ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 405–418
Published: 22 May 2005

Infinitely many two-variable generalisations of the Alexander-Conway polynomial

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Abstract We show that the Alexander-Conway polynomial Δ is obtainable via a particular one-variable reduction of each two-variable Links–Gould invariant $LG^{m,1}$, where m is a positive integer. Thus there exist infinitely many two-variable generalisations of Δ. This result is not obvious since in the reduction, the representation of the braid group generator used to define $LG^{m,1}$ does not satisfy a second-order characteristic identity unless $m = 1$. To demonstrate that the one-variable reduction of $LG^{m,1}$ satisfies the defining skein relation of Δ, we evaluate the kernel of a quantum trace.

AMS Classification 57M25, 57M27; 17B37, 17B81

Keywords Link, knot, Alexander-Conway polynomial, quantum superalgebra, Links–Gould link invariant

1 Introduction

The type I Lie superalgebras $sl(m|n)$ and $osp(2|2n)$ have the distinguishing property that they admit nontrivial one-parameter families of representations, and these representations extend to their quantum deformations $U_q[sl(m|n)]$ and $U_q[osp(2|2n)]$. Consequently, the link invariants derived from such representations are two-variable invariants [7, 14]. In the simplest case $sl(1|1)$, the invariant reduces to a one-variable invariant which is precisely the Alexander-Conway polynomial Δ [13]. The simplest nontrivial example of a two-variable invariant comes from $sl(2|1) \cong osp(2|2)$ [5, 8, 9, 11, 14]. For this case it has recently been shown [10] that a certain one-variable reduction recovers Δ. Whilst it may appear that the origin of this result may lie in the quantum superalgebra embedding $U_q[sl(1|1)] \subset U_q[sl(2|1)]$, in fact Δ is recovered only when the variable q assumes specific roots of unity. It is also well known that Δ occurs as a one-variable reduction of the two-variable HOMFLY polynomial [6]. The result of [10] thus shows that the extension of Δ to a two-variable quantum invariant is not unique.

In this paper we extend the result of [10] to higher rank superalgebras. Specifically we employ $U_q[gl(m|1)]$, which differs from $U_q[sl(m|1)]$ by the addition of a central element; that is $U_q[gl(m|1)] = U_q[u(1) \oplus sl(m|1)]$. This yields the same link invariant, but conveniently makes the representation theory easier to handle. For the minimal one-parameter family of representations of dimension 2^m, we construct a link invariant denoted $LG^{m,1}(\tau, q)$ which is a function of two independent variables q and $\tau \equiv q^{-\alpha}$. Here, α is the complex parameter which indexes the underlying representations. These invariants have been introduced and studied in [3, 4, 7].

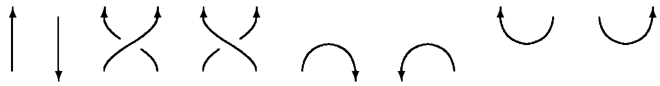
Our main result is Theorem 4.2 (originally conjectured in [10]), which is the following relation between $LG^{m,1}$ and Δ. For an oriented link L, we have:

$$LG_L^{m,1}(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m}). \qquad (1)$$

We prove this relation by showing that $LG^{m,1}(\tau, e^{\pi\sqrt{-1}/m})$ satisfies the skein relation defining $\Delta(\tau^{2m})$. To that end, we begin by recalling the method of construction for $LG^{m,1}$, following [3, 4]. Next, we demonstrate a couple of technical lemmas from the representation theory of $U_q[gl(m|1)]$. Using them, the key to the proof involves determining the kernel of a quantum trace, as per the method in [10]. We stress that the representation of the braid group generator used in the definition of $LG^{m,1}(\tau, e^{\pi\sqrt{-1}/m})$ does not satisfy a second-order characteristic identity. If this were the case, a proof would be trivial. We also stress that, as for the $m = 2$ case, our result does not directly arise from the quantum superalgebra embedding $U_q[sl(1|1)] \subset U_q[sl(m|1)]$.

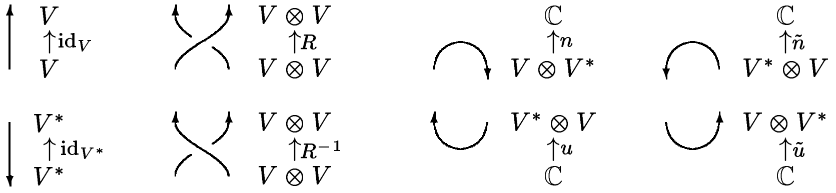
2 Quantum link invariants and $LG^{m,n}$

Any oriented tangle diagram can be expressed up to isotopy as a diagram composed from copies of the following elementary oriented tangle diagrams.

