ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 369–378
Published: 30 April 2005

All integral slopes can be Seifert fibered slopes for hyperbolic knots

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Abstract Which slopes can or cannot appear as Seifert fibered slopes for hyperbolic knots in the 3-sphere S^3? It is conjectured that if r-surgery on a hyperbolic knot in S^3 yields a Seifert fiber space, then r is an integer. We show that for each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot K_n in S^3 such that n-surgery on K_n produces a small Seifert fiber space.

AMS Classification 57M25, 57M50

Keywords Dehn surgery, hyperbolic knot, Seifert fiber space, surgery slopes

This paper is dedicated to Donald M. Davis on the occasion of his 60th birthday.

1 Introduction

Let K be a knot in the 3-sphere S^3 with a tubular neighborhood $N(K)$. Then the set of *slopes* for K (i.e., $\partial N(K)$-isotopy classes of simple loops on $\partial N(K)$) is identified with $\mathbb{Q} \cup \{\infty\}$ using preferred meridian-longitude pair so that a meridian corresponds to ∞. A slope γ is said to be *integral* if a representative of γ intersects a meridian exactly once, in other words, γ corresponds to an integer under the above identification. In the following, we denote by $(K;\gamma)$ the 3-manifold obtained from S^3 by Dehn surgery on a knot K with slope γ, i.e., by attaching a solid torus to $S^3 - \mathrm{int}N(K)$ in such a way that γ bounds a meridian disk of the filled solid torus. If γ corresponds to $r \in \mathbb{Q} \cup \{\infty\}$, then we identify γ and r and write $(K;r)$ for $(K;\gamma)$.

We denote by $\mathcal{L}$ the *set of lens slopes* $\{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is a lens space$\}$, where S^3 and $S^2 \times S^1$ are also considered as lens spaces. Then the cyclic surgery theorem [7] implies that $\mathcal{L} \subset \mathbb{Z}$. A result of Gabai [10, Corollary 8.3] shows that $0 \notin \mathcal{L}$, a result of Gordon and Luecke [14] shows that $\pm 1 \notin \mathcal{L}$. In [19] Kronheimer and Mrowka prove that $\pm 2 \notin \mathcal{L}$. Furthermore, a result of Kronheimer, Mrowka, Ozsváth and Szabó [20] implies that $\pm 3, \pm 4 \notin \mathcal{L}$. Besides, Berge [4, Table of Lens Spaces] suggests that if $n \in \mathcal{L}$, then $|n| \geq 18$ and not every integer n with $|n| \geq 18$ appears in $\mathcal{L}$. Fintushel and Stern [9] had shown that 18-surgery on the $(-2,3,7)$ pretzel knot yields a lens space.

Which slope (rational number) can or cannot appear in the set of Seifert fibered slopes $\mathcal{S} = \{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is Seifert fibered$\}$? It is conjectured that $\mathcal{S} \subset \mathbb{Z}$ [12].

The purpose of this paper is to prove:

Theorem 1.1 *For each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot K_n in S^3 such that $(K_n;n)$ is a small Seifert fiber space (i.e., a Seifert fiber space over S^2 with exactly three exceptional fibers).*

Remark Since K_n has tunnel number one, it is embedded in a genus two Heegaard surface of S^3 and strongly invertible [26, Lemma 5]. See [22, Question 3.1].

Theorem 1.1, together with the previous known results, shows:

Corollary 1.2 $\mathcal{L} \subsetneqq \mathbb{Z} \subset \mathcal{S}$.

Remarks

(1) For the *set of reducing slopes* $\mathcal{R} = \{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is reducible$\}$, Gordon and Luecke [13] have shown that $\mathcal{R} \subset \mathbb{Z}$. In fact, the cabling conjecture [11] asserts that $\mathcal{R} = \emptyset$.

(2) For the *set of toroidal slopes* $\mathcal{T} = \{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is toroidal$\}$, Gordon and Luecke [15] have shown that $\mathcal{T} \subset \mathbb{Z}/2$ (integers or half integers). In [28], Teragaito shows that $\mathbb{Z} \subset \mathcal{T}$ and conjectures that $\mathcal{T} \subsetneqq \mathbb{Z}/2$.

Acknowledgements We would like to thank the referee for careful reading and useful comments.
The first author was partially supported by Grant-in-Aid for Scientific Research (No. 15540095), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

2 Hyperbolic knots with Seifert fibered surgeries

Our construction is based on an example of a longitudinal Seifert fibered surgery given in [17].

