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Poincaré submersions

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Abstract We prove two kinds of fibering theorems for maps $X \to P$, where X and P are Poincaré spaces. The special case of $P = S^1$ yields a Poincaré duality analogue of the fibering theorem of Browder and Levine.

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1 Introduction

One of the early successes of surgery theory was the fibering theorem of Browder and Levine [B-L], which gives criteria for when a smooth map $f\colon M \to S^1$ is homotopic to a submersion. Here M is assumed to be a connected closed, smooth manifold of dimension ≥ 6, and we also require f to induce an isomorphism of fundamental groups. The Browder-Levine fibering theorem then says that f is homotopic to a submersion if and only if the homotopy groups of M are finitely generated in each degree.

The purpose of the current note is to prove fibering results in the Poincaré duality category. Note that a submersion of closed manifolds is a smooth fiber bundle with closed manifold fibers. Replacing the closed manifolds with finitely dominated Poincaré spaces and the fiber bundle with a fibration yields the notion of *Poincaré submersion:* this is a map between Poincaré spaces whose homotopy fibers are Poincaré spaces.

Our first result concerns the case when the target is acyclic (this includes the Browder-Levine situation). Let X be a connected, finitely dominated Poincaré duality space of (formal) dimension d and fundamental group π. Let

$$f\colon X \to B\pi$$

be the classifying map for the universal cover of X. We will be assuming that the classifying space $B\pi$ is a finitely dominated Poincaré space of dimension p.

Theorem A *Let F denote the homotopy fiber of f. Then F is a homotopy finite Poincaré duality space of dimension $d-p$ if and only if the homotopy groups of X are finitely generated in each degree.*

For our second result, let $f\colon X \to P$ be a map of orientable, finitely dominated and connected Poincaré duality spaces. Assume X has dimension d and P has dimension p. We will give criteria for deciding when the homotopy fiber F of f satisfies Poincaré duality.

Let $i\colon F \to X$ be the evident map. There is an *umkehr homomorphism*

$$i^!_*\colon H_*(X) \to H_{*-p}(F)$$

which is defined if $p \geq 3$ or if P is 1-connected (cf. §4). The pushforward of a fundamental class $[X] \in H_d(X)$ for X with respect to $i^!$ then gives a class

$$x_f := i^!_*([X]) \in H_{d-p}(F)\,.$$

This will be our candidate for a fundamental class of F.

Theorem B *Assume that f is 2-connected. Then the following are equivalent:*

(1) *$H_*(F) = 0$ in sufficiently large degrees.*[1]

(2) *F is homotopy finite.*

(3) *F is a Poincaré duality space.*

If in addition X is 1-connected, then the above are equivalent to the assertion that

(4) *the homomorphism*

$$\cap x_f\colon H^*(F) \to H_{d-p-*}(F)$$

is an isomorphism in all degrees.

Remark When $P = S^p$ is a sphere, (1) $\Rightarrow$ (3) overlaps with [C, lemma 1.1]. The implication (2) $\Rightarrow$ (3) is a consequence of [Kl1, theorem B].

We do not *a priori* assume that Poincaré duality spaces satisfy a finiteness condition, so the implication (3) $\Rightarrow$ (2) is non-trivial.

[1] Correction added June, 2005: If X is not 1-connected, one also requires the hypothesis that the homotopy groups of X are finitely generated. I am indebted to Jonathan Hillman for pointing out that a hypothesis was missing here. Hillman also communicated to me the following counterexample: take X to be the connected sum of $S^5 \times S^1$ with $S^3 \times S^3$ and let $f\colon X \to S^1$ classify the universal cover. Then $\pi_3(F)$ is infinitely generated.

Conventions A space is *homotopy finite* if has the homotopy type of a finite cell complex. A space is *finitely dominated* if it is the retract of a homotopy finite space.

