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The Johnson homomorphism and the second cohomology of IA_n

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Abstract Let F_n be the free group on n generators. Define IA_n to be group of automorphisms of F_n that act trivially on first homology. The Johnson homomorphism in this setting is a map from IA_n to its abelianization. The first goal of this paper is to determine how much this map contributes to the second rational cohomology of IA_n.

A descending central series of IA_n is given by the subgroups $K_n^{(i)}$ which act trivially on $F_n/F_n^{(i+1)}$, the free rank n, degree i nilpotent group. It is a conjecture of Andreadakis that $K_n^{(i)}$ is equal to the lower central series of IA_n; indeed $K_n^{(2)}$ is known to be the commutator subgroup of IA_n. We prove that the quotient group $K_n^{(3)}/\mathrm{IA}_n^{(3)}$ is finite for all n and trivial for $n = 3$. We also compute the rank of $K_n^{(2)}/K_n^{(3)}$.

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1 Introduction

Let $\mathrm{Aut}(F_n)$ denote the automorphism group of the free group F_n on generators $\{x_1, \ldots, x_n\}$. The IA-automorphism group of F_n, denoted by IA_n, is defined to be the kernel of the exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n) \longrightarrow \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1$$

induced by the action of $\mathrm{Aut}(F_n)$ on the abelianization of F_n.

Although Nielsen studied IA_n as early as 1918, it is still the case that little is understood about this group. For instance, while there is a well-known finite set of generators, discovered by Magnus in 1934 [M], it remains unknown whether IA_n is finitely presentable for $n > 3$. In 1997, Krstić-McCool proved that IA_3 is not finitely presentable [KM, Theorem 1].

Magnus's set of generators for IA_n is the collection of automorphisms

$$\{g_{ij}\colon\ 1 \le i,j \le n, i \ne j\} \cup \{f_{ijk}\colon\ 1 \le i,j,k \le n, j \ne i \ne k, i < j\}$$

where g_{ij} is defined by

$$g_{ij}(x_r) = \begin{cases} x_r & r \ne j \\ x_i x_j x_i^{-1} & r = j \end{cases}$$

and f_{ijk} is defined by

$$f_{ijk}(x_r) = \begin{cases} x_r & r \ne k \\ x_k[x_i, x_j] & r = k. \end{cases}$$

Recently Cohen-Pakianathan [CP], Farb [F], and Kawazumi [K] independently determined the first homology and cohomology groups of IA_n using a map based on Johnson's homomorphism for the Torelli group [J]. For a group G, let $G^{(1)} = G$, and let $G^{(k)}$ be kth term of the lower central series. Let $H_1(G)$ denote the integral homology of G, and $H_1(G)^*$ its dual. Cohen-Pakianathan [CP, Corollary 3.5], Farb [F, Theorem 1.1], and Kawazumi [K, Theorem 6.1] proved the following:

Theorem 1.1 *There exists a surjective homomorphism*

$$\tau\colon\ \mathrm{IA}_n \longrightarrow \textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*.$$

Furthermore, the induced map

$$\tau_*\colon\ H_1(\mathrm{IA}_n) \longrightarrow \textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*$$

is an isomorphism of $\mathrm{GL}(n,\mathbb{Z})$*-modules. In particular, the free abelian group* $H_1(\mathrm{IA}_n)$ *has rank*

$$\mathrm{rank}(H_1(\mathrm{IA}_n)) = n^2(n-1)/2\,.$$

See Section 2.3 for the definition of τ. Note that the theorem implies that $\mathrm{IA}_n^{(2)}$ is equal to the kernel of τ. Andreadakis [A] proved Theorem 1.1 for the case $n = 3$.

Recall that $H_1(F_n) \simeq F_n/F_n^{(2)} \simeq \mathbb{Z}^n$ has a standard basis $\{e_1, \ldots, e_n\}$ with dual basis $\{e_1^*, \ldots, e_n^*\}$. The image of the Magnus generators under τ is

$$\tau(g_{ij}) = (e_i \wedge e_j) \otimes e_j^*$$

and

$$\tau(f_{ijk}) = (e_i \wedge e_j) \otimes e_k^*.$$

The subgroup IA_n of $\mathrm{Aut}(F_n)$ is a free group analogue of the Torelli subgroup of the based mapping class group. The corresponding subgroup OA_n of the outer automorphism group $\mathrm{Out}(F_n)$, gives a free group analogue of the unbased Torelli group. The group of inner automorphisms of F_n is a normal subgroup of IA_n, so there is an exact sequence

$$1 \longrightarrow \mathrm{OA}_n \longrightarrow \mathrm{Out}(F_n) \longrightarrow \mathrm{GL}(n,\mathbb{Z}) \longrightarrow 1$$

and a surjective homomorphism $\overline{\tau}$ from OA_n to a quotient of $\bigwedge^2 H \otimes H^*$

$$\overline{\tau}\colon\ \mathrm{OA}_n \longrightarrow \big(\textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*\big)/H_1(F_n).$$

Almost nothing is known about higher cohomology of IA_n; it is not even known whether $H^2(\mathrm{IA}_n)$ is of finite or infinite dimension. Our main goal here is to determine the second rational cohomology of IA_n and OA_n coming from the homomorphisms τ. In papers by Hain [H] and Sakasai [Sak], the Johnson homomorphism is used to understand the second and third rational cohomology groups of the Torelli group. In this paper, we use their method to compute the kernel of the homomophism

$$\tau^*\colon\ H^2(\textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*, \mathbb{Q}) \longrightarrow H^2(\mathrm{IA}_n, \mathbb{Q})$$

and an analogue for OA_n.