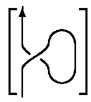


Furthermore any oriented tangle diagram can be expressed up to isotopy as a *sliced diagram* which is such a diagram sliced by horizontal lines such that each domain between adjacent horizontal lines contains either a single crossing or a single critical point.

Now let V be a finite-dimensional vector space, with dual space V^*. Using these, we assign an invertible endomorphism $R : V \otimes V \to V \otimes V$ and linear maps $n : V \otimes V^* \to \mathbb{C}$, $\tilde{n} : V^* \otimes V \to \mathbb{C}$, $u : \mathbb{C} \to V^* \otimes V$ and $\tilde{u} : \mathbb{C} \to V \otimes V^*$ to the elementary oriented tangle diagrams, as follows.



Corresponding to an oriented tangle diagram D, we then obtain a linear map $[D]$ by composing tensor products of copies of the linear maps associated with the elementary tangle diagrams in D. For example:



$$= (\mathrm{id}_V \otimes n)(R \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes u). \tag{2}$$

A quantum link invariant may then be defined as follows. Set V as the module associated with an irreducible, finite-dimensional representation π of some ribbon Hopf (super)algebra, for instance a quantum superalgebra. We then obtain the bracket [ ] by setting R as a representation of the braid group generator associated with the tensor product representation $\pi \otimes \pi$. This choice ensures the invariance of the bracket under the second Reidemeister move, due to the invertibility of R, and the third Reidemeister move, as R satisfies the Yang–Baxter equation (see (12) below). Note that at this point, we may freely use any scaling of R.

Now let the *quantum trace* be the linear map $\mathrm{cl} : \mathrm{End}(V^{\otimes(k+1)}) \to \mathrm{End}(V^{\otimes k})$ (where $k \geqslant 1$), which is defined for $X \in \mathrm{End}(V^{\otimes(k+1)})$ by:

$$\mathrm{cl}(X) = (\mathrm{id}_V^{\otimes k} \otimes n)(X \otimes \mathrm{id}_{V^*})(\mathrm{id}_V^{\otimes k} \otimes u).$$

Observe that (2) describes $\mathrm{cl}(R)$. Demanding that $\mathrm{cl}(R) = \mathrm{cl}(R^{-1}) = \mathrm{id}_V$ ensures the invariance of the bracket under the first Reidemeister move. This requirement determines the scaling of R, and also the choice of the mappings n, $\tilde{n}$, u and $\tilde{u}$. Specifically, representation-theoretic considerations mean that these mappings may be defined in terms of the representation of an element of the Cartan subalgebra of the underlying (super)algebra (see [3]).

For any given oriented tangle T, we thus obtain a map $[D_T]$, where D_T is an oriented tangle diagram corresponding to T, and the map $[D_T]$ is invariant under ambient isotopy of T. For notational convenience, we shall generally write $[T]$ for $[D_T]$, and this is meaningful as the evaluation of the invariant is independent of the choice of diagram D_T. By construction, the maps R, R^{-1}, n, $\tilde{n}$, u and $\tilde{u}$ are invariant with respect to the action of the Hopf (super)algebra. Consequently, the map $[T]$ is also invariant with respect to this action. Specifically, where T is an oriented $(1,1)$-tangle, the choice of V as irreducible means that Schur's Lemma ensures that $[T]$ is a scalar map (that is, a scalar multiple of id_V). This scalar is then a quantum link invariant of the link $\widehat{T}$ formed by the closure of T (see [16, 19]); in particular the scalar is unity when $\widehat{T}$ is the unknot.

Now fix positive integers m and n, and consider the quantum superalgebra $U_q[gl(m|n)]$, a quantum deformation of the universal enveloping algebra of the Lie superalgebra $gl(m|n)$. The two-variable Links–Gould invariant $LG^{m,n}(\tau, q)$ may then be obtained by specialising the above framework to the case of the minimal 2^{mn}-dimensional $U_q[gl(m|n)]$ representation π bearing a free parameter α (for details, see [3, 5]). In that case, where V is the module associated with π, we explicitly write:

$$[T] = LG^{m,n}_{\widehat{T}}(\tau, q)\,\mathrm{id}_V, \tag{3}$$

where we have used the variable $\tau = q^{-\alpha}$ instead of α; below we freely interchange use of the variables α and τ. Note that we have $LG^{m,n}_{\bigcirc}(\tau, q) = 1$.

