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On several varieties of cacti and their relations

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Abstract Motivated by string topology and the arc operad, we introduce the notion of quasi-operads and consider four (quasi)-operads which are different varieties of the operad of cacti. These are cacti without local zeros (or spines) and cacti proper as well as both varieties with fixed constant size one of the constituting loops. Using the recognition principle of Fiedorowicz, we prove that spineless cacti are equivalent as operads to the little discs operad. It turns out that in terms of spineless cacti Cohen's Gerstenhaber structure and Fiedorowicz' braided operad structure are given by the same explicit chains. We also prove that spineless cacti and cacti are homotopy equivalent to their normalized versions as quasi-operads by showing that both types of cacti are semi-direct products of the quasi-operad of their normalized versions with a re-scaling operad based on $\mathbb{R}_{>0}$. Furthermore, we introduce the notion of bi-crossed products of quasi-operads and show that the cacti proper are a bi-crossed product of the operad of cacti without spines and the operad based on the monoid given by the circle group S^1. We also prove that this particular bi-crossed operad product is homotopy equivalent to the semi-direct product of the spineless cacti with the group S^1. This implies that cacti are equivalent to the framed little discs operad. These results lead to new CW models for the little discs and the framed little discs operad.

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Introduction

The cacti operad was introduced by Voronov [V] descriptively as treelike configurations of circles in the plane to give an operadic interpretation of the string bracket and Batalin-Vilkovisky (BV) structure found by Chas and Sullivan [CS] on the loop space of a compact manifold. The key tool connecting the two is an "Umkehr" map in homology by using a Thom-Pontrjagin construction [V, CJ]. Studying combinatorial models of the moduli space of bordered surfaces, we constructed the $\mathcal{A}rc$ operad, which is an operad built on surfaces with arcs, and showed that this operad naturally carries compatible structures of a Gerstenhaber (G) and a BV algebra up to homotopy on the chain level [KLP]. Moreover the structures above were given by explicit generators for the operations and explicit homotopies for the relations. The structure of these generators and relations for the BV operations bear formal resemblance to those of the cacti operad. With the help of an additional analysis, we were indeed able to give a map of the operad of cacti into the $\mathcal{A}rc$ which embeds the former as a suboperad (up to an overall re-scaling) [KLP] and embeds $\mathcal{C}acti$ into the operad $\mathcal{D}\mathcal{A}rc = \mathcal{A}rc \times \mathbb{R}_{>0}$. This result defined the topology of $\mathcal{C}acti$ in terms of metric ribbon graphs. A brief review of this construction of [KLP] is contained in Appendix B. The operations defining the Gerstenhaber bracket lie in a very naturally defined smaller suboperad of $\mathcal{A}rc$ than the one corresponding to cacti. Moreover, adding a generator to this suboperad which is the element that becomes the BV operator we obtain the suboperad corresponding to cacti. The construction of these spaces and the relation between them is also briefly reviewed in Appendix B. Going back to string-topology, one can expect to be able to find a suboperad of cacti responsible for the G-bracket. This is the operad of spineless cacti.

Setting these observation in relation to the theorems of F. Cohen [C1, C2] and E. Getzler [Ge] —which state that Gerstenhaber algebras coincide with algebras over the homology of the little discs operad and BV algebras coincide with algebras over the framed little discs operad— leads us to a comparison of the operad of cacti and the suboperad mentioned above to the framed little discs operad and its suboperad of little discs. The equivalence of spineless cacti and the little discs operad is one of the main points of this paper. Furthermore, we also give a proof for the Theorem announced by Voronov [V, SV] that cacti and the framed little discs are equivalent.

One further striking fact about the explicit chain homotopies used in [KLP] is that they have one more restriction in common, that of a normalization. This property is, however, not stable under composition.

The considerations above prompt us to define several different species of cacti and to study their relations to each other and their relation to the little discs and the framed little discs operad. These different species consist of the original cacti, cacti without additionally marked local zeros which we call cacti without spines and lastly for both versions their normalized counterparts, which are made up of circles of radius one. It is actually a little surprising that one can go through the whole theory with normalized cacti. For the normalized versions the gluing rules are slightly different though and are only associative up to homotopy.

To systematically treat these objects, we introduce the notion of quasi-operads and define direct, semi-direct and bi-crossed products of quasi-operads. In this setting, the normalized versions of cacti and spineless cacti are homotopy associative quasi-operads, so that their homology quasi-operads are in fact operads.

In order to define the topological spaces underlying spineless cacti and cacti, we use a reformulation of the original approach of [V] in terms of graphs and trees [K1]. In this setting, the spaces for normalized spineless cacti are constructed as CW-complexes whose cells are indexed by trees. The underlying spaces for the other versions of cacti are then in turn given as products of these CW complexes with circles and lines. Moreover the quasi-operad structure of normalized spineless cacti induced on the level of cellular chains is already associative and provides an operad structure [K1].

As to the relation of (spineless) cacti and their normalized versions, the exact statement is that the non-normalized versions of cacti are isomorphic as operads to the quasi-operadic semi-direct product of the normalized version with a scaling operad. The scaling operad is defined on the spaces $\mathbb{R}^n_{>0}$ and controls the radii of the circles. We also show that this semi-direct product is homotopic to the direct product as quasi-operads. This makes the normalized and non-normalized versions of (spineless) cacti homotopy equivalent as spaces, but furthermore the products are also compatible up to homotopy, so that the two versions are equivalent as quasi-operads.

One main result we prove is that the spineless cacti are equivalent in the sense of [F2] to the little discs operad using the recognition principle of Fiedorowicz [F2]. For the proof, we take up the idea of [SV] to use the map contracting the $n+1$-st lobe of a cactus with $n+1$ lobes. We analyze this map further and prove that although it is not a fibration that it is a quasi-fibration. This is done using the Dold-Thom criterium [DT].

In this way, the cellular chains of normalized spineless cacti provide a model of the chains of the little discs operad. This fact together with indexing of the chains by trees is the basis for a natural topological solution to Deligne's conjecture on the Hochschild cohomology of an associative algebra [K1]. Furthermore, in the same spirit the cellular chains of normalized cacti form an operad which is a model for the framed little discs which can in turn be used to prove a cyclic version of Deligne's conjecture. The content of this Theorem [K2] is that there is a cell model of the framed little discs which acts on the Hochschild complex of a Frobenius algebra.

The theorem about the equivalence of spineless cacti and the little discs operad and its proof also nicely tie together the results of [C1, C2] and [F1, F2] in a geometrical setting on the chain level. By the results of F. Cohen [C1, C2] the algebras over the homology of the little discs operad are Gerstenhaber algebras and by the recognition principle of Fiedorowicz [F2] an operad is equivalent to the little discs operad if its universal cover is a contractible braided operad with a free braid group action. In our realization, the Gerstenhaber bracket is made explicit on the chain level. It is in fact given by the signed commutator of a non-commutative product $*$ which is defined by a path between point and its image under a transposition under the action of the symmetric group. This path is also the path needed to lift the symmetric group action to a braid action. Moreover, the odd Jacobi identity of the Gerstenhaber bracket is proved by a relation for the associator of $*$. In terms of the paths this equation is the same equation as the braid relation needed to ensure that the universal cover is a braided operad.

The relationship between framed little discs and little discs is that the framed little discs are a semi-direct product of the little discs with the operad built on the circle group S^1 [SW]. Actually, for this construction, which we review below, one only needs a monoid. This example is a special case of a semi-direct product of quasi-operads.

The relationship for the cacti and spineless cacti is more involved. To this end, we define the notion of bi-crossed products of quasi-operads which is an extension of the bi-crossed product of groups [Ka, Tak]. We show that cacti are a bi-crossed product of spineless cacti with an operad built on S^1. By analyzing the construction of the operads built on monoids in a symmetric tensor category where the tensor product is a product, we can relate the particular bi-crossed structure of cacti to the semi-direct product with the circle group. To be precise we show that the particular bi-crossed product giving rise to cacti is homotopy equivalent to the semi-direct product of cacti without spines and the group operad built on S^1.

The characterization above allows us to give a proof of the theorem announced by Voronov [V, SV] which states that the operad of cacti is equivalent to the framed little discs operad. This is done via the equivariant recognition principle [SW]. Vice-versa the theorem mentioned above together with the characterization of *Cacti* as a bi-crossed product imply that cacti without spines are homotopy equivalent to the little discs operad.

Along the way, we give several other pictorial realizations of the various types of cacti including trees, ribbon graphs and chord diagrams, which might be useful to relate this theory to other parts of mathematics. In particular the trees with grafting are reminiscent of the Connes-Kreimer operads [CK] and in fact as we have shown in [K1] they are intimately related, see also section 2.5 below. The chord diagram approach is close to Kontsevich's graph realization of the Chern-Simons theory (cf. e.g. [BN]) and to Goncharov's algebra of chord diagrams [Go].

Interpreting the above results inside the $\mathcal{A}rc$ operad, one obtains that the bi-crossed product corresponding to cacti is realized as the suboperad corresponding to cacti without spines and a Fenchel-Nielsen type twist.

The paper is organized as follows:

In the first section, we introduce the notion of quasi-operads and the operations of forming direct, semi-direct and bi-crossed products of quasi-operads which we need to describe the cacti operads and their relations. In section 2 we then define all the varieties of cacti we wish to consider, cacti with and without spines and normalized cacti with and without spines. In addition, we provide several pictorial realizations of these objects, which are useful for their study and relate them to other fields of mathematics. The third section contains the proof that the operad of spineless cacti is equivalent to the little discs operad. In paragraph 4, we collect examples and constructions which we generalize in paragraph 5 in order to study the relations between the various varieties of cacti. We start by introducing an operad called operad of spaces which can be defined in any symmetric tensor category with products such as topological spaces with Cartesian product. This operad lends itself to the description of the semi-direct product with a monoid whose construction we also review. In the last section, section 5 we then prove that the non-normalized versions of the (spineless) cacti operads are the semi-direct products of their normalized version and a re-scaling operad built on $\mathbb{R}^n_{>0}$. Moreover we show that this semi-direct product is homotopy equivalent to a direct product. This section also contains the result that cacti are a bi-crossed product of cacti without spines and the operad built on the group S^1. Moreover, we show that this bi-crossed product in turn is homotopic to the semi-direct product of these operads. These results are then used to give a proof that cacti are equivalent to the framed little discs.

We also provide two appendices. Appendix A is a compilation of the relevant notions of graphs and gives the interpretation of cacti as marked treelike ribbon graphs with a metric. In Appendix B, we briefly recall the $\mathcal{A}rc$ operad and the suboperads corresponding to the various cacti operads and show how to map cacti to elements of $\mathcal{A}rc$ and vice-versa. This short presentation slightly differs in style from [KLP], since we use the language of graphs of Appendix A to simplify the constructions in the situation at hand. As such it might also be useful to a reader acquainted with [KLP]. Furthermore, the $\mathcal{A}rc$ operad provides a straightforward generalization of cacti to higher genus and even allows to additionally introduce punctures.

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1 Quasi-operads and direct, semi-direct and bi-crossed products

In our analysis of the various types of cacti, we will need a structure which is slightly more relaxed than operads. In fact, in the normalized versions of cacti the compositions will fail to be associative on the nose, although they are associative up to homotopy. This leads us to define and study quasi-operads. These quasi-operads afford certain constructions such as semi-direct products and bi-crossed products which are not necessarily defined for operads. On the other hand semi-direct products and bi-crossed products of quasi-operads may yield operads. If one is mainly interested in the homology operads, it is natural to consider quasi-operads which are associative up to homotopy. Lastly, in certain cases quasi-operads can already provide operads on the chain level as our normalized spineless cacti below [K1].

1.1 Quasi-operads

We fix a strict monoidal category $\mathcal{C}$ and denote by $\mathbb{S}_n$ the symmetric group on n letters. A quasi-operad is an operad where the associativity need not hold. More precisely:

1.1.1 Definition A quasi-operad $\mathcal{C}$ is a collection of objects $\mathcal{O} := \{O(n) : O(n) \in \mathcal{C}, n \geq 1\}$ together with an $\mathbb{S}_n$ action on $O(n)$ and maps called compositions

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1), i \in \{1, \dots m\} \tag{1.1}$$

which are $\mathbb{S}_n$-equivariant: if $op_m \in O(m)$ and $op_n \in O(n)$

$$\sigma_m(op_m) \circ_i \sigma'_n(op_n) = \sigma_m \circ_i \sigma'_n(op_m \circ_{\sigma_m(i)} op_n) \tag{1.2}$$

where $\sigma_m \circ_i \sigma'_n \in \mathbb{S}_{m+n-1}$ is the permutation that the block or iterated permutation

$$\begin{aligned}(1, 2, \dots, i-1, (1', \dots, m'), i+1 \dots, n) \mapsto \\ \sigma_n(1, 2, \dots, i-1, \sigma'_m(1', \dots, m'), i+1 \dots, n)\end{aligned} \tag{1.3}$$

induces on $(1'', \dots, (m+n-1)'')$ where

$$j'' = \begin{cases} j & 1 \leq j \leq i-1 \\ j-i+1' & i \leq j \leq i+n-1 \\ j-n & i+n \leq j \leq m+n-1 \end{cases}$$

A quasi-operad is called unital if an element $id \in O(1)$ exists which satisfies for all $op_n \in O(n), i \in \{1, \dots, n\}$

$$\circ_i(op_n, id) = \circ_1(id, op_n) = op_n$$

1.1.2 Remark If a quasi-operad in the topological category is homotopy associative then its homology has the structure of an operad. In certain cases, like the ones we will consider, the structure of an operad already exists on the level of a chain model.

1.1.3 Definition A morphism of quasi-operads is a map which preserves all structures.

An isomorphism of quasi-operads is an invertible morphism of quasi-operads. This will be denoted by $\cong$.

A quasi-operad morphism $A \to B$ is said to be an equivalence if for each $k \geq 0$, $A(k) \to B(k)$ is a $\mathbb{S}_k$-equivariant homotopy equivalence. This relation will be denoted by $\simeq$.

1.1.4 Definition Two quasi-operad structures $\circ_i$ and $\circ'_i$ on a fixed collection of $\mathbb{S}_n$-spaces $O(m)$ are called homotopy equivalent (through quasi-operads) denoted by $\sim$ if there is a homotopy of maps

$$\circ_i(t) : O(m) \otimes O(n) \to O(m+n-1) \tag{1.4}$$

for $t \in [0,1]$ such that $\circ_i(0) = \circ_i, \circ_i(1) = \circ'_i$ and such that for any fixed t the $\circ_i(t)$ give the $O(n)$ the structure of a quasi-operad.

1.1.5 Remark Two homotopy equivalent quasi-operads induce isomorphic structures on the homology level.

1.1.6 Definition An operad is a quasi-operad for which associativity holds, i.e. for $op_k \in O(k), op'_l \in O(l)$ and $op''_m \in O(m)$

$$(op_k \circ_i op'_l) \circ_j op''_m = \begin{cases} (op_k \circ_j op''_m) \circ_{i+m-1} op'_l & \text{if } 1 \le j < i \\ op_k \circ_i (op'_l \circ_{j-i+1} op''_m) & \text{if } i \le j < i+l \\ (op_k \circ_{i-l+1} op'_l) \circ_j op''_m & \text{if } i+l \le j \end{cases} \tag{1.5}$$

An operad morphism is a map of collections preserving all the operad structures.

1.1.7 Remark Note that our operads correspond to the pseudo-operads of [MSS]. In case a unit exists these two notions coincide [MSS]. We drop the "pseudo" in our nomenclature in order to avoid confusion between quasi- and pseudo-operads. In a strict sense, our quasi-operads are quasi-pseudo-operads which is certainly an expression we wish to avoid.

We will use the following terminology of [F2].

1.1.8 Definition An operad morphism $A \to B$ is said to be an equivalence if for each $k \ge 0$, $A(k) \to B(k)$ is a $\mathbb{S}_k$-equivariant homotopy equivalence. We say that an operad A is E_n $(n = 1,2,3,\dots,\infty)$ if there is a chain of operad equivalences (in either or both directions) connecting A to the Boardman-Vogt little n-cubes operad C_n (cf. [BV]).

1.2 Direct products

1.2.1 Definition Given two quasi-operads $C(n)$ and $D(n)$ in the same category, we define their direct product $C \times D$ to be given by $(C \times D)(n) := C(n) \times D(n)$ with the diagonal $\mathbb{S}_n$ action, i.e. the action of $\mathbb{S}_n$ induced by the diagonal map $\mathbb{S}_n \to \mathbb{S}_n \times \mathbb{S}_n$, and the compositions

$$(c,d) \circ_{i,C\times D} (c',d') := (c \circ_{i,C} c', d \circ_{i,D} d')$$

Since the compositions are componentwise it follows that:

1.2.2 Proposition *The direct product of two operads is an operad.*

1.3 Semi-direct products

1.3.1 Definition Fix two quasi-operads $C(n)$ and $D(n)$ in the same category together with a collection of morphisms:

$$\begin{aligned} \circ_i^D : C(n) \times D(n) \times C(m) &\to C(n+m-1) \quad i = 1, \dots n \\ (c, d, c') &\mapsto \circ_i^D(c, d, c') =: c \circ_i^d c' \end{aligned} \tag{1.6}$$

which satisfy the analog of equation (1.2) for the action of $\mathbb{S}_n \times \mathbb{S}_m$ with $\mathbb{S}_n$ acting diagonally on the first two factors.

Note that we used the superscripts to indicate that we view the dependence of the map on D as a perturbation of the original quasi-operad structure.

We define the semi-direct product $C \rtimes D$ with respect to the $\circ_i^D$ to be given by the collection $(C \times D)(n) := C(n) \times D(n)$ with diagonal $\mathbb{S}_n$ action and compositions

$$\begin{aligned} \circ_i : (C \times D)(n) \times (C \times D)(m) &= C(n) \times D(n) \times C(m) \times D(m) \\ &\longrightarrow C(n+m-1) \\ (c, d) \circ_i (c', d') &= (c \circ_i^d c', d \circ_i d') \end{aligned} \tag{1.7}$$

where we use the upper index on the operations to show that we use the universal maps (1.6) with fixed middle argument.

In case we are dealing with unital (quasi) operads we will also require that $1_C \times 1_D$ is a unit in the obvious notation, and that $\circ_i^{1_D} = \circ_i$.

1.3.2 Remark In general the semi-direct product of two operads need not be an operad. This depends on the choice of the $\circ_i^D$. Of course the direct product of two operads is a semi-direct product of quasi-operads which is an operad. If the quasi-operad D fails to be associative, then any semi-direct product $C \rtimes D$ will fail to be an operad. However if D is an operad and C is just a quasi-operad, it is possible that there are maps $\circ_i^D$ s.t. $C \rtimes D$ will be an operad.

1.3.3 Definition We call a quasi-operad C *normal* with respect to an operad D and maps of the type (1.6) if the semi-direct product $C \rtimes D$ with respect to these maps yields an operad, i.e. is associative.

1.3.4 Examples Examples of this structure are usually derived if the operad D acts on the quasi-operad C and the twisted operations are defined by first applying this action. The semi-direct product of an operad with a monoid [SW] is such an example (see below) as are the semi-direct products of [KLP] and the (spineless) cacti with respect to their normalized versions (see below).

A sometimes useful criterion is:

1.3.5 Lemma *Consider two operads C and D with left actions of $D(n)$ on $C(m)$,*

$$\rho_i : D(n) \times C(m) \to C(m) \quad i \in \{1, \dots, n\}$$

s.t. the maps

$$\begin{array}{rcl} \circ_i^D : C(n) \times D(n) \times C(m) & \to & C(m) : \\ (c, d, c') & \mapsto & c \circ_i \rho_i(d) c' \end{array}$$

are $\mathbb{S}_n \times \mathbb{S}_m$ equivariant in the sense of 1.1.1 with $\mathbb{S}_n$ acting diagonally on the first two factors and for $c' \in C(l), i \leq j < i + l$

$$\rho_i(d)(c') \circ_{j-i+1} (\rho_j(d \circ_i d')c'') = \rho_i(d)[c' \circ_{j-i+1} \rho_{j-i-1}(c'')] \tag{1.8}$$

then $C \rtimes D$ with respect to $c \circ_i^d c'$ is an operad.