A *Poincaré space* of formal dimension d is a space X for which there exists a pair $(\mathcal{L}, [X])$ consisting of a rank one abelian system of local coefficients $\mathcal{L}$ on X and a (fundamental) class $[X] \in H_d(X;\mathcal{L})$ such that the cap product homomorphism

$$\cap[X]\colon\ H^*(X;\mathcal{A}) \to H_{d-*}(X;\mathcal{L}\otimes\mathcal{A})$$

is an isomorphism, for all local coefficient modules $\mathcal{A}$ on X (cf. [W1], [Kl2]). If X is connected, then it is enough to establish the isomorphism when $\mathcal{A}$ is the integral group ring of the fundamental group of X. When the local system $\mathcal{L}$ is constant, we say that X is *orientable.* We do not at assume any finiteness conditions in the definition of Poincaré space appearing here. However, in the 1-connected case, homotopy finiteness is actually a consequence of Poincaré duality (see 3.2 below).

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2 Proof of Theorem A

We first prove the 'only if' part. Assume that F is a homotopy finite Poincaré space. Since F is 1-connected and homotopy finite, we infer that its homology is finitely generated. Apply the mod $\mathcal{C}$ Hurewicz theorem (with $\mathcal{C}$ = the Serre class of finitely generated abelian groups) to see that the homotopy groups of F are finitely generated [S, corollary 9.6.16].

We now prove the 'if' part. Note that F has the homotopy type of the universal cover of X, so F is homotopy finite dimensional bacause X is. By the long exact homotopy sequence and the fact that $\pi_*(X)$ is degreewise finitely generated, we infer that $\pi_*(F)$ is degreewise finitely generated. Since F is simply connected, the mod $\mathcal{C}$ Hurewicz theorem shows that the homology groups of F are finitely generated. By a result of Wall [W2], we see that F is homotopy finite.

We now know that each space in the homotopy fiber sequence

$$F \to X \to B\pi$$

is finitely dominated. It follows directly from [Kl1, theorem B] (see also [G]) that F satisfies Poincaré duality and has formal dimension $d-p$. This completes the proof of Theorem A. $\square$

3 Duality and finiteness

A chain complex C of abelian groups is said to be *dualizable* if there is chain complex D and a map

$$d\colon \mathbb{Z} \to C \otimes D$$

($\otimes =$ derived tensor product, and d is allowed to be degree shifting) such that, for all P, we get that the induced map of complexes

$$\hom(C,P) \to \hom(\mathbb{Z}, P \otimes D)$$

(derived hom) given by $f \mapsto (f \otimes 1_D) \circ d$ induces an isomorphism on homology, where 1_D denote the identity map of D.

A chain map $C \to D$ is said to be a *weak equivalence* if it induces an isomorphism in homology. More generally C and D are said to be *weak equivalent* if there is a finite sequence of weak equivalences starting at C and ending at D. A chain complex is *(chain) homotopy finite* if it is weak equivalent to a finite chain complex, i.e., a complex of finite rank free abelian groups with finitely many non-trivial degrees. A chain complex is *finitely dominated* if is a retract up to homotopy of a finite chain complex. It is well-known chain complex over $\mathbb{Z}$ is homotopy finite if and only if it is finitely dominated (see [W2]).

Lemma 3.1 *If C is dualizable, then it is homotopy finite over $\mathbb{Z}$.*

Proof Since $\mathbb{Z}$ is "compact," there exists a finite chain complex C_0, a map $i\colon C_0 \to C$ and a map $d_0\colon \mathbb{Z} \to C_0 \otimes D$ such that

$$\mathbb{Z} \xrightarrow{d_0} C_0 \otimes D \xrightarrow{i \otimes 1} C \otimes D$$

is homotopic to d. Consider the homotopy commutative diagram

$$\begin{array}{ccc}
\hom(C,C) & \xrightarrow[\simeq]{(-\otimes 1_C)\circ d} & \hom(\mathbb{Z}, C \otimes D) \\
{\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\
\hom(C,C_0) & \xrightarrow[(-\otimes 1_C)\circ d]{\simeq} & \hom(\mathbb{Z}, C_0 \otimes D)
\end{array}$$

The map d_0 lives in the lower right corner and maps to d under the right vertical map. The map 1_C maps to d under the top horizontal map. Since the lower horizontal map is an equivalence, we get a map $j\colon C \to C_0$ such that $i_*(j) = j \circ i$ is homotopic to 1_C. We conclude that the identity map of C factors up to homotopy through the finite object C_0. □

Note now if X^d is a 1-connected space which is equipped with a chain level fundamental class $[X]$ for which Poincaré duality holds, then $C(X)$ = the singular chains on X is dualizable using the maps

$$\mathbb{Z} \xrightarrow{[X]} C(X) \xrightarrow{\text{diagonal}} C(X \times X) \simeq C(X) \otimes C(X)\,,$$

where the first map is the homomorphism (of degree d) induced by a choice of fundamental class. By the above lemma, we infer that $C(X)$ is homotopy finite.

