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A stably free nonfree module and its relevance for homotopy classification, case Q_{28}

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Abstract The paper constructs an "exotic" algebraic 2–complex over the generalized quaternion group of order 28, with the boundary maps given by explicit matrices over the group ring. This result depends on showing that a certain ideal of the group ring is stably free but not free. As it is not known whether the complex constructed here is geometrically realizable, this example is proposed as a suitable test object in the investigation of an open problem of C.T.C. Wall, now referred to as the D(2)–problem.

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Keywords Algebraic 2–complex, Wall's D(2)–problem, geometric realization of algebraic 2–complexes, homotopy classification of 2–complexes, generalized quaternion groups, partial projective resolution, stably free nonfree module

The main topic of this paper is the construction of an "exotic" algebraic 2–complex over Q_{28}, the generalized quaternion group of order 28. The result provides significantly more detail than mere existence proofs, as the boundary maps are given by explicit matrices over the group ring (Section 4). This example should serve as a suitable test object in the investigation of an open problem of C.T.C. Wall [15, p.57], now [9] referred to as the D(2)–problem.

D(2)–problem Suppose X is a finite three-dimensional connected CW–complex (with universal cover $\widetilde{X}$) such that $H_3(\widetilde{X},\mathbb{Z}) = 0$ and $H^3(X,\mathcal{B}) = 0$ for all local coefficient systems $\mathcal{B}$ on X. Is X homotopy equivalent to a finite 2–complex?

F.E.A. Johnson [7], [9] has shown that for finite base groups this question has an affirmative answer if, and only if, every algebraic 2–complex is geometrically realizable.

The work of K.W. Gruenberg and P.A. Linnell [5, (2.6)] on minimal resolutions of lattices shows that stably equivalent lattices over $\mathbb{Z}Q_{32}$, despite stringent minimality conditions, need not have the same number of generators. These lattices can be viewed as the second homology modules of algebraic 2–complexes. Johnson [8], [9] shows the existence of algebraic 2–complexes over Q_{4n}, $n \geq 6$, with "exotic" second homology modules and emphasizes their relevance for the D(2)–problem. It is not known whether these examples are geometrically realizable. It is difficult to determine whether these cases are actual counterexamples to C.T.C. Wall's D(2)–problem since the maps and the free bases of the relevant modules are not explicitly given.

Our joint work with M. Paul Latiolais [1] indicated to us that an explicit potential counterexample could be constructed over a binary polyhedral group G which allows stably free nonfree $\mathbb{Z}G$–modules. Our example and those listed above depend on Swan's work [13] on the existence of such projectives. We give a specific example for Q_{28} in Section 3.

The particular construction of an algebraic 2–complex presented here grew out of a series of talks and workshops given by us on geometrical realizability in Luttach (1997, 2001), Chelyabinsk (1999), and Portland (2002). We would like to thank Micheal Dyer, Cameron Gordon, Jens Harlander, Cynthia Hog-Angeloni, Paul Latiolais, and Wolfgang Metzler for questions and discussions that clarified the technical difficulties in developing an explicit example. Several of these contacts were facilitated by NSF Grant INT–9603282 and a companion grant by the DAAD for U.S.–Germany Cooperative Research.

1 Preliminaries

Given a group G, an *algebraic 2–complex* over $\mathbb{Z}G$ is a partial projective resolution of $\mathbb{Z}$, $C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$, where the C_i are finitely generated, stably free $\mathbb{Z}G$–modules. We will usually write this as an exact sequence of (left) $\mathbb{Z}G$–modules

$$A\colon \quad 0 \to M \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0\,,$$

where $M = \mathit{ker}\ \partial_2$ is the second homology module and $M \to C_2$ is inclusion. A familiar example of an algebraic 2–complex is the cellular chain complex $C(\widetilde{K})$ of the universal cover $\widetilde{K}$ of a finite two-dimensional CW–complex K with fundamental group G. Such a 2–complex is always homotopy equivalent to some presentation complex for G. An algebraic 2–complex which is chain homotopy equivalent to the cellular chain complex of the universal cover of some presentation complex for G is called *geometrically realizable.*

One may also create new algebraic 2–complexes by beginning with a known algebraic 2–complex A. For the reader's convenience, we summarize a construction due to Swan [12, §6]. Suppose that there exists a $\mathbb{Z}G$–module M' such that $M' \oplus \mathbb{Z}G^k \cong M \oplus \mathbb{Z}G^k$ for some $k > 0$.