Proof Since the action is compatible with the $\mathbb{S}_n$ actions it remains to check the associativity. The first and third case of the equation (1.5) clearly hold and the second case follows from the equation (1.8). □

1.3.6 The right semi-direct product There is a right version of the semi-direct products using maps

$$\begin{array}{rcl} \circ_i^C : D(n) \times C(m) \times D(m) & \to & D(n + m - 1) \quad i = 1, \dots n \\ (d, c, d') & \mapsto & \circ_i^C(d, c, d') =: d \circ_i^c d' \end{array} \tag{1.9}$$

which again define compositions on the products $(C \times D)(n) := C(n) \times D(n)$ via

$$\begin{array}{rcl} \circ_i : (C \times D)(n) \times (C \times D)(m) & = & C(n) \times D(n) \times C(m) \times D(m) \\ & \longrightarrow & C(n + m - 1) \\ (c, d) \circ_i (c', d') & = & (c \circ_i c', d \circ_i^{c'} d') \end{array} \tag{1.10}$$

We call a quasi-operad D *normal* with respect to C if there are maps of the type (1.9) such that the maps (1.10) give an operad structure to the product $C \times D$. Given an operad C an a quasi-operad D normal with respect to C, we call the product together with the operad structure (1.7) the *right semi-direct product* of C and D which we denote $C \ltimes D$.

This structure typically appears when one has a right action of the operad C on the quasi-operad D.

1.4 Bi-crossed products

1.4.1 Definition Consider two quasi-operads $C(n)$ and $D(n)$ together with a collection of maps:

$$\begin{aligned} \circ_i^D : C(n) \times D(n) \times C(m) &\to C(n+m-1) \quad i = 1, \dots n \\ (c, d, c') &\mapsto \circ_i^D(c, d, c') =: c \circ_i^d c' \qquad (1.11) \\ \circ_i^C : D(n) \times C(m) \times D(m) &\to C(n+m-1) \quad i = 1, \dots n \\ (d, c, d') &\mapsto \circ_i^C(d, c, d') =: d \circ_i^c d' \qquad (1.12) \end{aligned}$$

where the operations (1.11) satisfy the analog of equation (1.2) for the action of $\mathbb{S}_n \times \mathbb{S}_m$ with $\mathbb{S}_n$ acting diagonally on the first two factors and the operations (1.12) satisfy the analog of equation (1.2) for the action of $\mathbb{S}_n \times \mathbb{S}_m$ with $\mathbb{S}_m$ acting diagonally on the second two factors.

Again we used the superscripts to indicate that we view the dependence on the other quasi-operad as a perturbation of the original quasi-operad structure.

We define the bi-crossed product $C \bowtie D$ with respect to the operations $\circ_i^D, \circ_j^C$ to be given by the collection $(C \times D)(n) := C(n) \times D(n)$ with diagonal $\mathbb{S}_n$ action and compositions

$$\begin{aligned} \circ_i \, (C \times D)(n) \times (C \times D)(m) &= C(n) \times D(n) \times C(m) \times D(m) \\ &\longrightarrow C(n+m-1) \\ (c, d) \circ_i (c', d') &= (c \circ_i^d c', d \circ_i^{c'} d') \qquad (1.13) \end{aligned}$$

where we use the upper index on the operations to show that we use the universal maps (1.11) with fixed middle argument.

In the case we are dealing with unital operads we will also require that the perturbed compositions are such that $1_C \times 1_D$ is a unit in the obvious notation, and that $\circ_i^{1_C} = \circ_i$ and $\circ_i^{1_D} = \circ_i$.

1.4.2 Remark Again it depends on the choice of the $\circ_i^D, \circ_j^C$ if the bi-crossed product of two operads is an operad. In the case that $\circ_i^D, \circ_j^C$ are given by actions as in Lemma 1.3.5 then this is guaranteed if the condition (1.8) and its right analog hold.

Furthermore it is possible that the bi-crossed product of two quasi-operads is an operad.

1.4.3 Definition We call two quasi-operads C and D *matched* with respect to maps of the type (1.11) and (1.12) if the quasi-operad $C \bowtie D$ is an operad.

1.4.4 Examples

1) Factoring the multiplication maps through the first and third projection and using the structure maps of the two operads we obtain the direct product.
2) If D is the operad based on a monoid and choosing the maps $\circ_i^C$ to be unperturbed and defining the maps $\circ_i^D$ as in (4.6) we obtain the semi-direct product.
3) If C and D are concentrated in degree 1 and happen to be groups then the bi-crossed product is that of matched groups [Ka, Tak]. Otherwise we obtain that of matched monoids.
4) Below we will show that the cacti operad is a bi-crossed product of the cactus operad and the operad built on S^1 as a monoid and thus these operads are matched with respect to the specific perturbed multiplications given below.

2 Several Varieties of Cacti

2.1 Introduction

The operad of *Cacti* was first introduced by Voronov in [V] as pointed treelike configurations of circles. In the following we will first take up this description and introduce Cacti and several versions of related operads in this fashion. This approach is historical and lends itself to describe actions on the loop space of a compact manifold [CJ, V] of cacti. For other purposes, especially giving the topology on the space of cacti, other descriptions of a more combinatorial nature are more convenient. In the following section, we will both recall the traditional approach as well the sometimes more practical definition in terms of graphs.

2.2 General setup for configurations of circles

2.2.1 Notation By an S^1 in the plane we will mean a map of the standard $S^1 \subset \mathbb{R}^2$ with the induced metric and orientation which is an orientation preserving embedding $f : S^1 \to \mathbb{R}^2$. A configuration of S^1's in the plane is given by a collection of finitely many of these maps which have at most finitely many intersection points in the image. I.e. if $f_i : 1 \leq i \leq n$ is such a collection, then if $i \neq j : f_i(\theta_i) = f_j(\theta_j)$ for only finitely many points θ_i.

By an S^1_r in the plane we will mean a map of the standard circle of radius r: $S^1_r \subset \mathbb{R}^2$ with the induced metric and orientation which is an orientation preserving embedding $f : S^1_r \to \mathbb{R}^2$. A configuration of S^1_rs in the plane is a collection of finitely many of these maps which have at most finitely many intersection points in the image.

2.2.2 Re-parameterizations Notice that a circle in a plane comes with a natural parameter. This is inherited from the natural parameter θ_r of the standard parametrization of $S^1_r : (r\cos(\theta_r), r\sin(\theta_r))$. Sometimes we have to re-parameterize a circle, so that its length changes. To be precise let $f : S^1_r \to \mathbb{R}^2$ be a parametrization of a circle in the plane. Then $f_R : S^1_R \to \mathbb{R}^2$, called the re-parametrization to length R, is defined to be the map $f_R = f \circ rep^R_r$ with $rep^R_r : S^1_R \to S^1_r$ given by $\theta_R \mapsto \theta_r$.

2.2.3 Dual black and white graph Given a configuration of S^1s in the plane, we can associate to it a dual graph in the plane. This is a graph with two types of vertices, white and black. The first set of vertices is given by replacing each circle by a white vertex. The second set of vertices is given by replacing the intersection points with black vertices. The edges run only from white to black vertices, where we join two such vertices if the intersection point corresponding to the black vertex lies on the circle represented by the white vertex.

We remark that all the S^1s are pointed by the image of $0 = (0,1)$. On any given S^1 in the plane we will call this point and its image local zero or base point.

If we are dealing with circles of radii different from one we will label the vertices of the trees by the radius of the respective circle.

2.2.4 Cacti and trees The configurations corresponding to cacti will all have trees as their dual graphs. We would like to point out that these trees are planar trees, i.e. they are realized in the plane or equivalently they have a cyclic order of each of the sets of edges emanating from a fixed vertex.

Moreover we would like to consider rooted trees, i.e. trees with one marked vertex called root. Recall that specifying a root induces a natural orientation for the tree and a height function on vertices. The orientation of edges is toward the root and the height of a vertex is the number of edges traversed by the unique shortest path from the vertex to the root. Due to the orientation we can speak of incoming and outgoing edges, where the outgoing edge is unique and points toward the root. Naturally the edges point from the higher vertices to the lower vertices. A rooted planar tree has a cyclic order for all edges adjacent to a given vertex and a linear order on the adjacent edges to any given vertex except the root. A planar rooted tree with a linear order of the incoming edges at the root is called a planted tree. The leaves of a rooted tree are the vertices which only have outgoing edges.

We call a configuration of S^1s in the plane rooted if one of the circles is marked by a point, we also call this point the global zero. In this case, we include a black vertex in the dual graph for this marked point and make this the root, so that the dual graph of a rooted configuration is a rooted tree. This tree is actually also planted by the linear order of the incoming edges of the root provided by making the component on which the root lies the smallest element in the linear order.

Given such a rooted configuration of S^1_rs in the plane we call the images of the 0s $(f_i(0))$ together with the image of the marked point and the intersection points the special points of the configuration. We also call the connected components of the image minus the special points the arcs of the configuration. If $f_i|(\theta_1,\theta_2) = a$ then we define $|a| := \frac{1}{2\pi}(\theta_2-\theta_1)$ to be the length of a. In the same situation we define $\bar{a} := f_i|[\theta_1,\theta_2]$ to be the closure of a.

2.2.5 Definition Given a configuration of S^1s in the plane whose dual graph is a tree, we say that an S^1 is contained in another S^1 if the image first circle is a subset of the disc bounded by the image of the second circle.

A configuration of S^1s in the plane is called *tree-like* if its dual graph is a connected tree and no S^1 is contained in another S^1.

2.2.6 The perimeter or outside circle and the global zero For a marked tree-like configuration of S^1_rs in the plane let $R = \sum r_i$ then there is a surjective map of S^1_R to the image of this configuration which is a local embedding. This map is given by starting at the marked point or global zero of the root of the configuration going around this circle in the positive sense until one hits the first intersection point and then starting to go around the next circle in the cyclic order again in the positive direction until the next intersection point and so on until one again reaches the zero of the root.

We will call this map and, by abuse of notation, its image the perimeter or the outside circle of the configuration. We will also call the zero of the perimeter the global zero.

2.3 Normalized cacti without spines

We will now introduce normalized spineless cacti as configurations. Later, to give a topology, we will reinterpret these configurations in terms of graphs.

2.3.1 Definition We define *normalized cacti without spines* to be labelled rooted collections of parameterized S^1s grafted together in a tree like fashion with the gluing points being the zeros of the S^1. More precisely we set:

$\mathcal{C}act^1(n)$:= {rooted tree-like configurations of n labelled S^1s in the plane such that the root (global zero) coincides with the marked point (zero) of the component it lies on, the points of intersection are such that the circles of greater height all have zero as their point of intersection.} / isotopies preserving the incidence conditions.

Here and below preserving the incidence conditions means that if $f_{i,t} : S^1_{r_i} \times I$ are the isotopies and $f_{i,0}(p) = f_i(p) = f_j(q) = f_{j,0}(q)$ then for all t: $f_{i,t}(p) = f_{j,t}(q)$ and vice-versa if $f_{i,0}(p) = f_i(p) \neq f_j(q) = f_{j,0}(q)$ then for all t: $f_{i,t}(p) \neq f_{j,t}(q)$.

We will take the conditions "without spines" and "spineless" to be synonymous.

2.3.2 Remark We would like to point out that there is only one zero which is not necessarily an intersection point, namely that of the root. It can however also be an intersection point. Moreover, one could rewrite the condition of having a dual black and white graph that is a tree in the form: given any two circles their intersection is at most one point.

2.3.3 Remark There are several ways to give a topology to this space. One way to give it a topology is by describing the degenerations of the above configurations, as was done originally in [V]. This is done by allowing the intersection points and the root to move in such a way that they may collide, and "pass" each other moving along on the outside circle. If an intersection point collides with the marked point from the positive direction, i.e. the length of the arc going counterclockwise from the root to the intersection point goes to zero, then the root passes to the new component.

The quickest way is to give the topology to the spaces $\mathcal{C}act^1$ as subspaces of the operad $\mathcal{DA}rc$ as defined in [KLP]. A brief review of the necessary constructions is given in the Appendix B below for the reader's convenience.

Lastly, one can define the topology in terms of combinatorial data by gluing of products of simplices indexed by trees as first explained in [K1]. For definiteness, we will use this construction to fix our definitions.

2.3.4 Notation Recall that the dual tree of a cactus is a bi-colored (b/w) bi-partite planar planted tree. Such a tree has a natural orientation towards the root. We call an edge white if it points from a black to a white vertex in this orientation and call the set of these white edges E_w. Let V_w be the set of white vertices. For a vertex v we let $|v|$ be the set of incoming edges, which is equal to the number of white edges incident to v and is also equal to the total number of edges incident to v minus one. We call $\mathcal{T}(n)$ the set of planar planted bi-partite trees with white leaves, black root, and n white vertices which are labelled from 1 to n.

2.3.5 Definitions The topological type of a spineless normalized cactus in $\mathcal{C}act^1(n)$ is defined to be the tree $\tau \in \mathcal{T}(n)$ which is its dual b/w planar planted tree together with the labelling of the white vertices induced from the labels of the cactus.

We define $\mathcal{T}(n)^k$ to be the elements of $\mathcal{T}(n)$ with $|E_w| = k$.

Let Δ^n denote the standard n-simplex, $|\Delta^n|$ its standard realization in $\mathbb{R}^{n+1}$ as $\{(t_1, \dots t_{n+1}) | \sum_i t_i = 1\}$. We denote the interior of $|\Delta^n|$ by $|\dot{\Delta}^n|$.

For $\tau \in \mathcal{T}$ we define

$$\Delta(\tau) := \times_{v \in V_w(\tau)} |\Delta^{|v|}| \tag{2.1}$$

Notice that $\dim(\Delta(\tau)) = |E_w(\tau)|$ and that the set E_w has a linear order which defines an orientation of $\Delta(\tau)$.

We also let

$$\dot{\Delta}(\tau) := \times_{v \in V_w(\tau)} |\dot{\Delta}^{|v|}| \tag{2.2}$$

2.3.6 Lemma *A normalized spineless cactus is uniquely determined by its topological type and the length of the arcs.*

Proof It is clear that each normalized spineless cactus gives rise to the described data. Vice versa given the data, one can readily construct a representative of a spineless cactus with the underlying data. A quick recipe is as follows. Realize the given tree in the plane. Blow up the white vertices to circles which do not intersect and do not contain any of the black points. Mark the segments of the circles between the edges by the label of the second edge bounding the arc in the counterclockwise orientation. Mark the intersection point of the first edge in the linear order of the root with the unique circle it intersects. Delete the part of the edges inside these circles. Now contract the edges and if necessary deform the circles during the contraction such that they do not touch. This is only a finite problem and thus such choices can be made. We will call the images of the circles lobes. The lobes are labelled from 1 to n by the label of the vertices. There are obvious maps of S^1 onto each lobe, which have lengths of arcs between the special points (intersection or marked) corresponding to the labelling. This gives a representative. Since the data is invariant under isotopy preserving the intersections, we have constructed the desired cactus and hence shown the bijection. □

2.3.7 Lemma *For a normalized spineless cactus the lengths of the segments lying on a given lobe represented by a vertex v are in 1-1 correspondence with points of the open simplex $\dot{\Delta}^{|v|}$.*

Proof The lengths of the arcs have to sum up to the radius of the lobe which is one and the number of arcs on a given lobe is $|v|+1$. □

2.3.8 Remark The Lemma above also gives an identification of the arcs of a cactus with the edges of the tree τ specifying its topological type. Here we fix that an edge e incident to a white vertex v_w and a black vertex v_b corresponds to the arc on the lobe of v_w running from the special point (intersection or root) preceding v_b to v_b.

2.3.9 Proposition *As sets $\mathcal{C}act^1(n) = \amalg_{\tau\in\mathcal{T}(n)}\dot{\Delta}(\tau)$.*

Proof Immediate by the preceding two Lemmas. □

2.3.10 Degenerations By the above we can identify the vertices of $\Delta(\tau)$ and therefore the coordinates of points of $\Delta(\tau)$ with the arcs of a cactus of topological type τ.

Given a cactus c and an arc a with length $|a|<1$ of c we define the degeneration of c with respect to a to be the configuration of S^1s obtained by a homotopy contracting the closure of the arc $\bar{a}$ to a point p, but preserving all other incidence conditions, together with the following root. If a is not the first arc on the outside circle, then the root remains unchanged. If a is the first arc of the outside circle and a' is the second arc of the outside circle which lies in the image of the map f_j, then the new root is defined to be the point which is the pre-image of p under the map f_j, i.e. $f_j^{-1}(p)$.

2.3.11 Degeneration of trees There is also a purely combinatorial way to describe the degeneration of the b/w bi-partite planar planted tree by cutting and re-grafting [K1]. An abbreviated non-technical version is as follows. Given $\tau \in \mathcal{T}(n)$ and an edge e in τ incident to a white vertex v_w with $|v_w| > 0$ the contraction of τ with respect to e is given by the following procedure. First let v_w, v_b be the white and black vertices e is incident to and let e' be the edge immediately preceding e in the cyclic order at v_w. Let v_w and v'_b be the vertices of e'. The contraction of τ with respect to e is given by the tree in which the edge e and the vertex v_b are removed and the remaining branches of v_b are grafted to v'_b in such a way that they keep their linear order and immediately precede the edge e' in the cyclic order at v'_b.

2.3.12 Remark The reader can readily verify that the degeneration of cacti in the arc and combinatorial interpretations agree.

2.3.13 A CW-Complex Given a cell $\Delta(\tau)$ and a vertex v of any of the constituting simplices of $\Delta(\tau)$ we define the v-th face of $\Delta(\tau)$ to be the subset of $\Delta(\tau)$ whose points have v-th coordinate equal to zero.

We let $K(n)$ be the CW complex whose k-cells are indexed by $\tau \in \mathcal{T}(n)^k$ with the cell $C(\tau) = |\Delta(\tau)|$ and the attaching maps e_τ defined as follows. We identify the v-th face of $\Delta(\tau)$ with $\Delta(\tau')$ where τ' is the topological type of the cactus c' which is the degeneration of a cactus c of topological type τ with respect to the arc a that represents the vertex v.

We denote by $\dot{e}_\tau$ the restriction of e_τ to the interior of $\Delta(\tau)$. Notice that $\dot{e}_\tau$ is a bijection.

2.3.14 Theorem *The elements of $\mathcal{C}act^1(n)$ are in bijection with the elements of the CW complex $K(n)$.*

Proof Immediate from the Proposition 2.3.9 above. □

2.3.15 Definition We will use the above theorem to give $\mathcal{C}act^1$ the topology induced by the above bijection, that is we define the topological space $\mathcal{C}act^1(n)$ as

$$\mathcal{C}act^1(n) := K(n).$$

2.3.16 The action of $\mathbb{S}_n$ There is an action of $\mathbb{S}_n$ on $\mathcal{C}act^1(n)$ which acts by permuting the labels.

2.3.17 Gluing We define the following operations

$$\circ_i : \mathcal{C}act^1(n) \times \mathcal{C}act^1(m) \to \mathcal{C}act^1(n+m-1) \tag{2.3}$$

by the following procedure: given two normalized cacti without spines we reparameterize the i-th component circle of the first cactus to have length m and glue in the second cactus by identifying the outside circle of the second cactus with the i-th circle of the first cactus.

These gluings do not endow the normalized spineless cacti with the structure of an operad, but with the slightly weaker structure of a quasi-operad of section 1.

By straightforward computation we have the following:

2.3.18 Proposition *The glueings make the spaces $\mathcal{C}act^1(n)$ into a topological quasi-operad.*

2.3.19 Remark The above gluing operations are indeed not strictly associative as the example in Figure 1 shows. As in this example, the gluings are associative up to homotopy in general, as we will discuss below.

