfaces are detected in the complements the Kinoshita–Terasaka knot and the figure eight knot, as well as the so–called sister manifold of the latter.

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2 Tentatively mutable representations

Our standard references for character varieties are [3, 1]. We recall some definitions and facts. The $SL_2(\mathbb{C})$*-representation variety* of a finitely generated group Γ is $\mathfrak{R}(\Gamma) = \text{Hom}(\Gamma, SL_2(\mathbb{C}))$. Each $\rho \in \mathfrak{R}(\Gamma)$ defines a *character* $\chi_\rho : \Gamma \to \mathbb{C}$ by $\chi_\rho(\gamma) = \text{tr}\,\rho(\gamma)$, and the set of characters $\mathfrak{X}(\Gamma)$ is the $SL_2(\mathbb{C})$*–character variety.* Both varieties are regarded as affine algebraic sets, and there is a regular map $\text{t} : \mathfrak{R}(\Gamma) \to \mathfrak{X}(\Gamma)$. If Γ is the fundamental group of a topological space M, we write $\mathfrak{R}(M)$ and $\mathfrak{X}(M)$ instead of $\mathfrak{R}(\Gamma)$ and $\mathfrak{X}(\Gamma)$ respectively.

A representation is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant under its image are trivial. Otherwise it is *reducible.* Irreducible representations are determined by characters up to inner automorphisms of $SL_2(\mathbb{C})$. Let $\mathfrak{R}^i(\Gamma)$ denote the closure of the set of irreducible representations, then the images $\mathfrak{X}^r(\Gamma) = \text{t}(\mathfrak{Red}(\Gamma))$ and $\mathfrak{X}^i(\Gamma) = \text{t}(\mathfrak{R}^i(\Gamma))$ are closed algebraic sets, and we have $\mathfrak{X}^r(\Gamma) \cup \mathfrak{X}^i(\Gamma) = \mathfrak{X}(\Gamma)$. The variety $\mathfrak{X}^r(\Gamma)$ is completely determined by the abelianisation of Γ.

There is a character variety arising from representations into $PSL_2(\mathbb{C})$, and the relevant objects are denoted by placing a bar over the previous notation. As with the $SL_2(\mathbb{C})$–character variety, the surjective map $\overline{\text{t}} : \overline{\mathfrak{R}}(\Gamma) \to \overline{\mathfrak{X}}(\Gamma)$ is constant on conjugacy classes and if $\overline{\rho}$ is an irreducible representation, then $\overline{\text{t}}^{-1}(\overline{\text{t}}(\overline{\rho}))$ is the orbit of $\overline{\rho}$ under conjugation. The natural map $\text{q} : \mathfrak{X}(\Gamma) \to \overline{\mathfrak{X}}(\Gamma)$ is finite–to–one, but in general not onto. It is the quotient map corresponding to the action of $\text{Hom}(\Gamma, \mathbb{Z}_2)$ on $\mathfrak{X}(M)$. This action is not free in general.

2.1 Tentatively mutable in $\mathbf{SL_2}(\mathbb{C})$

Given a mutation surface (S, τ), we define a subvariety in $\mathfrak{R}(S)$ by

$$\mathfrak{R}_\tau(S) = \{\rho \in \mathfrak{R}(S) \mid \text{tr}\,\rho(\gamma) = \text{tr}\,\rho(\tau_*\gamma) \text{ for all } \gamma \in \pi_1(S)\}.$$

This subvariety descends to the character variety, and we let $\mathfrak{X}_\tau(S) = \text{t}(\mathfrak{R}_\tau(S))$. In fact, τ induces a polynomial automorphism of $\mathfrak{X}(S)$, and $\mathfrak{X}_\tau(S)$ is the set of its fixed points. If $\mathfrak{R}_\tau(S)$ contains an irreducible representation, then the subvariety of reducible representations has positive codimension. For the symmetric surfaces, one obtains the following result:

Lemma 6 [9] *Let (S,τ) be a symmetric surface as described in Figure 1. If $S=T_1$ or $S=G_2$, then $\mathfrak{R}(S)=\mathfrak{R}_\tau(S)$. Otherwise the character of $\rho\in\mathfrak{R}(S)$ is invariant under τ if and only if it satisfies the following equations:*

- *if $S=S_3$, $\pi_1(S_3)=\langle a,b\rangle$ then $\operatorname{tr}\rho(a)=\operatorname{tr}\rho(b)$,*
- *if $S=S_4$, $\pi_1(S_4)=\langle a,b,c\rangle$ then $\operatorname{tr}\rho(a)=\operatorname{tr}\rho(b)$ and $\operatorname{tr}\rho(c)=\operatorname{tr}\rho(abc)$,*
- *if $S=T_2$, $\pi_1(T_2)=\langle a,b,c\rangle$ then $\operatorname{tr}\rho(c)=\operatorname{tr}\rho(c^{-1}[a,b])$.*

The subvariety of reducible representations in $\mathfrak{R}_\tau(S)$ has codimension one. Moreover, this property is preserved under t.

If (S,τ) is a mutation surface in a 3–manifold M, we call a representation $\rho\in\mathfrak{R}(M)$ *tentatively mutable with respect to* (S,τ) if its character restricted to S is invariant under τ. The set $\mathfrak{R}_\tau(M)$ of these representations is a subvariety of $\mathfrak{R}(M)$. Let $\mathrm{t}(\mathfrak{R}_\tau(M))=\mathfrak{X}_\tau(M)$. The S–reducible characters form a closed set in $\mathfrak{X}_\tau(M)$, which we denote by $\mathfrak{F}(M)$. Let the closure of $\mathfrak{X}_\tau(M)-\mathfrak{F}(M)$ in $\mathfrak{X}_\tau(M)$ be $\mathfrak{M}(M)$.[1] Then $\mathfrak{M}(M)\subseteq\mathfrak{X}^i(M)$ is the union of irreducible components of $\mathfrak{X}_\tau(M)$ which contain the character of a S–irreducible representation.

In particular, if (S,τ) is one of the symmetric surfaces (T_1,τ) or (G_2,τ), we have $\mathfrak{X}_\tau(M)=\mathfrak{X}(M)$, so $\mathfrak{X}^r(M)\subseteq\mathfrak{F}(M)$, and the same is true for M^τ. In general, it is not true that $\mathfrak{X}_\tau(M)=\mathfrak{X}(M)$ implies $\mathfrak{X}_\tau(M^\tau)=\mathfrak{X}(M^\tau)$.