Form a new complex

$$0 \to M \oplus \mathbb{Z}G^k \xrightarrow{[i,id]} C_2 \oplus \mathbb{Z}G^k \to C_1 \to C_0 \to \mathbb{Z} \to 0\,. \tag{1.1}$$

Since $\mathbb{Z}G$ is weakly injective [3], [12], there is another direct sum decomposition $C_2 \oplus \mathbb{Z}G^k \cong Q_2 \oplus \mathbb{Z}G^k$ such that

$$0 \to M' \oplus \mathbb{Z}G^k \to Q_2 \oplus \mathbb{Z}G^k \to C_1 \to C_0 \to \mathbb{Z} \to 0 \tag{1.2}$$

is exact and the map restricted to $\mathbb{Z}G^k$ is the identity. Note that Q_2 is stably free. The summand $\mathbb{Z}G^k$ may now be cancelled to give

$$A'\colon \quad 0 \to M' \to Q_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0\,.$$

(To meet the requirement that $M' \to Q_2$ is inclusion, modify Q_2 by renaming the elements in the image of M'.) If the original complex A is the chain complex of the universal cover of a 2–complex K, then (1.1) corresponds to $\overline{K} = K \vee \bigvee_{i=1}^{k} S^2$. Since the summand $\mathbb{Z}G^k$ to be cancelled in (1.2) represents elements of $\pi_2(\overline{K})$, attaching 3–balls to $\overline{K}$ gives a 3–complex K^3 such that $C(\widetilde{K^3})$ is chain equivalent to A'. It is unknown whether A' is geometrically realizable.

It is this question that remains unanswered in C.T.C. Wall's work on the problem of determining whether a CW–complex is homotopy equivalent to a finite one of lower dimension.

Consider the well-known presentation of the generalized quaternion group of order $4n$: $Q_{4n} = \langle x, y \mid x^n y^{-2}, y^2(xy)^{-2}\rangle$. For the purposes of our construction, we will use the following Q^{**}–equivalent presentation:

$$Q_{4n} = \langle x, y \mid x^n y^{-2}, yxyx^{-n+1}\rangle.$$

Let K be the associated presentation complex. The relations of this presentation may be represented by two disjoint simple closed curves on a genus 2 handlebody, and determine a Heegaard splitting of a 3–manifold [6, §3.1] with fundamental group Q_{4n}. After an appropriate thickening, K may be embedded in this manifold as a spine. Consequently $\pi_2(K) \cong \mathbb{Z}Q_{4n}/\langle N\rangle$, where $N \in \mathbb{Z}Q_{4n}$ is the sum of the group elements. Schanuel's Lemma implies that any module M, such that $M \oplus \mathbb{Z}Q_{4n}^k \cong \mathbb{Z}Q_{4n}/\langle N\rangle \oplus \mathbb{Z}Q_{4n}^k$ for some $k > 0$, is a candidate for the second homology group of an algebraic 2–complex with minimal Euler characteristic over $\mathbb{Z}Q_{4n}$.

Suppose P is a $\mathbb{Z}Q_{4n}$–projective with

$$P \oplus \mathbb{Z}Q_{4n} \cong \mathbb{Z}Q_{4n} \oplus \mathbb{Z}Q_{4n}\,;$$

that is, P is *stably free.* This isomorphism may be modified by a basis change to give $(NP, 0) \mapsto (N, 0)$. Thus

$$P/NP \oplus \mathbb{Z}Q_{4n} \cong \mathbb{Z}Q_{4n}/\langle N\rangle \oplus \mathbb{Z}Q_{4n}\,.$$

The construction earlier in this section yields an algebraic 2–complex with second homology module P/NP and having the same Euler characteristic as that of a 3–manifold spine.

If P/NP is not isomorphic to $\mathbb{Z}Q_{4n}/\langle N\rangle$, this algebraic 2–complex is either chain homotopy equivalent to the cellular chain complex of the universal cover of an *exotic* presentation complex for Q_{4n} (i.e., not homotopy equivalent to a 3–manifold spine), or is a counterexample to Wall's D(2)–problem. This motivates the search for a suitable example of a stably free nonfree module.

2 A class of projective modules over $\mathbb{Z}Q_{4n}$

Given integers a and b, let P be the left ideal in $\mathbb{Z}Q_{4n}$ generated by $a+by$ and $x+1$. Since $(x+1)y = yx^{-1}(x+1)$ and $(a+by)x = x(a+by)+by(1-x^{-1})(x+1)$, P is actually a two-sided ideal.