Theorem 1.2 *The kernel of the homomorphism*

$$\tau^*\colon\ H^2(\textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*, \mathbb{Q}) \longrightarrow H^2(\mathrm{IA}_n, \mathbb{Q})$$

decomposes into simple $\mathrm{GL}(n,\mathbb{Q})$*-modules as*

$$\ker(\tau^*) \simeq \Phi_{0,\dots,0,1,0,-1} \oplus \Phi_{1,0,\dots,0,1,1,-2}.$$

The kernel of the homomorphism

$$\overline{\tau}^*\colon\ H^2((\textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)^*)/H_1(F_n), \mathbb{Q}) \longrightarrow H^2(\mathrm{OA}_n, \mathbb{Q})$$

is the simple module

$$\ker(\overline{\tau}^*) \simeq \Phi_{1,0,\dots,0,1,1,-2}.$$

The notation requires some explanation. Briefly, each Φ_α represents an isomorphism class of a simple $\mathrm{GL}(n,\mathbb{Q})$-module; the class is uniquely determined by the subscript α. This notation is described in more detail in Section 2.1 below and is explained comprehensively in Fulton-Harris [FH].

A key input is a “higher” Johnson homomorphism $\tau^{(2)}$, defined on $\ker(\tau)$:

Theorem 1.3 *There exists a homomorphism*

$$\tau^{(2)}\colon\ \mathrm{IA}_n^{(2)} \longrightarrow \big(\textstyle\bigwedge^2 H_1(F_n) \otimes H_1(F_n)/\bigwedge^3 H_1(F_n)\big) \otimes H_1(F_n)^*$$

and a homomorphism

$$\overline{\tau}^{(2)}\colon \mathrm{OA}_n^{(2)} \longrightarrow \big(\big(\textstyle\bigwedge^2 H_1(F_n)\otimes H_1(F_n)/\bigwedge^3 H_1(F_n)\big)\otimes H_1(F_n)^*\big)/\bigwedge^2 H_1(F_n).$$

The kernel of $\tau^{(2)}$ *(respectively,* $\overline{\tau}^{(2)}$*) contains* $\mathrm{IA}_n^{(3)}$ *(respectively,* $\mathrm{OA}_n^{(3)}$*) as a subgroup of finite index. Tensoring each target space with* $\mathbb{Q}$*, the image of* IA_n *in*

$$\big(\textstyle\bigwedge^2 H_1(F_n)\otimes H_1(F_n)/\bigwedge^3 H_1(F_n)\big)\otimes H_1(F_n)^*\otimes\mathbb{Q}$$

is isomorphic as a $\mathrm{GL}(n,\mathbb{Q})$*-representation to*

$$\Phi_{0,1,0,\dots,0}\oplus\Phi_{1,1,0,\dots,0,1,-1}$$

and the image of OA_n *in*

$$\big(\big(\textstyle\bigwedge^2 H_1(F_n)\otimes H_1(F_n)/\bigwedge^3 H_1(F_n)\big)\otimes H_1(F_n)^*/\bigwedge^2 H_1(F_n)\big)\otimes\mathbb{Q}$$

is isomorphic to $\Phi_{1,1,0,\dots,0,1,-1}$.

Theorem 1.3 is an analogue of a result of Morita's for the Torelli group [Mo]. Satoh has determined the image of $\tau^{(2)}$ over $\mathbb{Z}$ in recent work [Sat].

An interesting filtration $K_n^{(i)}$ of IA_n is given by the subgroups

$$K_n^{(i)}=\ker(\mathrm{Aut}(F_n)\longrightarrow\mathrm{Aut}(F_n/F_n^{(i)}))$$

Following from the definitions and Theorem 1.1, we have $K_n^{(1)}=\mathrm{IA}_n$ and $K_n^{(2)}=[\mathrm{IA}_n,\mathrm{IA}_n]$. Andreadakis proved that $K_n^{(i)}$ is equal to the lower central series in the case $n=2$ [A, Theorem 6.1], and that the first three terms of these series coincide in the case $n=3$ [A, Theorem 6.2]. This led him to conjecture that these two series might be equal for all n. We reprove his result for $n=3$ and show that the quotient $K_n^{(3)}/\mathrm{IA}_n^{(3)}$ is finite for all n.

Corollary 1.4 *The quotient* $K_n^{(3)}/\mathrm{IA}_n^{(3)}$ *is finite for all* n *and trivial for* $n=3$*. The rank of the free abelian group* $K_n^{(2)}/K_n^{(3)}$ *is*

$$\mathrm{rank}(K_n^{(2)}/K_n^{(3)})=\frac{1}{3}n^2(n^2-4)+\frac{1}{2}n(n-1).$$

Andreadakis computed the rank of $K_n^{(2)}/K_n^{(3)}$ only for the case $n=3$, where he determined the rank to be 18 [A, Theorem 6.2].

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2 Preliminaries

2.1 Notation

The reader is referred to Fulton-Harris [FH] for the representation theory in this paper. Of particular relevance are Sections 1, 15 and 24.

Any finite dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ decomposes as a tensor product of a representation of $\mathrm{SL}(n,\mathbb{C})$ with a one-dimensional representation. Let $\mathfrak{gl}(n,\mathbb{C})$ (respectively, $\mathfrak{sl}(n,\mathbb{C})$) be the Lie algebra of $\mathrm{GL}(n,\mathbb{C})$ (respectively, $\mathrm{SL}(n,\mathbb{C})$). The finite-dimensional representations of $\mathrm{SL}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$ are in natural bijection. Let L_i be the element in the dual of the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}(n,\mathbb{C})$ which sends a matrix to its ith diagonal entry. Any irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ is determined up to isomorphism by a unique highest weight for $\mathrm{SL}(n,\mathbb{C})$ and a power of the determinant. This information corresponds to a sum

$$a_1L_1 + a_2(L_1+L_2) + \cdots + a_n(L_1+L_2+\cdots+L_n)$$

where a_i is a non-negative integer, $1 \le i \le n-1$, and $a_n \in \mathbb{Z}$. We let $\Phi_{a_1,a_2,\ldots,a_n}$ denote the representation corresponding to this sum. For example, the standard representation of $\mathrm{GL}(n,\mathbb{C})$ is given by $\Phi_{1,0,\ldots,0}$, and its dual is given by $\Phi_{0,\ldots,0,1,-1}$.