Let $k \cup c$ be a 2-bridge link given in Figure 1, and let K_n be a knot obtained from k by $\frac{1}{-n+4}$-surgery along c.

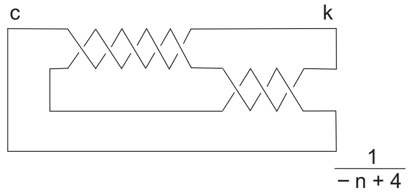

Figure 1: K_n

We shall say that a Seifert fiber space is of *type* $S^2(n_1, n_2, n_3)$ if it has a Seifert fibration over S^2 with three exceptional fibers of indices n_1, n_2 and n_3 $(n_i \geq 2)$. Since K_4 is unknotted, $(K_4; 4)$ is a lens space $L(4,1)$. For the other n's, we have:

Lemma 2.1 *$(K_n; n)$ is a small Seifert fiber space of type $S^2(3, 5, |4n - 15|)$ for any integer $n \neq 4$.*

Proof Since the linking number of k and c is one (with suitable orientations), $(K_n; n)$ has surgery descriptions as in Figure 2.

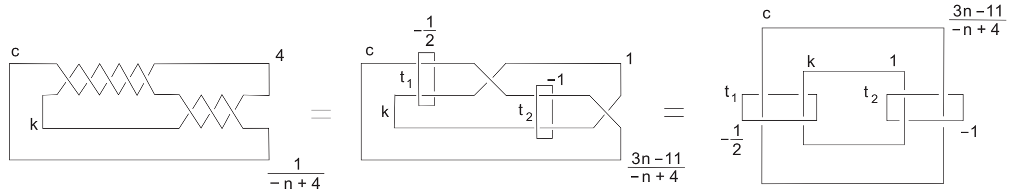

Figure 2: Surgery descriptions of $(K_n; n)$

Let us take the quotient by the strong inversion of S^3 with an axis L as shown in Figure 3.

Then we obtain a branch knot b' which is the image of the axis L. The Montesinos trick ([25], [6]) shows that $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-surgery on t_1, t_2, c and

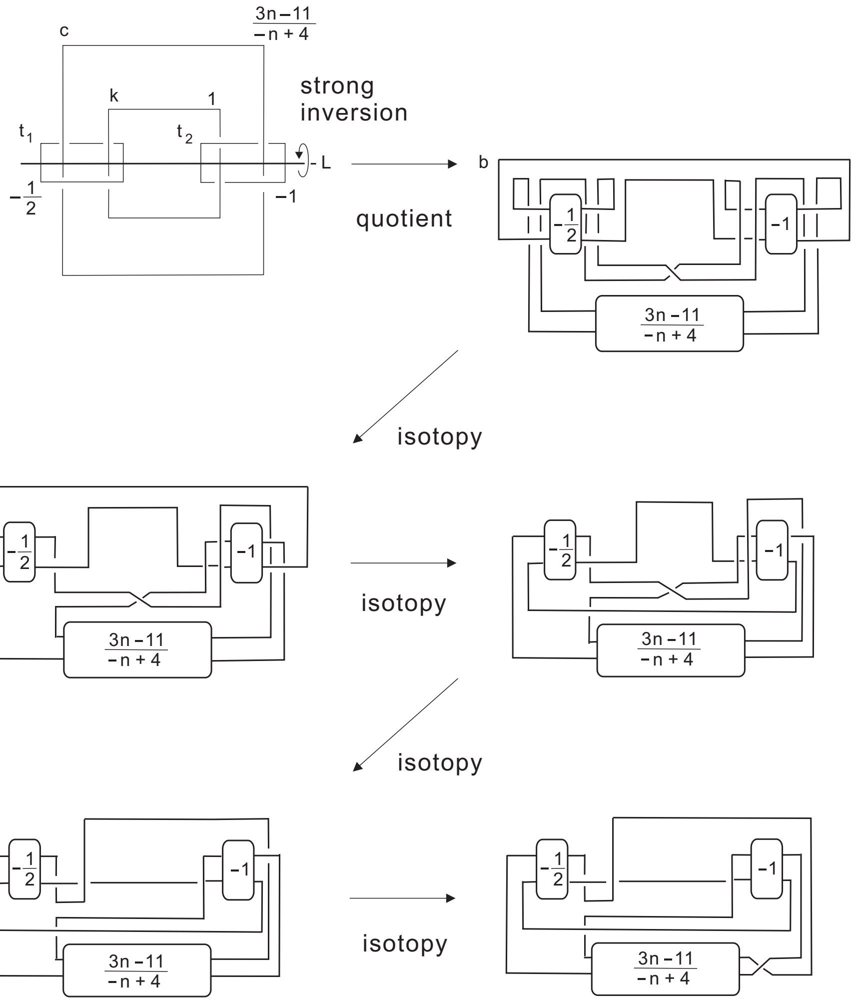

Figure 3

k in the upstairs correspond to $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-untangle surgery on b' in the downstairs, where an r*-untangle surgery* is a replacement of $\frac{1}{0}$-untangle by r-untangle. (We adopt Bleiler's convention [5] on the parametrization of rational tangles.) These untangle surgeries convert b' into a link b (Figure 3).