Proposition 2.1 *Let $k = a^2 + b^2$ for n odd, and $k = a^2 - b^2$ for n even. If $(k, 2n) = 1$ then P is projective.*

Proof Note that $k \in P$. Let $d = gcd(a, b)$ and $t = k/d$. Since another generating set for P is $\{t, x+1, a+by\}$, $\mathbb{Z}Q_{4n}/P \cong \mathbb{Z}/t\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ as abelian groups. Since the quotient has exponent relatively prime to $4n$, P is projective by [11, Prop. 7.1, p.570]. □

Towards an explicit example of a stably free module that is not free, from now on we assume n odd and later will specialize to $n = 7$.

Let n be odd. In the spirit of Swan [13, Ch. 10], the ring $\mathbb{Z}Q_{4n}$ may be decomposed by a series of Milnor squares [10, §2], [4, §42]:

$$\text{(I)}\qquad \begin{array}{ccc} \mathbb{Z}Q_{4n} & \longrightarrow & \mathbb{Z}Q_{4n}/\langle x^n+1\rangle \\ \downarrow & & \downarrow \\ \mathbb{Z}D_{2n} & \longrightarrow & \mathbb{F}_2 D_{2n} \end{array}$$

$$
\text{(II)} \qquad \begin{array}{ccc}
\mathbb{Z}Q_{4n}/\langle x^n+1\rangle & \rightarrow & \mathbb{Z}Q_{4n}/\langle \psi_{2n}\rangle \\
\downarrow & & \downarrow \\
\mathbb{Z}Q_{4n}/\langle x+1\rangle & \rightarrow & \mathbb{Z}_n[y]/\langle y^2+1\rangle
\end{array}
$$

where D_{2n} is the dihedral group of order $2n$ and $\psi_{2n} = 1-x+x^2-x^3+\cdots+x^{n-1}$. Note that for an odd prime p, $\mathbb{Z}_p[y]/\langle y^2+1\rangle$ is isomorphic to the Galois field $\mathbb{F}_{p^2}$ of order p^2 when $p \equiv 3 \ mod\ 4$, and isomorphic to $\mathbb{F}_p \times \mathbb{F}_p$ when $p \equiv 1 \ mod\ 4$. Every $\mathbb{Z}Q_{4n}$–projective is isomorphic to some pullback

$$P(\, P_1, P_2, \alpha\,) = \{(a,b) \in P_1 \oplus P_2 \mid \overline{a} = \alpha\overline{b} \in \mathbb{F}_2 D_{2n}\},$$

where P_1 and P_2 are projectives over $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle$ and $\mathbb{Z}D_{2n}$, respectively, and $\alpha \in Aut(\mathbb{F}_2 D_{2n})$, cf. [4, (42.11)]. Similarly, every $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle$–projective is isomorphic to a pullback constructed from projectives over the quotient rings in square (II). Thus projective modules over $\mathbb{Z}Q_{4n}$ are built (up to isomorphism) by a sequence of pullbacks involving related projectives over quotient rings. For certain odd primes p, Swan [13, Ch. 10] has shown that suitable choices in this building process lead to a $\mathbb{Z}Q_{4p}$–projective that is stably free and not free.

Proposition 2.2 *Let n be odd and let $(a^2+b^2, 2n) = 1$. Let P be the left ideal $\langle a+by, x+1\rangle$ in $\mathbb{Z}Q_{4n}$. Then $P/\langle x^n+1\rangle$ is isomorphic to*

$$
\begin{aligned}
\overline{P} &= P(\, \mathbb{Z}Q_{4n}/\langle\psi_{2n}\rangle,\ \mathbb{Z}Q_{4n}/\langle x+1\rangle,\ a+by\,) \\
&= \{(e,f) \in \mathbb{Z}Q_{4n}/\langle\psi_{2n}\rangle \oplus \mathbb{Z}Q_{4n}/\langle x+1\rangle \mid \overline{e} = (a+by)\overline{f} \in \mathbb{Z}_n[y]/\langle y^2+1\rangle\}
\end{aligned}
$$

associated with square (II)*, and P is isomorphic to*

$$P' = P(\, P/\langle x^n+1\rangle,\ \mathbb{Z}D_{2n},\ 1\,) = \{(c,d) \in P/\langle x^n+1\rangle \oplus \mathbb{Z}D_{2n} \mid \overline{c} = \overline{d} \in \mathbb{F}_2 D_{2n}\}$$

associated with square (I).