Next, we present an important symmetry of these invariants. To that end, firstly note that $U_q[gl(m|n)]$ is defined (see [18]) in terms of a fixed invariant bilinear form on the weight space of $gl(m|n)$. We adopt the convention that the form is positive definite for $gl(m)$ roots and negative definite for $gl(n)$ roots. It may be deduced from the definition that, under this convention, the following superalgebra isomorphism holds:

$$U_q[gl(m|n)] \cong U_{q^{-1}}[gl(n|m)]. \tag{4}$$

We then note that the substitution $\alpha \to -(\alpha + m - n)$ maps the $U_q[gl(m|n)]$ representation π to its dual π^*. This, together with (4) allows us to deduce that, for any oriented link L, we have:

$$LG^{m,n}_L(\tau, q) = LG^{n,m}_L(\tau, q^{-1}). \tag{5}$$

We shall be interested below in the case $LG^{m,1}$ and the substitution of the root of unity $e^{\pi\sqrt{-1}/m}$ for q; importantly, the structure of the representation does not change at this particular root of unity. We also emphasise that under this substitution, we intend τ to remain independent; that is, we do *not* express it as $e^{-\alpha\pi\sqrt{-1}/m}$.

3 Some $U_q[gl(m|1)]$ representation theory

The construction of the mappings R, R^{-1}, n, $\tilde{n}$, u and $\tilde{u}$ determining $LG^{m,1}$ can be described in terms of the representation theory of $U_q[gl(m|1)]$. In this section, we establish notational conventions and provide the necessary representation-theoretic results needed to deduce our main result, relation (1).

We begin with the fact that every irreducible finite-dimensional $U_q[gl(m|1)]$ module $V(\Lambda)$ is uniquely labelled by its highest weight $\Lambda = (\Lambda_1, \ldots, \Lambda_m|\Lambda_{m+1})$. Moreover, each $V(\Lambda)$ is completely reducible with respect to the even subalgebra $U_q[gl(m) \oplus gl(1)]$ such that we may write

$$V(\Lambda) = \bigoplus_k V^0(\mu_k),$$

where each $V^0(\mu_k)$ is an irreducible $U_q[gl(m) \oplus gl(1)]$ module with highest weight μ_k. Here, we are in fact only interested in a subclass of these $U_q[gl(m|1)]$ modules, that is those whose highest weights are of the form

$$\Lambda(i,j,\alpha) \triangleq (0_{m-i-j}, -1_i, -2_j \,|\, \alpha + i + 2j),$$

where the subscripts indicate the number of times each entry is repeated in the weight, and α is an arbitrary complex parameter. We set $V(i,j,\alpha)$ as the irreducible module with highest weight $\Lambda(i,j,\alpha)$, and we also let $V^0(i,j,\alpha)$ denote the irreducible $U_q[gl(m) \oplus gl(1)]$ module with the same highest weight.

Specifically, $LG^{m,1}$ is defined in terms of the representation associated with the module $V(0,0,\alpha)$. We have the following decompositions [7]:

$$V(0,0,\alpha) = \bigoplus_{i=0}^{m} V^0(i,0,\alpha), \tag{6}$$

$$V(0,0,\alpha) \otimes V(0,0,\alpha) = \bigoplus_{i=0}^{m} V(i,0,2\alpha). \tag{7}$$

As each submodule $V(i,0,2\alpha)$ in (7) is *typical*, applying the Kac induced module construction [12], we may similarly deduce the following decomposition:

$$V(i,0,2\alpha) = \bigoplus_{j=0}^{i} \bigoplus_{k=i}^{m} V^0(k-j,j,2\alpha). \tag{8}$$

In [7] the decompositions (6) and (7) were deduced for generic values of α and real, positive q. It is important to stress that (6)–(8) remain valid when $q = e^{\pi\sqrt{-1}/m}$. We comment further on this aspect in the proof of Lemma 3.1 (below).