All of these representations are defined over $\mathbb{Q}$, so they can be considered as irreducible representations of $\mathrm{GL}(n,\mathbb{Q})$ and $\mathfrak{gl}(n,\mathbb{Q})$.

2.2 Deriving an exact sequence

The following exact sequence will be used in the computations underlying Theorem 1.2. While this sequence is implicit in Sullivan's paper [Su], we include its derivation here for completeness.

Lemma 2.1 *For any group G with finite rank $H^1(G,\mathbb{Q})$, the following sequence is exact:*

$$0 \longrightarrow \mathrm{Hom}(G^{(2)}/G^{(3)},\mathbb{Q}) \longrightarrow \textstyle\bigwedge^2 H^1(G,\mathbb{Q}) \xrightarrow{\cup} H^2(G,\mathbb{Q})$$

The left hand map is induced by the surjective map

$$\textstyle\bigwedge^2 G_{ab} \longrightarrow G^{(2)}/G^{(3)}$$

$$g_1 \wedge g_2 \mapsto [g_1,g_2]G^{(3)}$$

and the isomorphism

$$\textstyle\bigwedge^2 H^1(G,\mathbb{Q}) \simeq \mathrm{Hom}(\bigwedge^2 G_{ab},\mathbb{Q}).$$

The right hand map of the sequence comes from the cup product on cohomology.

Given an exact sequence of groups

$$1 \longrightarrow G^{(2)} \longrightarrow G \longrightarrow G_{ab} \longrightarrow 0$$

the Hochschild-Serre spectral sequence yields the five-term exact sequence

$$0 \longrightarrow H^1(G_{ab}, H^0(G^{(2)},\mathbb{Q})) \longrightarrow H^1(G,\mathbb{Q}) \longrightarrow H^0(G_{ab}, H^1(G^{(2)},\mathbb{Q})) \longrightarrow \\ \longrightarrow H^2(G_{ab}, H^0(G^{(2)},\mathbb{Q})) \longrightarrow H^2(G,\mathbb{Q}).$$

(For details, consult Section 7.6 of Brown's book [B], particularly Corollary 6.4.) Now $H^1(G_{ab},\mathbb{Q})$ and $H^1(G,\mathbb{Q})$ are isomorphic and finite rank, so the sequence immediately reduces to

$$0 \longrightarrow H^0(G_{ab}, H^1(G^{(2)},\mathbb{Q})) \longrightarrow H^2(G_{ab},\mathbb{Q}) \longrightarrow H^2(G,\mathbb{Q}).$$

We also have $H^2(G_{ab},\mathbb{Q}) = \bigwedge^2 H^1(G,\mathbb{Q})$, so we are done once we interpret $H^0(G_{ab}, H^1(G^{(2)},\mathbb{Q}))$. By definition, it is the set of invariants of $H^1(G^{(2)},\mathbb{Q})) = \mathrm{Hom}(G^{(2)},\mathbb{Q})$ under the action of G_{ab}. This action is given by lifting G_{ab} to G and acting by conjugation. Thus $H^0(G_{ab}, H^1(G^{(2)},\mathbb{Q}))$ is the set of homomorphisms of $\mathrm{Hom}(G^{(2)},\mathbb{Q})$ which are trivial on $G^{(3)}$, i.e.

$$H^0(G_{ab}, H^1(G^{(2)},\mathbb{Q})) = \mathrm{Hom}(G^{(2)}/G^{(3)},\mathbb{Q}).$$

2.3 A higher Johnson homomorphism

Unless otherwise stated, $n \geq 3$ will be fixed, IA will denote IA_n, and F will denote F_n.

In this subsection, we define a map

$$\tau^{(2)}\colon \mathrm{IA}^{(2)} \longrightarrow \big(\textstyle\bigwedge^2 H_1(F) \otimes H_1(F)/\bigwedge^3 H_1(F)\big) \otimes H_1(F)^*$$

which is analogous to the Johnson homomorphism for IA. A generalization of this map to all $\mathrm{IA}^{(n)}$ and to some related classes of groups is discussed in [CP], [K], and [Sat].

We begin with a map

$$\delta^{(2)}\colon \mathrm{IA}^{(2)} \longrightarrow \mathrm{Hom}(H_1(F), F^{(3)}/F^{(4)})$$

defined as follows:

For $f \in \mathrm{IA}^{(2)}$, map f to $(x \mapsto f(\tilde{x})\tilde{x}^{-1})$ where $\tilde{x}$ is a lift of $x \in F/F^{(2)}$ to $F/F^{(4)}$. From the fact that $\mathrm{IA}^{(2)}$ acts as the identity on both $F/F^{(2)}$ and $F^{(2)}/F^{(4)}$, it is straightfoward to verify that $\delta^{(2)}(f)$ is a well-defined homomorphism from $H_1(F)$ to $F^{(2)}/F^{(4)}$. Furthermore, since $F/F^{(4)}$ factors through $F/F^{(3)}$, and since $\mathrm{IA}^{(2)}$ also acts as the identity on $F/F^{(3)}$, it follows that $f(\tilde{x})\tilde{x}^{-1}$ actually lies in $F^{(3)}/F^{(4)}$. Therefore we have $\delta^{(2)}(f) \in \mathrm{Hom}(H_1(F), F^{(3)}/F^{(4)})$.