Proof Consider square (II). Note that $a+by$ is a unit in $\mathbb{Z}_n[y]/\langle y^2+1\rangle$. Define $\eta_1\colon P/\langle x^n+1\rangle \to \mathbb{Z}Q_{4n}/\langle\psi_{2n}\rangle$ by the projection $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle \to \mathbb{Z}Q_{4n}/\langle\psi_{2n}\rangle$ restricted to $P/\langle x^n+1\rangle$. Any element $\rho = r(a+by) + s(x+1)$ of P uniquely determines $r \ mod\ \langle x+1\rangle$, because $(a+by)$ is not a zero divisor in $\mathbb{Z}Q_{4n}/\langle x+1\rangle$, the Gaussian integers. Thus $\eta_2(\rho) = r + \langle x+1\rangle$ gives a well-defined homomorphism $\eta_2\colon P/\langle x^n+1\rangle \to \mathbb{Z}Q_{4n}/\langle x+1\rangle$. Since $\overline{\eta_1(\rho)} = (a+by)\overline{\eta_2(\rho)}$ in $\mathbb{Z}_n[y]/\langle y^2+1\rangle$, this pair of maps uniquely determines a homomorphism $\eta\colon P/\langle x^n+1\rangle \to \overline{P}$ by the pullback property.

A generating set for $\overline{P}$ in $\mathbb{Z}Q_{4n}/\langle\psi_{2n}\rangle \oplus \mathbb{Z}Q_{4n}/\langle x+1\rangle$ is given by

$$\{(a+by,1), (0,\psi_{2n}), (x+1,0), (0,n), (n,0)\}.$$

Since $n \equiv \psi_{2n}$ *mod* $\langle x+1\rangle$, and $(0,\psi_{2n}) = \psi_{2n}(a+by,1)$, this set may be reduced to the two generators $(a+by,1),(x+1,0)$ which are the η–images of $a+by, x+1$. Thus $\eta = (\eta_1,\eta_2)$ is an epimorphism. Since $\mathbb{Z}_n[y]/\langle y^2+1\rangle$ is finite, the rank of $\overline{P}$ as a free abelian group is $2n$. Since $P/\langle x^n+1\rangle$ has the same rank, η is an isomorphism.

Consider square (I). Since one of a or b is even and the other is odd, $P/\langle x^n+1\rangle$ maps onto $\mathbb{F}_2 D_{2n}$. Restricting the top homomorphism of square (I) to $P \to P/\langle x^n+1\rangle$ gives a commutative diagram like square (I) with $\mathbb{Z}Q_{4n}$ replaced by P. This defines a homomorphism $\tau\colon P \to P'$, the pullback. Now P' is generated by

$$\{(a+by,a+by),(x+1,x+1),(0,2)\} \subseteq P/\langle x^n+1\rangle \oplus \mathbb{Z}D_{2n}.$$

The first two generators are obviously in $im\,\tau$, and $(0,2) = \tau(x^n+1)$. As before, the epimorphism τ is an isomorphism, since P and P' have the same rank as abelian groups. □

Corollary 2.3 *If $\overline{P}$ is not free over $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle$, then P is not free over $\mathbb{Z}Q_{4n}$.*

Proof As $\langle x^n+1\rangle P = \langle x^n+1\rangle$, this follows from

$$\overline{P} \cong P/\langle x^n+1\rangle \cong \mathbb{Z}Q_{4n}/\langle x^n+1\rangle \otimes_{\mathbb{Z}Q_{4n}} P.$$ □

Corollary 2.4 *If $\overline{P}$ is not free over $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle$, P/NP is not isomorphic to $\mathbb{Z}Q_{4n}/\langle N\rangle$.*

Proof If $\overline{P}$ is not free over $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle$,

$$\overline{P} \cong P/\langle x^n+1\rangle \cong \mathbb{Z}Q_{4n}/\langle x^n+1\rangle \otimes_{\mathbb{Z}Q_{4n}} P/NP$$

is not isomorphic to $\mathbb{Z}Q_{4n}/\langle x^n+1\rangle \cong \mathbb{Z}Q_{4n}/\langle x^n+1\rangle \otimes_{\mathbb{Z}Q_{4n}} \mathbb{Z}Q_{4n}/\langle N\rangle$. □

3 A stably free module which is not free

In this section we specialize to the case where P is the left ideal $\langle -3+4y, x+1\rangle$ of $\mathbb{Z}Q_{28}$.