To simplify notation, we shall write V for $V(0,0,\alpha)$. With respect to (7), setting V_i as $V(i,0,2\alpha)$, let P_i be the projector mapping $V\otimes V$ onto V_i, so that we have:

$$P_iP_j=\delta_{ij}P_i,\qquad\qquad P_0+\cdots+P_m=\mathrm{id}_{V\otimes V}. \tag{9}$$

Then, from [4], we have:

$$R=\sum_{i=0}^{m}\xi_iP_i,\qquad\qquad R^{-1}=\sum_{i=0}^{m}\xi_i^{-1}P_i, \tag{10}$$

where

$$\xi_i=(-1)^iq^{i(2\alpha+i-1)-m\alpha}\equiv(-1)^i\tau^{m-2i}q^{i(i-1)}. \tag{11}$$

Note that the scaling of R has been chosen such that $\mathrm{cl}(R)=\mathrm{cl}(R^{-1})=\mathrm{id}_V$. The grading of the underlying vector space V means that R as defined in (10) actually satisfies a *graded* Yang–Baxter equation [19]. However, by insertion of factors of -1 into some of the components of R (as described in [4]) it is made to satisfy the usual ungraded Yang–Baxter equation:

$$(R\otimes\mathrm{id}_V)(\mathrm{id}_V\otimes R)(R\otimes\mathrm{id}_V)=(\mathrm{id}_V\otimes R)(R\otimes\mathrm{id}_V)(\mathrm{id}_V\otimes R). \tag{12}$$

It is clear from (10) that R satisfies the characteristic identity of order $m+1$:

$$\prod_{i=0}^{m}(R-\xi_i\,\mathrm{id}_{V\otimes V})=0\,\mathrm{id}_{V\otimes V}. \tag{13}$$

For any linear map X we denote $X|_{q=e^{\pi\sqrt{-1}/m}}$ by $\overline{X}$. Similarly, for any vector space W over $\mathbb{C}[q,q^{-1},\tau,\tau^{-1}]$, we denote by $\overline{W}$ the vector space over $\mathbb{C}[\tau,\tau^{-1}]$ obtained from W by setting $q=e^{\pi\sqrt{-1}/m}$. It is necessary to affirm that the mappings R, R^{-1}, n, $\tilde{n}$, u, $\tilde{u}$ and each P_i are well-defined in the substitution $q=e^{\pi\sqrt{-1}/m}$.

Lemma 3.1 *The mappings $\overline{R}$, $\overline{R^{-1}}$, $\overline{n}$, $\overline{\tilde{n}}$, $\overline{u}$, $\overline{\tilde{u}}$ and each $\overline{P_i}$ are well-defined, that is, all matrix elements of R, R^{-1}, n, $\tilde{n}$, u, $\tilde{u}$ and each P_i have no pole at $q=e^{\pi\sqrt{-1}/m}$.*

Proof We begin by recalling from §3 of [15] the $U_q[gl(m|1)]$ central element $\Gamma\triangleq(v\otimes v)\Delta(v^{-1})$, where Δ is the coproduct and v is the ribbon element in the centre of $U_q[gl(m|1)]$. Each projector P_i may be expressed as a polynomial function of the representation of Γ via:

$$P_i=\prod_{j\neq i}\frac{\Gamma-\gamma_j\,\mathrm{id}_{V\otimes V}}{\gamma_i-\gamma_j},$$

where γ_i denotes the eigenvalue of Γ on V_i. In fact, $\gamma_i = \xi_i^2$, where ξ_i is as introduced in (11). Note that $\overline{\gamma_i} \neq \overline{\gamma_j}$, for $i \neq j$. If we rewrite $P_i = N_i/D_i$, where:

$$N_i \triangleq \prod_{j\neq i} (\Gamma - \gamma_j \,\mathrm{id}_{V\otimes V}), \qquad \text{and} \qquad D_i \triangleq \prod_{j\neq i} (\gamma_i - \gamma_j),$$

then we see that $\overline{D_i}$ is nonzero. We next show that $\overline{N_i}$ is well-defined.

To that end, Γ may be expressed as a power series over $\mathbb{C}[q, q^{-1}]$ of the simple $U_q[gl(m|1)]$ generators [15, §3]. As the weights $\Lambda(i, j, \alpha)$ are generically *essentially typical*, for general q, expressions are known [17] for the matrix elements of the simple generators in a Gel'fand–Zetlin basis. The matrix elements of the even simple generators are well-defined when $q = e^{\pi\sqrt{-1}/m}$. This follows as condition (3.2) of [1] is satisfied for all the modules $V^0(i, 0, \alpha)$ of (6). Thus, (6), and by the same reasoning (8), remains valid for $q = e^{\pi\sqrt{-1}/m}$. The matrix elements of the odd simple generators for $U_q[gl(m|1)]$ are given by formulae (27,28) of [17]. Unlike the situation for the even generators, these formulae explicitly depend on the variable α. This means that they are well-defined when $q = e^{\pi\sqrt{-1}/m}$, since their denominators are nonvanishing for generic values of α.