2.2 Tentatively mutable in $\mathbf{PSL_2}(\mathbb{C})$

Given a mutation surface (S,τ), we define a subvariety in $\overline{\mathfrak{R}}(S)$ by

$$\overline{\mathfrak{R}}_\tau(S)=\{\overline{\rho}\in\overline{\mathfrak{R}}(S)\mid\chi_{\overline{\rho}}=\chi_{\overline{\rho}\tau_*}\}.$$

This subvariety descends to the character variety, and we let $\overline{\mathfrak{X}}_\tau(S)=\overline{\mathrm{t}}(\overline{\mathfrak{R}}_\tau(S))$. For the symmetric surfaces, we have the following lemma.[2]

Lemma 7 *Let (S,τ) be a symmetric surface as described in Figure 1 and $\overline{\rho}$ be a $PSL_2(\mathbb{C})$–representation of $\pi_1(S)$. If S is one of the surfaces with boundary, then $\chi_{\overline{\rho}}=\chi_{\overline{\rho}\tau_*}$ if and only if there is a lift ρ of $\overline{\rho}$ such that $\chi_\rho=\chi_{\rho\tau_*}$.*

[1] The varieties $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{F}$ of [9] are here denoted by $\mathfrak{R}_\tau$, $\mathfrak{X}_\tau$ and $\mathfrak{F}^i$ respectively.

[2] The author thanks Steven Boyer for pointing out that an earlier version of this lemma was incorrect.

If $S = G_2$, then $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(G_2)$ either if $\overline{\rho}$ lifts to a $SL_2(\mathbb{C})$–representation, or if

$$(\operatorname{tr}\overline{\rho}(ad^{-1}))^2 = (\operatorname{tr}\overline{\rho}(bc^{-1}))^2 = (\operatorname{tr}\overline{\rho}(abd^{-1}))^2 = (\operatorname{tr}\overline{\rho}(b^{-1}cd))^2 = (\operatorname{tr}\overline{\rho}(acd))^2 = 0.$$

Proof We use Lemma 3.1 of [1] throughout this proof, and assume familiarity with the notation used there; in particular, ϵ denotes a homomorphism into the group $\{\pm 1\}$.

Let $\overline{\rho} \in \overline{\mathfrak{R}}(S)$, and assume that there is a lift ρ of $\overline{\rho}$ such that $\rho \in \mathfrak{R}_\tau(S)$. It then follows that $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, by choosing $\rho' = \rho\tau_*$ and $\epsilon = \mathrm{id}$.

Since the fundamental groups of the surfaces with boundary are free, every $PSL_2(\mathbb{C})$–representation of these surfaces lifts to a $SL_2(\mathbb{C})$-representation. We now verify the statement of the lemma for these surfaces.

Case $S = T_1$ Since $\overline{\mathfrak{R}}(T_1) = \mathrm{q}(\mathfrak{R}(T_1))$ and $\mathfrak{R}(T_1) = \mathfrak{R}_\tau(T_1)$, there is nothing to prove. In particular, we have $\overline{\mathfrak{R}}(T_1) = \overline{\mathfrak{R}}_\tau(T_1)$.

Case $S = T_2$ Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. We have $\chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}$ if and only if there is $\epsilon \in \operatorname{Hom}(\pi_1(S), \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau_*}$. Now $\epsilon(a)\operatorname{tr}\rho(a) = \operatorname{tr}\rho\tau(a) = \operatorname{tr}\rho(a^{-1})$ forces $\epsilon(a) = 1$. Similarly, $\epsilon(b)\operatorname{tr}\rho(b) = \operatorname{tr}\rho\tau(b) = \operatorname{tr}\rho(ab^{-1}a^{-1})$ forces $\epsilon(b) = 1$. Then $\epsilon(bc)\operatorname{tr}\rho(bc) = \operatorname{tr}\rho\tau(bc) = \operatorname{tr}\rho(bc)$ yields $\epsilon(c) = \epsilon(b)$. Thus, $\epsilon = \mathrm{id}$, and the claim follows.

Case $S = S_3$ Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. Since $\tau(a) = b$, we have $(\operatorname{tr}\overline{\rho}(a))^2 = (\operatorname{tr}\overline{\rho}(b))^2$. If $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$, then $\rho \in \mathfrak{R}_\tau(S)$. If $\operatorname{tr}\rho(a) = -\operatorname{tr}\rho(b)$, then define $\sigma(a) = \rho(a)$ and $\sigma(b) = -\rho(b)$. Then σ is a lift of $\overline{\rho}$ and $\sigma \in \mathfrak{R}_\tau(S)$. This completes the proof in this case.

Case $S = S_4$ Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. We have $\chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}$ if and only if there is a homomorphism $\epsilon \in \operatorname{Hom}(\pi_1(S), \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau_*}$. As above, considering the action of τ yields $\epsilon(a) = \epsilon(b) = \epsilon(c)$. If ϵ is trivial, then $\rho \in \mathfrak{R}_\tau(S)$. Otherwise, the character of the lift σ defined by $\sigma(a) = \rho(a)$, $\sigma(b) = \rho(b)$ and $\sigma(c) = -\rho(c)$ is invariant under τ.

Now consider $S = G_2$. It follows from Theorem 5.1 in [4] that $\overline{\mathfrak{X}}(G_2)$ has two topological components with the property that every representation in one of the components lifts to $SL_2(\mathbb{C})$, and every representation in the other does not. We only have to consider the latter component since $\mathfrak{R}(G_2) = \mathfrak{R}_\tau(G_2)$.

Assume that $\overline{\rho}$ is a $PSL_2(\mathbb{C})$–representation of G_2 with representative matrices A, B, C, D for the generators a, b, c, d, such that $[A, B][C, D] = -E$. Then $\overline{\rho}$ does not lift to $SL_2(\mathbb{C})$. Now assume that $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and define a representation $\rho \in \mathfrak{R}(\mathfrak{F}_4)$ by $\rho(\alpha) = A$, $\rho(\beta) = B$, $\rho(\gamma) = C$ and $\rho(\delta) = D$.