Theorem 3.1 *Let P be the left ideal $\langle -3+4y, x+1\rangle$ of $\mathbb{Z}Q_{28}$. Then P is stably free and not free.*

The remainder of this section deals with the proof of this theorem. Generalizing this result involves a number of special cases. General criteria for a projective $P = \langle a+by, x+1\rangle$ over $\mathbb{Z}Q_{4n}$ to be stably free and not free are given in [2].

Theorem 3.2 *Let P be the left ideal $\langle -3+4y, x+1\rangle$ of $\mathbb{Z}Q_{28}$. Then P is not free, and P/NP is not isomorphic to $\mathbb{Z}Q_{28}/\langle N\rangle$.*

Proof Consider square (II) and associated pullbacks of the form

$$P(\,\mathbb{Z}Q_{28}/\langle\phi_{14}\rangle,\, \mathbb{Z}Q_{28}/\langle x+1\rangle,\, \alpha\,) \\ = \{(a,b) \in \mathbb{Z}Q_{28}/\langle\phi_{14}\rangle \oplus \mathbb{Z}Q_{28}/\langle x+1\rangle \bigm| \overline{a} = \alpha\overline{b} \in \mathbb{F}_7[y]/\langle y^2+1\rangle\},$$

where α varies over $(\mathbb{F}_7[y]/\langle y^2+1\rangle)^* \cong \mathbb{F}_{49}^*$. In the proof of [13, Lemma 10.13], Swan shows that there are four isomorphism classes which correspond to the cosets $(\mathbb{F}_7[y]/\langle y^2+1\rangle)^*/\langle 3, y\rangle$, with the free class represented by the trivial coset $[1]$. These cosets form a cyclic group of order four generated by the coset $[1+2y]$. Since $[-3+4y] = [1+2y]^2$ is a nontrivial coset, $\overline{P}$ is not free over $\mathbb{Z}Q_{28}/\langle x^7+1\rangle$. The assertion now follows from Corollaries 2.3 and 2.4. □

The work of Swan [13, pp. 110–111] can be interpreted to give that the set of stably free modules over $\mathbb{Z}Q_{28}$ corresponds to the subgroup of order two in the above construction. While this approach leads to a proof that P is stably free, the following alternative proof has the advantage of giving an explicit basis for $\mathbb{Z}Q_{28} \oplus P$ as a free module. A computational method for constructing similar isomorphisms over $\mathbb{Z}Q_{4n}$ is given in [2].

Theorem 3.3 *Let P be the left ideal $\langle -3+4y, x+1\rangle$ of $\mathbb{Z}Q_{28}$. As a submodule of $\mathbb{Z}Q_{28} \oplus \mathbb{Z}Q_{28}$, $\mathbb{Z}Q_{28} \oplus P$ is free with basis $\{(\Phi_{11}, \Phi_{12}), (\Phi_{21}, \Phi_{22})\}$, where*

$$\begin{aligned}
\Phi_{11} &= x^7[1+(1-x^7)y][1+x^{-5}] - [7-7x^7-\Sigma^-][1+(x^{-3}+x^3)x^5y],\\
\Phi_{12} &= [1+(1-x^7)y][1-(x^{-3}+x^3)y] + [7-7x^7][1+x^5],\\
\Phi_{21} &= x^7[7-7x^7][1+x^{-5}] - [19-20x^7][1-(1-x^7)y][1+(x^{-3}+x^3)x^5y] + 14\Sigma^-,\\
\Phi_{22} &= [7-7x^7-\Sigma^-][1-(x^{-3}+x^3)y] + [19-20x^7][1-(1-x^7)y][1+x^5],
\end{aligned}$$

and $\Sigma^- = 1 - x + x^2 - x^3 + \cdots + x^{12} - x^{13}$. Thus P is stably free.