The $\mathrm{GL}(n,\mathbb{Z})$-module $\mathrm{Hom}(H_1(F), F^{(3)}/F^{(4)})$ is isomorphic to

$$\left(\bigwedge^2 H_1(F) \otimes H_1(F)/\bigwedge^3 H_1(F)\right) \otimes H_1(F)^*.$$

We define the map $\tau^{(2)}$ to be the composition of $\delta^{(2)}$ with this isomorphism.

Since $\mathrm{IA}^{(3)}$ acts as the identity on $F/F^{(4)}$, it lies in the kernel of $\tau^{(2)}$.

3 The kernel of the Johnson homomorphism on second cohomology

There are three main steps to the proof of Theorem 1.2, and we will first restrict our attention to IA, postponing OA until subsection 3.4. First we decompose $(H_1(F) \bigwedge H_1(F)) \otimes H_1(F)^*$ into simple $\mathrm{GL}(n,\mathbb{Q})$-submodules. The kernel of τ is a $\mathrm{GL}(n,\mathbb{Q})$-submodule, so is a direct sum of a subset of these simple submodules. Next we use the exact sequence of Section 2.2 to identify isomorphism classes of simple modules contained in the kernel of τ^*. Finally, we use the method of Hain [H] and Sakasai [Sak] to find classes which are not in the kernel. Steps 2 and 3 exhaust the set of classes in the decomposition of $(H_1(F) \bigwedge H_1(F)) \otimes H_1(F)^*$, showing that the terms identified in the step 2 are the only ones in the kernel.

Once the computations underlying Theorem 1.2 are complete, Theorem 1.3 and Corollary 1.4 are straightforward. The fourth subsection below contains their proofs.

3.1 Module decompositions

We begin with a decomposition of the relevant $\mathrm{GL}(n,\mathbb{Q})$-modules into direct sums of simple submodules. Let $H = H_1(F)$ and

$$U = \bigwedge^2 H \otimes H^*.$$

Let $H_{\mathbb{Q}}$ and $U_{\mathbb{Q}}$ denote the tensor product with $\mathbb{Q}$ of the spaces H and U, respectively. The action of $\mathrm{GL}(n,\mathbb{Q})$ on $H^*_{\mathbb{Q}}$ will be dual to that on $H_{\mathbb{Q}}$.

Direct computation yields the following two lemmas.

Lemma 3.1 *$U^*_{\mathbb{Q}}$ decomposes into a direct sum of simple $\mathrm{GL}(n,\mathbb{Q})$-submodules as:*

$$U^*_{\mathbb{Q}} \simeq \begin{cases} \Phi_{2,0,-1} \oplus \Phi_{0,1,-1} & n=3 \\ \Phi_{1,0,\ldots,0,1,0,-1} \oplus \Phi_{0,\ldots,0,1,-1} & n \geq 3 \end{cases}$$

Lemma 3.2 *As a $\mathrm{GL}(n,\mathbb{Q})$-module, $\bigwedge^2 U^*_{\mathbb{Q}}$ decomposes as:*

$$\bigwedge^2 U^*_{\mathbb{Q}} \simeq \begin{cases} 2\Phi_{1,0,-1} \oplus 2\Phi_{2,1,-2} & n=3 \\ \Phi_{2,0,0,-1} \oplus 3\Phi_{0,1,0,-1} \oplus \Phi_{0,3,0,-2} \oplus \Phi_{3,0,1,-2} \oplus 2\Phi_{1,1,1,-2} & n=4 \\ 2\Phi_{1,1,0,0,-1} \oplus 3\Phi_{0,0,1,0,-1} \oplus \Phi_{0,1,2,0,-2} \oplus & \\ \qquad \oplus \Phi_{2,1,0,1,-2} \oplus 2\Phi_{1,0,1,1,-2} & n=5 \end{cases}$$

For $n=6$:

$$\bigwedge^2 U^*_{\mathbb{Q}} \simeq \Phi_{0,2,0,0,0,-1} \oplus 2\Phi_{1,0,1,0,0,-1} \oplus 3\Phi_{0,0,0,1,0,-1} \oplus \\ \oplus \Phi_{0,1,0,2,0,-2} \oplus \Phi_{2,0,1,0,1,-2} \oplus 2\Phi_{1,0,0,1,1,-2}$$

For $n \geq 7$:

$$\bigwedge^2 U^*_{\mathbb{Q}} \simeq \Phi_{0,1,0,\ldots,0,1,0,0,0,-1} \oplus 2\Phi_{1,0,\ldots,0,1,0,0,-1} \oplus 3\Phi_{0,\ldots,0,1,0,-1} \oplus \\ \oplus \Phi_{0,1,0,\ldots,0,2,0,-2} \oplus \Phi_{2,0,\ldots,0,1,0,1,-2} \oplus 2\Phi_{1,0,\ldots,0,1,1,-2}$$

3.2 A lower bound on $\ker(\tau^*)$

The key to finding terms in the kernel of τ^* comes from the following diagram:

$$\begin{tikzcd} & & \bigwedge^2 H^1(U,\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "\tau^*"] & H^2(U,\mathbb{Q}) \arrow[d, "\tau^*"] \\ 0 \arrow[r] & \mathrm{Hom}(\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)},\mathbb{Q}) \arrow[r] & \bigwedge^2 H^1(\mathrm{IA},\mathbb{Q}) \arrow[r, "\cup"] & H^2(\mathrm{IA},\mathbb{Q}) \end{tikzcd}$$

The bottom line is the sequence of Lemma 2.1. The right hand horizontal maps are the cup product on cohomology; the top one is an isomorphism since U is abelian. The diagram commutes by naturality of the cup product. The vertical maps are induced by τ on second cohomology. The left vertical map is an isomorphism by Theorem 1.1, so there exists an injective homomorphism from

$\mathrm{Hom}(\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)}, \mathbb{Q})$ to $\bigwedge^2 H^1(U, \mathbb{Q})$. Every map is equivariant with respect to the action of $\mathrm{Aut}(F)$. We can deduce from the diagram that the kernel of

$$\tau^*\colon\ H^2(U, \mathbb{Q}) \longrightarrow H^2(\mathrm{IA}, \mathbb{Q})$$

is precisely

$$\ker(\tau^*) = \mathrm{image}\big(\mathrm{Hom}(\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)}, \mathbb{Q})\big) \subset \textstyle\bigwedge^2 H^1(U, \mathbb{Q})$$

a $\mathrm{GL}(n, \mathbb{Q})$-submodule of $\bigwedge^2 H^1(U, \mathbb{Q})$.