Following the sequence of isotopies in Figures 3 and 4, we obtain a Montesinos link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$.

Since $(K_n; n)$ is the double branched cover of S^3 branched over the Montesinos

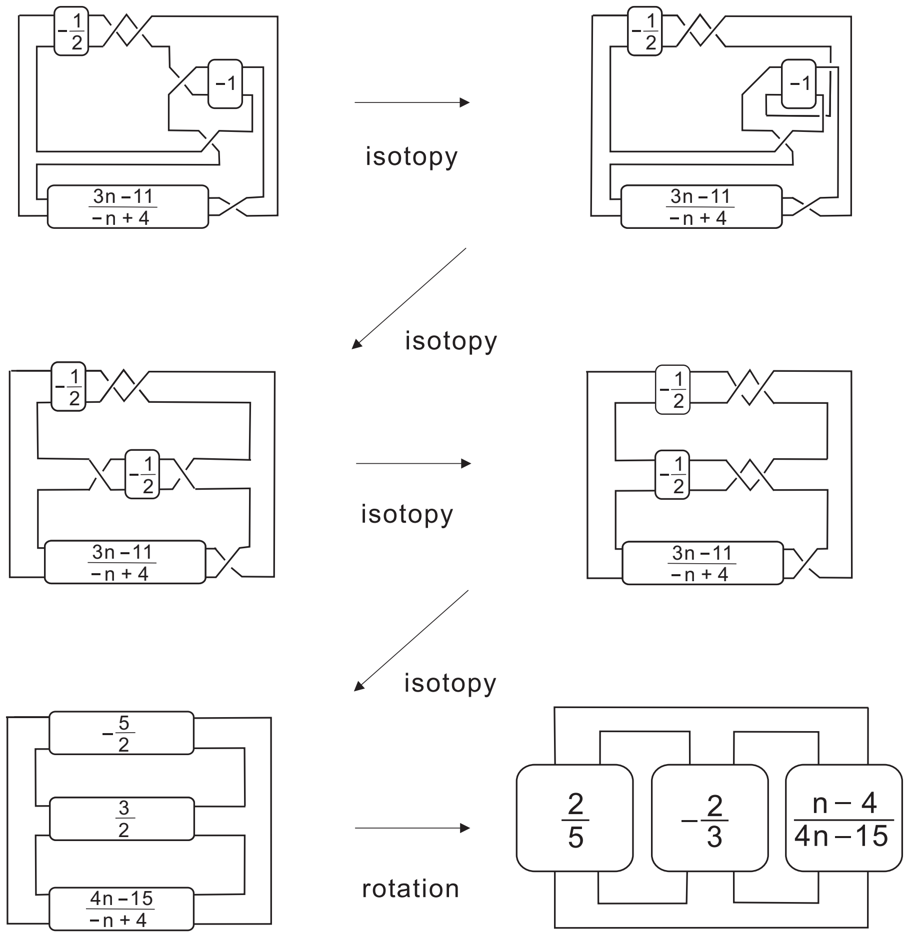

Figure 4: Continued from Figure 3

link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$, $(K_n; n)$ is a Seifert fiber space of type $S^2(3, 5, |4n-15|)$ as desired. □

Lemma 2.2 *The knot K_n is hyperbolic if $n \neq 3, 4, 5$.*

Proof Note that the 2-bridge link given in Figure 1 is not a $(2, p)$-torus link, and hence by [23] it is a hyperbolic link. If $n \neq 3, 4, 5$, then $|-n+4| > 1$ and it follows from [1, Theorem 1] (also [3, Theorem 1.2]) that K_n is a hyperbolic knot. See also [16, Corollary A.2], [24, Theorem 1.2] and [2, Theorem 1.1]. □

Remark It follows from [21], [18] that K_n is a nontrivial knot except when $n = 4$. An experiment using Weeks' computer program "SnapPea" [31] suggests that K_3 and K_5 are hyperbolic, but we will not use this experimental results.