Thus, each $\overline{N_i}$, hence each $\overline{P_i}$ is well-defined, and consequently so are $\overline{R}$ and $\overline{R^{-1}}$. The fact that the mappings $\overline{n}$, $\overline{\tilde{n}}$, $\overline{u}$ and $\overline{\tilde{u}}$ are also well-defined follows from their definitions in terms of the representation of an element of the $U_q[gl(m|1)]$ Cartan subalgebra. □

We remark that this proof also demonstrates that the decomposition of (7) remains valid in the reduction $q = e^{\pi\sqrt{-1}/m}$, since the projectors remain well-defined.

Lemma 3.2 *For each $i = 0, \ldots, m$ the expression $\overline{\mathrm{cl}(P_i)}$ is a well-defined scalar multiple of $\mathrm{id}_{\overline{V}}$; in fact $\overline{\mathrm{cl}(P_i)} = 0\,\mathrm{id}_{\overline{V}}$ for $i = 1, \ldots, m-1$.*

Proof Theorem 1 of [7], specified to our situation, reads:

$$\mathrm{cl}(P_i) = (-1)^i \prod_{j=1}^{i} \frac{q^{m-j+1} - q^{-(m-j+1)}}{q^{i-j+1} - q^{-(i-j+1)}} \cdot \frac{\tau q^{-(j-1)} - \tau^{-1} q^{j-1}}{\tau^2 q^{-(i+j-2)} - \tau^{-2} q^{i+j-2}}$$
$$\times \prod_{j=i+1}^{m} \frac{\tau q^{-(j-1)} - \tau^{-1} q^{j-1}}{\tau^2 q^{-(i+j-1)} - \tau^{-2} q^{i+j-1}}\, \mathrm{id}_V;$$

note that in the cases $i = 0, m$, the formula reduces to the following:

$$\begin{aligned}
\mathrm{cl}(P_0) &= \prod_{j=1}^{m} \frac{\tau q^{-(j-1)} - \tau^{-1} q^{j-1}}{\tau^2 q^{-(j-1)} - \tau^{-2} q^{j-1}} \, \mathrm{id}_V \\
\mathrm{cl}(P_m) &= (-1)^m \prod_{j=1}^{m} \frac{\tau q^{-(j-1)} - \tau^{-1} q^{j-1}}{\tau^2 q^{-(m+j-2)} - \tau^{-2} q^{m+j-2}} \, \mathrm{id}_V .
\end{aligned}$$

In these formulae, we intend $\tau \equiv q^{-\alpha}$ to be restricted so that the complex variable α is not an integer. (By an analytic continuation argument, this restriction does not affect our final result.) Now observe that the denominator of $\mathrm{cl}(P_i)$ never contains any factors of $q^m - q^{-m}$; this means that $\overline{\mathrm{cl}(P_i)}$ is always well-defined. However, if $i \neq 0, m$, its numerator always contains a factor of $q^m - q^{-m}$, meaning that $\overline{\mathrm{cl}(P_i)} = 0 \, \mathrm{id}_{\overline{V}}$. □

Now let V have a weight basis $\{e_0, \ldots, e_{2^m - 1}\}$. Since the weight spectrum of V is multiplicity-free, we can choose the labelling such that for $i = 0, \ldots, m$, the vector e_i has weight $\Lambda(i, 0, \alpha)$. In terms of this basis, any $A \in \mathrm{End}(V \otimes V)$ may be written in component form via $A(e_k \otimes e_l) = \sum_{ij} A_{kl}^{ij} (e_i \otimes e_j)$.

Lemma 3.3 $(P_i)_{jj}^{jj} = \delta_{ij}$*, for all* $i, j = 0, \ldots, m$.