A first approximation of this image is achieved by considering the map $\tau^{(2)}$ defined in Section 2.3. The target space $\mathrm{Hom}(H, F^{(2)}/F^{(3)}) \otimes \mathbb{Q}$ is isomorphic as a $\mathrm{GL}(n, \mathbb{Q})$-module to

$$\big(\textstyle\bigwedge^2 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}/\bigwedge^3 H_{\mathbb{Q}}\big) \otimes H_{\mathbb{Q}}^*$$

which decomposes as

$$\Phi_{0,1,0,\ldots,0} \oplus \Phi_{1,1,0,\ldots,0,1,-1} \oplus \Phi_{2,0,\ldots,0}.$$

Consider the element $[g_{12}, g_{21}]$ in $\mathrm{IA}^{(2)}$. This acts on the generators of F by

$$x_r \mapsto \begin{cases} [[x_1, x_2], x_r]x_r & r \in \{1, 2\} \\ x_r & r \notin \{1, 2\}. \end{cases}$$

Thus $[g_{12}, g_{21}]$ maps by $\tau^{(2)}$ to (the equivalence class of)

$$(e_1 \wedge e_2) \otimes e_1 \otimes e_1^* + (e_1 \wedge e_2) \otimes e_2 \otimes e_2^*$$

in $\big(\bigwedge^2 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}/\bigwedge^3 H_{\mathbb{Q}}\big) \otimes H_{\mathbb{Q}}^*$.

Now consider the $\mathrm{GL}(n, \mathbb{Q})$-equivariant map

$$\big(\textstyle\bigwedge^2 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}/\bigwedge^3 H_{\mathbb{Q}}\big) \otimes H_{\mathbb{Q}}^* \longrightarrow H_{\mathbb{Q}}^{\otimes 3} \otimes H_{\mathbb{Q}}^*$$

defined by

$$(a \wedge b) \otimes c \otimes d^* \mapsto$$
$$a \otimes b \otimes c \otimes d^* - b \otimes a \otimes c \otimes d^* + c \otimes b \otimes a \otimes d^* - b \otimes c \otimes a \otimes d^*$$

and the $\mathrm{GL}(n, \mathbb{Q})$-equivariant map

$$H_{\mathbb{Q}}^{\otimes 3} \otimes H_{\mathbb{Q}}^* \longrightarrow \textstyle\bigwedge^2 H_{\mathbb{Q}}$$

defined by $\quad a \otimes b \otimes c \otimes d^* \mapsto d^*(c) a \wedge b.$

Applying the composition of these maps to

$$(e_1 \wedge e_2) \otimes e_1 \otimes e_1^* + (e_1 \wedge e_2) \otimes e_2 \otimes e_2^*$$

gives $$6(e_1 \wedge e_2)$$

which is a highest weight vector for the module $\Phi_{0,1,0,\dots,0,0}$.

Let E_{ij} be the matrix in $\mathfrak{gl}(n,\mathbb{Q})$ with a 1 in the ijth place and 0's everywhere else. Applying to the vector

$$(e_1 \wedge e_2) \otimes e_1 \otimes e_1^* + (e_1 \wedge e_2) \otimes e_2 \otimes e_2^*$$

the map

$$(a \wedge b) \otimes c \otimes d^* \mapsto \\ a \otimes b \otimes c \otimes d^* - b \otimes a \otimes c \otimes d^* + c \otimes b \otimes a \otimes d^* - b \otimes c \otimes a \otimes d^*$$

followed by E_{1n}, and then by

$$a \otimes b \otimes c \otimes d^* \mapsto (a \wedge b) \otimes c \otimes d^*$$

we obtain $$-4(e_1 \wedge e_2) \otimes e_1 \otimes e_n^*$$

This is a highest weight vector for the irreducible representation corresponding to $\Phi_{1,1,0,\dots,0,1,-1}$.

A simple $\mathrm{GL}(n,\mathbb{Q})$-module is equal to the orbit of its highest weight by the action of $\mathrm{GL}(n,\mathbb{Q})$. Since we have found highest weights for $\Phi_{0,1,0,\dots,0,0}$ and $\Phi_{1,1,0,\dots,0,1,-1}$ in the image of the $\mathrm{GL}(n,\mathbb{Q})$-invariant map $\tau^{(2)}$, both modules are contained in the image of $\tau^{(2)}$. Therefore the dual module

$$\Phi_{0,\dots,0,1,0,-1} \oplus \Phi_{1,0,\dots,0,1,1,-2}$$

is contained in the kernel of τ^*.

3.3 An upper bound on $\ker(\tau^*)$

In this subsection, we make use of the fact that elements in the image of $\tau_*\colon H_2(\mathrm{IA},\mathbb{Q}) \longrightarrow H_2(U,\mathbb{Q})$ correspond to elements in the dual $H^2(\mathrm{IA},\mathbb{Q})$ which lie outside of the kernel of τ^*. This will give us an upper bound on $\ker(\tau^*)$.