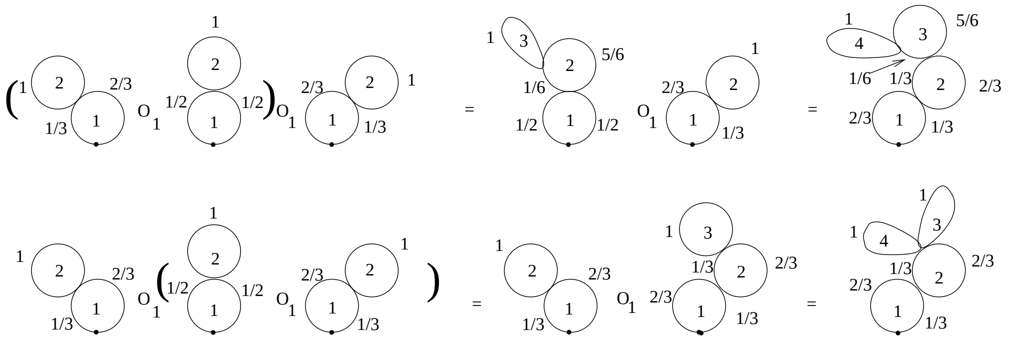

Figure 1: An example for non-associativity in $\mathcal{C}act^1$

2.3.20 Remark It is shown in [K1] that there is an operad structure on normalized spineless cacti on the cellular chains of K.

This together with the Theorem 3.2.1 provides the basis for a proof of Deligne's conjecture on the Hochschild cohomology of an associative algebra [K1].

2.4 Cacti without spines

2.4.1 Definition We define *cacti without spines* by an analogous procedure to that of normalized cacti only this time taking S^1_r s, i.e. circles of different radii.

As a set $\mathcal{C}act(n) := \{$rooted tree-like configurations of n labelled S^1_r s in the plane such that the root (global zero) coincides with the marked point (zero) of the component it lies on and that the points of intersection are such that the circles of greater height all have zero as their point of intersection $\}/$ isotopies preserving the incidence conditions.

Again $\mathbb{S}_n$ acts via permuting the labels.

2.4.2 Lemma $\mathcal{C}act(n) = \mathcal{C}act(n)^1 \times \mathbb{R}^n_{>0}$.

Proof As in the previous case of normalized spineless cacti, such a configuration is given bijectively by its topological type and the lengths of its arcs. Each arc belongs to a unique lobe and the sum of the lengths of the arcs belonging to a lobe is the radius of the given S^1. Let $\mathbf{l}_i := (l_{i_1}, \dots, l_{i_s})$ be the collections of lengths of the arcs of the lobe i whose radius is $r_i = \sum_j l_{i_j}$ then $\mathbf{l}_i$ corresponds to a unique point in $\Delta^{s-1} \times \mathbb{R}_{>0}$ given by $((l_{i_1}/r_i, \dots, l_{i_s}/r_i), r_i)$. This establishes the claimed bijection. □

2.4.3 Definition As a topological space, we define $\mathcal{C}act(n)$ to be

$$\mathcal{C}act(n) := \mathcal{C}act(n)^1 \times \mathbb{R}^n_{>0}$$

with the product topology.

2.4.4 Remark A description of the topology on this space is given, by allowing the intersection points and the global zero to move and collide and pass each other along the outside circle as before, with the same rule for the global zero as before and also letting the radii vary. This topology agrees with the product topology $\mathcal{C}act(n) = \mathcal{C}act^1(n) \times \mathbb{R}^n_{>0}$ above.

It also agrees with the one induced by the embedding of the operad into $\mathcal{DA}rc$ [KLP], see also Appendix B.

2.4.5 Gluing We define the following operations

$$\circ_i : \mathcal{C}act(n) \times \mathcal{C}act(m) \to \mathcal{C}act(n+m-1) \tag{2.4}$$

by the following procedure: given two cacti without spines we re-parameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i-th circle of the first cactus.

Notice that this gluing differs from the one above, since now a whole cactus and not just a lobe is re-scaled.

2.4.6 Proposition *The gluing endows the spaces $\mathcal{C}act(n)$ with the structure of a topological operad.*

Proof Straightforward calculation. □

2.4.7 Remark There is an obvious map from normalized spineless cacti to spineless cacti. This map is not a map of operads, since the gluing procedures differ. There is however a homotopy of one gluing to the other by moving the intersection points around the outside circle of the cactus which is glued in, so that the two structures of quasi-operads do agree up to homotopy. This means that the spaces $\mathcal{C}act^1$ form a homotopy associative quasi-operad and thus the homology of this quasi-operad is an operad. On the homology level normalized spineless cacti are thus a sub-operad of spineless cacti and moreover, since the factors of $\mathbb{R}^n$ are contractible this sub-operad coincides with the homology operad of spineless cacti, as we show below.

The fact mentioned before, that the cellular chains of $\mathcal{C}act^1$ form an operad [K1] can be seen from the discussion of the inclusion of $\mathcal{C}act^1$ into $\mathcal{C}act$ mentioned above which is explained in detail below.

2.5 Different pictorial realizations

As exhibited in the previous paragraph, there are two pictorial descriptions of $\mathcal{C}act^1$ and $\mathcal{C}act$ given by circles in the plane and the dual black and white planar planted tree whose edges are marked by positive real numbers - the lengths of the arcs. There are more pictorial realizations for (normalized) spineless cacti, which are useful.

2.5.1 The tree of a cactus without spines In the case that the configuration of circles is a cactus without spines there is a dual tree that we can associate to it that is a regular tree with markings that is not black and white, but is just planar and planted.

This is done as follows. The vertices correspond to the circles. They are labelled by the radius of the respective circle. We will draw an edge between two vertices if the circles have a common point and if one circle is higher than the other in the height of the dual graph. We will label the edge by the length of the arc on the lower circle between the intersection point and the previous intersection point where we now also allow the length of the arcs to be zero if these two points coincide. Here we also consider the global zero as an intersection point. In this procedure we give the edges the cyclic order that is dictated by the perimeter. This means that now the labels on the edges are in $\mathbb{R}_{\geq 0}$ with the restriction that at each vertex the label (radius) of that vertex is strictly greater than the sum of the labels (weights) of the incoming edges. For normalized spineless cacti the labels on the vertices are all 1 and can be omitted. Using this structure we can view the space of normalized cacti as a sort of "blow up of a configuration space". The "open part" is the part with only double points. In this case, the weight on the edges are restricted by the equations $0 < \sum w_i < 1$. Allowing intersections of more than two components at a time amounts to letting $w_i \to 0$. In the limit $\sum w_i \to 1$ the tree is identified with the tree where the last incoming edge is transplanted to the other vertex of the outgoing edge in such a way that it is the next edge in the cyclic order of that vertex. Lastly if the weight on the first edge of the root goes to zero, the root vertex will be the other vertex of that edge.

If we do not want to use the height function of the black and white tree, we can still define a height function via the outside circle. Start at height zero for the root. If the perimeter hits a component for the first time, increase the height by one and assign this height to the component. Each time you return to a component decrease the height by one.

Given a planar planted tree whose vertices and edges are labelled in the above fashion, it gives a prescription on how to grow a cactus. Start at the root and draw a based loop of length given by the label of the root. For the first edge mark the point at the distance given by the label of the edge along the loop. Then mark a second point by travelling the distance of the label of the second edge and so on. Now at the next level of the tree draw a loop based at the marked point of the previous level and again mark points on it according to the outgoing edges. This will produce a cactus without spines.

Lastly, we wish to point out that now the composition looks like the grafting of trees into vertices as in the Connes-Kreimer [CK] tree operads. In fact, we have recently shown [K1] that indeed there is a cell decomposition of spineless normalized cacti whose cellular chains form an operad and whose symmetric top dimensional cells are isomorphic as an operad to the operad of rooted trees whose Hopf algebra is that of Connes and Kreimer [K1].

2.5.2 The chord diagram of a cactus There is yet another representation of a cactus. If one regards the outside loop, then this can be viewed as a collection of points on an S^1 with an identification of these points, plus a marked point corresponding to the global zero. We can represent this identification scheme by drawing one chord for each pair of points being identified as the beginning and end of a circle this chord is oriented from the beginning point of the lobe to the end point of the lobe. Note that one of the two segments of the outside loop defined by the chord corresponds to the lobe. There is a special case for the chord diagram which is given if there is a closed cycle of chords. This happens if two or more lobes intersect at the global zero. Here one can delete the first chord, if so desired, we call this the reduced chord diagram.

The chord diagram comes equipped with a decoration of its arcs by their length thus giving a map of S^1_R to the outside circle. Here $R = \sum_i r_i$ where the r_i are the radii of the lobes. To obtain a cactus from such a diagram, one simply has to collapse the chords.

This kind of representation is reminiscent of Kontsevich's formalism of chord diagrams (cf. eg. [BN]) as well as the shuffle algebras and diagrams of Goncharov [Go]. We wish to point out that although the multiplication is similar to Kontsevich's and also could be interpreted as cutting the circle at the global zero resp. the local zero, it is not quite the same. However, the exact relationship and the co-product deserve further study.

Lastly, we can recover the a planar rooted tree above as the dual tree of the chord diagram. This is the dual tree on the surface which is given by the disc whose boundary is the outside circle. The chords on the surface then divide the disc up into chambers — the connected components of the complement of the chords. The dual tree on this surface has one vertex for each such chamber and an edge for each pair of chambers separated by a common chord. If the global zero lies on only one lobe the root of the tree is the vertex of the complementary region whose boundary includes the global zero. If there are two components meeting at the global zero the root of the tree is given by the vertex whose chamber has the global as left boundary on the outside circle. In a special case for the chord diagram which is given if there is a closed cycle of chords, i.e. three or more lobes intersect at the global zero, the root vertex will be the unique vertex inside the closed cycle. These trees are in fact planted due to the linear order they inherit from the embedding of the chord diagram. The planar tree is the tree obtained from the bi-partite planted planar tree by removing the black vertices with the exception of the root.

A representation of a cactus without spines in all possible ways including its image in the $\mathcal{DA}rc$ operad can be found in Figure 2.

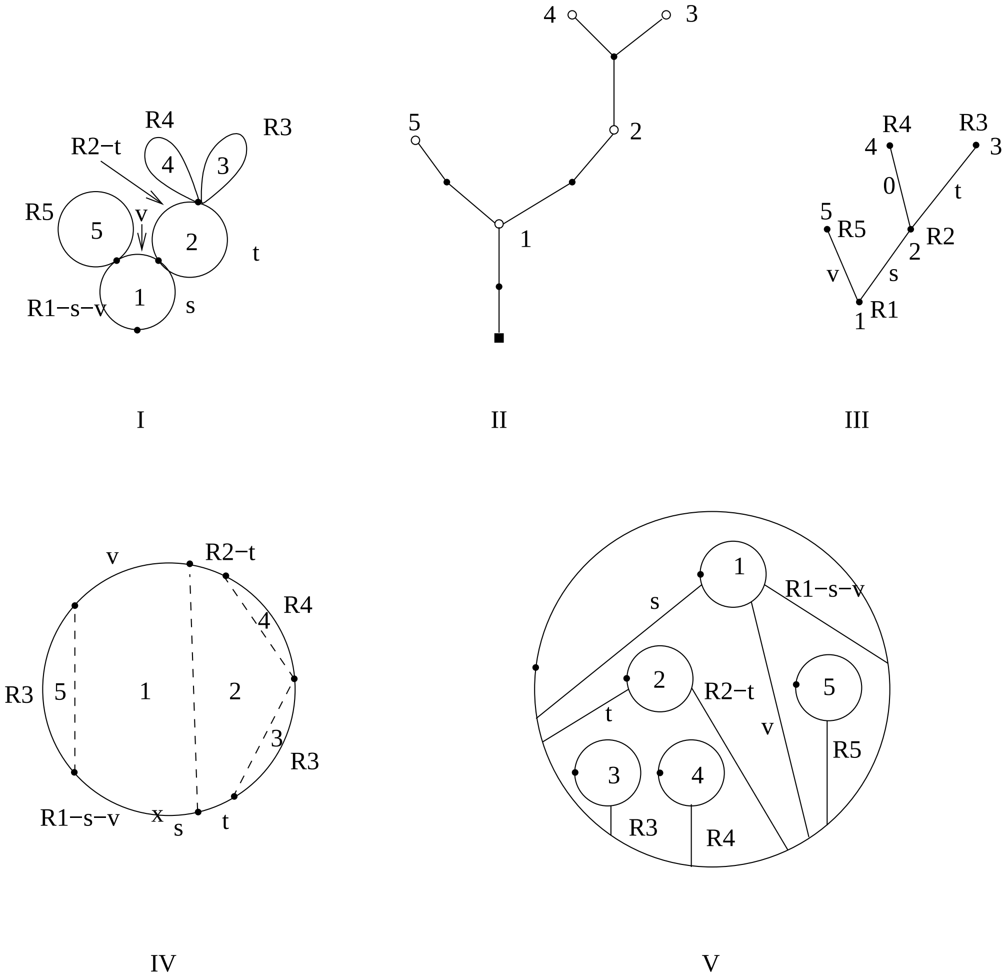

Figure 2: I: A cactus without spines II: Its black and white tree III: Its dual tree IV: Its chord diagram V: Its image in $\mathcal{DA}rc$

2.6 Cacti with spines

The following definition is the original definition of cacti due to Voronov.

2.6.1 Definition [V] Define *Voronov cacti or cacti with spines or simply cacti* in the same fashion as cacti without spines, but without requiring that the zeros be the intersection points.

In addition, and this is key, we add a global zero/base point to the configuration, which means that we mark a circle and a point on that circle. The circle with the global base point will be the root. We call the n-th component of this operad $\mathcal{C}acti(n)$.

As a set $\mathcal{C}act(n) := \{$rooted tree-like configurations of n labelled S^1_rs in the plane$\}/$ isotopies preserving the incidence conditions.

The perimeter or outside circle will be given by the same procedure as 2.2.6 by starting at the global zero.

2.6.2 Remark To define the topology we remark that the cacti with spines are as a set in bijective correspondence to spineless cacti times a product of S^1s: $\mathcal{C}acti(n) \stackrel{1-1}{\longleftrightarrow} \mathcal{C}act(n) \times (S^1)^{\times n}$.

The bijection is given by mapping the underlying spineless cactus, which is obtained by forgetting all local zeros and the induced coordinates of the local zeros, and fixing a coordinate on S^1 for every lobe which gives the length of the arc starting at the unique intersection point with the lobe of lower height (or the root) going counter-clockwise to the local zero.

2.6.3 Definition As a topological space we set

$$\mathcal{C}acti(n) := \mathcal{C}act(n) \times (S^1)^{\times n}.$$

2.6.4 Remark Originally the topology was introduced by describing that the lobes, the special points and the root can move with the caveat that the root passes to a new component if the intersection point of the lobe collides with the root from the right — just as for spineless cacti. Of course these two descriptions are compatible. Again one can also realize $\mathcal{C}acti \subset \mathcal{DA}rc$ and obtain the same topology as above in this way.

2.6.5 Gluing We define the following operations

$$\circ_i : \mathcal{C}acti(n) \times \mathcal{C}acti(m) \to \mathcal{C}acti(n+m-1) \tag{2.5}$$

by the following procedure which differs slightly from the above: given two cacti without spines we re-parameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i-th circle of the first cactus. We stress that now the local zero of the i-th circle is identified with the global zero. viz. the starting point of the outside circle. This local zero need not coincide with the intersection point with the lobe of lower height (or the global zero).

2.6.6 Proposition [V] *The cacti form a topological operad.*

2.7 Normalized cacti

2.7.1 Definition We define the spaces of *normalized cacti* denoted by $\mathcal{C}acti^1(n) \subset \mathcal{C}acti(n)$ to be the subspaces of cacti with the restriction that all circles have radius one.

As spaces

$$\mathcal{C}acti^1(n) = \mathcal{C}act^1(n) \times (S^1)^{\times n}.$$

2.7.2 Glueings We define the compositions by scaling as for normalized spineless cacti and then gluing in the second cactus into the i-lobe of the first, but now using the identification of the outside circle of the second cactus with the circle of the i-th lobe by matching the local zero of the i-th lobe of the first cactus with the global zero of the second.

2.7.3 Proposition *Together with the $\mathbb{S}_n$ action permuting the labels and the glueings above normalized cacti form a topological quasi-operad.*

Proof Straightforward computation. □

2.7.4 Remark The contents of Remark 2.4.7 applies analogously in the cacti situation.

2.7.5 Remark There are natural forgetful morphisms from cacti to cacti without spines forgetting all the local zeros. We arrange the map in such a way, that the global zero becomes the base-point of the spineless cactus. This works for the normalized version as well. These maps are not maps of operads. The precise relationship between the different varieties is that of a bi-crossed product of section 1.4, see Theorem 5.3.4 below.

There is, however, an embedding of spineless cacti into cacti as a suboperad by considering the global zero to be the zero of the root and by making the first intersection point at which the perimeter reaches a lobe of the cactus the local zero of that circle.

2.8 Different pictorial realizations

2.8.1 The tree of a cactus The missing information of a cactus without spines relative to a cactus proper is the location of the local zeros. We just add this information as a second label on each vertex. Notice that the local zero of the root component then need not be the global zero. The label we associate to the root is the position of the local zero with respect to the global zero.

2.8.2 The chord diagram The chord diagram of a cactus again is the chord diagram of a cactus without spines, where the location of the spines is additionally marked on the S^1. There is a choice if the local zero coincides with an intersection point. Just to fix notation we will mark the first occurrence of the endpoint of a chord, where first means in the natural orientation starting at the global zero.

A representation of a cactus (with spines) in all possible ways including its image in $\mathcal{DA}rc$ can be found in Figure 3.

3 Spineless cacti and the little discs operad

In this section, we will show that $\mathcal{C}act$ is an E_2 operad using the recognition principle of Fiedorowicz [F2]. To assure the needed assumptions are met we mimic the construction of [F1] which shows that the universal covers of the little discs operad naturally form a B_∞ operad.

3.1 The E_1 structure

3.1.1 Definition A spineless corolla cactus (SCC) is a spineless cactus whose points of intersection all coincide with the global zero.

Since the condition of being an SCC is preserved when composing two spineless corolla cacti:

3.1.2 Lemma *Spineless Corolla Cacti are a suboperad of spineless Cacti.*

We define $\mathcal{SCC}(n) \subset \mathcal{C}act(n)$ to be the subset of spineless corolla cacti and denote the operad constituted by the $\mathcal{SCC}(n)$ with the permutation action of $\mathbb{S}_n$ and the induced gluing by $\mathcal{SCC}(n)$.

3.1.3 Theorem *The suboperad $\mathcal{SCC}(n)$ of corolla cacti is an E_1 operad.*

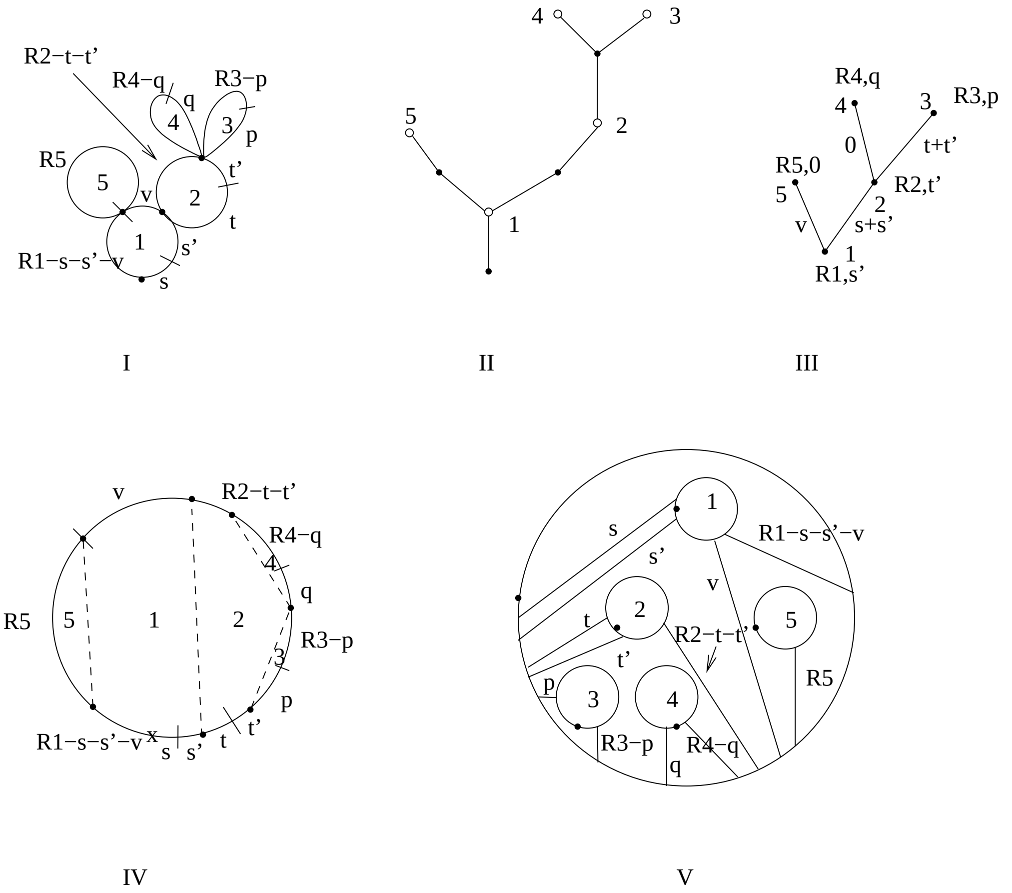

Figure 3: I: A cactus (with spines) II: Its black and white tree III: Its dual tree IV: Its chord diagram V: Its image in $\mathcal{D}\mathcal{A}rc$

Proof We will use the recognition principle of Boardman-Vogt [BV]. First notice that we have a free action of $\mathbb{S}_n$. If the lobes are all grafted together at the root then the only parameters are the sizes of the lobes. These sizes together with the labelling fixes a unique spineless corolla. Two spineless corollas lie in the same path component if and only if the sequence of the labels of the lobes as read off from the outside circle agree. Thus $\mathcal{S}CC(n) = \amalg_{\sigma \in \mathbb{S}_n} \mathbb{R}^n_{>0}$. And thus each path component is contractible and thus the action of $\mathbb{S}_n$ is free and transitive on $\pi_0(\mathcal{S}CC(n))$. □

3.1.4 Remark The Theorem above has immediate applications to operads built from moduli spaces (see Appendix B) giving them an A_∞-structure.