Lemma 2.3 *The knot K_n has tunnel number one for any integer $n \neq 4$.*

Proof Since the link $k \cup c$ is a two-bridge link, the tunnel number of $k \cup c$ is one with unknotting tunnel τ; A regular neighborhood $N(k \cup c \cup \tau)$ is a genus two handlebody and $S^3 - \mathrm{int}N(k \cup c \cup \tau)$ is also a genus two handlebody, see Figure 5.

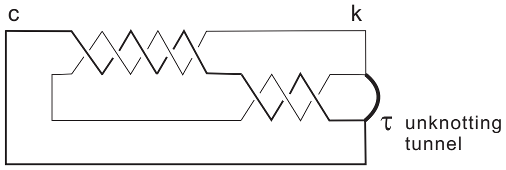

Figure 5

Then the general fact below (in which $k \cup c$ is not necessarily a two-bridge link) shows that the tunnel number of K_n is less than or equal to one. Since our knot K_n $(n \neq 4)$ is knotted in S^3, the tunnel number of K_n is one. □

Claim 2.4 *Let $k \cup c$ be a two component link in S^3 which has tunnel number one. Assume that c is unknotted in S^3. Then every knot obtained from k by twisting along c has tunnel number at most one.*

Proof Let τ be an unknotting tunnel and V a regular neighborhood of $k \cup c \cup \tau$ in S^3; V is a genus two handlebody. Since τ is an unknotting tunnel for $k \cup c$, by definition, $W = S^3 - \mathrm{int}V$ is also a genus two handlebody. Take a small tubular neighborhood $N(c) \subset \mathrm{int}V$ and perform $-\frac{1}{n}$-surgery on c using $N(c)$. Then we obtain a knot k_n as the image of k and obtain a genus two handlebody $V(c; -\frac{1}{n})$. Note that $V(c; -\frac{1}{n})$ and W define a genus two Heegaard splitting of S^3, see Figure 6, where c_n^* denotes the core of the filled solid torus.

Then it is easy to see that an arc τ_n given by Figure 6 is an unknotting tunnel for k_n as desired. □

Now we are ready to prove Theorem 1.1. Lemmas 2.1, 2.2 and 2.3 show that our knots K_n enjoy the required properties, except for $n = 3, 4, 5$. To prove

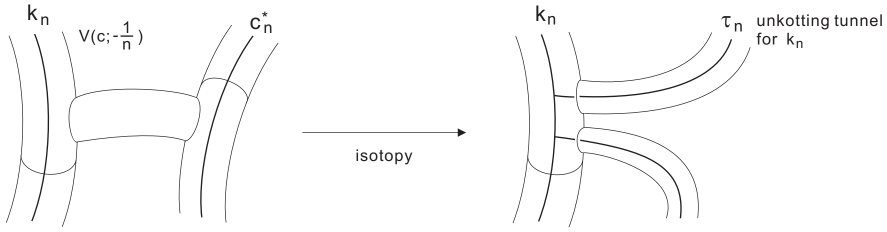

Figure 6

Theorem 1.1, we find hyperbolic knots K'_n so that $(K'_n; n)$ is Seifert fibered for $n = 3, 4, 5$ (instead of showing that K_3, K_5 are hyperbolic). As the simplest way, let K'_3, K'_4 and K'_5 be the mirror image of K_{-3}, K_{-4} and K_{-5}, respectively. Since K_{-3}, K_{-4} and K_{-5} are tunnel number one, hyperbolic knots by Lemmas 2.2 and 2.3, their mirror images K'_3, K'_4 and K'_5 are also tunnel number one, hyperbolic knots. It is easy to observe that $(K'_3; 3)$ (resp. $(K'_4; 4)$, $(K'_5; 5)$) is the mirror image of $(K_{-3}; -3)$ (resp. $(K_{-4}; -4)$, $(K_{-5}; -5)$). By Lemma 2.1, $(K_{-3}; -3)$, $(K_{-4}; -4)$ and $(K_{-5}; -5)$ are Seifert fibered, and hence $(K'_3; 3)$, $(K'_4; 4)$ and $(K'_5; 5)$ are also Seifert fibered. Putting K_n as K'_n for $n = 3, 4, 5$, we finish a proof of Theorem 1.1. □

3 Identifying exceptional fibers

In [24], Miyazaki and Motegi conjectured that if K admits a Seifert fibered surgery, then there is a trivial knot $c \subset S^3$ disjoint from K which becomes a Seifert fiber in the resulting Seifert fiber space, and verified the conjecture for several Seifert fibered surgeries [24, Section 6], see also [8]. Furthermore, computer experiments via "SnapPea" [31] suggest that such a knot c is realized by a short closed geodesic in the hyperbolic manifold $S^3 - K$, for details see [24, Section 9], [27].