Proof From (6), we know that e_i is a $U_q[gl(m) \oplus gl(1)]$ highest weight vector. Therefore $v_i \triangleq e_i \otimes e_i$ is also a $U_q[gl(m) \oplus gl(1)]$ highest weight vector, of weight $\Lambda(0, i, 2\alpha)$. Now looking at (8), we see that this $U_q[gl(m) \oplus gl(1)]$ highest weight only occurs in $V(i, 0, 2\alpha)$. Thus, v_i generates the irreducible module $V(i, 0, 2\alpha)$, and moreover, for each $V(j, 0, 2\alpha)$ there exists a $v_j \equiv e_j \otimes e_j$ which generates it. Thus, for each projector P_i we have $P_i(e_j \otimes e_j) = \delta_{ij}(e_j \otimes e_j)$, hence we conclude $(P_i)_{jj}^{jj} = \delta_{ij}$. □

4 The relation

In this section we show the following relation:

$$LG_L^{m,1}(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m}),$$

where $\Delta_L(t)$ is the Alexander-Conway polynomial which is defined by the following relations:

$$\Delta_{\bigcirc}(t) = 1, \tag{14}$$

$$\Delta(t) - \Delta(t) = (t^{1/2} - t^{-1/2}) \Delta(t). \tag{15}$$

Lemma 4.1 *Where T is an oriented $(2,2)$-tangle, $[T]$ may be expressed as:*

$$[T] = \sum_{i=0}^{m} a_i^T P_i, \tag{16}$$

where the coefficients a_i^T are such that each $\overline{a_i^T}$ is well-defined.

Proof Firstly, note that $[T]$ is a product of $U_q[gl(m|1)]$-invariant mappings, and $\{P_0, \ldots, P_m\}$ is a basis for the space of such mappings on $V \otimes V$. Thus, $[T]$ is necessarily of the form (16). Recall from Lemma 3.1 that the mappings R, R^{-1}, n, $\tilde{n}$, u and $\tilde{u}$ are well-defined in the substitution $q = e^{\pi\sqrt{-1}/m}$. Thus, as $[T]$ is defined in terms of these mappings, it is also well-defined in the substitution. Then, using Lemma 3.3, we have:

$$[T]_{jj}^{jj} = \sum_{i=0}^{m} a_i^T (P_i)_{jj}^{jj} = a_j^T, \qquad \text{for } j = 0, \ldots, m,$$

and conversely, for each index $i = 0, \ldots, m$, we have $a_i^T = [T]_{ii}^{ii}$. Thus, as $\overline{[T]}$ is well-defined, so is $\overline{a_i^T}$. $\square$

Before moving on to our main result, we emphasise that $\overline{R}$ does not satisfy a second-order characteristic identity (unless $m = 1$). In particular, the following identity:

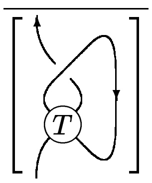
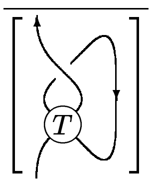
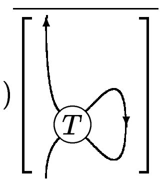
$$\overline{R} - \overline{R^{-1}} = (\tau^m - \tau^{-m})\, \mathrm{id}_{\overline{V} \otimes \overline{V}}, \tag{17}$$

only holds for $m = 1$. If (17) held for arbitrary m, the proof of our main result would be trivial.

Theorem 4.2 *For any oriented link L, there holds:*

$$LG_L^{m,1}(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m}).$$

Proof For any oriented $(2,2)$-tangle T, we have:

 $-$  $- (\tau^m - \tau^{-m})$ 

$$\begin{aligned}
&= \overline{\mathrm{cl}(R \circ [T])} - \overline{\mathrm{cl}(R^{-1} \circ [T])} - (\tau^m - \tau^{-m})\overline{\mathrm{cl}([T])} \\
&= \overline{\textstyle\sum_{i=0}^{m} \xi_i a_i^T \mathrm{cl}(P_i)} - \overline{\textstyle\sum_{i=0}^{m} \xi_i^{-1} a_i^T \mathrm{cl}(P_i)} - (\tau^m - \tau^{-m})\overline{\textstyle\sum_{i=0}^{m} a_i^T \mathrm{cl}(P_i)}
\end{aligned}$$

$$\begin{aligned}
&= \textstyle\sum_{i=0}^{m}\overline{\xi_i}\;\overline{a_i^T}\;\overline{\mathrm{cl}(P_i)} - \sum_{i=0}^{m}\overline{\xi_i^{-1}}\;\overline{a_i^T}\;\overline{\mathrm{cl}(P_i)} - (\tau^m - \tau^{-m})\sum_{i=0}^{m}\overline{a_i^T}\;\overline{\mathrm{cl}(P_i)} \\
&= \textstyle\sum_{i=0}^{m}\left((\overline{\xi_i} - \overline{\xi_i^{-1}}) - (\tau^m - \tau^{-m})\right)\overline{a_i^T}\;\overline{\mathrm{cl}(P_i)} \\
&= 0\,\mathrm{id}_{\overline{V}},
\end{aligned}$$