3.1.5 Corollary *The operad of the decorated moduli space of bordered, punctured surfaces with marked points on the boundary $\widetilde{M}^s_{g,r}$ which is proper homotopy equivalent to $\mathcal{A}rc_{\#}$ contains an E_1 operad. Thus so does the operad $\mathcal{A}rc$.*

The same is true for the operad of the moduli spaces $M^n_{g,n}$ of genus g curves with n punctures and a choice of tangent vector at each puncture and its restriction to genus 0.

Finally the spaces $M_{g,n}$ form a partial operad which is an E_1 operad.

Proof By the Appendix B there is an operad map $\mathcal{C}acti \to \mathcal{A}rc_{\#} \subset \mathcal{A}rc$ which is an equivalence onto its image. Furthermore it is shown that $\mathcal{A}rc_{\#}$ is proper homotopy equivalent to the mentioned moduli space in [P]. This establishes the first part.

The second claim follows from the identification of the suboperad of bordered surfaces with marked points on the boundary and no further punctures $\mathcal{A}rc^0_{\#}$ with $M^n_{g,n}$ via marked ribbon graphs [K3].

The last statement comes from the fact that the SCCs are ribbon graphs and as such index cells of $M_{g,n}$. The operad structure of SCCs thus defines a partial operad structure on $M_{g,n}$. □

This also means that on the chain level algebras over these operads will be A_∞ operads.

3.2 The E_2 structure

The main result of this section is the following.

3.2.1 Theorem *$\mathcal{C}act$ is an E_2 operad.*

We will use the recognition principle of Fiedorowicz [F2] to prove this theorem (see also [SW]). For this one needs the notion of a braid operad, which is given by replacing the symmetric groups in the definition of operads by braid groups (see [F1] or [SW]).

3.2.2 Definition [F1] A collection $B(n)$ a B_∞ operad if the $B(n)$ form a braid operad in the sense of [F1] with the properties

i) the spaces $B(n)$ are contractible.

ii) The braid group action on each $B(n)$ is free.

3.2.3 Proposition [F2] *An operad $\mathcal{A}$ is an E_2 operad if and only if each space $\mathcal{A}(k)$ is connected and the collection of covering spaces $\{\tilde{A}(k)\}$ form a B_∞ operad.* □

Adapting the proof of [F1] that the universal covers of the little disc operad form a B_∞ operad one arrives at the following proposition, which is essentially contained in [F1] and in spirit in [MS].

Let $\tau_i \in \mathbb{S}_n$ denote the transposition which transposes i and $i+1$.

3.2.4 Proposition *Suppose we are given an operad $D(n)$ with the properties:*

i) *The $\mathbb{S}_n$ action on each $D(n)$ is free.*
ii) *D affords a morphism of non-Σ operads $I : C_1 \to D$ where C_1 is an E_1 operad.*
iii) *$D(n)/\mathbb{S}_n$ is a $K(Br_n, 1)$ for the braid group Br_n where the braid action covers the symmetric group action.*
iv) *The spaces are $D(n)$ are homotopy equivalent to CW complexes.*

Then the collection of universal covers $\tilde{D}(n)$ is a B_∞ operad and hence D is equivalent as an operad to C_2, the little 2-cubes operad.

Proof Let $p : \tilde{D}(n) \to D(n)$ be the universal cover. We have to show that the spaces $\tilde{D}(n)$ form a braid operad and that they are contractible. The latter fact is true since by iii) the spaces $\tilde{D}(n)$ are weakly contractible and by assumption iv) the $D(n)$ are homotopic to a CW complex, so that the $\tilde{D}(n)$ are indeed contractible. For each n choose a component of $p^{-1}(I(C_1(n))$ which we call $\tilde{C}_1$.

The $\tilde{C}_1$ allow to lift the operad composition maps by letting $\tilde{\gamma}$

$$\begin{array}{ccc} \tilde{D}(k) \times \tilde{D}(j_1) \cdots \times \tilde{D}(j_k) & \xrightarrow{\tilde{\gamma}} & \tilde{D}(j_1 + \cdots + j_k) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ D(k) \times D(j_1) \cdots \times D(j_k) & \xrightarrow{\gamma} & D(j_1 + \cdots + j_k) \end{array}$$

be the unique lift which takes $\tilde{C}_1(k) \times \tilde{C}_1(j_1) \cdots \times \tilde{C}_1(j_k)$ to $\tilde{C}_1(j_1 + \cdots + j_k)$. To write out the braid action fix a point $c_n \in C_1(n)$ and for each i a path α_i from $I(c_n)$ to $\tau_i I(c_n)$ which lifts a non-null homotopic path of $D(n)/\mathbb{S}_n$. Notice that these satisfy the conditions that for each n and i the paths $\tau_i\tau_{i+1}(\alpha_i)\cdot\tau_i(\alpha_{i+1})\cdot\alpha_i$ and $\tau_{i+1}\tau_i(\alpha_{i+1})\cdot\tau_{i+1}(\alpha_i)\cdot\alpha_{i+1}$ are path homotopic (where $\cdot$ denotes concatenation of paths) due to condition iii). The explicit paths τ_i then provide the Br_n action on $\tilde{D}(\mathrm{n})$ again by using $\tilde{C}_1(n)$ as "base-points" to lift the $\mathbb{S}_n$ action. It is now a straightforward computation that the compositions $\tilde{\gamma}$ and the braid group action define a braid operad. Furthermore the braid group actions are free by iii) and thus the $\tilde{D}(n)$ form a B_∞ operad. □

Proof of Theorem 3.2.1 As announced we will check the conditions of Proposition 3.2.4. In our case the operad D will be the operad of spineless cacti $\mathcal{C}act$. The condition i) is obvious, since $\mathbb{S}_n$ acts freely on the labels. We showed above that $\mathcal{S}CC$ is an E_1 operad which is a suboperad of $\mathcal{C}act$. This establishes ii). The condition iii) follows from Proposition 3.3.19 below. Lastly, the condition iv) follows from the definition of the spaces $\mathcal{C}act(n) = \mathcal{C}act^1(n) \times \mathbb{R}^n_{>0}$. □

3.3 The forgetful quasi-fibration

This section is devoted to showing that the spaces $\mathcal{C}act(n)/\mathbb{S}_n$ are $K(Br_n, 1)$.

3.3.1 Definition The completed chord diagram of a cactus c without spines is the topological space obtained as follows. Cut the outside circle at the global zero, mark the two endpoints and add a chord a_z between them. If a chord started at the global zero, then the new starting point will be the right endpoint of a_z, if ended on the root then the new endpoint will be the left endpoint of a_z.

Identify each marked point (that is the added endpoints of a_z and the endpoints of the chords) of the circle with a 0-simplex and each arc connecting two marked points with a 1-simplex joining the two 0-simplices. Now for any sequence of chords connecting k points of the outside circle glue in a $k-1$ simplex, by identifying the vertices of the simplex with these points. For any chord including a_z this means that the sub-one-simplex given by the two endpoints of the chord can be identified with the chord. We let the diagram have the co-induced topology.

For an example of a completed diagram, see Figure 4.

3.3.2 Definition We define the spine of a completed chord diagram to be the following subspace. For each maximal k-simplex, fix the barycenter. First connect the barycenter to all the vertices of the simplex by a straight line, then connect the vertices of the simplices by the arcs of the outside circle to obtain the spine.

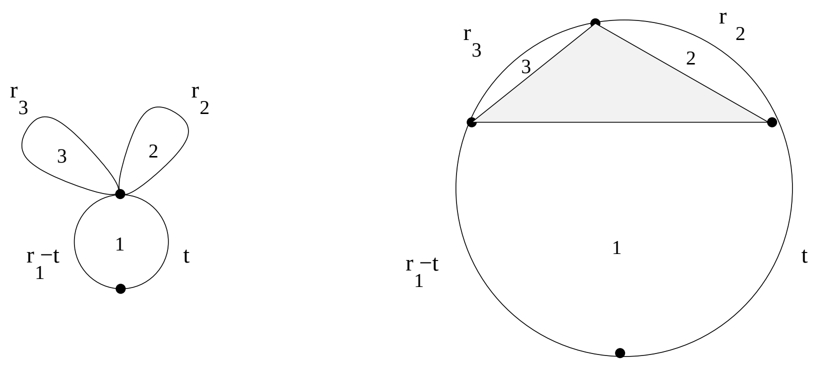

Figure 4: A cactus without spines and its completed chord diagram

3.3.3 Lemma *A completed chord diagram of a cactus c is homotopy equivalent to its spine which is homotopy equivalent to the image of c.*

Proof By retracting to the spine, the first claim follows. For the second claim in one direction we contract of the straight lines of the spine to retrieve the underlying cactus. For the homotopy inverse identify the vertices of the cactus with the barycenters. Each arc of the cactus a corresponds to a unique arc a' on the outside circle of the chord diagram. Let v_1, v_2 be the starting- and the endpoint of the directed arc. Now map each arc a of the cactus between two vertices to the path between the barycenters representing these vertices which first goes from the barycenter to the vertex v_1, then along the arc a', and finally from v_2 to the second barycenter. . □

Now we will consider the surjective map $p_{n+1} : \mathcal{C}act(n+1) \to \mathcal{C}act(n)$, which contracts the $n+1$-st lobe of the cactus. We call the image of the contracted lobe the marked point. If the root happens to lie on the component $n+1$ then the root after the contraction is fixed to be the marked point.

3.3.4 Forgetful maps Define a map

$$p^T : \mathcal{T}(n+1) \to \mathcal{T}(n)$$

by mapping a labelled tree $\tau \in \mathcal{T}(n+1)$ to $p^T(\tau) \in \mathcal{T}(n)$ which is the tree obtained from τ by coloring the vertex v_{n+1} labelled by $n+1$ black, forgetting the label and contracting all the edges incident to this vertex. If the image of the vertex v_{n+1} under the contraction only has one adjacent edge, we also delete this vertex and this edge to define $p^T(\tau)$.

This induces projection maps $p^{\Delta(\tau)} : \Delta(\tau) \to \Delta(p^T(\tau))$ projecting to the product of the first n simplices, i.e. forgetting the coordinates of the flags of v. Formally, let $E(\tau)$ be the edges of τ, and $E(v)$ be the edges incident to v. We map the point with coordinates $(x_e), e \in E(\tau)$ in $\Delta(\tau)$ to the point with coordinates $(x'_e = x_e), e \in E(p^T(\tau))$ in $\Delta(p^T(\tau))$, where we identified the non-contracted edges of τ with those of $p^T(\tau)$.

Now we define a map $p' : \mathcal{C}act^1(n+1) \to \mathcal{C}act^1(n)$ as follows. For $c' \in \mathcal{C}act^1$ let τ be its topological type. Set

$$p'(c') := e_{p^T(\tau)} \circ p^{\Delta(\tau)} \circ \dot{e}_\tau^{-1}(c)$$

Finally let $c = (c, (r_1, \dots, r_{n+1})) \in \mathcal{C}act_{n+1}$. We define

$$p(c', (r_1, \dots, r_{n+1})) = (p'(c'), (r_1, \dots, r_n))$$

This defines the map $p : \mathcal{C}act(n+1) \to \mathcal{C}act(n)$ mentioned above.

3.3.5 Proposition *The fiber of the map p over a spineless cactus c is homeomorphic to the completed chord diagram of c times $\mathbb{R}_{>0}$. The fiber of the map p' over a normalized spineless cactus c is homeomorphic to the completed chord diagram of c.*

Proof This follows directly from the above description of the map p' as projecting out the simplex for the vertex v_{n+1} labelled by $n+1$.

A detailed description is as follows: Fix a cactus $c \in \mathcal{C}act(n)$. The fiber over it can then be characterized in the following way. First there is a factor of $\mathbb{R}_{>0}$ which fixes the radius of the cactus. Then there is an interval which parameterizes the cacti where the lobe $n+1$ has no lobe above it and does not contain the root. The parametrization is via a marked point on the outside circle. In the case that the root is only on one lobe this interval is glued together with another interval to form a circle. The second interval parameterizes the pre-images obtained by gluing the $n+1$st lobe to the root and then moving the root around that lobe. If the root is moved all the way around, the limit is the same configuration as the one in which the lobe has been moved all the way around the outside circle. Also keeping the root at the intersection point is the same configuration as the initial point of the first interval. Thus we obtain a circle glued from two intervals.

If the pre-image is such that the $n+1$st lobe has higher lobes attached to it, then the marked point is necessarily a point of intersection on the cactus. First assume that this intersection point is not the global zero. This means that we can blow up this intersection point to a circle and arrange the lobes attached to it keeping their order according to the outside circle. Such a configuration is determined by the length of the arcs between the attached circles. These lengths add up to the total radius and thus are parameterized by points in a k-simplex, if the number of lobes meeting at this point is $k+1$. At the vertices of this simplex each lobe is again attached to the common intersection point. These configurations coincide with the points of the chord diagram to which the chords are attached. In the case that the global zero is at the intersection of $k>1$ lobes, then we again "blow up" the global zero to a k-simplex. One of the edges of this simplex is the interval in which the $n+1$ lobe is attached to the global zero and the global zero is moved around this lobe as discussed above and identified with the arc replacing the global zero when completing the chord diagram. The $k-2$ vertices which are not on this edge are then identified with the $k-2$ vertices of the closed sequence of chords excluding the global zero. □

3.3.6 Corollary *The fiber of the map p over a spineless cactus c is homotopy equivalent to the image of the cactus c in $\mathbb{R}^2$ and is thus homotopy equivalent to a bouquet of n circles $\bigvee_n S^1$.*

Proof The first equivalence follows from Lemma 3.3.3. The second equivalence is straightforward. □

3.3.7 Remark Let $\mathcal{UC}act(n)$ be the set obtained from $\mathcal{C}act(n+1)$ by contracting the simplices of the completed chord diagrams in each fiber of the map $p: \mathcal{C}act(n+1) \to \mathcal{C}act(n)$. Let $\rho_{n+1}: \mathcal{C}act(n+1) \to \mathcal{UC}act(n)$ be the induced surjection and endow $\mathcal{UC}act(n)$ with the quotient topology. The map p factors though ρ, that is $p_{n+1} = \tilde{p}_n \circ \rho_{n+1}$ where $\tilde{p}_n: \mathcal{UC}act(n) \to \mathcal{C}act(n)$ is the universal map whose fiber over a spineless cactus c is the image of that spineless cactus and whose total space is $\mathcal{UC}act(n) = \bigcup_{c \in \mathcal{C}act(n)} Im(c)$. A point in this space is a cactus together with an additional marked point on the cactus. The map $\tilde{p}$ forgets this point.

Then $p: \mathcal{C}act(n+1) \to \mathcal{C}act(n)$ is fiberwise homotopy equivalent to the universal map $\tilde{p}_n: UCact(n) \to \mathcal{C}act(n)$.

3.3.8 Remark The maps p and p' are not fibrations. We will show that they are, however, quasi-fibrations.

3.3.9 Definition Let $c, c' \in \mathcal{C}act^1$ and τ, τ' be their topological types. We say that c' can be derived from c and also that τ' can be derived from τ

if $e_{\tau'}(\Delta(\tau')) \subset e_\tau(\Delta(\tau)) \subset \mathcal{C}act^1$. Here $\Delta(\tau)$ is the product of simplices as defined in equation (2.1).

If the inclusion is proper, we say that c' is a degeneration of c and also say τ' is a degeneration of τ.

3.3.10 Remark The notion of degeneration induces a partial order on $\mathcal{T}$ where $\tau' \prec \tau$ if τ' is a degeneration of τ.

3.3.11 Definition If there is a τ'' s.t. $c, c' \in e_{\tau''}(\Delta(\tau''))$ we say c, c' share the common type τ'' and also say that τ, τ' share the common type τ''.

In case c and c' share a common type τ'', we let $d_{\tau''}(c, c')$ be the distance between their lifts into $\Delta(\tau'')$.

3.3.12 Definition For $c \in \mathcal{C}act(n)$, τ with $c \in e_\tau(\Delta(\tau))$ and $\epsilon > 0$ we define

$$\begin{aligned} U(c,\epsilon,\tau) &:= \{c' \in e_\tau(\Delta(\tau)) | d_\tau(c,c') < \epsilon\} \text{ and} \\ U(c,\epsilon) &:= \bigcup_{\tau : c \in e_\tau(\Delta(\tau))} U(c,\epsilon,\tau) \end{aligned} \tag{3.1}$$

It is clear that the $U(c,\epsilon)$ are open.

We call ϵ small for c if $c' \in U(c,\epsilon)$ implies that c is a degeneration of c'.

3.3.13 Remark The set of small ϵ for a fixed c is non-empty. For instance if A is the set of arcs of c, any $\epsilon < \frac{1}{2}\min(|a| : |a| \neq 0)$ will do, since one cannot move the root or a lobe more than the length of any arc and hence cannot create new degenerations without going beyond the distance ϵ.

3.3.14 Lemma

i) *The sets $U(c,\epsilon)$ with ϵ small for c are open and contractible.*
ii) *The sets $U(c,\epsilon)$ with $c \in \mathcal{C}act^1(n)$ cover $\mathcal{C}act^1(n)$.*
iii) *If $c'' \in U(c,\epsilon) \cap U(c',\epsilon')$ then there exists an ϵ'' s.t. $c'' \in U(c'',\epsilon'') \subset U(c,\epsilon) \cap U(c',\epsilon')$.*

Proof The fact that these sets are open and cover is immediate. For the contraction we define $h : U(c,\epsilon) \times I \to U(c,\epsilon)$ as follows: for $c' \in U(c,\epsilon)$ with topological type τ', we set $c'(t)$ to be the image of the point of $\Delta(\tau')$ which is at distance $\epsilon - t/\epsilon$ from the point corresponding to c in $\Delta(\tau')$ along the unique line joining these two points. This map is easily seen to be continuous and contracts $U(c,\epsilon)$ onto c.

For part ii) let τ, τ', τ'' be the respective topological types. If $d_{\tau''}(c, c'') = d_1$ and $d_{\tau''}(c', c'') = d_2$ fix any $\epsilon'' < \min(\sqrt{\epsilon^2 - d_1^2}, \sqrt{\epsilon'^2 - d_2^2})$. Then the inclusion follows from the fact that $\prec$ is a partial order, i.e. if $\tau'' \preceq \tau'''$ then also $\tau \preceq \tau'''$ and $\tau' \preceq \tau'''$ since $\tau \preceq \tau''$ and $\tau' \preceq \tau''$. □

3.3.15 Remark The contraction simultaneously contracts all those arcs of c' which do not appear in c, i.e. those which correspond to the edges of τ' which are contracted to obtain τ.