If two elements g_1 and g_2 of a group G commute, there is a homomorphism

$$\zeta\colon \mathbb{Z}^2 \longrightarrow G$$

defined by mapping the two standard generators of $\mathbb{Z}^2$ to g_1 and g_2, inducing a map

$$\mathbb{Z} \simeq H_2(\mathbb{Z}^2) \longrightarrow H_2(G).$$

The image of the generator $1 \in H_2(\mathbb{Z}^2)$ in $H_2(G)$ is known as an *abelian cycle*, for which we write

$$\{g_1, g_2\} := \text{image}(\zeta_*(1)) \in H_2(G).$$

In general, this element may be zero. In our case, we identify some abelian cycles in $H_2(\mathrm{IA}, \mathbb{Q})$ and compute their image in $H_2(U, \mathbb{Q})$. These turn out to be nonzero, giving us nonzero cycles in the image of τ_* in $H_2(\mathrm{IA}, \mathbb{Q})$. We list below a set of $\mathrm{GL}(n, \mathbb{Q})$-equivariant homomorphisms which, pre-composed with τ, will detect these cycles.

First define $\mathrm{GL}(n, \mathbb{Q})$-homomorphisms:

- $f_1\colon \bigwedge^2(\bigwedge^2 H_\mathbb{Q} \otimes H_\mathbb{Q}^*) \longrightarrow (\bigwedge^2 H_\mathbb{Q} \otimes H_\mathbb{Q}^*)^{\otimes 2}$
  $(a_1 \wedge b_1 \otimes c_1^*) \wedge (a_2 \wedge b_2 \otimes c_2^*) \mapsto$
  $((a_1 \wedge b_1 \otimes c_1^*) \otimes (a_2 \wedge b_2 \otimes c_2^*)) - ((a_2 \wedge b_2 \otimes c_2^*) \otimes (a_1 \wedge b_1 \otimes c_1^*))$
- $f_2\colon (\bigwedge^2 H_\mathbb{Q} \otimes H_\mathbb{Q}^*)^{\otimes 2} \longrightarrow (H_\mathbb{Q}^{\otimes 2} \otimes H_\mathbb{Q}^*)^{\otimes 2}$
  $(a_1 \wedge b_1 \otimes c_1^*) \otimes (a_2 \wedge b_2 \otimes c_2^*) \mapsto$
  $((a_1 \otimes b_1 - b_1 \otimes a_1) \otimes c_1^*) \otimes ((a_2 \otimes b_2 - b_2 \otimes a_2) \otimes c_2^*))$

and let f be the composition:

$$f = f_2 \circ f_1\colon \textstyle\bigwedge^2 U_\mathbb{Q} \longrightarrow (H_\mathbb{Q}^{\otimes 2} \otimes H_\mathbb{Q}^*)^{\otimes 2}$$

Define a vector $v \in (H_\mathbb{Q}^{\otimes 2} \otimes H_\mathbb{Q}^*)^{\otimes 2}$ by

$$v = (a_1 \otimes b_1 \otimes c_1^*) \otimes (a_2 \otimes b_2 \otimes c_2^*).$$

The following set of $\mathrm{GL}(n, \mathbb{Q})$-homomorphisms are defined on the image of f:

- $g_1(v) = c_1^*(b_2) c_2^*(b_1)(a_1 \wedge a_2) \ \in\ \bigwedge^2 H_\mathbb{Q}$
- $g_2(v) = c_1^*(b_1) c_2^*(b_2)(a_1 \wedge a_2) \ \in\ \bigwedge^2 H_\mathbb{Q}$
- $h_1(v) = c_1^*(b_1)(a_1 \otimes a_2 \otimes b_2 \otimes c_2^*) \ \in\ H_\mathbb{Q}^{\otimes 3} \otimes H_\mathbb{Q}^*$
- $h_2(v) = c_1^*(b_2)(a_1 \otimes a_2 \otimes b_1 \otimes c_2^*) \ \in\ H_\mathbb{Q}^{\otimes 3} \otimes H_\mathbb{Q}^*$
- $k(v) = (a_1 \wedge b_1) \otimes (a_2 \wedge b_2) \otimes (c_1^* \wedge c_2^*) \ \in\ (\bigwedge^2 H_\mathbb{Q})^{\otimes 2} \otimes \bigwedge^2 H_\mathbb{Q}^*$
- $l(v) = (a_1 \wedge a_2 \wedge b_1) \otimes b_2 \otimes c_1^* \otimes c_2^* \ \in\ \bigwedge^3 H_\mathbb{Q} \otimes H_\mathbb{Q} \otimes (H_\mathbb{Q}^*)^{\otimes 2}$
- $m(v) = c_1^*(b_1)(a_1 \otimes (a_2 \wedge b_2) \otimes c_2^*) \ \in\ H_\mathbb{Q} \otimes \bigwedge^2 H_\mathbb{Q} \otimes H_\mathbb{Q}^*$
- $n(v) = (a_1 \wedge a_2 \wedge b_1 \wedge b_2) \otimes (c_1^* \wedge c_2^*) \ \in\ \bigwedge^4 H_\mathbb{Q} \otimes \bigwedge^2 H_\mathbb{Q}^*$

This completes the list of required homomorphisms. We work through proving that there are two terms in the isomorphism class of $\Phi_{0,\dots,0,1,0,-1}$ in some detail. For the remaining terms, only the abelian cycles and the homomorphisms needed to perform the computations are given.