In this section, we verify the conjecture for Seifert fibered surgeries given in Theorem 1.1.

Recall that K_n is obtained from k by $\frac{1}{-n+4}$-surgery on the trivial knot c (i.e., $(n-4)$-twist along c), see Figure 1. Denote by c_n the core of the filled solid torus. Then $K_n \cup c_n$ is a link in S^3 such that c_n is a trivial knot.

Lemma 3.1 *After n-surgery on K_n, c_n becomes an exceptional fiber of index $|4n - 15|$ in the resulting Seifert fiber space $(K_n; n)$.*

Proof Following the sequences given by Figures 3 and 4, we have a Montesinos link with three arcs γ, τ_1 and τ_2 as in Figure 7, where $n = 1$ in the final Montesinos link, and γ, τ_1, τ_2 and κ are the images of c, t_1, t_2 and k, respectively.

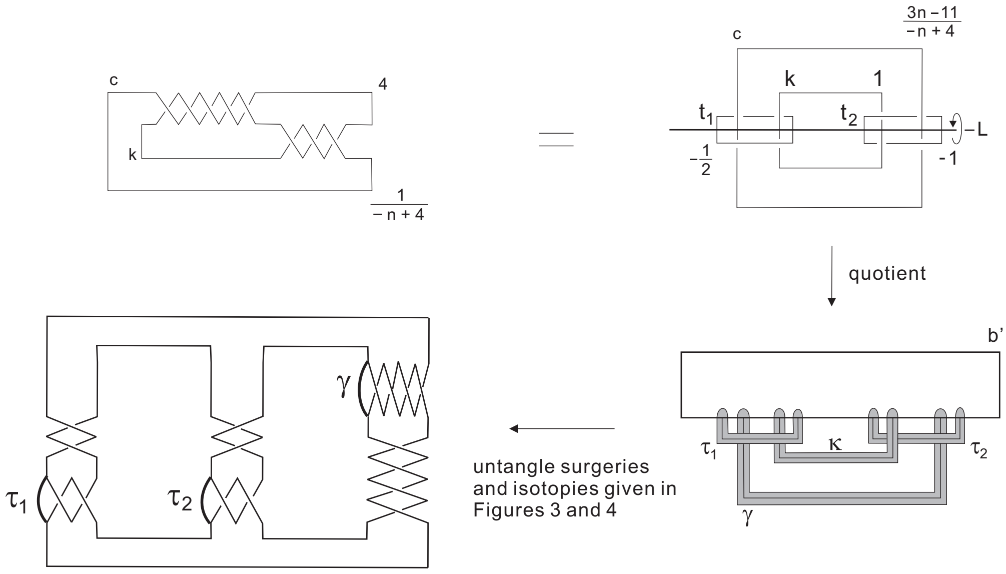

Figure 7: Positions of exceptional fibers

From Figure 7 we recognize that t_1, t_2 and c become exceptional fibers of indices 5, 3 and $|4n-15|$, respectively in $(K_n; n)$. □

For $n \neq 3, 4, 5$, c_n becomes an exceptional fiber of index $|4n-15|$, which is the unique maximal index, in $(K_n; n)$. Experiments via "SnapPea" [31] suggest that c_n is a shortest closed geodesic in $S^3 - K_n$ $(n \neq 3, 4, 5)$. For sufficiently large $|n|$, hyperbolic Dehn surgery theorem [29], [30] shows that c_n is the unique shortest closed geodesic in $S^3 - K_n$.

Let us assume that $n = 3, 4, 5$. Then we have put K_n as the mirror image of K_{-n} in the proof of Theorem 1.1. Let $k' \cup c'$ be the mirror image of the link $k \cup c$. Then K_n is obtained also from k' by $\frac{1}{-n-4}$-surgery on c' (i.e., $(n+4)$-twist along c'); we denote the core of the filled solid torus by c'_n. Note that there is an orientation reversing diffeomorphism from $(K_{-n}; -n)$ to $(K_n; n)$ sending c_{-n} (regarded as a fiber in $(K_{-n}; -n)$) to c'_n (regarded as a fiber in $(K_n; n)$). Thus the above observation implies that c'_n becomes an exceptional fiber of index $|4n+15|$, which is the unique maximal index, in $(K_n; n)$ $(n = 3, 4, 5)$.

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Received: 10 March 2005 Revised: 25 March 2005