3.3.16 Lemma *The pair* $({p'}^{-1}(U(c,\epsilon)), {p'}^{-1}(c))$ *is homotopy equivalent to* $({p'}^{-1}(c), {p'}^{-1}(c))$.

Proof We define the homotopy

$$H : ({p'}^{-1}(U(c,\epsilon)), {p'}^{-1}(c)) \times I \to ({p'}^{-1}(U(c,\epsilon)), {p'}^{-1}(c))$$

as follows. Given $\hat{c}' \in {p'}^{-1}(U(c,\epsilon))$ write it as the tuple $(c', ch(c'))$ where $c' = p'(\hat{c}')$ and $ch(\hat{c}')$ is the point in the fiber over c'. By Proposition 3.3.5 $ch(\hat{c}')$ is a unique point of the completed chord diagram $Chord(c')$ of c'. We let $H(t)(c') = (c'(t), ch(c')(t))$ where $c'(t) := h(c', t)$ is the cactus as in Lemma 3.3.14 and $ch(\hat{c}')(t) \in Chord(c'(t))$ is defined as follows. First notice that during the homotopy h the topological type τ' of c' does not change as long as $t \neq 1$ and therefore there are natural homeomorphisms $h_{chord}(t) : Chord(c') \to Chord(c'(t))$ obtained by a homogeneous re-scaling of the arcs of the outside circle by factors $x_a(c)(t)/x_a(c)$ -where again the x_a are the coordinates in $\Delta(\tau)$. For $t \neq 1$ we set $ch(\hat{c}')(t) := h_{chord}(t)(ch(\hat{c}'))$.

In order to extend to $t = 1$ notice that the chords of $Chord(c')$ and those of $Chord(c)$ are in 1-1 correspondence as 1-simplices, as they correspond to the labels 1 through n. Therefore the simplices of $Chord(c')$ uniquely correspond to faces of the simplices or simplices of $Chord(c)$. We let $h_{chord}(1) : Chord(c') \to Chord(c)$ be the map that first contracts the arcs of the outside circle which are indexed by arcs a with $x_a(c) = 0$ and then identifies the result of this contraction with a subset of $Chord(c)$ by identifying the arcs of the outside circle with the same labels and identifying the simplices of $Chord(c')$ with the respective faces of $Chord(c)$. Finally set $ch(\hat{c}')(1) = h_{chord}(1)(ch(\hat{c}'))$. It is now easy to check that the defined map is indeed a homotopy. □

3.3.17 Remark The effect of the contraction above is to contract the arcs not belonging to c while keeping the lobe $n+1$ in its relative place.

3.3.18 Proposition *$p' : \mathcal{C}act^1(n+1) \to \mathcal{C}act^1(n)$ and $p : \mathcal{C}act(n+1) \to \mathcal{C}act(n)$ are quasi-fibrations.*

Proof First let's handle p': $p'|p'^{-1}(U(c,\epsilon))$ is a quasi-fibration by the Lemma 3.3.16 above. This fact together with Lemma 3.3.14 shows that the conditions of the Dold-Thom criterium [DT][Satz 2.2] are met and hence p' is a quasi-fibration. Now fix some base-point $c = (c', \vec{r})$, then the claim follows from the following equalities:

$$\begin{aligned}\pi_i(\mathcal{C}act(n+1), p^{-1}(c)) &= \pi_i(\mathcal{C}act^1(n+1) \times \mathbb{R}^{n+1}_{>0}, p^{-1}(c', \vec{r})) \\ &= \pi_i(\mathcal{C}act^1(n+1), p'^{-1}(c')) = \pi_i(\mathcal{C}act^1(n), c') = \pi_i(\mathcal{C}act(n), (c, \vec{r})) \quad (3.2)\end{aligned}$$

where the first equality holds by definition, the second holds since the pair $(\mathcal{C}act^1(n+1) \times \mathbb{R}^{n+1}_{>0}, p^{-1}(c', \vec{r}))$ is homotopy equivalent to the pair $(\mathcal{C}act^1(n+1), p'^{-1}(c'))$ by contracting the factors $\mathbb{R}_{>0}$ to the point 1, the third equation holds, since p' is a quasi-fibration and finally the last equation holds since again by contraction of the factors $\mathbb{R}_{>0}$ the pair $(\mathcal{C}act^1(n), c')$ is homotopy equivalent to the pair $(\mathcal{C}act(n), (c, \vec{r}))$. □

3.3.19 Proposition *The spaces $\mathcal{C}act(n)$ are $K(PBr_n, 1)$ spaces and the spaces $\mathcal{C}act(n)/\mathbb{S}_n$ are $K(Br, 1)$ spaces.*

Proof Since by Proposition 3.3.18 p is a quasi-fibration, we have the long exact sequence of homotopy groups [DT]

$$\to \pi_{i+1}(\mathcal{C}act(n)) \to \pi_i(\bigvee_n S^1) \to \pi_i(\mathcal{C}act(n+1)) \to \pi_i(\mathcal{C}act(n))$$

where we inserted $\pi_i(p^{-1}(c)) = \pi_i(\bigvee_n S^1)$.

The fibration p admits a section, for instance attaching the (n+1)st lobe at the root and letting the root lie on the new (n+1)st lobe. Thus the long exact sequence for this quasi-fibration splits.

First notice that since $\mathcal{C}act(1) = *, \mathcal{C}act(2) = S^1$ by induction $\pi_i(\mathcal{C}act(n)) = 0$ for $k \geq 2$ and so also $\pi_i(\mathcal{C}act(n)/\mathbb{S}_n) = 0$ for $k \geq 2$.

For the first homotopy group, we fix some data. Choose the spineless corolla cactus with radii all equal to one and labelling $1, 2, \dots, n$ as the base point c_n of $\mathcal{C}act(n)$ and choose the paths α_i as indicated in Figure 5 which makes the braid action explicit.

Note that the braid condition that for each n and i the paths $\tau_i\tau_{i+1}(\alpha_i) \cdot \tau_i(\alpha_{i+1}) \cdot \alpha_i$ and $\tau_{i+1}\tau_i(\alpha_{i+1}) \cdot \tau_{i+1}(\alpha_i) \cdot \alpha_{i+1}$ are path homotopic (where $\cdot$

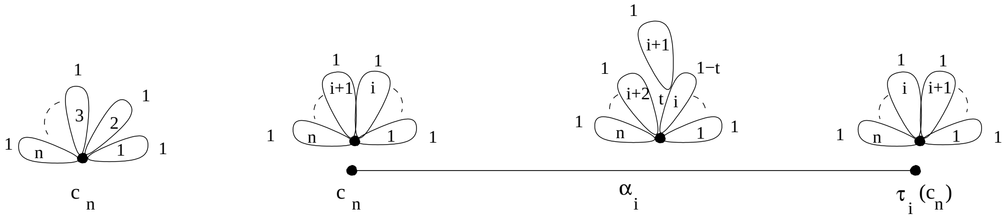

Figure 5: The point c_n and the path α_i

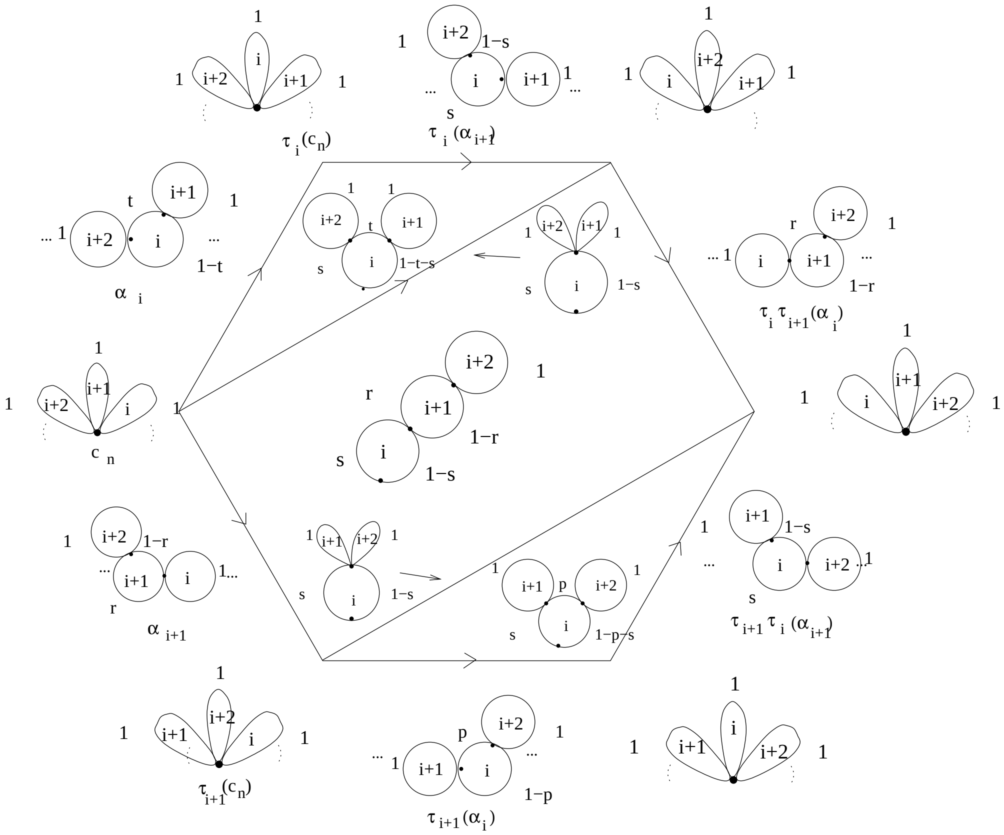

Figure 6: The braid homotopy

denotes concatenation of paths) is verified explicitly in Figure 6 were we have only drawn the relevant three lobes and indicated the other lobes by dots.

Now we proceed by induction on n assuming $\pi_1(\mathcal{C}act(n)) = PBr_n$ where PBr_n is the pure braid group and $\pi_1(\mathcal{C}act(n)/\mathbb{S}_n) \simeq Br_n$ with the explicit maps $Br_n \to \pi_1(\mathcal{C}act(n)/\mathbb{S}_n)$ is given by $b_i \mapsto \alpha_i$. Notice $\pi_1(\mathcal{C}act(2)) \simeq \mathbb{Z} = PBr_2$

generated by $\tau_1(\alpha_1)\cdot\alpha_1$ and $\pi_1(\mathcal{C}act(2)/\mathbb{S}_2)\simeq Br_2\simeq\mathbb{Z}$ generated by α_1.

First we treat $\pi_1(\mathcal{C}act(n+1))$. For this consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\bigvee_n S^1) \arrow[r] \arrow[d, equal] & PBr_{n+1} \arrow[r] \arrow[d] & PBr_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\bigvee_n S^1) \arrow[r] & \pi_1(\mathcal{C}act(n+1)) \arrow[r] & \pi_1(\mathcal{C}act(n)) \arrow[r] & 1
\end{tikzcd}$$

Here the second line follows from the long exact sequence and the first line is a classic fact. It follows e.g. from regarding the long exact sequence for the forgetful map between the configuration spaces of $n+1$ and n ordered points in $\mathbb{R}^2$ which forgets the $n+1$st point. By induction we know the map $Br_n\to\pi_1(\mathcal{C}act_n/\mathbb{S}_n)$ sending the generator b_i which maps to the transposition τ_i in $\mathbb{S}_n$ to $[\alpha_i]\in\mathcal{C}act(n)/\mathbb{S}_n$ is an isomorphism. The right down arrow is its restriction to PBr_n which is the isomorphism sending the generators $\xi_{ij}=b_ib_{i+1}\dots b_{j-1}b_j^2b_{j-1}^{-1}\dots b_i^{-1}\,:\,1\leq i<j\leq n$ of PBr_n to the class $[\alpha_{ij}]\in\pi_1(\mathcal{C}act(n))$ where α_{ij} is the closed path $\tau_{i+1}\dots\tau_{j-1}\tau_j^2\tau_{j-1}^{-1}\dots\tau_i^{-1}(\alpha_i)\cdot\dots\cdot\tau_j\dots\tau_i(\alpha_j)\cdot\tau_{j-1}\dots\tau_i(\alpha_j)\cdot\dots\cdot\tau_i(\alpha_{i+1})\alpha_i^{-1}$. Choosing the base point of the fiber to be the cactus c_n we see that the generators of $\pi_1(Chord(c_n))$ can be identified with the paths α_{in+1} hence identifying the left isomorphism as ξ_{in+1} maps to $[\alpha_{in+1}]$. Hence we have a diagram of group extensions and the middle arrow which sends $\xi_{i,j}$ to $[\alpha_{ij}]$ for $1\leq i<j\leq n+1$ is also an isomorphism.

For the fact that $\pi_1(\mathcal{C}act(n+1)/\mathbb{S}_{n+1})=Br_{n+1}$ consider the following diagram of group extensions

$$\begin{tikzcd}
1 \arrow[r] & PBr_{n+1} \arrow[r] \arrow[d] & Br_{n+1} \arrow[r] \arrow[d] & \mathbb{S}_{n+1} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{C}act(n+1)) \arrow[r] & \pi_1(\mathcal{C}act(n+1)/\mathbb{S}_{n+1}) \arrow[r] & \mathbb{S}_{n+1} \arrow[r] & 1
\end{tikzcd}$$

where the left arrow was shown to be an isomorphism above and the commutativity follows from the explicit mapping of the upper to the lower row which sends b_i to the respective class of α_i. Hence the middle arrow is an isomorphism, proving the claim. $\square$

3.3.20 Remark In our proof, we chose the E_1 operad of spineless corollas. If we want to use little intervals there is an obvious map by assigning the lengths of the little intervals to be the sizes the lobes, but this map is only an operad map up to homotopy due to the different scalings. To remedy the situation, we could augment the spineless cacti operad to a larger homotopy equivalent model. In this model the outside circle will be additional data. It will have to be orientation preserving but not necessarily injective. The parameterizations will be allowed to have stops at the intersections points and the start (the global zero).

3.4 The Gerstenhaber Structure

Due to the theorem of Cohen [C1, C2] identifying Gerstenhaber algebras with algebras over the homology of the little discs operad and the Theorem 3.2.1 above, we know that algebras over the homology of the operad of spineless cacti are Gerstenhaber algebras.

3.4.1 The explicit presentation of the operations in normalized spineless cacti Parallel to [CS, V, KLP], we can give explicit generators for the operations on the chain level yielding the Gerstenhaber structure on the homology spineless cacti (see Figure 7).

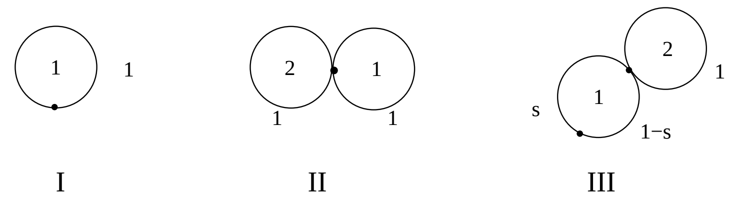

Figure 7: I: The identity II: The product $\cdot$ III: The operation $*$

We would like to emphasize that the product $\cdot$ is *associative* on the nose already on the chain level. As usual, the multiplication $*$ defines the bracket via the odd commutator.

$$\{a, b\} := a * b - (-1)^{(|a|+1)(|b|+1)} b * a \tag{3.3}$$

where we denoted the degree of a and b by $|a|$ and $|b|$. Its iterations are given in Figure 8 from which one can also read off the associator (pre-Lie) relation which guarantees the odd Jacobi identity.

Using the dual graph construction of Appendix B all the other chain homotopies can be made explicit by translating them from [KLP] to normalized cacti.

Just like Corollary 3.1.5 one obtains:

3.4.2 Corollary *The operad of the decorated moduli space of bordered, punctured surfaces with marked points on the boundary $\widetilde{M}^s_{g,r}$ which is proper*

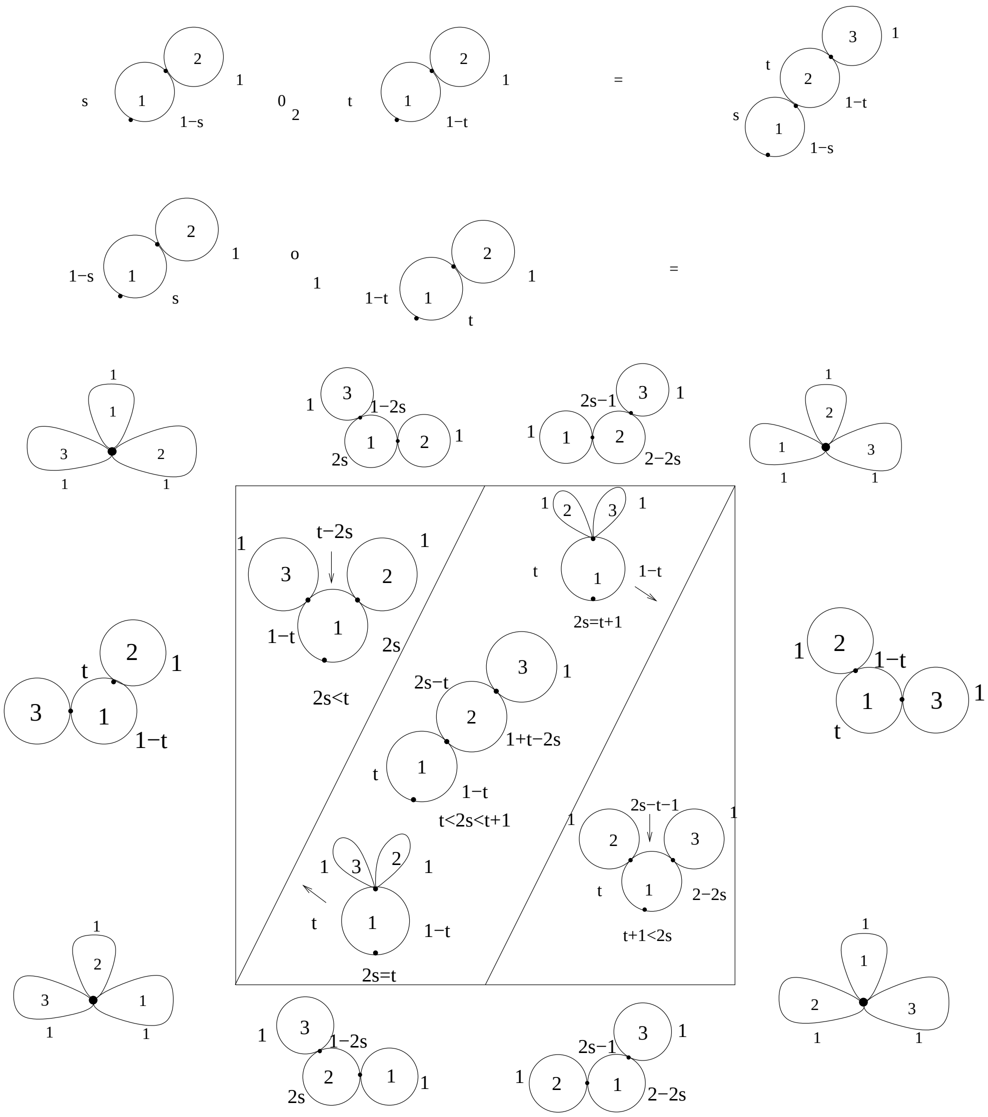

Figure 8: The associator in normalized spineless cacti

homotopy equivalent to $\mathcal{A}rc_{\#}$ contains an E_2 operad. Thus so does the operad $\mathcal{A}rc$.

The same is true for the operad of the moduli spaces $M^n_{g,n}$ of genus g curves with n punctures and n tangent direction and its restriction to genus 0.

Finally the spaces $M^1_{g,n}$ of surfaces of genus g, n marked points and a tangent *vector at the first marked point form a partial operad which is an E_2 operad.*

Proof The proof of all but the last statement is analogous to the proof of Corollary 3.1.5. For the last statement we need to use the fact that there is a cell model for $M^1_{g,n}$ by ribbon graphs with one marking (cf. [K3]) and that the rooted tree-like ribbon graphs given by $\mathcal{C}act$ form an operad structure which gives partial operad on the whole space. □

3.4.3 Remark Using Cohen's Theorem this means that on the cell level algebras over these operads will be Gerstenhaber algebras up to homotopy and on the homology level Gerstenhaber algebras. In particular the operads themselves possess these properties.