By assumption, there is $\epsilon \in \mathrm{Hom}(\mathfrak{F}_4, \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau}$, where $\rho\tau$ is defined by

$$\rho\tau(\alpha) = A^{-1}, \qquad \rho\tau(\gamma) = (B^{-1}CD)C^{-1}(B^{-1}CD)^{-1},$$
$$\rho\tau(\beta) = AB^{-1}A^{-1}, \qquad \rho\tau(\delta) = (B^{-1}C)D^{-1}(B^{-1}C)^{-1}.$$

Then $\epsilon(a)\,\mathrm{tr}\,A = \epsilon(a)\,\mathrm{tr}\,\rho(\alpha) = \mathrm{tr}\,\rho\tau(\alpha) = \mathrm{tr}\,A^{-1}$ forces $\epsilon(a) = 1$. We similarly obtain $1 = \epsilon(b) = \epsilon(c) = \epsilon(d)$. But then $\epsilon = \mathrm{id}$, and we have

$$\begin{aligned}
\mathrm{tr}(AD) &= \epsilon(\alpha\delta)\,\mathrm{tr}\,\rho(\alpha\delta) = \mathrm{tr}\,\rho\tau(\alpha\delta) \\
&= \mathrm{tr}(A^{-1}(B^{-1}CD^{-1}C^{-1}B)) && \mid \text{by definition of } \rho\tau \\
&= \mathrm{tr}(A)\,\mathrm{tr}(D) - \mathrm{tr}(AB^{-1}CD^{-1}C^{-1}B) && \mid \text{by } \mathrm{tr}\,X^{-1}Y = \mathrm{tr}\,X\,\mathrm{tr}\,Y - \mathrm{tr}\,XY \\
&= \mathrm{tr}(A)\,\mathrm{tr}\,\rho(D) + \mathrm{tr}(AB^{-1}D^{-1}ABA^{-1}) && \mid \text{by } [A,B][C,D] = -E \\
&= \mathrm{tr}(A)\,\mathrm{tr}(D) + \mathrm{tr}(D^{-1}A) \\
&= \mathrm{tr}(AD) + 2\,\mathrm{tr}(D^{-1}A) && \mid \text{by } \mathrm{tr}\,X^{-1}Y = \mathrm{tr}\,X\,\mathrm{tr}\,Y - \mathrm{tr}\,XY.
\end{aligned}$$

Thus, $\mathrm{tr}(AD^{-1}) = 0$, and therefore $(\mathrm{tr}\,\overline{\rho}(ad^{-1}))^2 = 0$. The other stated trace identities follow similarly. This completes the proof of the lemma. □

If $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$ is an irreducible representation, then there exists an element $\overline{X} \in PSL_2(\mathbb{C})$, such that $\overline{\rho} = \overline{X}^{-1}\overline{\rho}\tau\overline{X}$. The centraliser of an element $\overline{Y}$ in $PSL_2(\mathbb{C})$ is the quotient of its centraliser in $SL_2(\mathbb{C})$ unless $(\mathrm{tr}\,\overline{Y})^2 = 0$. Thus, if Γ is a finitely generated group, then the centraliser of an irreducible representation $\overline{\rho} \in \overline{\mathfrak{R}}(\Gamma)$ is trivial if $(\mathrm{tr}\,\overline{\rho}(\gamma_i))^2 \neq 0$ for all generators γ_i of Γ. Let $\mathfrak{F}_2$ be the free group in two elements $\langle \alpha, \beta \rangle$. We have $\mathfrak{X}(\mathfrak{F}_2) \cong \mathbb{C}^3$, and the map $\overline{\mathfrak{X}}(\mathfrak{F}_2) \to \mathbb{C}^3$ given by $\chi_{\overline{\rho}} \to ((\mathrm{tr}\,\overline{\rho}(\alpha))^2, (\mathrm{tr}\,\overline{\rho}(\beta))^2, (\mathrm{tr}\,\overline{\rho}(\alpha\beta))^2)$ is a $2:1$ covering map.

Lemma 8 *Consider the above two–to–one parameterisation of the $PSL_2(\mathbb{C})$–character variety of $\mathfrak{F}_2 = \langle \alpha, \beta \rangle$ by the points $((\mathrm{tr}\,\overline{\rho}\alpha)^2, (\mathrm{tr}\,\overline{\rho}\beta)^2, (\mathrm{tr}\,\overline{\rho}\alpha\beta)^2)$ in $\mathbb{C}^3$. Then the set of irreducible representations with non–trivial centraliser is contained in the union of the three coordinate axes.*

Proof Assume that $\overline{\rho}$ is an irreducible representation of $\mathfrak{F}_2$ with non–trivial centraliser in $PSL_2(\mathbb{C})$. According to the above discussion at least one of $(\mathrm{tr}\,\overline{\rho}\alpha)^2$ or $(\mathrm{tr}\,\overline{\rho}\beta)^2$ is equal to zero. Assume that $(\mathrm{tr}\,\overline{\rho}\alpha)^2 = 0$. Direct calculation shows that the centraliser of $\overline{\rho}(\mathfrak{F}_2)$ is non–trivial if and only if $(\mathrm{tr}\,\overline{\rho}\beta)^2 = 0$ or $(\mathrm{tr}\,\overline{\rho}\alpha\beta)^2 = 0$. If both are equal to zero, then the image of $\overline{\rho}$ is a Kleinian four group in $PSL_2(\mathbb{C})$ and equal to its centraliser, and if one of the traces is not equal to zero, then the centraliser has order equal to two. □

It follows that if (S,τ) is a mutation surface and $\overline{\mathfrak{R}}_\tau(S)$ contains an irreducible representation with trivial centraliser, then the set of reducible representations and the set of irreducible representations with non–trivial centraliser are contained in subvarieties of positive codimension. In particular:

Lemma 9 *Let (S,τ) be a symmetric surface. The set of reducible representations in $\overline{\mathfrak{R}}_\tau(S)$ and the set of representations in $\overline{\mathfrak{R}}_\tau(S)$ with non–trivial centralisers are contained in a finite union of subvarieties, each of which has codimension one. Moreover, this property is preserved under $\overline{\mathfrak{t}}$.*

Proof The subvariety of reducible representations has codimension one since the proof of Lemma 6 (Lemma 2.1.3 in [9]) applies again. The set of irreducible representations with non–trivial centralisers are contained in a union of subvarieties each of which is defined by stating that two coordinates are equal to zero. Each of these subvarieties is easily observed to have codimension at least one in $\overline{\mathfrak{R}}_\tau(S)$ for each of the symmetric surfaces. □

We can now define $\overline{\mathfrak{M}}(M)$ to be the union of the irreducible components of $\overline{\mathfrak{X}}_\tau(M)$ which contain the character of an S–irreducible representation such that the image of $\operatorname{im}(\pi_1(S) \to \pi_1(M))$ has trivial centraliser.