Proof Consider the map $\Phi\colon \mathbb{Z}Q_{28} \oplus \mathbb{Z}Q_{28} \to \mathbb{Z}Q_{28} \oplus \mathbb{Z}Q_{28}$ given by multiplication of row vectors on the right by the matrix $\left[\begin{smallmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{smallmatrix}\right]$. In the rational group ring, this matrix factors as

$$\begin{bmatrix} 1+(1-x^7)y & 7-7x^7-\Sigma^- \\ 7-7x^7-\Sigma^- & (19-20x^7)[1-(1-x^7)y] \end{bmatrix} \times$$
$$\begin{bmatrix} -x^7(1+x^{-5}) & 1-(x^{-3}+x^3)y \\ 1+(x^{-3}+x^3)x^5y & 1+x^5 \end{bmatrix} \begin{bmatrix} \frac{14}{195}\Sigma^- -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Multiplying this product on the left first by

$$\begin{bmatrix} -1-(x^{-3}+x^3)y & -x^7(1+x^{-5}) \\ -(1+x^5) & 1-(x^{-3}+x^3)x^5y \end{bmatrix} \times$$
$$\begin{bmatrix} (19-20x^7)(1-(1-x^7)y) & \Sigma^- -7+7x^7 \\ \Sigma^- -7+7x^7 & 1+(1-x^7)y \end{bmatrix},$$

and then by

$$(x+1)(x^2-x^7+x^{12}),$$

gives $(x+1)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. This shows that both $\begin{bmatrix} x+1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & x+1 \end{bmatrix}$ are in the image of Φ. Then $im\ \Phi\ mod\ \langle x+1 \rangle$ is generated by

$$\begin{bmatrix} 0 & (1+2y)(1+2y) \end{bmatrix} \equiv \begin{bmatrix} 0 & -3+4y \end{bmatrix},$$
$$\begin{bmatrix} -39(1-2y)(1+2y)+(14)(14) & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Thus $im\ \Phi = \mathbb{Z}Q_{28} \oplus P \subseteq \mathbb{Z}Q_{28} \oplus \mathbb{Z}Q_{28}$. Since $\mathbb{Z}Q_{28} \oplus P$ and $\mathbb{Z}Q_{28} \oplus \mathbb{Z}Q_{28}$ have the same rank as free abelian groups, Φ restricts to an isomorphism. □

4 The construction

Let n be odd and let $(a^2+b^2, 2n) = 1$. Let P be the left ideal $\langle a+by, x+1 \rangle$ in $\mathbb{Z}Q_{4n}$. The following construction gives a partial projective resolution which is an algebraic 2–complex whenever P is stably free.

Let K be the presentation complex associated with

$$Q_{4n} = \langle x, y \mid x^n y^{-2}, yxyx^{-n+1} \rangle .$$

Denote the relators by $R_1 = x^n y^{-2}$ and $R_2 = yxyx^{-n+1}$. Let $C(\widetilde{K})$ be the cellular chain complex of the universal cover of K:

$$0 \to H_2(\widetilde{K}) \to \mathbb{Z}Q_{4n} \oplus \mathbb{Z}Q_{4n} \xrightarrow{\partial_2} \mathbb{Z}Q_{4n} \oplus \mathbb{Z}Q_{4n} \xrightarrow{\partial_1} \mathbb{Z}Q_{4n} \to \mathbb{Z} \to 0 . \tag{4.1}$$

Lemma 4.2 *$H_2(\widetilde{K})$ is generated by*

$$\sigma = (1-y)\widetilde{R}_1 + (1-yx)\widetilde{R}_2\,,$$

and isomorphic to $\pi_2(K) \cong \mathbb{Z}Q_{4n}/\langle N\rangle$ where $1+\langle N\rangle \mapsto \sigma$. This isomorphism induces $P/NP \cong P\sigma$.

The authors originally found the formula for σ by inspecting the Heegaard splitting determined by these relators. Here is an alternative algebraic proof.

Proof By [1, Prop. 1.2], $H_2(\widetilde{K}) \cong \pi_2(K) \cong \mathbb{Z}Q_{4n}/\langle N\rangle$. By inspection, $\sigma \in H_2(\widetilde{K})$. If σ does not generate $H_2(\widetilde{K})$, then there is a nonzero solution to

$$\begin{aligned} A[1+x+x^2+\cdots+x^{n-1}] + B[y-1-x-x^2-\cdots-x^{n-2}] &= 0 \\ A[-y-1] + B[1+yx] &= 0 \end{aligned}$$

with $B = B_1(x)+B_2(x)y$ and A of the form $A = A_1(x)$. Comparing coefficients with respect to y gives

$$\begin{aligned} A_1(x)[1+x+x^2+\cdots+x^{n-1}] + B_1(x)[-1-x-x^2-\cdots-x^{n-2}] + B_2(x)x^n &= 0 \\ B_1(x) - B_2(x)[1+x^{-1}+x^{-2}+\cdots+x^{-n+2}] &= 0 \\ -A_1(x) + B_1(x) + B_2(x)x^{n+1} &= 0 \\ -A_1(x) + B_1(x)x^{-1} + B_2(x) &= 0. \end{aligned}$$