The operad $M_{g,n}$ is not included in Corollary 3.4.2, since we do need the markings of a global root on the ribbon graphs in order to define the gluing. For spineless cacti one such marking is enough, however.

3.4.4 Remark We wish to point out that the chain defining the product $*$ is exactly the path α_1 for $\mathcal{C}act(2)$ and diagram for the associator Figure 8 coincides up to re-parametrization with the braid relation for the paths α_1 and α_2 in $\mathcal{C}act(3)$.

One can obtain all paths α and all braid relation by taking quasi-operadic products with spineless corolla cacti whose radii are all one. These can be viewed as a quasi-sub-operad of the quasi-operad of normalized spineless cacti. In fact, the base-points c_n suffice for this, i.e the path α_i in $\mathcal{C}act(n)$ is $c_{n-2} \circ_i \alpha_1$ and likewise one obtains the braid relation.

This nicely ties together the point of view of [F1] and [C1, C2] in relating the Gerstenhaber structure directly to the braided structure.

4 Examples and constructions

In this section, we collect constructions and results which we will modify and use in section 5 to relate our different varieties of cacti in terms of semi-direct and bi-crossed products.

4.1 Operads of spaces

The following procedure is motivated by topological spaces with Cartesian product, but actually works in any strict symmetric monoidal category where the monoidal product is a product (i.e. we have projection maps).

Let X be a topological space, then we can form the iterated Cartesian product $X \times \cdots \times X$. We simply denote the n-fold product by $X(n)$. This space has an action of $\mathbb{S}_n$ by permutation of the factors. We denote the corresponding morphism also by elements of $\mathbb{S}_n$. Given a subset $I \subset \{1, \dots, n\}$ we denote the projection $\pi_I : X(n) \to X(I) = \times_{i \in I} X$.

$$\bar{\circ}_i : X(n) \times X(m) \stackrel{\pi_{\{1,\dots,n\}\setminus i}}{\to} X(n-1) \times X(m) \to X(m+n-1) \stackrel{\sigma}{\to} X(m+n-1) \quad (4.1)$$

where $\sigma \in \mathbb{S}_{m+n-1}$ is the permutation that shuffles the last m factors into the place i. I.e.

$$((x_1, \dots, x_n)\bar{\circ}_i(x'_1, \dots, x'_m)) = (x_1, \dots x_{i-1}, x'_1, \dots, x'_m, x_{i+1}, \dots, x_n). \quad (4.2)$$

4.1.1 The cyclic version Using $X((n)) = X(n+1)$ with the $\mathbb{S}_{n+1}$ action and the gluing above, one obtains a cyclic version of the construction.

4.2 Operads built on monoids

Let S be a monoid with associative multiplication $\mu : S \times S \to S$. For simplicity we will denote this multiplication just by juxtaposition: $s, s' \in S; ss' := \mu(s, s')$. We will take S to be an object in a strict symmetric monoidal category.

We set $S(n) := S^{\times n}$ and endow it with the permutation action.

4.2.1 An operad defined by a monoid We consider the following maps:

$$\begin{array}{rcl} \circ_i : S(n) \times S(m) & \to & S(n+m-1) \\ ((s_1, \dots, s_n), (s'_1, \dots, s'_m)) & \mapsto & (s_1, \dots s_{i-1}, s_i s'_1, \dots, s_i s'_m, s_{i+1}, \dots, s_n) \end{array}$$

It is straightforward to check that these maps define an operad in the same category as S. This operad is unital if S is unital.

4.2.2 Examples

1) One standard example is that of a Lie group in topological spaces.

2) Another nice example is that of a field k. Then $k(n) = k^n$ and the gluing is plugging in vectors into vectors scaled by scalar multiplication. There are $\mathbb{Z}$-graded and super versions of this given by including the standard supersign for the permutation action where in the $\mathbb{Z}$-graded version one uses the induced $\mathbb{Z}/2\mathbb{Z}$-grading.
3) The example S^1 is particularly nice. In this case (see e.g. [KLP]), we can see that for the monoid S^1 the homology operad $H_*(S^1(n), \mathbb{Z}/2\mathbb{Z})$ of its induced operad $S^1(n)$ is isomorphic, though not naturally, to the operad built on $\mathbb{Z}/2\mathbb{Z}$ and that for a field k, $H_*(S^1(n), k)$ is isomorphic to the direct product of the operads $Comm$ and the operad built on the monoid $\mathbb{Z}/2\mathbb{Z}$.

4.2.3 Remark There are several other natural versions of operads and cyclic operads which can be defined analogously to the operads built on circles which are presented in [KLP].

4.3 Semi-direct products with monoids

We now turn to the situation where the monoid S acts on all the components of an operad.

I.e. Let $Op(n)$ be an operad in a symmetric monoidal category $\mathcal{C}$ and let S be a monoid in the same category such that S acts on $Op(n)$.

$$S \times Op(n) \stackrel{\rho}{\to} Op(n)$$

s.t. the following diagrams are commutative

$$\begin{array}{ccc} S \times S \times Op(n) & \xrightarrow{\mu \times id} & S \times Op(n) \\ {\scriptstyle id\times\rho}\uparrow & & \downarrow{\scriptstyle \rho} \\ S \times Op(n) & \xrightarrow{\rho} & Op(n) \end{array}$$

$$\begin{array}{ccccc} S \times Op(m) \times Op(n) & \xrightarrow{id\times\circ_i} & S \times Op(m+n-1) & \xrightarrow{\rho} & Op(m+n-1) \\ {\scriptstyle \Delta\times id\times id}\downarrow & & & & \uparrow{\scriptstyle \circ_i} \\ S\times S\times Op(m)\times Op(n) & \xrightarrow{\sigma_{23}} & S\times Op(m)\times S\times Op(n) & \xrightarrow{\rho\times\rho} & Op(m)\times Op(n) \end{array}$$

Consider the action

$$\begin{aligned} \rho_i : S(n) \times Op(m) &\to Op(m) \\ ((s_1, \dots, s_n), o) &\mapsto \rho(s_i)(o) \end{aligned} \tag{4.3}$$

and the twisted multiplications

$$\begin{aligned} \circ_i^s : Op(n) \times Op(m) &\to Op(n+m-1) && (4.4)\\ (o, o') &= (O \circ_i \rho(s, O')) && (4.5) \end{aligned}$$

It is straightforward to check that the action ρ satisfies the conditions of Lemma 1.3.5.

4.3.1 Definition We define the semi-direct product $Op \rtimes S$ of an operad Op with a monoid S in the same category to be given by the spaces

$$(Op \rtimes S)(n) := Op(n) \times S(n)$$

with diagonal $\mathbb{S}_n$ action and the compositions

$$\begin{aligned} \circ_i^{\rtimes} : (Op \rtimes S)(n) \times (Op \rtimes S)(m) &\to (Op \rtimes S)(n+m-1)\\ ((O, s), (O', s')) &:= (O \circ_i^{\pi_i(s)} O', s \circ_i s') && (4.6) \end{aligned}$$

where π_i is the projection to the i-th component.

4.3.2 Proposition *Given an action ρ and the multiplications $\circ i^{\rtimes}$ defining the quasi-operad structure of the semi-direct product as above, the semi-direct product quasi-operad is an operad.*

4.3.3 Example The operad of framed little discs is the semi-direct product of the little discs operad with the operad based on the monoid given by the circle group S^1 [SW].

4.3.4 The action If we are in a category that satisfies the conditions of 4.1, we can break down the operad structure based on a monoid into two parts. The first is the structure of operads of spaces and the second is the diagonal action.

More precisely let $\Delta : S \to S(n)$ be the diagonal and $\mu : S \times S \to S$ be the multiplication:

$$\rho_\Delta : S \times S(n) \stackrel{\Delta}{\to} S(n) \times S^n \stackrel{\mu^n}{\to} S(n)$$

Here we denote by μ^n the diagonal multiplication:

$$\mu^n((s_1, \dots, s_n), (s'_1, \dots, s'_n)) = (s_1 s'_1, \dots, s_n s'_n)$$

$$\begin{aligned} \circ_i : S(n) \times S(m) &\stackrel{(id\times\pi_i)(\Delta)\times id}{\longrightarrow} S(n) \times S \times S(m)\\ &\stackrel{id\times\rho_\Delta}{\longrightarrow} S(n) \times S(m) \stackrel{\bar{\circ}_i}{\to} S(n+m-1) \quad (4.7) \end{aligned}$$

where $\bar{\circ}_i$ is the operation of the operad of spaces and

$$(id \times \pi_i)(\Delta) : S(n) \stackrel{\Delta}{\to} S(n) \times S(n) \stackrel{id\times\pi_i}{\to} S(n) \times S.$$

5 The relations of the cacti operads

5.1 The relation between normalized (spineless) cacti and (spineless) cacti

5.1.1 The scaling operad We define the scaling operad $\mathcal{R}_{>0}$ to be given by the spaces $\mathcal{R}_{>0}(n) := \mathbb{R}^n_{>0}$ with the permutation action by $\mathbb{S}_n$ and the following products

$$(r_1,\dots,r_n)\circ_i(r'_1,\dots,r'_m) = (r_1,\dots r_{i-1},\frac{r_i}{R}r'_1,\dots,\frac{r_i}{R}r'_m,r_{i+1},\dots r_n)$$

where $R=\sum_{k=1}^m r'_k$. It is straightforward to check that this indeed defines an operad.

5.2 The perturbed compositions

We define the perturbed compositions

$$\circ_i^{\mathcal{R}_{>0}} : \mathcal{C}acti^1(n)\times\mathcal{R}_{>0}(m)\times\mathcal{C}acti^1(m)\to\mathcal{C}acti^1(n+m-1) \tag{5.1}$$

via the following procedure: Given $(c,\vec{r}',c')$ we first scale c' according to $\vec{r}'$, i.e. scale the j-th lobe of c' by the j-th entry r_i of $\vec{r}$ for all lobes. Then we scale the i-th lobe of the cactus c by $R=\sum_j r_j$ and glue in the scaled cactus. Finally we scale all the lobes of the composed cactus back to one.

We also use the analogous perturbed compositions for $\mathcal{C}act^1$.

5.2.1 The perturbed multiplications in terms of an action We can also describe, slightly more technically, the above compositions in the following form. Fix an element $\vec{r}:=(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$ and set $R=\sum_i r_i$ and a normalized cactus c with n lobes. Denote by $\vec{r}(c)$ the cactus where each lobe has been scaled according to $\vec{r}$, i.e. the j-th lobe by the j-th entry of $\vec{r}$. Now consider the chord diagram of the cactus $\vec{r}(c)$. It defines an action on S^1 via

$$\rho : S^1 \xrightarrow{rep^1_R} S^1_R \xrightarrow{cont_{\vec{r}(c)}} S^1_n \xrightarrow{rep^n_1} S^1 \tag{5.2}$$

Where $cont_{\vec{r}}$ acts on S^1_R in the following way. Identify the pointed S^1_R with the pointed outside circle of the chord diagram of $\vec{r}(c)$. Now contract the arcs belonging to the i-th lobe homogeneously with a scaling factor $\frac{1}{r_i}$.

We think of the i-lobe of a normalized (spineless) cactus as an S^1 with base point given by the local zero together with additional marked points; the additional marked points are the intersection points. Using the map above on the i-lobe we thus obtain maps:

$$\begin{aligned} \rho_i : \mathcal{C}act^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}act^1(m) &\to \mathcal{C}act^1(n) \\ \rho_i : \mathcal{C}acti^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}acti^1(m) &\to \mathcal{C}acti^1(n) \end{aligned} \tag{5.3}$$

What this action effectively does is move the lobes and if applicable the root of the cactus c which are attached to the i-th lobe according to the cactus $\vec{r}(c')$ in a manner that depends continuously on $\vec{r}$ and c'.

With this action we can write the perturbed multiplication as:

$$\begin{aligned} \circ_i^{\mathcal{R}_{>0}} : \mathcal{C}acti^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}acti^1(m) \\ \xrightarrow{id \times id \times \Delta} \mathcal{C}acti^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}acti^1(m) \times \mathcal{C}acti^1(m) \\ \xrightarrow{\rho_i \times id} \mathcal{C}acti^1(n) \times \mathcal{C}acti^1(m) \xrightarrow{\circ_i} \mathcal{C}acti^1(n+m-1) \end{aligned} \tag{5.4}$$

5.2.2 Theorem *The operad of spineless cacti is isomorphic to the operad given by the semi-direct product of their normalized version with the scaling operad. The latter is homotopy equivalent (through quasi-operads) to the direct product as a quasi-operad. The direct product is in turn equivalent as a quasi-operad to $\mathcal{C}act^1$. The same statements hold true for cacti.*

$$\begin{aligned} \mathcal{C}act &\cong \mathcal{R}_{>0} \ltimes \mathcal{C}act^1 \sim \mathcal{C}act^1 \times \mathcal{R}_{>0} \simeq \mathcal{C}act^1 \\ \mathcal{C}acti &\cong \mathcal{R}_{>0} \ltimes \mathcal{C}acti^1 \sim \mathcal{C}acti^1 \times \mathcal{R}_{>0} \simeq \mathcal{C}acti^1 \end{aligned} \tag{5.5}$$

here the semi-direct product compositions are given by:

$$(\vec{r}, c) \circ_i (\vec{r}', c') = (\vec{r} \circ_i \vec{r'}, c \circ_i^{\vec{r'}} c') \tag{5.6}$$

Proof By definition the space $\mathcal{C}act(n) = \mathcal{C}act^1(n) \times \mathbb{R}^n_{>0} = \mathcal{C}act^1(n) \times \mathcal{R}_{>0}(n)$. To establish that the operad structure of $\mathcal{C}act$ is that of a semi-direct product, first notice that the behavior of the radii under gluing is given by the scaling operad. Second we notice that the global incidence conditions, i.e. the positions of the intersection points on the outer circle, are shifted under the scaling in a way that is compensated by perturbed multiplication. This means that when gluing, we do not use the outside circle of the normalized cactus, but the outside circle of the original cactus which is recovered form the outside circle of the normalized cactus and the radii by the action of the scaling operad.

Secondly, the perturbed multiplications are homotopic to the unperturbed multiplications: a homotopy is for instance given by a the choice of a paths $\vec{r}_t$ given by the line segment from $\vec{r}$ to $(1,\dots,1)$ and the following homotopy for the quasi-operadic compositions

$$(\vec{r},c)\circ_{i,t}(\vec{r}',c') = (\vec{r}\circ_i \vec{r'}, c\circ_i^{\vec{r}'_t} c')$$

and the composition $\circ_i^{\vec{r}'_t}$ uses the action chord diagram of $\vec{r}'_t(C')$, i.e. the cactus C' scaled by $\vec{r}'_t$, for the gluing.

Lastly, the $\mathcal{R}(n)=\mathbb{R}^n_{>0}$ are contractible and the contraction induces an equivalence of quasi-operads.

The analogous arguments hold for $\mathcal{C}acti$. □

5.2.3 Corollary *Cacti without spines are homotopy equivalent to normalized cacti without spines as quasi-operads. The quasi-operad of homology normalized spineless cacti is an operad which is isomorphic to the homology of the spineless cacti operad.*

Also, cacti and normalized cacti are homotopy equivalent as quasi-operads and the homology quasi-operad of normalized cacti is an operad which is isomorphic to the homology of the cacti operad.

Proof We first remark that there is a homotopy of quasi-operads to the direct product, then there is a homotopy of quasi-operads from the scaling operad to the operad of spaces. Finally there is a homotopy between the operad of spaces built on $\mathbb{R}_{>0}$ and a point in each degree. □

From the previous analysis, we obtain:

5.2.4 Corollary *The quasi-operads of normalized cacti and normalized spineless cacti are both homotopy associative.*

5.2.5 Remark It is shown in [K1] that the quasi-operad structure induced on the cellular chains $CC_*(\mathcal{C}act^1)$ is actually an operad structure. This can be seen from the explicit description of the semi-direct product above. It follows from the Theorem above and Theorem 3.2.1 that $CC_*(\mathcal{C}act^1)$ is a cell model for the little discs operad.

The fact that the cells are indexed by trees then yields a quick proof of Deligne's conjecture [K1] which states that the Hochschild cochains of an associative algebra are an algebra over a cell model operad of the little discs.

5.3 The relation between spineless cacti and cacti

We would now like to specialize the monoid of section 4 to $S = S^1$. We already showed that the normalized and non-normalized versions of the different species of cacti are homotopic and moreover that they are related by taking the direct product with the scaling operad. Below we will see that cacti with spines are a bi-crossed product of the cacti without spines and the operad built on S^1. Furthermore we show that this bi-crossed product is homotopic to the semi-direct product. Thus we see that the relation of cacti with and without spines is analogous to the relation of framed little discs and little discs.

5.3.1 The action of S^1 and the twisted gluing There is an action of S^1 on $\mathcal{C}act(n)$ given by rotating the base point *clockwise* around the perimeter. We denote this action by

$$\rho^{S^1} : S^1 \times \mathcal{C}act(n) \to \mathcal{C}act(n).$$

With this action we can define the twisted gluing:

$$\begin{array}{rcl} \circ_i^{S^1} : \mathcal{C}act(n) \times S^1(n) \times \mathcal{C}act(m) & \to & \mathcal{C}act(n+m-1) \\ (C, \theta, C') & \mapsto & C \circ \rho^{S^1}(\theta_i, C') =: C \circ_i^{\theta_i} C' \end{array} \quad (5.7)$$

5.3.2 The homotopy diagonal defined by a spineless cactus Given a cactus without spines $C \in \mathcal{C}act(n)$ the orientation reversed perimeter (i.e. going around the outer circle *clockwise*) gives a map

$$\Delta_C : S^1 \to (S^1)^n. \quad (5.8)$$

As one goes around the perimeter the map goes around each circle once and thus the map Δ_C is homotopic to the diagonal

$$\Delta_C(S^1) \sim \Delta(S^1). \quad (5.9)$$

A picture of the image of Δ_C for a two component cactus is depicted in Figure 9.

5.3.3 The action based on a cactus We can use the map Δ_C to give an action of S^1 and $(S^1)^{\times n}$.

$$\rho^C : S^1 \times (S^1)^{\times n} \stackrel{\Delta_C}{\to} (S^1)^{\times n} \times (S^1)^{\times n} \stackrel{\mu^n}{\to} (S^1)^{\times n} \quad (5.10)$$

And furthermore using concatenations with projections we can define maps

$$\begin{array}{rl} \circ^C : (S^1)^{\times n} \times (S^1)^{\times m} & \stackrel{(id \times \pi_i)(\Delta) \times id}{\longrightarrow} (S^1)^{\times n} \times S^1 \times (S^1)^{\times m} \\ & \stackrel{id \times \rho^C}{\longrightarrow} (S^1)^{\times n} \times (S^1)^{\times m} \stackrel{\bar{\circ}_i}{\longrightarrow} (S^1)^{\times n+m-1} \end{array} \quad (5.11)$$

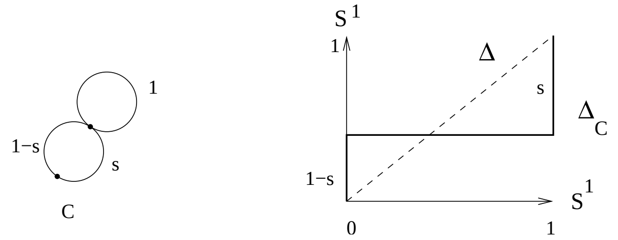

Figure 9: The image of Δ_C for a two component cactus

5.3.4 Theorem *The (quasi-)operad of (normalized) cacti is isomorphic to the bi-crossed product of the operad of (normalized) spineless cacti with the operad $\mathcal{S}^1$ based on S^1 with respect to the compositions of (5.7) and (5.11) and furthermore this bi-crossed product is homotopy equivalent as a quasi-operad to the semi-direct product of the operad of cacti without spines with the circle group S^1 considered as a monoid.*

$$\mathcal{C}acti \cong \mathcal{C}act \bowtie \mathcal{S}^1 \sim \mathcal{C}act \rtimes \mathcal{S}^1 \tag{5.12}$$

with respect to the operations of (5.7) and (5.11). Similarly

$$\mathcal{C}acti^1 \cong \mathcal{C}act^1 \bowtie \mathcal{S}^1 \sim \mathcal{C}act \rtimes \mathcal{S}^1 \tag{5.13}$$

Proof As spaces the operad of cacti is the direct product of spineless cacti and the operad built on the monoid S^1

$$\mathcal{C}acti(n) = \mathcal{C}act(n) \times S^1(n)$$

where for the identification we use intersection points and the global zero to define the parameterizations of the S^1s constituting the cactus. Then the local zeros are specified by their coordinate, i.e. all zeros are fixed by a point on $S^1(n)$.