A result of Wall says that a 1-connected space is homotopy finite if and only if its chain complex is (chain) homotopy finite (see [W3]). Hence,

Corollary 3.2 *Let X be a 1-connected space which satisfies Poincaré duality. Then X is also homotopy finite.*

4 The umkehr homomorphism

According to [W1, theorem 2.4], if $\dim P \geq 3$ is a Poincaré duality space, then there is a homotopy equivalence

$$P \;\simeq\; P_0 \cup_\alpha D^p\,,$$

in which P_0 is a CW complex of dimension $\leq p{-}1$. If P is 1-connected, then P_0 has the homotopy type of a CW complex of dimension $\leq p{-}2$. If P has dimension ≤ 2, then $P \simeq S^p$, and the above decomposition is also available.

Furthermore, once an orientation for P has been chosen, the above cell decomposition is unique up to oriented homotopy equivalence. ¿From now on, we fix an identification $P := P_0 \cup D^p$, where $\dim P_0 \leq p{-}1$.

Without loss in generality, let us assume that $f\colon X \to P$ has been converted into a Hurewicz fibration. Let $X_0 = f^{-1}(P_0)$. Then we obtain a pushout square

$$\begin{array}{ccc} f^{-1}(S^{p-1}) & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ f^{-1}(D^p) & \longrightarrow & X\,. \end{array}$$

Using the homotopy lifting property, we see that the pair $(f^{-1}(D^p), f^{-1}(S^{p-1}))$ has the homotopy type of the pair $(F \times D^p, F \times S^{p-1})$. Taking vertical cofibers in the diagram, we get an umkehr map

$$i^!\colon X \longrightarrow X/X_0 = f^{-1}(D^p)/f^{-1}(S^{p-1}) \simeq F_+ \wedge S^p$$

The *umkehr homomorphism*

$$i^!_*\colon H_*(X) \to H_{*-p}(F)$$

is the effect of applying singular homology to $i^!$, and using the suspension isomorphism to perform the degree shift.

5 Proof of Theorem B

(1) $\Rightarrow$ (2) By the long exact homotopy sequence of the fibration, we see that $\pi_*(F)$ is degreewise finitely generated. By the mod $\mathcal{C}$ Hurewicz theorem, we infer that $H_*(F)$ is finitely generated. Then F is homotopy finite by [W2].

(2) $\Rightarrow$ (3) Follows from [Kl1, theorem B].

(3) $\Rightarrow$ (1) This follows from 3.2.

For the remainder of the proof of the theorem, we suppose that X is 1-connected. Then so are F and P.

(3) $\Rightarrow$ (4) It will be enough to show that the class x_f is a generator of $H_{d-p}(F) \cong \mathbb{Z}$. By definition of x_f, this is equivalent to knowing that the homomorphism

$$i^!_*\colon H_d(X) \to H_{d-p}(F)$$

is of degree ± 1.

This can be seen as follows: the space X_0 is the pullback of the fibration $f\colon X \to P$ along a CW complex P_0 of dimension $\leq p-2$ (this uses the fact that P is 1-connected, cf. §4). As F has formal dimension $\leq d-p$, it is straightforward to check that X_0 has the homotopy type of a CW complex of dimension $\leq d-2$. Using the homotopy cofiber sequence

$$X_0 \longrightarrow X \xrightarrow{i^!} F_+ \wedge S^p$$

and the fact that the homology of X_0 vanishes above degree $d-2$, we see that $i^!$ induces an isomorphism in homology in degree d.

(4) $\Rightarrow$ (3) Trivial. $\square$

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