Consider the abelian cycles

$$\begin{aligned}\omega_1 &= \{g_{12}g_{1n}, g_{2n}\}\\ \omega_2 &= \{g_{13}, g_{2n}\}\\ \omega_3 &= \{f_{12n}, g_{3(n-1)}\}\end{aligned}$$

of $H_2(\mathrm{IA},\mathbb{Q})$. For ω_2 and ω_3, we must have $n>3$. By direct computation, we have

$$\tau_*(\omega_1) = (e_1\wedge e_2\otimes e_2^* + e_1\wedge e_n\otimes e_n^*)\wedge(e_2\wedge e_n\otimes e_n^*).$$

Applying $g_1\circ f$, we obtain $2(e_1\wedge e_2)$, a highest weight for the representation $\Phi_{0,1,0,\dots,0,0}$, implying that the dual representation $\Phi_{0,\dots,0,1,0,-1}$ is not in the kernel of τ^*.

Now compute:

$$\tau_*(\omega_2) = (e_1\wedge e_3\otimes e_3^*)\wedge(e_2\wedge e_n\otimes e_n^*)$$

Notice that this vector is in the kernel of $g_1\circ f$. Applying $g_2\circ f$ to $\tau_*(\omega_2)$, we again obtain $2(e_1\wedge e_2)$, giving us a second submodule of $H^2(U,\mathbb{Q})$ in the class $\Phi_{0,\dots,0,1,0,-1}$ not in $\ker(\tau^*)$.

The simple module $\Phi_{1,0,\dots,0,1,0,0,-1}$ appears with multiplicity two in $H^2(U,\mathbb{Q})$, so we can use

$$h_2\circ f\circ E_{3n}\circ\tau_*(\omega_1)$$

and

$$h_1\circ f\circ\tau_*(\omega_3)$$

to show that both of these terms are in the image of τ_*.

We list here four more $\mathrm{GL}(n,\mathbb{Q})$-homomorphisms and abelian cycles which give highest weights of submodules whose dual modules are not in $\ker(\tau^*)$:

- $k\circ f\circ E_{2(n-1)}\circ E_{1n}\circ\tau_*(\omega_1)$ is a highest weight for the submodule of $H_2(U,\mathbb{Q})$ whose dual is $\Phi_{0,1,0,\dots,0,2,0,-2}$ in $H^2(U,\mathbb{Q})$.
- $l\circ f\circ E_{3n}\circ E_{1n}\circ\tau_*(\omega_2)$ is a highest weight for the module whose dual is $\Phi_{2,0,\dots,0,1,0,1,-2}$.
- $m\circ f\circ E_{14}\circ E_{4n}\circ\tau_*(\omega_1)$ is a highest weight for the module whose dual is $\Phi_{1,0,\dots,0,1,1,-2}$.
- $n\circ f\circ E_{3(n-1)}\circ E_{4n}\circ\tau_*(\omega_2)$ is a highest weight for the module whose dual is $\Phi_{0,1,0,\dots,0,1,0,0,0,-1}$.

3.4 Conclusion of the proofs

The computations needed to prove the theorems for IA are complete. Those of the previous subsection show that the submodule

$$2\Phi_{0,\dots,0,1,0,-1} \oplus 2\Phi_{1,0,\dots,0,1,0,0,-1} \oplus \Phi_{0,1,0,\dots,0,2,0,-2} \oplus \\ \oplus \Phi_{2,0,\dots,0,1,0,1,-2} \oplus \Phi_{1,0,\dots,0,1,1,-2} \oplus \Phi_{0,1,0,\dots,0,1,0,0,0,-1}$$

of $H^2(U,\mathbb{Q})$ is not in the kernel of τ^*. Recall that in subsection 3.2, we proved that the submodule

$$\Phi_{0,\dots,0,1,0,-1} \oplus \Phi_{1,0,\dots,0,1,1,-2}$$

is in the kernel. Comparing this with Lemma 3.2, this accounts for all terms of the decomposition of $H^2(U,\mathbb{Q})$. This completes the proof of Theorem 1.2 for IA.

Recall that the map

$$\tau^{(2)} \colon \mathrm{IA}^{(2)} \longrightarrow \mathrm{Hom}(H, F^{(3)}/F^{(4)})$$

factors through $\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)}$. The proof of Therem 1.2 together with the computations of subsection 3.2 imply that as $\mathbb{Q}$-vector spaces:

$$\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)} \otimes \mathbb{Q} \simeq \Phi_{0,1,0,\dots,0} \oplus \Phi_{1,1,0,\dots,0,1,-1} \subset \tau^{(2)}(\mathrm{IA}^{(2)}/\mathrm{IA}^{(3)}) \otimes \mathbb{Q}$$

Therefore the map

$$\tau^{(2)} \otimes id \colon \mathrm{IA}^{(2)}/\mathrm{IA}^{(3)} \otimes \mathbb{Q} \longrightarrow \mathrm{Hom}(H, F^{(3)}/F^{(4)}) \otimes \mathbb{Q}$$

is injective. Tensoring with $\mathbb{Q}$ kills only torsion, so this completes the proof of Theorem 1.3 for IA.

From the definitions, the kernel of $\tau^{(2)}$ is exactly $K_n^{(3)}$, so it follows from Theorem 1.3 that $K_n^{(3)}$ contains $\mathrm{IA}^{(3)}$ as a finite index subgroup. That they are equal when $n = 3$ follows from computations to come in Section 3.5. As $K_n^{(2)}/K_n^{(3)}$ is a free abelian group, its rank is equal to the dimension of the image of $\tau^{(2)} \otimes id$ in $\mathrm{Hom}(H, F/F^{(3)}) \otimes \mathbb{Q}$, which can be computed using the Weyl character formula (see [FH, page 400]). This concludes the proof of Corollary 1.4.