The product on the cacti operad in this identification is given by

$$(C,\theta) \circ_i (C',\theta') = (C \circ_i^{\theta_i} C', \theta \circ_i^{C'} \theta') \tag{5.14}$$

where we used the operations of (5.7) and (5.11).

This comes from the observation that gluing the global zero to the local zero is the same as first using the S^1 action that rotates the global zero around the perimeter in a clockwise fashion by the amount which is given by the coordinate of the local zero on the cactus to be glued in and then using the standard gluing.

During the movement of the global zero the coordinates of the local zeros change according to the map ρ_C. This proves the first assertion.

Now, in the semi-direct product the multiplication is given by:

$$(C,\theta)\circ_i(C',\theta') = (C \circ_i^{\theta_i} C', \theta\circ_i\theta') \tag{5.15}$$

Now by (5.9) and (5.11) $\circ_i^C \sim \circ_i$ are homotopic and $\circ_i^C$ depends continuously on C. Choosing the following simultaneous homotopies of the Δ_C with Δ, we obtain a homotopy equivalence of quasi-operads. The homotopy is given by straightening out ΔC. For $\Delta_C(\theta) = (\theta_1,\dots,\theta_n)$ notice that $\theta = \frac{1}{n}\sum_{i=1}^n \theta_i$; we set

$$h_t(\theta) = ((1-t)\theta_1 + t\theta, \dots, (1-t)\theta_n + t\theta) \quad t\in[0,1].$$

This proves the second statement. On the level of homology we therefore obtain an isomorphism of operads, so that the map of $\mathcal{C}acti$ to the semi-direct product is a quasi-isomorphism and hence an equivalence. To obtain the version of the theorem in the normalized situation, we remark that the bi-crossed structure does not depend on the size of the lobes. □

5.3.5 Corollary *The operads of cacti and the semi-direct product of spineless cacti with the operad $\mathcal{S}^1$ are homotopy equivalent as quasi-operads.*

To connect these results to the literature, we recall a theorem of Voronov announced in [V, SV]:

5.3.6 Theorem *The cacti operad is equivalent to the framed little discs operad.*

5.3.7 Proposition *Theorem 3.2.1 and the Corollary 5.3.5 above in conjunction with the equivariant recognition principle of [SW] imply Theorem 5.3.6.*

Proof For this first remark that by what we have proven above the spaces $\mathcal{C}acti$ already have the right weak homotopy type, they are $K(PRBr,1)$. Here $PRBr$ is the pure ribbon braid group. Next, it is clear from the decomposition of the spaces $\mathcal{C}acti(n) = \mathcal{C}act(n)\times(S^1)^n$, the results on the quotient $\mathcal{C}act(n)/\mathbb{S}_n$ and the fact that $\mathbb{S}_n$ acts by permutation on the factors S^1 that $\mathcal{C}acti(n)/\mathbb{S}_n$ is a $K(RBr,1)$. Here RBr is the ribbon braid group. The construction of the ribbon braid operad on the covers is done by using a lifting of the appropriate paths to define the operad compositions and an action of the pure ribbon braid group which covers the symmetric group action. Here SCC again act as basepoints. This will endow the universal covers of $\mathcal{C}acti$ with a ribbon braid group operad structure. This can be checked directly. It also follows from the existence of the homotopies to the semi-direct product, since the same holds true for the semi-direct product of the little discs with S^1 and the little discs are equivalent to $\mathcal{C}act$. Also, the covers are contractible and the ribbon braid group action is free so that the covers form an R_∞ operad and hence by [SW] $\mathcal{C}acti$ is weakly homotopy equivalent to the framed little discs. The fact that $\mathcal{C}acti$ are homotopy equivalent to a CW complex then allows one to upgrade from the weak equivalence to an honest one. □

Vice-versa using the recognition principle of [SW] and starting with characterization of $\mathcal{C}acti$ in terms of $\mathcal{C}act$ above:

5.3.8 Proposition *The Theorem 5.3.6 together with the Theorem 5.3.4 imply Theorem 3.2.1, namely that the operad of (normalized) cacti without spines is equivalent to the little discs operad.*

Sketch of a proof The main idea is that the operad of little discs is embedded into the framed little discs by setting the coordinates of the factors S^1 equal to zero in the semi-direct product. The argument is as follows. Following [SW] one can show that if the universal cover of a semi-direct product $fD = D \rtimes S^1$ is an R_∞ operad then the universal cover of the suboperad $D(n) \times 0^n$ is a B_∞ operad. Since by the above theorem $\mathcal{C}acti$ are weakly homotopy equivalent to the framed little discs and also the semi-direct product of $\mathcal{C}act$ with S^1 is equivalent with $\mathcal{C}acti$ it follows by [SW] that the universal cover of the semi-direct product is an R_∞ operad, and hence the universal cover of the natural inclusion of $\mathcal{C}act$ into the semi-direct product is a B_∞ operad which implies that $\mathcal{C}act$ is an E_2 operad. □

5.3.9 Remark It is shown in [K2] that the quasi-operad on the cellular chains $CC_*(\mathcal{C}acti^1)$ is an operad. Due to the above result, this operad agrees on the homology level with the semi-direct product of $\mathcal{C}act$ with S^1 which in turn agrees with the semi-direct product of the little discs operad with S^1. Thus on the homology level we obtain an operad isomorphic to the homology of the framed little discs operad and thus $CC_*(\mathcal{C}acti^1)$ gives a chain level operad model for the framed little discs.

Again, the combinatorial description of the cells allows one to prove a theorem in the spirit of Deligne's conjecture. Namely [K2], the Hochschild cochains of a Frobenius algebra are an algebra over a cell level model of the framed little discs operad.

Appendix A: Graphs

In this appendix, we formally introduce the graphs and the operations on graphs used in our analysis of cacti. This language is useful for describing the $\mathcal{A}rc$ operad which is done in Appendix B. We also give a supplemental result characterizing cacti as a certain type of ribbon graph. A cactus is a marked treelike ribbon graph with a metric.

A.1 Graphs

A graph Γ is a tuple $(V_\Gamma, F_\Gamma, \imath_\Gamma : F_\Gamma \to F_\Gamma, \partial_\Gamma : F_\Gamma \to V_\Gamma)$ where $\imath_\Gamma$ is an involution $i_\Gamma^2 = id$ without fixed points. We call V_Γ the vertices of Γ and F_Γ the flags of Γ. The edges E_Γ of Γ are the orbits of the flags under the involution. A directed edge is an edge together with an order of the two flags which define it.

In case there is no risk of confusion we will drop the subscripts Γ.

Notice that $f \mapsto (f, \imath(f))$ gives a bijection between flags and directed edges.

We also call $F_\gamma(v) := \partial^{-1}(v) \subset F_\Gamma$ the set of flags of the vertex v and call $|F_\Gamma(v)|$ the valency of v and denote it by $|v|$.

The geometric realization of a graph is given by considering each flag as a half-edge and gluing the half-edges together using $\imath$. This yields a one-dimensional CW complex whose realization we call the realization of the graph.

A.2 Trees

A graph is connected if its realization is. A graph is a tree if it is connected and its realization is contractible.

A rooted tree is a pair (τ, v_0) where τ is a tree and $v_0 \in V_\tau$ is a distinguished vertex.

In a rooted tree there is a natural orientation for edges, in which the edge points toward the root. That is we say $(f, \imath(f))$ is naturally oriented if $\partial(\imath(f))$ is on the unique shortest path from $\partial(f)$ to the root.

A bi-colored or black and white tree is a tree τ together with a map $cr : V \to \mathbb{Z}/2\mathbb{Z}$. Such a tree is called bipartite if for all $f \in F_\tau : cr(\partial(f)) + cr(\partial(\imath(f))) = 1$, that is edges are only between black and white vertices. We call the set $E_w := cr^{-1}(1)$ the white vertices. If $(f, \imath(f))$ is a naturally oriented edge, we call the edge white if $\partial(\imath(f)) \in E_w$.

A.3 Planar trees and Ribbon graphs

A ribbon graph is a connected graph whose vertices are of valency at least two together with a cyclic order of the set of flags of the vertex v for every vertex v.

A tree with a cyclic order of the flags at each vertex is called planar.

A graph with a cyclic order of the flags at each vertex gives rise to bijections $N_v : F_v \to F_v$ where $N_v(f)$ is the next flag in the cyclic order and since $F = \amalg F_v$ to a map $N : F \to F$.

The orbits of the map $N \circ \imath$ are called the cycles or the boundaries of the graph. These sets have the induced cyclic order.

Notice that each boundary can be seen as a cyclic sequence of directed edges. The directions are as follows. Start with any flag f in the orbit. In the geometric realization go along this half-edge starting from the vertex $\partial(f)$, continue along the second half-edge $\imath(f)$ until you reach the vertex $\partial(\imath(f))$ then continue starting along the flag $N(\imath(f))$ and repeat.

A planar tree has only one cycle c_0.

A planted planar tree is a rooted planar tree (τ, v_0) together with a linear order of the set of flags at v_0. Such a tree has a linear order of all flags as follows, let f be the smallest element of $\partial^{-1}(v_0)$, then every flag appears in c_0 and defining the flag f to be the smallest gives a linear order on the set of all flags. This linear order induces a linear order on all oriented edges and on all oriented edges, by restricting to the edges in the orientation opposite the natural orientation i.e. pointing away from the root.

The genus $g(\Gamma)$ of a ribbon graph Γ is given by

$$2g(\Gamma) + 2 = |V_\Gamma| - |E_\Gamma| + \#cycles$$

The surface $\Sigma(\Gamma)$ of a ribbon graph Γ is the surface obtained from the realization of Γ by thickening the edges to ribbons. I.e. replace each 0-simplex v by a closed oriented disc $D(v)$ and each 1-simplex e by $e \times I$ oriented in the standard fashion. Now glue the boundaries of $e \times [-1, 1]$ to the appropriate discs in their cyclic order according to the orientations. Notice that the genus of $\Sigma(\Gamma)$ is $g(\Gamma)$ and that Γ is naturally embedded as the spine of this surface.

A.4 Treelike and marked ribbon graphs

A ribbon graph together with a distinguished cycle c_0 is called treelike if the graph is of genus 0 such that for all other cycles $c_i \neq c_0$ if $f \in c_i$ then $\imath(f) \in c_0$. In other words each edge is traversed by the cycle c_0. Therefore there is a cyclic order on all (non-directed) edges, namely the cyclic order of c_0.

A marked ribbon graph is a ribbon graph together with a map $\mathrm{mk} : \{cycles\} \to F_\Gamma$ satisfying the conditions:

i) For every cycle c the directed edge $\mathrm{mk}(c)$ belongs to the cycle.

ii) All vertices of valence two are in the image of mk, that is $\forall v, |v| = 2$ implies $v \in \partial(Im(\mathrm{mk}))$.

Notice that on a marked treelike ribbon graph there is a linear order on each of the cycles c_i. This order is defined by upgrading the cyclic order to the linear order $\prec_i$ in which $\mathrm{mk}(c_i)$ is the smallest element.

A marked treelike ribbon graph is called spineless, if:

i) There is at most one vertex of valence 2. If there is such a vertex v_0 then $\partial(mk(c_0)) = v_0$.

ii) The induced linear orders on the c_i are compatible with that of c_0, i.e. $f \prec_i f'$ if and only if $\imath(f') \prec_0 \imath(f)$.

A metric w_Γ for a graph is a map $E_v \to \mathbb{R}_{>0}$.

A.5 Graphs with a metric

A projective metric for a graph is a class of metrics equivalent under re-scaling, i.e. $w \sim w'$ if $\exists \lambda \in \mathbb{R}_{>0} \forall e \in E : w(e) = \lambda w'(e)$.

The length of a cycle is the sum of the length of its edges $length(c) = \sum_{f \in c} w(\{f, \imath(f)\})$.

A metric for a treelike ribbon graph is called normalized if the length of each non-distinguished cycle is 1.

A projective metric for a treelike ribbon graph is called normalized if it has a normalized representative.

A.6 Marked ribbon graphs with metric and maps of circles.

For a marked ribbon graph with a metric, let c_i be its cycles, let $|c_i|$ be their image in the realization and let r_i be the length of c_i. Then there are natural maps $S^1_{r_i} \to |c_i|$ which map S^1 onto the cycle by starting at the vertex $v_i := \partial(\mathrm{mk}(c_i))$ and go around the cycle mapping each point $\theta \in S^1$ to the point at distance $\frac{\theta}{2\pi} r_i$ from v_i along the cycle c_i.

A.7 Contracting edges

The contraction $(\bar{V}_\Gamma, \bar{F}_\Gamma, \bar{\imath}, \bar{\partial})$ of a graph $(V_\Gamma, F_\Gamma, \imath, \partial)$ with respect to an edge $e = \{f, \imath(f)\}$ is defined as follows. Let $\sim$ be the equivalence relation induced by $\partial(f) \sim \partial(\imath(f))$. Then let $\bar{V}_\Gamma := V_\Gamma / \sim$, $\bar{F}_\Gamma = F_\Gamma \setminus \{f, \imath(f)\}$ and $\bar{\imath} : \bar{F}_\Gamma \to \bar{F}_\Gamma, \bar{\partial} : \bar{F}_\Gamma \to \bar{V}_\Gamma$ be the induced maps.

For a marked ribbon graph, we define the marking of $(\bar{V}_\Gamma, \bar{F}_\Gamma, \bar{\imath}, \bar{\partial})$ to be $\overline{\mathrm{mk}}(\bar{c}) = \overline{\mathrm{mk}(c)}$ if $\mathrm{mk}(c) \notin \{f, \imath(f)\}$ and $\overline{\mathrm{mk}}(\bar{c}) = \overline{N \circ \imath(\mathrm{mk}(c))}$ if $\mathrm{mk}(c) \in \{f, \imath(f)\}$, viz. the image of the next flag in the cycle.

A.8 Labelling graphs

By a labelling of the edges of a graph Γ by a set S, we simply mean a map $E_\Gamma \to S$.

A labelling of a ribbon graph Γ by a set S is a map $\{\text{cycles of } \Gamma\} \to S$.

By a labelling of a black and white tree by a set S we mean a map $E_w \to S$.

A.9 Cacti as ribbon graphs

By considering its image a cactus is naturally a marked treelike ribbon graph, with a metric. Vice-versa, given such a graph, one obtains a cactus.

A.9.1 Proposition *A cactus with n lobes is equivalent to an $\{0, 1, \dots, n\}$ labelled marked treelike ribbon graph with a metric. I.e. The set $\mathcal{C}acti(n)$ is in bijection with the respective set of graphs. The conditions of being normalized and/or spineless are compatible with this bijection.*

Proof Given a cactus, its image is a ribbon graph. The vertices are the marked points and the edges are the arcs. The flags being pairs (v, a) of a marked point v and an arc a ending at v with the obvious involution and map ∂. At each vertex there is a cyclic order induced from the one in the plane. This graph has $n + 1$ cycles. First each lobe marked by i yields a cycle c_i (going clockwise) and secondly the outside circle (going counter-clockwise) is a cycle which we will call c_0. Furthermore the outside circle goes through each edge and hence the ribbon graph is treelike for the distinguished cycle c_0. The labelling of the cycles is implicit in this description. The marking is given by marking the unique flag (v, f) on the cycle c_i for which v is the local zero, for $i > 0$ and the unique flag (v, f) on c_0 for which v is the global zero. Finally the metric is given by associating to an edge representing an arc, the length of that arc.

In the reverse direction, given a treelike marked ribbon graph with a metric Γ, consider the surface $\Sigma(\Gamma)$ of Γ. This is a surface of genus 0 with $n+1$ boundary components. We embed this surface into the plane as a disc with holes where the outside circle of the disc corresponds to the cycle c_0. In this embedding, Γ realized as the spine of $\Sigma(\Gamma)$ is a cactus by considering the maps $S^1_{r_i} \to c_i$ for $i > 0$ as above.

It is clear that these maps induce bijections between the sets $\mathcal{C}acti(n)$ and the respective set of graphs.

To be normalized means that $r_i = 1$ for $i > 0$ in both cases. Finally to be spineless in both cases also means the same. For this notice that condition i) says that there is a global zero which may or may not be an intersection point and that all local zeros are at the intersection points or at the global zero. Furthermore the root component of the cactus is fixed as the unique cycle c_i s.t. $\imath(\mathrm{mk}(c_0)) \in c_i$. The condition b then insures that the local zero of the root component is the global zero. The reverse direction is immediate. □

Appendix B: The arc operad

In this appendix, we would like to briefly recall some facts from [KLP] about the arc operad. We will make use of some reformulations given in [K3] of the results of [KLP] which use the language of graphs, since this will simplify the exposition.

B.1 The Arc Operad

We would like to recall some definitions of [KLP]. For this we will fix an oriented surface $F^s_{g,r}$ of genus g with s punctures and r boundary components which are labelled from 0 to $r-1$, together with marked points on the boundary, one for each boundary component. We call this data F for short if no confusion can arise.

We recall from [KLP] that the space $A^s_{g,n}$ is the CW complex whose cells are indexed by graphs on the surface F up to the action of the pure mapping class group PMC which is the group of elements of the mapping class group which fixes the boundaries pointwise. A quick review in terms of graphs is as follows.

B.1.1 Embedded graphs By an embedding of a graph Γ onto a surface F, we mean an embedding $i: |\Gamma| \to F$ with the conditions:

i) Γ has at least one edge.
ii) The vertices map bijectively to the marked points on the boundaries.
iii) No images of two edges are homotopic to each other.
iv) No image of an edge is homotopic to a part of the boundary.

Two embeddings are equivalent if there is a homotopy of embeddings of the above type from one to the other. Note that such a homotopy is necessarily constant on the vertices.

The images of the edges are called arcs. And the set of connected components of $F \setminus j(\Gamma)$ are called complementary regions.

Changing representatives in a class yields a natural bijection of the sets of arcs and connected components, we will therefore associate arcs and connected components also with a class of embeddings.

B.1.2 Definition By a graph on a surface we mean a triple $(F, \Gamma, [i])$ where $[i]$ is an equivalence class of embeddings of Γ.

B.1.3 A linear order on arcs Notice that due to the orientation of the surface the graph inherits an induced linear order of all the flags at every vertex $F(v)$ from the embedding. Furthermore there is even a linear order on all flags by enumerating the flags first according to the boundary components on which their vertex lies and then according to the linear order at that vertex. This induces a linear order on all edges by enumerating the edges by the first appearance of a flag of that edge.

B.1.4 The poset structure The set of such graphs on a fixed surface F is a poset. The partial order is given by calling $(F,\Gamma',[i']) \prec (F,\Gamma,[i])$ if Γ' is a subgraph of Γ with the same vertices and $[i']$ is the restriction of $[i]$ to Γ'. In other words, the first graph is obtained from the second by deleting some arcs.

We associate a simplex $\Delta(F,\Gamma,[i])$ to each such graph. Δ is the simplex whose vertices are given by the set of arcs/edges enumerated in their linear order. The face maps are then given by deleting the respective arcs. This allows us to construct a CW complex out of this poset.

B.1.5 Definition Fix $F = F^s_{g,n}$. The space $A'^s_{g,n}$ is the space obtained by gluing the simplices $\Delta(F,\Gamma',[i'])$ for all graphs on the surface according to the face maps.

The pure mapping class group (PMC) which is the part of the mapping class group that preserves the boundary pointwise naturally acts on $A'^s_{g,n}$. It actually has finite isotropy [KLP].