2.3 Extension lemma

Let A be a finitely generated group and $\varphi : A_1 \to A_2$ be an isomorphism between finitely generated subgroups of A. Define

$$\overline{\mathfrak{R}}_\varphi(A) := \{\overline{\rho} \in \overline{\mathfrak{R}}(A) \mid \chi_{\overline{\rho}}|_{A_1} = \chi_{\overline{\rho}\varphi}|_{A_1}\},$$

and $\overline{\mathfrak{t}}(\overline{\mathfrak{R}}_\varphi(A)) = \overline{\mathfrak{X}}_\varphi(A)$. Let $\Gamma = \langle A, k \mid k^{-1}ak = \varphi(a)\,\forall a \in A_1\rangle$ be a HNN–extension of A. Assume that $\overline{\rho} \in \overline{\mathfrak{R}}_\varphi(A)$ has the property that $\overline{\rho}|_{A_1}$ is irreducible with trivial centraliser. Then there exists a unique $\overline{\rho}' \in \overline{\mathfrak{R}}(\Gamma)$ such that $\overline{\rho}'|_A = \overline{\rho}$: the assignment $\overline{\rho}'(k)$ is the unique element of $PSL_2(\mathbf{C})$ which conjugates $\overline{\rho}$ to $\overline{\rho}\varphi$.

Lemma 10 *Let Γ and $\overline{\mathfrak{X}}_\varphi(A)$ be as defined above. Let V be an irreducible component of $\overline{\mathfrak{X}}(\Gamma)$ containing the character of a representation which restricted to A_1 is irreducible and has trivial centraliser. Then the restriction map $r : \overline{\mathfrak{X}}(\Gamma) \to \overline{\mathfrak{X}}_\varphi(A)$ is a birational equivalence between V and $\overline{r(V)}$.*

Proof The restriction map is a polynomial map, and hence $W := \overline{r(V)}$ is an irreducible component of $\overline{\mathfrak{X}}_\varphi(A)$. If follows from Lemma 8, Lemma 4.1 of [1] and the fact that irreducible representations with the same character are equivalent, that the above construction of $PSL_2(\mathbb{C})$–representations of Γ from $PSL_2(\mathbb{C})$–representations of A is a well–defined 1–1 correspondence of $PSL_2(\mathbb{C})$–characters in V and W apart from a subvariety of codimension at least one. Thus, r has degree one and is therefore a birational isomorphism onto its image. □

2.4 Proofs of the main results

Proof of Proposition 2 The following construction is taken from [2]. Given a separating mutation surface (S, τ), we obtain a decomposition

$$\pi_1(M) \cong \pi_1(M_-) \star_{\pi_1(S)} \pi_1(M_+).$$

The varieties $\mathfrak{R}(M)$ and $\mathfrak{R}(M^\tau)$ can be viewed as a subsets of $\mathfrak{R}(M_-) \times \mathfrak{R}(M_+)$, and the inclusion map is the restriction to the respective subgroups. Let $\rho \in \mathfrak{R}_\tau(M)$ be an S–irreducible representation. Since $\rho_-\tau$ is equivalent to ρ_- on $\pi_1(S)$, there is an element $X \in SL_2(\mathbb{C})$ such that $\rho_- = X^{-1}\rho_-\tau X$ on $\pi_1(S)$, and X is defined up to sign. We can now define a representation ρ^τ of M^τ as follows: Let $\rho^\tau_+ = \rho_+$ on $\pi_1(M_+)$ and $\rho^\tau_- = X^{-1}\rho_-\tau X$ on $\pi_1(M_-)$. Then $\rho^\tau = (\rho^\tau_-, \rho_+) \in \mathfrak{R}(M^\tau)$ is well defined, since both definitions agree on the amalgamating subgroup, and the map $\rho \to \rho^\tau$ only depends upon the inner automorphism induced by X. Both ρ and ρ^τ are irreducible and $\rho^\tau \in \mathfrak{R}_\tau(M^\tau)$. It is shown in [9] that this construction yields an isomorphism $\mu : \mathfrak{M}(M) \to \mathfrak{M}(M^\tau)$ defined everywhere apart from the subvariety $\mathfrak{F}^i(M)$ of characters of irreducible representations which are reducible on $\pi_1(S)$. Moreover, it is shown on pages 567-568 of [9] that μ is a birational equivalence between irreducible components (since they contain a S–irreducible character). □

Proof of Proposition 1 Assume that S is separating. The previous construction of representations also works for projective representations with trivial centraliser, and the argument in the above mentioned proof goes through if one uses Lemma 4.1 of [1] instead of Proposition 1.1.1 of [3].

Thus, let S be a non–separating mutation surface. The boundary of $M - S$ contains two copies S_+ and S_- of S. Let $A = \text{im}(\pi_1(M - S) \to \pi_1(M))$, $A_1 = \text{im}(\pi_1(S_+) \to \pi_1(M)) \le A$ and $A_2 = \text{im}(\pi_1(S_-) \to \pi_1(M)) \le A$. Then $\pi_1(M)$ is an HNN–extension of A by some $k \in \pi_1(M)$ across A_1 and A_2:

$$\pi_1(M) = \langle A, k \mid k^{-1}A_1k = A_2\rangle.$$

The action of k is determined by the gluing map $S_+ \to S_-$, and the mutation changes the gluing map by τ. We thus obtain a presentation of $\pi_1(M^\tau)$:

$$\pi_1(M^\tau) = \langle A, k \mid k^{-1}\tau(A_1)k = A_2 \rangle.$$

Let $\overline{\rho}$ be a $PSL_2(\mathbb{C})$–representation of M. Note that $\overline{\rho}(k)$ is only determined up to the centraliser of $\overline{\rho}(A_1)$. Assume that $\overline{\rho}$ is tentatively mutable and $\overline{\rho}(A_1)$ is irreducible and has trivial centraliser. Then $\overline{\rho}(k)$ is uniquely determined by $\overline{\rho}(A_1)$ and the gluing map. Furthermore, $\overline{\rho}(a)$ is conjugate to $\overline{\rho}\tau(a)$ via some uniquely determined $X \in PSL_2(\mathbb{C})$ for all $a \in A_1$. It follows that $\overline{\rho}(k)^{-1}\overline{\rho}(a_1)\overline{\rho}(k) = \overline{\rho}(a_2)$ is equivalent to $\overline{\rho}(k)^{-1}X^{-1}\overline{\rho}\tau(a_1)X\overline{\rho}(k) = \overline{\rho}(a_2)$. Define a representation $\overline{\rho}^\tau \in \overline{\mathfrak{R}}(M^\tau)$ by $\overline{\rho}^\tau(a) := \overline{\rho}(a)$ for all $a \in A$, and $\overline{\rho}^\tau(k) = X\overline{\rho}(k)$. Denote the corresponding map by $\overline{\mu}$. Since we can define an inverse map, we have a natural 1–1 correspondence of A_1–irreducible representations (with non–trivial centraliser on A_1) in $\overline{\mathfrak{R}}_\tau(M)$ and $\overline{\mathfrak{R}}_\tau(M^\tau)$. Moreover, this map is well–defined for equivalence classes of representations, and hence for the corresponding characters in $\overline{\mathfrak{M}}(M)$ and $\overline{\mathfrak{M}}(M^\tau)$.

Let C be an irreducible component of $\overline{\mathfrak{M}}(M)$, i.e. a component of $\overline{\mathfrak{X}}_\tau(M)$ which contains the character of a S–irreducible $PSL_2(\mathbb{C})$–representation such that the image of $\pi_1(S)$ has trivial centraliser. By definition, the restriction maps $r : \overline{\mathfrak{M}}(M) \to \overline{\mathfrak{X}}_\varphi(A)$ and $r^\tau : \overline{\mathfrak{M}}(M^\tau) \to \overline{\mathfrak{X}}_{\varphi\tau}(A)$ have range in a subvariety of $\overline{\mathfrak{X}}_\varphi(A) \cap \overline{\mathfrak{X}}_{\varphi\tau}(A)$. The construction of $\overline{\mu}$ gives $r(\chi) = r^\tau(\overline{\mu}\chi)$, whenever applicable. Since $\overline{\mu}$ is defined on a dense subset of C, Lemma 10 implies that it is the composition $(r^\tau)^{-1} \circ r$. □

Proof of Corollary 4 Assume that M is a finite volume hyperbolic 3–manifold and S is a separating symmetric surface and either T_2 or G_2. If $\mathfrak{X}_0(M) \subseteq \mathfrak{X}_\tau(M)$, then $\overline{\mathfrak{X}}_0(M) \subseteq \overline{\mathfrak{X}}_\tau(M)$, since $\mathrm{q}(\mathfrak{X}_0) = \overline{\mathfrak{X}}_0$ and Lemma 7 applies. The two boundary components of any separating incompressible T_2 have to lie on the same boundary component of M, hence Lemma 6 implies that $\mathfrak{X}_\tau(M) = \mathfrak{X}(M)$. Since ρ_0 is torsion free and S–irreducible, both $\overline{\mathfrak{X}}_0(M)$ and $\mathfrak{X}_0(M)$ satisfy the hypotheses of Propositions 1 and 2 (where applicable). It follows from [5] that the birational equivalence takes the complete representation of M to the complete representation of M^τ, and hence it restricts to a birational equivalence between the two Dehn surgery components. □

Proof of Corollary 5 Since $S = T_1$ or $S = G_2$, we have $\mathfrak{X}_0(M) \subseteq \mathfrak{X}_\tau(M)$ and $\mathfrak{X}_0(M^\tau) \subseteq \mathfrak{X}_\tau(M^\tau)$. The same arguments as in the proof of Corollary 4 now yield the conclusion. □

Remark The proofs of Propositions 1 and 2 show that we have birational equivalences $\mu : \mathfrak{M}(M) \to \mathfrak{M}(M^\tau)$ and $\overline{\mu} : \overline{\mathfrak{M}}(M) \to \overline{\mathfrak{M}}(M^\tau)$. Since every knot group abelianises to $\mathbb{Z}$, this in particular implies:

Proposition 11 *Let $\mathfrak{k}$ and $\mathfrak{k}^\tau$ be Conway mutant knots. If every component of $\mathfrak{X}(\mathfrak{k})$ and $\mathfrak{X}(\mathfrak{k}^\tau)$ which contains the character of an irreducible representation contains the character of a S–irreducible representation, then $\mathfrak{X}(\mathfrak{k})$ and $\mathfrak{X}(\mathfrak{k}^\tau)$ are birationally equivalent.*

3 Surfaces and ideal points

We build on the construction by Culler and Shalen [7, 1] to give a method to determine whether a connected essential surface is associated to an ideal point. This method is then applied to two pairs of mutative manifolds in conjunction with the respective birational equivalences.

3.1 Surface associated to the action

Let M be an orientable, irreducible 3–manifold, and assume that $\mathcal{T}_v$ is Serre's tree associated to an ideal point ξ of a curve C in $\mathfrak{X}(M)$ or $\overline{\mathfrak{X}}(M)$. A surface *associated to the action* of $\pi_1(M)$ on $\mathcal{T}_v$ is defined by Culler and Shalen using a construction due to Stallings. If the given manifold is not compact, replace it by a compact core. Choose a triangulation of M and give the universal cover $\tilde{M}$ the induced triangulation. One can then construct a simplicial, $\pi_1(M)$–equivariant map f from $\tilde{M}$ to $\mathcal{T}_v$. The inverse image of midpoints of edges is a surface in $\tilde{M}$ which descends to a non–empty, 2–sided surface S in M. The map f is changed by a homotopy (if necessary) so that S is incompressible and has no boundary parallel or sphere components. We then say that S is *essential*. The associated surface S depends upon the choice of triangulation of M and the choice of the map f. An associated surface often contains finitely many parallel copies of one of its components. They are somewhat redundant, and we implicitly discard them, whilst we still call the resulting surface associated.