where the second equality follows from (9), (10) and (16), the third from Lemmas 3.2 and 4.1, and the last from Lemma 3.2 and the observation from (11) that

$$\overline{\xi_0} = -\overline{\xi_m^{-1}} = \tau^m. \tag{18}$$

In view of (3), we thus have the following skein relation:

$$\overline{LG^{m,1}_{\times}} - \overline{LG^{m,1}_{\times}} = (\tau^m - \tau^{-m})\overline{LG^{m,1}_{)(}},$$

so $\overline{LG^{m,1}}$ satisfies (15). It also satisfies (14), as $\overline{LG^{m,1}_{\bigcirc}} = 1$. Thus, for any oriented link L, we have $LG^{m,1}_L(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m})$. □

Now note that the proof of Theorem 4.2 remains valid when $\overline{X}$ is instead regarded as $X|_{q=e^{\pi\sqrt{-1}\,r/m}}$, where r is any integer such that r and m are relatively prime. This follows since Lemmas 3.1, 3.2 and 4.1, and also (18) remain valid in this case. We thus have the following.

Theorem 4.3 *For any oriented link L, there holds:*

$$LG^{m,1}_L(\tau, e^{\pi\sqrt{-1}\,r/m}) = \Delta_L(\tau^{2m}),$$

where r is any integer such that r and m are relatively prime.

In particular, via the choice $r = -1$ and the use of symmetry (5), we immediately deduce the following.

Corollary 4.4 *For any oriented link L, there holds:*

$$LG^{1,m}_L(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m}).$$

5 Extensions

To conclude, we believe that Theorem 4.2 can be extended to a similar statement for $LG^{m,n}$.

Conjecture 5.1 *For any oriented link L, there holds:*

$$LG_L^{m,n}(\tau, e^{\pi\sqrt{-1}/m}) = \Delta_L(\tau^{2m})^n, \tag{19}$$

and equivalently, by the symmetry (5):

$$LG_L^{m,n}(\tau, e^{\pi\sqrt{-1}/n}) = \Delta_L(\tau^{2n})^m. \tag{20}$$

Thus, for a given invariant $LG^{m,n}$, there are two distinct reductions which recover Δ; note that Theorem 4.2 and Corollary 4.4 are particular cases of Conjecture 5.1. We mention that the considerations leading to Theorem 4.3 also lead to the obvious generalisation of Conjecture 5.1.

These relations are initially surprising in that neither is symmetric in m and n; however, we have a range of evidence to support them. For instance, we can verify (20) for $LG^{2,1}$ for closed 2-braids $\widehat{\sigma^k}$, where σ is the generator for the braid group B_2. To that end, with reference to (10), we have for $LG^{2,1}$:

$$R = q^{-2\alpha}P_0 - P_1 + q^{2\alpha+2}P_2 = \tau^2 P_0 - P_1 + \tau^{-2}q^2 P_2,$$

thus:

$$R^k = \tau^{2k}P_0 + (-1)^k P_1 + \tau^{-2k}q^{2k}P_2.$$

Specialising the formulae of Lemma 2, we have:

$$\begin{aligned} \mathrm{cl}(P_0) &= \frac{\tau - \tau^{-1}}{(\tau q + \tau^{-1}q^{-1})(\tau^2 q - \tau^{-2}q^{-1})}\,\mathrm{id}_V, \\ \mathrm{cl}(P_1) &= \frac{-(q+q^{-1})}{(\tau + \tau^{-1})(\tau q + \tau^{-1}q^{-1})}\,\mathrm{id}_V, \\ \mathrm{cl}(P_2) &= \frac{\tau q - \tau^{-1}q^{-1}}{(\tau + \tau^{-1})(\tau^2 q - \tau^{-2}q^{-1})}\,\mathrm{id}_V. \end{aligned}$$