Every solution of this system is a multiple of

$$(1-x^n)\widetilde{R}_1 + (1-x^{n+1}+[1-x^{-1}]y)\widetilde{R}_2 = (1+y)\sigma\,.$$

Since $\langle a+by, x+1\rangle$ always contains an element with augmentation 1, $NP = \langle N\rangle$. Thus the kernel of the composite map

$$P \hookrightarrow \mathbb{Z}Q_{4n} \twoheadrightarrow \mathbb{Z}Q_{4n}/\langle N\rangle \cong \langle\sigma\rangle$$

is $\langle N\rangle$ and the image is $P\sigma$. Therefore $P/NP \cong P\sigma$. □

Note that $P\sigma$ is generated by

$$\begin{aligned} (x+1)\sigma &= (x+1)[(1-y)\widetilde{R}_1 + (1-yx)\widetilde{R}_2] \\ \text{and} \qquad (a+by)\sigma &= (a+by)[(1-y)\widetilde{R}_1 + (1-yx)\widetilde{R}_2]\,. \end{aligned}$$

Consider the subcomplex of (4.1) given by

$$0 \to ker\ \partial_2' \to \mathbb{Z}Q_{4n} \oplus P \xrightarrow{\partial_2'} \mathbb{Z}Q_{4n} \oplus \mathbb{Z}Q_{4n} \xrightarrow{\partial_1} \mathbb{Z}Q_{4n} \to \mathbb{Z} \to 0\,, \tag{4.3}$$

where ∂_2' is the restriction.

Proposition 4.4 *The kernel of ∂_2' equals $P\sigma$ and the sequence* (4.3) *is exact.*

Proposition 4.4 and Lemma 4.2 combine to imply that the second homology module $ker\ \partial_2'$ of (4.3) is isomorphic to P/NP.

Proof (a) To verify that $ker\ \partial_2' = P\sigma$: Since P is a two-sided ideal of $\mathbb{Z}Q_{4n}$, $P\sigma \subseteq \mathbb{Z}Q_{4n} \oplus P \subseteq C_2(\widetilde{K})$. Since $ker\ \partial_2$ is generated by

$$\sigma = (1-y)\widetilde{R}_1 + (1-yx)\widetilde{R}_2\,,$$

an element $C\sigma \in ker\ \partial_2$ lies in $\mathbb{Z}Q_{4n} \oplus P$ if, and only if, there exist $A, B \in \mathbb{Z}Q_{4n}$ with $A(x+1) + B(a+by) = C(1-yx)$. Since n is odd, reducing this problem $mod\ \langle x+1\rangle$ gives the condition:

$$(B_1 + B_2 y)(a+by) \equiv (C_1 + C_2 y)(1+y) \text{ where } B_1, B_2, C_1, C_2 \in \mathbb{Z}\,.$$

Thus

$$2\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} a+b & b-a \\ a-b & a+b \end{bmatrix}.$$

This equation mod 2 implies $B_2 - B_1$ is even. For an integral solution, $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ must be an integral linear combination of $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} a-b & a+b \end{bmatrix}$. Since $(a-b)+(a+b)y \equiv (y+1)(a+by)\ mod\ \langle x+1\rangle$, C is a $\mathbb{Z}Q_{4n}$–linear combination of $a+by$ and $x+1$. Thus $C\sigma \in ker\ \partial_2'$ implies $C \in P$ and $C\sigma \in P\sigma$.