3.2 Surface detected by an ideal point

We now describe associated surfaces satisfying certain non–triviality conditions. An essential surface S in M gives rise to a graph of groups decomposition of

$\pi_1(M)$. Let $t_1, ..., t_k$ be the generators of the fundamental group of the graph of groups arising from HNN–extensions. Let $M_1, ..., M_m$ be the components of $M - S$, let $\mathcal{T}_S$ be the dual tree to $\tilde{S}$ in $\tilde{M}$ and $\mathcal{G}_S$ be the dual graph to S in M. For each component M_i of $M - S$, fix a representative Γ_i of the conjugacy class of $\text{im}(\pi_1(M_i) \to \pi_1(M))$ as follows. Let $\mathcal{T}' \subset \mathcal{T}_S$ be a tree of representatives, i.e. a lift of a maximal tree in $\mathcal{G}_S$, and let $\{s_1, \ldots, s_m\}$ be the vertices of $\mathcal{T}'$, labelled such that s_i maps to M_i under the composite mapping $\mathcal{T}_S \to \mathcal{G}_S \to M$. Then let Γ_i be the stabiliser of s_i.

Assume that S does not contain parallel copies of one of its components. Then S is *detected by an ideal point* of the character variety with Serre tree $\mathcal{T}_v$ if

- **S1** every vertex stabiliser of the action on $\mathcal{T}_S$ is included in a vertex stabiliser of the action on $\mathcal{T}_v$,
- **S2** every edge stabiliser of the action on $\mathcal{T}_S$ is included in an edge stabiliser of the action on $\mathcal{T}_v$,
- **S3** if M_i and M_j, where $i \neq j$, are identified along a component of S, then there are elements $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ such that $\gamma_i\gamma_j$ acts as a loxodromic on $\mathcal{T}_v$,
- **S4** each of the generators t_i can be chosen to act as a loxodromic on $\mathcal{T}_v$.

Lemma 12 *Let M be an orientable, irreducible 3–manifold. An essential surface S in M which is detected by an ideal point ξ of a curve C in $\mathfrak{X}(M)$ is associated to the action of $\pi_1(M)$ on the Serre tree $\mathcal{T}_v$.*

Proof Choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S, and such that the intersection of any edge in the triangulation with S consists of at most one point. Give $\tilde{M}$ the induced triangulation. We may assume that the retraction $\tilde{M} \to \mathcal{T}_S$ is simplicial, and we now define a map $\mathcal{T}_S \to \mathcal{T}_v$.

The vertices $\{s_1, \ldots, s_m\}$ of the tree of representatives are a complete set of orbit representatives for the action of $\pi_1(M)$ on the 0–skeleton of $\mathcal{T}_S$. Condition S3 implies that we may choose vertices $\{v_1, \ldots, v_m\}$ of $\mathcal{T}_v$ such that v_i is stabilised by Γ_i, and if $M_i \neq M_j$, then $v_i \neq v_j$. Define a map f^0 between the 0–skeleta of $\mathcal{T}_S$ and $\mathcal{T}_v$ as follows. Let $f^0(s_i) = v_i$. For each other vertex s of $\mathcal{T}_S$ there exists $\gamma \in \pi_1(M)$ such that $\gamma s_i = s$ for some i. Then let $f^0(s) = \gamma f^0(s_i)$. We thus obtain a $\pi_1(M)$–equivariant map from $\mathcal{T}_S^0 \to \mathcal{T}_v^0$, which extends uniquely to a map $f^1 : \mathcal{T}_S \to \mathcal{T}_v$, since the image of each edge is determined by the images of its endpoints. Since $v_i \neq v_j$ for $i \neq j$, and since each t_k acts as a loxodromic on $\mathcal{T}_v$, the image of each edge of $\mathcal{T}_S$ is a path of length greater or equal to one in $\mathcal{T}_v$.

If f^1 is not simplicial, then there is a subdivison of $\mathcal{T}_S$ giving a tree $\mathcal{T}_{S'}$ and a $\pi_1(M)$–equivariant, simplicial map $f : \mathcal{T}_{S'} \to \mathcal{T}_v$. There is a surface S' in M which is obtained from S by adding parallel copies of components such that $\mathcal{T}_{S'}$ is the dual tree of $\tilde{S}'$.

As before, choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S', and such that the intersection of any edge in the triangulation with S' consists of at most one point, and give $\tilde{M}$ the induced triangulation. The composite map $\tilde{M} \to \mathcal{T}_{S'} \to \mathcal{T}_v$ is $\pi_1(M)$–equivariant and simplicial, and the inverse image of midpoints of edges descends to the surface S' in M. Thus, S' is associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$. □

We now wish to decide whether a given essential surface S in M is detected by an ideal point of a curve in $\mathfrak{X}(M)$. Denote the components of $M - S$ by $M_1, \ldots, M_m$. If S is detected by an ideal point, then the limiting character restricted to each M_i is finite. There is a natural map from $\mathfrak{X}(M)$ to $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_m)$ by restricting to the respective subgroups. Splittings along S which are detected by ideal points of curves in $\mathfrak{X}(M)$ correspond to points $(\chi_1, \ldots, \chi_m)$ in the cartesian product satisfying the following necessary conditions:

- **C1** $\chi_i \in \mathfrak{X}(M_i)$ is finite for each $i = 1, \ldots, m$.
- **C2** For each component of S, let $\varphi : S^+ \to S^-$ be the gluing map between its two copies arising from the splitting, and assume that $S^+ \subset \partial M_i$ and $S^- \subset \partial M_j$, where i and j are not necessarily distinct. Denote the homomorphism induced by φ on fundamental group by φ_*. Then for each $\gamma \in \text{im}(\pi_1(S^+) \to \pi_1(M_i))$, $\chi_i(\gamma) = \chi_j(\varphi_* \gamma)$.
- **C3** For each $i = 1, \ldots, m$, the restriction of χ_i to any component of S in ∂M_i is reducible.
- **C4** There is an ideal point ξ of a curve C in $\mathfrak{X}(M)$ and a connected open neighbourhood U of ξ on C such that the image of U under the map to the cartesian product contains an open neighbourhood of $(\chi_1, \ldots, \chi_m)$ on a curve in $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_m)$, but not $(\chi_1, \ldots, \chi_m)$ itself.