So, in the substitution $q = -1$, denoting $X|_{q=-1}$ by $\overline{\overline{X}}$:

$$\overline{\overline{\mathrm{cl}(R^k)}} = \tau^{2k}\overline{\overline{\mathrm{cl}(P_0)}} + (-1)^k\overline{\overline{\mathrm{cl}(P_1)}} + \tau^{-2k}\overline{\overline{\mathrm{cl}(P_2)}},$$

where $\overline{\overline{\mathrm{cl}(P_0)}} = -\frac{1}{2}\overline{\overline{\mathrm{cl}(P_1)}} = \overline{\overline{\mathrm{cl}(P_2)}} = (\tau + \tau^{-1})^{-2}\,\mathrm{id}_{\overline{\overline{V}}}$, and so:

$$LG^{2,1}_{\widehat{\sigma^k}}(\tau, -1) = \left(\frac{\tau^k - (-\tau)^{-k}}{\tau + \tau^{-1}}\right)^2.$$

Then, for $\Delta(\tau^2) \equiv LG^{1,1}(\tau)$, we have $R = \tau P_0 - \tau^{-1}P_1$, where

$$\mathrm{cl}(P_0) = -\mathrm{cl}(P_1) = (\tau + \tau^{-1})^{-1}\,\mathrm{id}_V.$$

Hence $\Delta_{\widehat{\sigma^k}}(\tau^2) = (\tau^k - (-\tau)^{-k})/(\tau + \tau^{-1})$, and thus:

$$LG^{2,1}_{\widehat{\sigma^k}}(\tau, -1) = \Delta_{\widehat{\sigma^k}}(\tau^2)^2.$$

Similarly, we can verify Conjecture 5.1 for $LG^{2,2}$ for closed 2-braids $\widehat{\sigma^k}$, using formulae derived in [7] (specifically, formula (71) and explicit details described in later sections). That is, we have:

$$LG^{2,2}_{\widehat{\sigma^k}}(\tau, e^{\pi\sqrt{-1}/2}) = \Delta_{\widehat{\sigma^k}}(\tau^4)^2.$$

Lastly, we have also been able to computationally verify (20) for $LG^{2,1}$ for a range of prime knots using the state model method of evaluation for $LG^{2,1}$ described in [2]. Specifically, this has been done for a selection of 4310 prime knots of up to 14 crossings, including all prime knots of up to 10 crossings. Beyond that, using formula (71) of [7], we have also verified that (19) holds for $LG^{m,n}$ for all $m, n \leqslant 5$, for closed 2-braids $\widehat{\sigma^k}$ for $k = 2, \ldots, 6$ (and thereby, for all $0 \leqslant |k| \leqslant 6$).

Now, let GLZ^n denote the invariants proposed in [7] associated with the $U_q[osp(2|2n)]$ superalgebras. We can state a similar result to Conjecture 5.1.

Conjecture 5.2 *For any oriented link L, there holds*

$$GLZ^n_L(\tau, e^{\pi\sqrt{-1}/2}) = \Delta_L(\tau^4)^n.$$

As $osp(2|2) \cong sl(2|1)$, we have $GLZ^1 \equiv LG^{2,1}$, so this conjecture is true for $n = 1$. Further evidence for it is that via similar considerations to the above using results from [7], we have confirmed that:

$$GLZ^2_{\widehat{\sigma^k}}(\tau, e^{\pi\sqrt{-1}/2}) = \Delta_{\widehat{\sigma^k}}(\tau^4)^2.$$

The difficulty in proving these conjectures lies in the fact that in general $\Delta(t)^n$ satisfies higher-order skein relations. One could begin by establishing that $LG^{m,n}$ and GLZ^n, at the appropriate values of q, satisfy the same skein relations as $\Delta(t)^n$. For example, for $LG^{2,1}$, two such skein relations are known, and these may be used to evaluate the invariant for all algebraic links [9]. For $q = -1$, we have checked that both skein relations reduce to ones which are satisfied by $\Delta(t)^2$, which confirms that (20) holds for $LG^{2,1}$ for a vast class of links. However, it is not clear that these two skein relations are sufficient to determine $LG^{2,1}$ for any arbitrary link. More generally, for either $LG^{m,n}$ or GLZ^n, the only easily-determined skein relation is that corresponding to the characteristic identity satisfied by R (illustrated for the $LG^{m,1}$ case in (13)). Additional skein relations are generally not known.

Acknowledgements

We thank Michael Gagen and Mark Gould for helpful comments. Jon Links thanks the Australian Research Council for support through an Australian Research Fellowship.

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Received: 21 January 2005 Revised: 14 April 2005