We now turn our attention to OA, beginning with an analogue of Theorem 1.1. Recall that OA is equal to $\mathrm{IA}/\mathrm{Inn}(F)$, the quotient of IA by the inner automorphisms of F. The equivariance of τ with respect to the action of $\mathrm{Aut}(F)$ implies that $\mathrm{Inn}(F)$ maps to a $\mathrm{GL}(n,\mathbb{Z})$-submodule of $\bigwedge^2 H \otimes H^*$. Thus we obtain an $\mathrm{Aut}(F)$-equivariant map $\overline{\tau}$ defined on OA having as its image the quotient of $\bigwedge^2 H \otimes H^*$ by the image of $\mathrm{Inn}(F)$. Note that $\mathrm{Inn}(F)$ is generated by the elements

$$g_{k1} g_{k2} \cdots g_{kn} \qquad 1 \le k \le n$$

and so its image in $\bigwedge^2 H \otimes H^*$ is generated by the elements

$$\sum_{i=1}^{n} e_k \wedge e_i \otimes e_i^* \qquad 1 \leq k \leq n.$$

There is an injective homomorphism

$$H \longrightarrow \bigwedge^2 H \otimes H^*$$

given by mapping the generators e_k to

$$e_k \mapsto \sum_{i=1}^{n} e_k \wedge e_i \otimes e_i^* \qquad 1 \leq k \leq n.$$

So $\overline{\tau}$ is the homomorphism

$$\overline{\tau}\colon \mathrm{OA} \longrightarrow \big(\bigwedge^2 H \otimes H^*\big)/H$$

with kernel $[\mathrm{OA}, \mathrm{OA}]$.

Similarly there is a "higher" Johnson homomorphism for OA. The homomorphism $\tau^{(2)}$ maps $\mathrm{Inn}(F)$ to the submodule of $\big(\bigwedge^2 H \otimes H/\bigwedge^3 H\big) \otimes H^*$ generated by (equivalence classes) of the form

$$\sum_{i=1}^{n} (e_j \wedge e_k \otimes e_i) \otimes e_i^*$$

There is an injective homomorphism

$$\bigwedge^2 H \longrightarrow \big(\bigwedge^2 H \otimes H/\bigwedge^3 H\big) \otimes H^*$$

$$e_j \wedge e_k \mapsto \sum_{i=1}^{n} (e_j \wedge e_k \otimes e_i) \otimes e_i^*$$

and so there is a homomorphism

$$\overline{\tau}^{(2)}\colon \mathrm{OA}^{(2)} \longrightarrow \Big(\big(\bigwedge^2 H \otimes H/\bigwedge^3 H\big) \otimes H^*\Big)/\bigwedge^2 H.$$

Now arguments previously applied to IA complete the proofs of Theorems 1.2 and 1.3.

3.5 Towards computing the kernel of τ^* on $H^3(U, \mathbb{Q})$.

At this stage, it should be possible to find cocyles in $H^3(\mathrm{OA})$ using Theorem 1.2 and the method of Sakasai [Sak]. However we are not yet equipped to take this step for IA because, although we know the isomorphism classes of terms in the kernel of $\tau^*\colon H^2(U, \mathbb{Q}) \to H^2(\mathrm{IA}, \mathbb{Q})$, these classes appear with multiplicity greater than one in the decomposition of $\bigwedge^2 U^*_{\mathbb{Q}}$. To follow Sakasai's method for computing the kernel in $H^3(U,\mathbb{Q})$ of τ^*, we would first need to find explicit highest weights for the simple modules in the kernel. We do this for the case $n=3$.

The set of nine Magnus generators of IA_3 gives a set of 36 generators for the finite rank abelian group $\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)}$. For distinct $i,j,k\in\{1,2,3\}$, the following elements of $\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)}$ are equal to 0:

$$\begin{aligned}
&[g_{ij},g_{ik}]\\
&[g_{ij},g_{ki}]+[g_{ij},g_{kj}]\\
&[f_{ijk},g_{ij}]+[f_{ijk},g_{ik}]\\
&[f_{ijk},g_{ji}]+[f_{ijk},g_{jk}]\\
&[f_{ijk},g_{ki}]+[f_{ijk},g_{kj}]+[g_{ij},g_{ji}]+[g_{ik},g_{ji}]
\end{aligned}$$

where the "+" notation denotes multiplication in $\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)}$. These relations imply that a minimal generating set for $\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)}$ has no more than 18 elements. Since $K_3^{(3)}$ is equal to the kernel of $\tau^{(2)}$, the formula of Corollary 1.4 implies that 18 is the dimension of the image of $\mathrm{IA}_3^{(2)}$ in $\mathrm{Hom}(H,F_3/F_3^{(3)})\otimes\mathbb{Q}$. As $\mathrm{IA}_3^{(3)}$ is contained in $K_3^{(3)}$, the list of relations must be complete. Notice that this implies the last claim of Corollary 1.4, that $K_3^{(3)}=\mathrm{IA}_3^{(3)}$, which was originally shown by Andreadakis [A].

Now we can identify explicit vectors in the image of $\mathrm{Hom}(\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)},\mathbb{Q})$ in $\bigwedge^2 H^1(U,\mathbb{Q})$. For example, the element

$$(e_2^*\wedge e_3^*\otimes e_3)\wedge(e_3^*\wedge e_2^*\otimes e_2)+(e_2^*\wedge e_3^*\otimes e_1)\wedge(e_1^*\wedge e_3^*\otimes e_3)$$

of $\bigwedge^2 H^1(U,\mathbb{Q})$ which sends each of

$$(e_2\wedge e_3\otimes e_3^*)\wedge(e_3\wedge e_2\otimes e_2^*)$$

and $$(e_2\wedge e_3\otimes e_1^*)\wedge(e_1\wedge e_3\otimes e_3^*)$$

to 1 and all other standard basis elements of $\bigwedge^2 H^1(U,\mathbb{Q})$ to 0 is in the image of $\mathrm{Hom}(\mathrm{IA}_3^{(2)}/\mathrm{IA}_3^{(3)},\mathbb{Q})$.

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