The first condition implies that $\text{im}(\pi_1(M_i) \to \pi_1(M))$ is contained in a vertex stabiliser for each $i = 1, \ldots, m$. The second defines a subvariety of the cartesian product containing the image of $\mathfrak{X}(M)$ under the restriction map. Condition C3 must be satisfied since it is shown in [7] that the limiting representation of every component of an associated surface is reducible. The last condition implies that the action of $\pi_1(M)$ on Serre's tree is non–trivial.

Lemma 13 *Let S be a connected essential surface in an orientable, irreducible 3-manifold M. Then S is associated to an ideal point of the character variety of M if and only if there are points in the cartesian product of the character varieties of the components of $M - S$ satisfying conditions C1–C4.*

Proof We need to show that the conditions are sufficient. Assume that S is non–separating. Let $A = \operatorname{im}(\pi_1(M - S) \to \pi_1(M))$, and denote the subgroups of A corresponding to the two copies of S in $\partial(M - S)$ by A_1 and A_2. Then $\pi_1(M)$ is an HNN–extension of A by some $t \in \pi_1(M)$ across A_1 and A_2. We may assume that $t^{-1}A_1t = A_2$.

Let ξ be the ideal point provided by C4, and denote Serre's tree associated to ξ by $\mathcal{T}_v$. C1 implies that the subgroup A stabilises a vertex Λ of $\mathcal{T}_v$, and hence condition S1 is satisfied.

Note that A is finitely generated. Condition C4 yields that the action of $\pi_1(M)$ on $\mathcal{T}_v$ is non–trivial, and Corollary 2 in Section I.6.5 of [6] implies that either t is loxodromic with respect to the action on $\mathcal{T}_v$ or there is $a \in A$ such that ta or at is loxodromic. In the first case, we keep A_1 and A_2 as they are; in the second case, we replace t by ta and A_2 by $a^{-1}A_2a$; and in the third case, we replace t by at and A_1 by aA_1a^{-1}. Thus, t satisfies condition S4.

Since A stabilises Λ, $t^{-1}At$ stabilises $t^{-1}\Lambda$, and since t acts as a loxodromic, we have $t^{-1}\Lambda \neq \Lambda$. In particular, A_2 fixes these two distinct vertices, and hence the path $[\Lambda, t^{-1}\Lambda]$ pointwise, which implies that it is contained in an edge stabiliser. Thus, condition S2 is satisfied, and the lemma is proven in the case where S is connected, essential and non–separating, since condition S3 does not apply.

The proof for the separating case is similar, and will therefore be omitted. □

The conditions are not sufficient when S has more than one component, since condition C4 does not rule out the possibility that the limiting character is finite on all components of $M - S'$ for a proper subsurface S' of S.

3.3 The Kinoshita–Terasaka knot

Let M and M^τ denote the complements of the Kinoshita–Terasaka knot and its Conway mutant respectively, and S the corresponding Conway sphere. In [9], the S–reducible–non–abelian representations in $\mathfrak{R}(M)$ and $\mathfrak{R}(M^\tau)$ are computed up to conjugacy, and a comparison thereof leads to the conclusion that a closed essential surface in M is associated to an ideal point of $\mathfrak{X}(M)$.

Lemma 13 together with the calculations in [9] implies that the Conway sphere as well as any surface obtained by joining boundary components of the sphere with annuli is a surface associated to the ideal points of $\mathfrak{X}(M)$ at which the holes in the eigenvalue variety occur. Two detected genus two surfaces and their involutions are shown in Figures 3(a) and 3(b) in [9].

3.4 The figure eight knot

The complement M of the figure eight knot $\mathfrak{k}$ in S^3 is a once–punctured torus bundle with fibre a Seifert surface of the knot. Mutation along this surface results in the so–called sister manifold. The mutation is detected by the first homology group, but also by the $SL_2(\mathbb{C})$–Dehn surgery components. We verify that the $PSL_2(\mathbb{C})$–Dehn surgery components are birationally equivalent, and use the mutation map to show that the fibres in both manifolds are detected by ideal points. This method may have non–trivial applications.

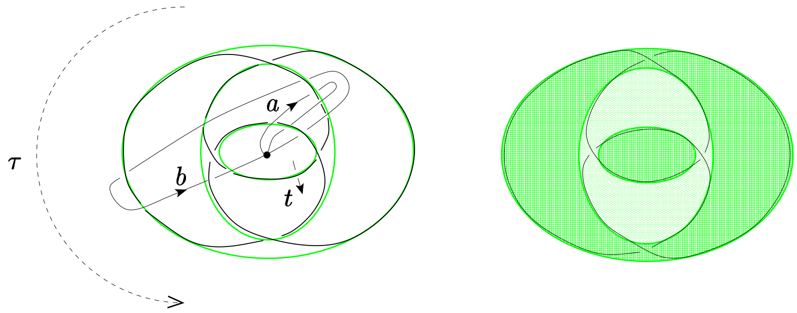

Figure 2: Mutation along the Seifert surface

A Seifert surface T_1 is shown in Figure 2. A base point and generators are chosen such that $\tau(a) = a^{-1}$, $\tau(b) = ab^{-1}a^{-1}$, and we compute the presentation $\Gamma = \langle t, a, b \mid t^{-1}at = aba, t^{-1}bt = ba\rangle$ for $\pi_1(M)$. The action of t corresponds to the isomorphism Φ induced by the monodromy of the fibre bundle. The isomorphism for the mutative manifold M^τ is $\Phi^\tau(a) := \tau(\Phi(a)) = b^{-1}a^{-2}$ and $\Phi^\tau(b) := \tau(\Phi(b)) = ab^{-1}a^{-2}$, which yields a presentation Γ^τ for $\pi_1(M^\tau)$. Both presentations can be simplified to:

$$\Gamma = \langle t, a \mid t^{-1}a^{-1}t^{-1}ata^{-2}ta = 1\rangle \quad \text{and} \quad \Gamma^\tau = \langle t, a \mid t^{-1}ata^2tat^{-1}a = 1\rangle.$$

Note that $H_1(M) \cong \mathbb{Z}$ and $H_1(M^\tau) \cong \mathbb{Z}_5 \oplus \mathbb{Z}$. Let $x = \operatorname{tr}\rho(t)$ and $y = \operatorname{tr}\rho(a)$. A computation reveals $\mathfrak{X}^r(M) = \{(x, y) \in \mathbb{C}^2 \mid 0 = (2 - y)\}$ and $\mathfrak{X}^r(M^\tau) = \{(x, y) \in \mathbb{C}^2 \mid 0 = (2 - y)(1 - y - y^2)\}$.