B.1.6 Definition The space $A^s_{g,r} := A'^s_{g,r}/PMC$.

B.1.7 Elements of the $A^s_{g,r}$ as projectively weighted graphs The space $A^s_{g,n}$ is a CW complex whose cells are indexed by graphs on the surface F up to the action of the pure mapping class group PMC. Moreover the cell for a given class of graphs is actually a simplex whose vertices correspond to the arcs in the order discussed above. The attaching maps are given by deleting edges. Due to the action of PMC some of the faces also might become identified by the gluing.

Using barycentric coordinates the elements of $A^s_{g,n}$ are graphs on the surface F up to the action of the pure mapping class group PMC together with a map w of the edges of the graph E_Γ to $\mathbb{R}_{>0}$ assigning a weight to each edge s.t. the sum of all weights is 1.

Equivalently we can regard the map $w: E_\Gamma \to \mathbb{R}_{>0}$ as an equivalence class under the equivalence relation of, i.e. $w \sim w'$ if $\exists \lambda \in \mathbb{R}_{>0} \forall e \in E_\Gamma\ w(e) = \lambda w'(e)$. That is w is a projective metric. We call the $w(e)$ the projective weights of the edge. In the limit, when the projective weight of an edge goes to zero, the edge/arc is deleted.

B.1.8 Example $A^0_{0,2} = S^1$. Up to PMC there is a unique graph with two edges. This gives a one simplex. The two subgraphs lie in the same orbit of PMC and thus the 0-simplices are identified to yield S^1. The fundamental cycle is given by δ of Figure 11.

B.1.9 Drawing pictures for arcs There are several pictures one can use to view elements in the arc operad. In order to draw elements of the $\mathcal{A}rc$ operad it is useful, to expand the marked point on the boundary to an interval, and let the arcs end on this interval according to the linear order. Equivalently, one can mark one point of the boundary and let the arcs end in their linear order anywhere but on this point.

B.1.10 Remark There is an operad structure on $\mathcal{A}rc$ which is obtained by gluing the surfaces along the boundaries and splitting the arcs according to their weights. We refer the reader to [KLP] for the details.

B.1.11 Definition of suboperads and $\mathcal{DA}rc$

Let $\mathcal{DA}rc^s_{g,r} := \mathcal{A}rc^s_{g,r} \times \mathbb{R}_{>0}$.

Let $\mathcal{A}rc^s_g(n) \subset A^s_{g,n+1}$ be the subspace of graphs on the surface with the condition that $\partial : F_\Gamma \to V_\Gamma$ is surjective. This means that each boundary gets hit by an arc.

Let $\mathcal{A}rc^s_{\#g}(n) \subset \mathcal{A}rc^s_g(n)$ be the subspace of elements whose complementary regions are polygons or once punctured polygons.

Set $\mathcal{A}rc_{cp}(n) := \mathcal{A}rc^0_0(n)$.

Let $\mathcal{T}ree_{cp}(n) \subset \mathcal{A}rc_{cp}(n)$ be the subspace in which all arcs run from 0 to some boundary i only.

Finally let $\mathcal{LT}ree_{cp} \subset \mathcal{T}ree_{cp}$ be the space in which the linear order of the arcs at the boundary 0 is anti-compatible with the linear order at each boundary, i.e. if $\prec_i$ is the linear order at i then if $f \prec_i f'$ $\imath(f') \prec_0 \imath(f)$.

B.1.12 Remarks

1) The elements of $\mathcal{DA}rc$ are graphs on surfaces with a metric, i.e. a function $w : E_\Gamma \to \mathbb{R}_{>0}$. And $\mathcal{DA}rc$ is an operad equivalent to $\mathcal{A}rc$ [KLP].
2) The subspaces above are actually suboperads [KLP].
3) Any suboperad $\mathcal{S}$ of the list above defines a suboperad $\mathcal{DS} := \mathcal{S} \times \mathbb{R}_{>0}$ of $\mathcal{DA}rc$ which is equivalent to $\mathcal{S}$.
4) One can also reverse the orientation at zero. This is in line with the usual cobordism point of view used in [KLP]. In this case the condition for $\mathcal{LT}ree$ is the compatibility of the orders.

In [KLP] we defined a map called $\mathcal{L}oop$ which is the suitable notion of a dual graph for a graph on a surface. This map is uses an interpretation of the graph as a partially measured foliation. If one restricts to the subspace $\mathcal{A}rc_\#$ though, this map has a simpler purely combinatorial description. This description will be enough for our purposes here, but we would like to emphasize that this description is only valid on the subspace $\mathcal{A}rc_\#$ and cannot be generalized to the whole of $\mathcal{A}rc$.

B.1.13 The dual graph Informally the dual graph of an element in $\mathcal{A}rc_\#$ is given as follows. The vertices are the complementary regions. Two vertices are joined by an edge if the complementary regions border the same arc. Due to the orientation of the surface this graph is actually a ribbon graph via the induced cyclic order. Moreover the marked points on the boundary make this graph into a marked ribbon graph. A more precise formal definition is given in the next few paragraphs.

B.1.14 Polygons and $\mathcal{A}rc^0_\#$ By definition, in $\mathcal{A}rc^0_\#$ the complementary regions are actually k-gons. Let $Poly(F,\Gamma,[i])$ be the set of these polygons and let $Sides(F,\Gamma,[i])$ be the disjoint union of sets of sides of the polygons. We define $\partial_{poly}: Sides(F,\Gamma,[i]) \to Poly(F,\Gamma,[i])$ to be the map which associates to a side s of a polygon p the polygon p. The sides are either given by arcs or the boundaries. We define the map lab : $Sides(F,\Gamma,[i]) \to E_\Gamma \bigcup V_\Gamma$ that associates the appropriate label. Notice that for $e \in E_\Gamma; |\mathrm{lab}^{-1}(e)| = 2$ and for $v \in V_\Gamma : |\mathrm{lab}^{-1}(v)| = 1$. Thus there is a fixed point free involution $\imath_{side}$ on the set $\mathrm{lab}^{-1}(E_\Gamma)$ of sides of the polygons marked by arcs which maps one side to the unique second side carrying the same label. This in turn defines an involution $\imath$ of pairs (p,s) of a polygon together with a side in $\mathrm{lab}^{-1}(E_\Gamma)$ by mapping s to $\imath_{side}(s)$ and taking the polygon p to the polygon $p' := \partial_{poly}(\imath(s))$ of which $\imath_{side}(s)$ is a side. Although p and p' might coincide the sides will differ making the involution $\imath$ fixed point free.

B.1.15 Definition For an element $(F,\Gamma,[i],w) \in \mathcal{A}rc_{\#g}(n)$ we define the dual graph to be the marked ribbon graph with a projective metric $(\hat{\Gamma}, ord, \hat{w}, \mathrm{mk})$ which is defined as follows. The vertices of $\hat{\Gamma}$ are the complementary regions of the arc graph (i.e. the polygons) and the flags are the pairs (p,s) of a polygon (vertex) together with a side of this polygon marked by an arc ($s \in \mathrm{lab}^{-1}(E_\Gamma)$). The map ∂ is defined by $\partial((p,s)) = p$ and the involution $\imath((p,s)) := (\partial_{poly}(\imath_{side}(s)), \imath_{side}(s))$.

Each polygonal complimentary region is oriented by the orientation induced by the surface, so that the sides of each polygon and thus the flags of $\hat{\Gamma}$ at a given vertex p have a natural induced cyclic order ord making $\hat{\Gamma}$ into a ribbon graph.

Notice that there is a one-one correspondence between edges of the dual graph and edges of Γ. This is given by associating to each edge $\{(p,s), \imath(p,s)\}$ the edge corresponding to the arc $\mathrm{lab}(s)$.

We define a projective metric $\hat{w}$ on this graph by associating to each edge $\{(p,s), \imath(p,s)\}$ the weight of the arc labelling the side s where w is the projective metric on the arc graph $\hat{w}(\{(p,s), \imath(p,s)\}) := w(\mathrm{lab}(s))$.

To define the marking, notice that the cycles of $\hat{\Gamma}$ correspond to the boundary components of the surface F. Let c_k be the cycle of the boundary component labelled by k. The k-th boundary component lies in a unique polygon $p = \partial_{poly}(\mathrm{lab}^{-1}(k))$. Let $\prec_p$ be the cyclic order on the set of sides of p, $\partial^{-1}(p)$. Let s_k be the side corresponding to the boundary and let $N(s_k)$ the element following s_k in $\prec_p$. We define $\mathrm{mk}(c_k) := (p, N(s_k))$.

B.1.16 Remark The above map will suffice for the purposes of this paper. For the general theory and the reader acquainted with the constructions of [KLP], the following Proposition will be helpful.

We do not, however, wish to go into technical details here on how a marked weighted ribbon graph defines a configuration in the sense of [KLP] and refer the reader to [K3] for details.

B.1.17 Proposition [K3] *For elements in $\mathcal{A}rc_{\#}$ the dual graph realizes the map $\mathcal{L}oop$.*

B.2 The suboperads of $\mathcal{A}rc$ defined by the cacti operads

In this section, we would like to recall that the cacti without spines and cacti can be embedded into the $\mathcal{A}rc$ operad up to an overall scaling factor as defined in [KLP]. Moreover there is an S^1 action given by the twist operator δ. For the complete details, we refer to [KLP].

In one direction the map is given by the dual graph discussed above. In the other direction, the embedding is basically constructed as follows: start by decomposing the cactus into the arcs of its perimeter, where the break point of a cactus with spines are the intersection points, the global zero and the local zeros. Then one runs an arc from each arc to an outside pointed circle which is to be drawn around the cactus configuration. The arcs should be embedded starting in a counterclockwise fashion around the perimeter of the circle. The marked points on the inside circles which are the lobes of the cactus are the local zeros for the cactus with spines and the global zero and the first intersection point for a cactus without spines.

An equivalent formal definition in terms of graphs is given below.

For more orientation, we include two figures: the Figure 10 shows the framing i.e. embedding of two cacti without spines and a cactus with spines into arc; Figure 11 shows the identity in arc and the family of weighted arcs corresponding to the twist which yields the BV operator. The Figures 2 V and 3 V depict more elaborate examples.

B.2.1 Arcs and cacti The main result of the arc picture is summed up in the following Theorem.

B.2.2 Theorem *There is an map of (spineless) cacti into $\mathcal{DA}rc$ which maps $\mathcal{C}acti$ bijectively onto $\mathcal{DT}ree_{cp}$ and $\mathcal{C}act$ bijectively onto $\mathcal{DLT}ree_{cp}$. When restricted to its image this map is an equivalence of operads.*

Furthermore the suboperad in $\mathcal{DA}rc$ generated by the Fenchel-Nielsen type twist δ and the image of spineless cacti is equal to the image the cacti operad.

There are operadic maps of the spineless cacti and the cacti operad into $\mathcal{A}rc$ which are equivalences when restricted to their image.

Partial proof We will prove this claim on the level of sets and refer to [KLP] for the operad structure.

The map in one direction is given by associating the dual graph. For an element in $\mathcal{DT}ree_{cp}$ this graph is a marked treelike ribbon graph with a metric, viz. a cactus. If the linear orders agree, it is spineless.

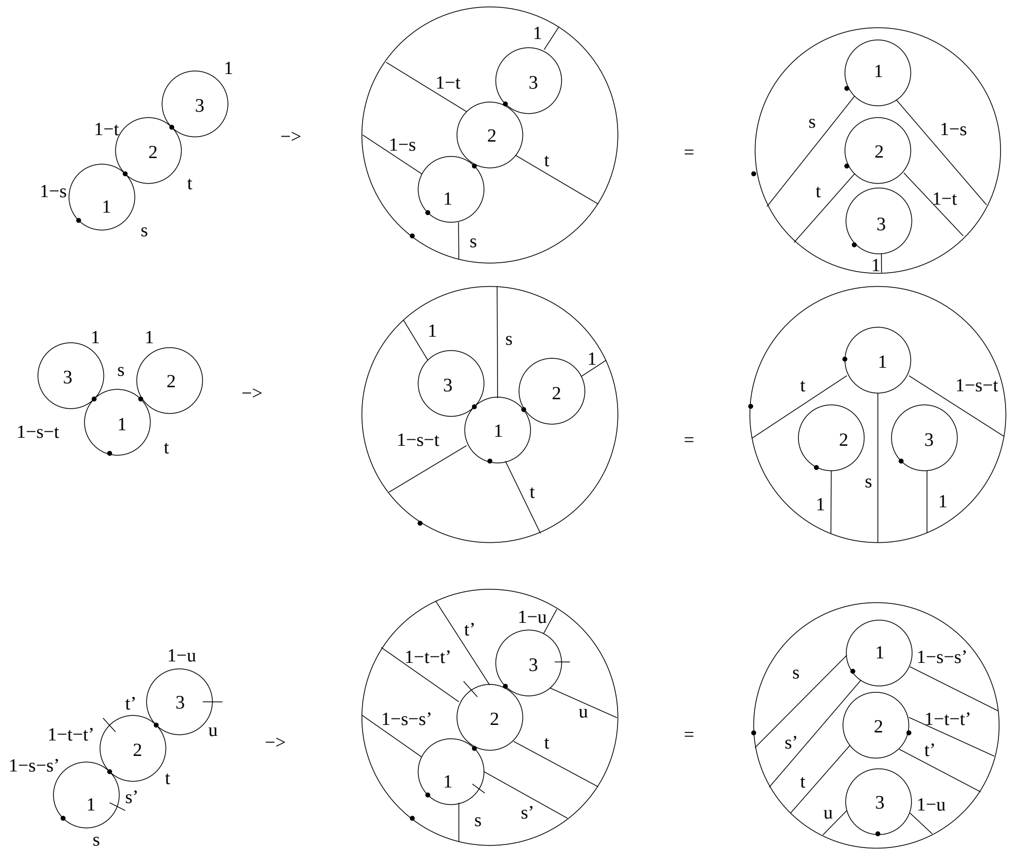

Figure 10: The embedding of cacti into arc

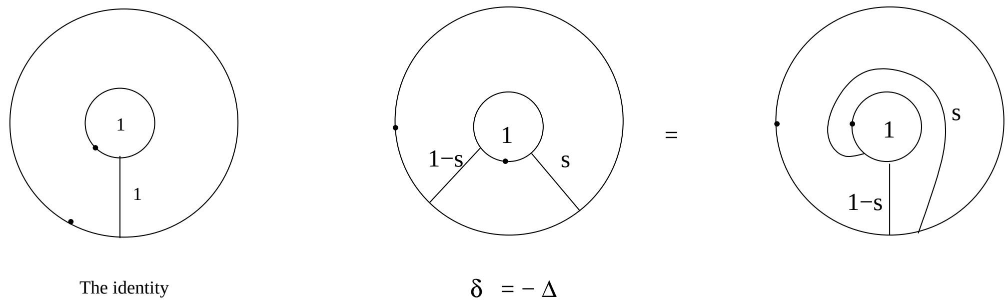

Figure 11: The identity and the twist δ yielding the BV operator

For the map in the reverse direction, we realize a cactus c as a marked treelike ribbon graph with a metric $\Gamma(c)$. The surface will be $\Sigma(\Gamma(c))$. The boundaries correspond to the lobes and the outside circle and are hence labelled from 0 to n. Let $m(e)$ be the midpoint of the edge e. Then we consider the arcs $m(e) \times [-1,1]$ on Σ. Notice that these arcs come in a linear order at each boundary component according to their linear order on the cactus and are labelled by $w(e) \in \mathbb{R}_{>0}$. Finally we mark off an interval on each boundary such that the arcs on a boundary all end on this interval and appear in the above linear order on the interval, where the orientation of the interval is induced by that of the surface. Contracting the interval to a point, we obtain the desired element in $\mathcal{DA}rc$.

The claim about $\mathcal{A}rc$ follows immediately from the equivalence of operads $\mathcal{DA}rc \to \mathcal{DA}rc/\mathbb{R}_{>0} = \mathcal{A}rc$ which contracts the factor $\mathbb{R}_{>0}$ of $\mathcal{DA}rc$. $\square$

B.2.3 Remark Alternatively, we can mark a point on each boundary such that the arcs appear in their order on the complement of this point. This alternative corresponds to the map called framing in [KLP] which we also used in our depictions in this paper.

References

[BN] D. Bar-Natan. *On the Vassiliev knot invariants.* Topology 34 (1995), 423–472.

[BV] J.M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347, Springer, 1973

[CS] M. Chas and D. Sullivan. *String Topology.* To appear in Ann. of Math. `arXiv:math.GT/9911159`

[C1] F. R. Cohen. The homology of C_{n+1}-spaces, $n \geq 0$. In The homology of iterated loop spaces., volume 533 of Lecture Notes in Mathematics. Springer, 1976.

[C2] F. R. Cohen. Artin's braid groups, classical homotopy theory, and sundry other curiosities. In Braids (Santa Cruz, CA, 1986), volume 78 of Contemp. Math., pages 167–206. Amer. Math. Soc., Providence, RI, 1988.

[CJ] R. L. Cohen and J.D.S. Jones *A homotopy theoretic realization of string topology* Math. Ann. 324 (2002), no. 4, 773–798.

[CK] A. Connes and D. Kreimer *Hopf Algebras, Renormalization and Noncommutative Geometry.* Commun. Math. Phys. 199 (1998) 203–242

[DT] A. Dold und R. Thom *Quasifaserungen und unendliche symmetrische Produkte.* Ann. of Math. (2) 67 (1958), 239–281.

[F1] Z. Fiedorowicz. *The symmetric Bar Construction.* Preprint, available at: `http://www.math.ohio-state.edu/~fiedorow/`

[F2] Z. Fiedorowicz. *Constructions of E_n operads.* `arXiv:math.AT/9808089`

[G] M. Gerstenhaber. *The cohomology structure of an associative ring*, Ann. of Math. 78 (1963), 267–288.

[Ge] E. Getzler. *Two-dimensional topological gravity and equivariant cohomology.* Commun. Math. Phys. 163 (1994), 473–489.

[Go] A.B. Goncharov *Multiple zeta-values, Galois groups, and geometry of modular varieties.* European Congress of Mathematics, Vol. I (Barcelona, 2000), 361–392, Progr. Math., 201, Birkhäuser, Basel, 2001

[Ka] C. Kassel. *Quantum groups.* Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[K1] R. M. Kaufmann. *On Spineless Cacti, Deligne's Conjecture and Connes-Kreimer's Hopf Algebra.* arXiv:math.QA/0308005

[K2] R. M. Kaufmann. *A proof of a cyclic version of Deligne's conjecture via Cacti.* arXiv:math.QA/0403340

[K3] R. M. Kaufmann. *Moduli spaces, Ribbon graphs and actions on the Hochschild complex.* In preparation.

[KLP] R. M. Kaufmann, M. Livernet and R. C. Penner. *Arc Operads and Arc Algebras* Geom. Topol. 7 (2003) 511–568.

[MS] J. E. McClure and J. H. Smith, Jeffrey H. *A solution of Deligne's Hochschild cohomology conjecture.* Recent progress in homotopy theory (Baltimore, MD, 2000), 153–193, Contemp. Math., 293, Amer. Math. Soc., Providence, RI, 2002.

[MSS] M. Markl, S. Shnider and J. Stasheff. *Operads in algebra, topology and physics.* Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp.

[P] R. C. Penner. *Decorated Teichmüller theory of bordered surfaces.* Preprint math.GT/0210326.

[SW] P. Salvatore and N. Wahl. *Framed discs operads and the equivariant recognition principle.* Preprint math.AT/0106242. *Framed discs operads and Batalin-Vilkovisky algebras.* Q. J. Math. 54 (2003), no. 2, 213–231.

[SV] D. Sullivan and A. A. Voronov. *Brane Topology.* In preparation.

[Tak] M. Takeuchi. *Matched pairs of groups and bismash products of Hopf algebras.* Comm. Algebra 9 (1981), 841–882.

[V] A. A. Voronov *Notes on universal algebra.* Graphs and Patterns in Mathematics and Theoretical Physics (M. Lyubich and L. Takhtajan, eds.), Proc. Sympos. Pure Math., vol. 73. AMS, Providence, RI, 2005, pp. 81–103,

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