(b) To verify that $im\ \partial_2' = im\ \partial_2 = ker\ \partial_1$: Observe that $C_2(\widetilde{K}) = \langle\sigma\rangle + (\mathbb{Z}Q_{4n} \oplus P)$. Since $\widetilde{R}_1 \in \mathbb{Z}Q_{4n} \oplus P$, it is only necessary to show that $\widetilde{R}_2 \in \langle\sigma\rangle + (\mathbb{Z}Q_{4n} \oplus P)$. The latter submodule contains the multiples of $\widetilde{R}_2$ with the coefficients $a+by$, $x+1$, $1-yx$, thus also with

$$\begin{aligned} y+1 &= y(x+1) + (1-yx), \\ a-b &= (a+by) - b(y+1), \\ 2 &= \psi_{2n}(x)\cdot(x+1) + (1-y)(y+1), \end{aligned}$$

and eventually $1\cdot\widetilde{R}_2$ since $a-b$ is odd. Since $\partial_2\sigma = 0$, the result follows. □

Now specialize to $n=7$ and $P = \langle -3+4y, x+1\rangle \subseteq \mathbb{Z}Q_{28}$. By Theorems 3.3 and 3.2, P is stably free, but P/NP is not isomorphic to $\mathbb{Z}Q_{28}/\langle N\rangle$. In this case our complex is an *exotic* algebraic 2–complex over $\mathbb{Z}Q_{28}$. That is, either this complex is geometrically realizable with fundamental group $\mathbb{Z}Q_{28}$ and is not homotopy equivalent to a 3–manifold spine, or is a counterexample to Wall's D(2)–problem. Viewing $\mathbb{Z}Q_{28} \oplus P$ as a free module, with the basis specified in Theorem 3.3, gives a computable example:

Theorem 4.5 *For $G = Q_{28}$ the following free algebraic 2–complex is exotic:*

$$\mathbb{Z}G^2 \xrightarrow{\partial_2} \mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0\,,$$

where ε is augmentation and the boundary maps are given by multiplying row vectors on the right by matrices. The matrix for ∂_1 is $\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$ and the matrix for ∂_2 is

$$\begin{bmatrix} \Phi_{11} \sum_{k=0}^{6} x^k + \Phi_{12}(y - \sum_{k=0}^{5} x^k) & -\Phi_{11}(1+y) + \Phi_{12}(1+yx) \\ \Phi_{21} \sum_{k=0}^{6} x^k + \Phi_{22}(y - \sum_{k=0}^{5} x^k) & -\Phi_{21}(1+y) + \Phi_{22}(1+yx) \end{bmatrix}$$

where the Φ_{ij} are defined in Theorem 3.3. □

References

[1] **F R Beyl**, **M P Latiolais**, **N Waller**, *Classifications of 2-complexes whose finite fundamental group is that of a 3-manifold*, Proc. Edinburgh Math. Soc. (2) 40 (1997) 69–84

[2] **F R Beyl**, **N Waller**, *Examples of stably free nonfree modules over Q_{4n}*, in preparation

[3] **H Cartan**, **S Eilenberg**, *Homological algebra*, Princeton University Press, Princeton, N. J. (1956)

[4] **C W Curtis**, **I Reiner**, *Methods of representation theory. Vol. II*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York (1987)

[5] **K W Gruenberg**, **P A Linnell**, *Minimal free resolutions of lattices over finite groups*, Illinois J. Math. 32 (1988) 361–374

[6] **C Hog-Angeloni**, **W Metzler**, **A Sieradski** (editors), *Two-dimensional homotopy and combinatorial group theory*, London Math. Soc. Lecture Note Series 197, CUP, Cambridge (1993)

[7] **F E A Johnson**, *Stable modules and Wall's $D(2)$-problem*, Comment. Math. Helv. 78 (2003) 18–44

[8] **F E A Johnson**, *Minimal 2-complexes and the* D(2)*-problem*, Proc. Amer. Math. Soc. 132 (2004) 579–586 (electronic)

[9] **F E A Johnson**, *Stable modules and the $D(2)$-problem*, London Math. Soc. Lecture Note Series 301, CUP, Cambridge (2003)

[10] **J Milnor**, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J. (1971)

[11] **R G Swan**, *Induced representations and projective modules*, Ann. of Math. (2) 71 (1960) 552–578

[12] **R G Swan**, *Periodic resolutions for finite groups*, Ann. of Math. 72 (1960) 267–291

[13] **R G Swan**, *Projective modules over binary polyhedral groups*, J. Reine Angew. Math. 342 (1983) 66–172

[14] **K Varadarajan**, *The finiteness obstruction of C. T. C. Wall*, Canad. Math. Soc. Series of Monographs and Advanced Texts, John Wiley & Sons Inc. New York (1989)

[15] **C T C Wall**, *Finiteness conditions for* CW*-complexes*, Ann. of Math. (2) 81 (1965) 56–69

[16] **C T C Wall**, *Finiteness conditions for* CW *complexes. II*, Proc. Roy. Soc. Ser. A 295 (1966) 129–139

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