It turns out that the character varieties have only one component containing the character of an irreducible representation:

$$\mathfrak{X}_0(M) = \{(x,y) \in \mathbb{C}^2 \mid 0 = 1 - y - y^2 + (-1+y)x^2\},$$
$$\mathfrak{X}_0(M^\tau) = \{(x,y) \in \mathbb{C}^2 \mid 0 = 1 + (-1+y)x^2\}.$$

The curve $\mathfrak{X}_0(M)$ has no singularities and no singularities at infinity. Its smooth projective completion is therefore a torus. The curve $\mathfrak{X}_0(M^\tau)$ is rational, and a smooth projective model is hence a sphere.

Each $PSL_2(\mathbb{C})$–representation lifts to $SL_2(\mathbb{C})$ for each example, and the quotient map is given by $\mathrm{q}(x,y) = (x^2, y)$. Thus:

$$\overline{\mathfrak{X}}(M) = \{(X,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2+(-1+y)X)\}$$
$$\overline{\mathfrak{X}}(M^\tau) = \{(X,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2)(1+(-1+y)X)\}$$

The rational maps between the Dehn surgery components induced by mutation show that $\overline{\mathfrak{X}}_0(M)$ and $\overline{\mathfrak{X}}_0(M^\tau)$ are in fact homeomorphic:

$$\overline{\mu} : \overline{\mathfrak{X}}_0(M) \to \overline{\mathfrak{X}}_0(M^\tau) \qquad (X,y) \to \left(\frac{1}{1-y}, y\right),$$
$$\overline{\mu}^{-1} : \overline{\mathfrak{X}}_0(M^\tau) \to \overline{\mathfrak{X}}_0(M) \qquad (X,y) \to \left(\frac{1-y-y^2}{1-y}, y\right).$$

The surfaces detected by the Dehn surgery components do not include the fibre, but one can recover curves of reducible representations as follows. There are only three points on each of the projective Dehn surgery components on which $\overline{\mu}$ and $\overline{\mu}^{-1}$ are not defined a priori, and they correspond to the intersection with $\{(2-y)(1-y-y^2) = 0\}$. The corresponding representations of M and M^τ are T_1–abelian and satisfy $\tau(\rho(\gamma)) = \rho(\gamma)^{-1}$. For each we can find a 1–parameter family of elements in $PSL_2(\mathbb{C})$ which realise the action of τ. Consider the following lift to $SL_2(\mathbb{C})$ of an irreducible $PSL_2(\mathbb{C})$–representation of M:

$$\rho(t) = \begin{pmatrix} i & 1 \\ 0 & i^{-1} \end{pmatrix} \quad \text{and} \quad \rho(a) = \begin{pmatrix} u & 0 \\ i(u^{-1} - u) & u^{-1} \end{pmatrix}$$

subject to $0 = 1 + u + u^2 + u^3 + u^4$. These are dihedral representations, and they are abelian on the fibre. Elements realising the involution are:

$$H(z) = \begin{pmatrix} iz & z \\ z - z^{-1} & i^{-1}z \end{pmatrix} \quad \text{for any } z \in \mathbb{C} - \{0\},$$

and we obtain the following representations $\rho_z \in \mathfrak{R}(M^\tau)$:

$$\rho_z(t) = H(z)\rho(t) = \begin{pmatrix} -z & 0 \\ i(z - z^{-1}) & -z^{-1} \end{pmatrix} \text{ and } \rho_z(a) = \rho(a).$$

These representations are abelian. The construction yields a map $\mathbb{C} - \{0\} \to \overline{\mathfrak{X}}(M^\tau)$, which is non–constant since $(\operatorname{tr} \rho_z(t))^2 = (z + z^{-1})^2$, and the image is a curve in $\overline{\mathfrak{X}}(M^\tau)$. At an ideal point of this curve, the conditions of Lemma 13 are satisfied with respect to the fibre in M^τ. One can do a similar construction for the other points in $\overline{\mathfrak{X}}^i(M) \cap \overline{\mathfrak{F}}(M)$.

Using characters in $\overline{\mathfrak{X}}^i(M^\tau) \cap \overline{\mathfrak{F}}(M^\tau)$, one only obtains a curve in $\overline{\mathfrak{X}}(M)$ for the point corresponding to the intersection with $\{y = 2\}$, whilst the points in the intersection with $\{1 = y + y^2\}$ yield a map $\mathbb{C} \to \overline{\mathfrak{X}}(M)$ which is constant.

References

[1] S. Boyer, X. Zhang: *On Culler–Shalen seminorms and Dehn filling*, Ann. Math., 148, 737-801 (1998).

[2] D. Cooper, D.D. Long: *Remarks on the A-polynomial of a Knot*, J. Knot Theory Ramifications, 5, 609-628 (1996).

[3] M. Culler, P.B. Shalen: *Varieties of group representations and splittings of 3-manifolds*, Ann. Math., 117, 109-146 (1983).

[4] W. Goldman: *Topological components of spaces of representations*, Invent. Math., 93, 557-607 (1988).

[5] D. Ruberman: *Mutation and volumes of knots in S^3*, Invent. Math., 90, 189-215 (1987).

[6] J.–P. Serre: *Trees*, Springer, Berlin, 1980.

[7] P.B. Shalen: *Representations of 3–manifold groups*, Handbook of geometric topology, 955-1044, Amsterdam, 2002.

[8] W.P. Thurston: *The geometry and topology of 3–manifolds*, Princeton Univ. Math. Dept. (1978). `http://msri.org/publications/books/gt3m/`

[9] S. Tillmann: *On the Kinoshita–Terasaka mutants and generalised Conway mutation*, J. Knot Theory Ramifications, 9, 557-575 (2000).

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