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Deformations of reducible representations of 3-manifold groups into $\mathrm{PSL}_2(\mathbb{C})$

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Abstract Let M be a 3-manifold with torus boundary which is a rational homology circle. We study deformations of reducible representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ associated to a simple zero of the twisted Alexander polynomial. We also describe the local structure of the representation and character varieties.

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Dedicated to the memory of Heiner Zieschang

1 Introduction

This article is a continuation of the work started in [14]. Let M be a connected, compact, orientable, 3-manifold such that ∂M is a torus. We assume that the first Betti number $\beta_1(M)$ is one, i.e. M is a rational homology circle. In particular, M is the exterior of a knot in a rational homology sphere.

Given a homomorphism $\alpha\colon \pi_1(M) \to \mathbb{C}^*$, we define an abelian representation $\rho_\alpha\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ as follows:

$$\rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha^{\frac{1}{2}}(\gamma) & 0 \\ 0 & \alpha^{-\frac{1}{2}}(\gamma) \end{pmatrix} \qquad \forall \gamma \in \pi_1(M) \tag{1}$$

where $\alpha^{\frac{1}{2}}\colon \pi_1(M) \to \mathbb{C}^*$ is a map (not necessarily a homomorphism) such that $(\alpha^{\frac{1}{2}}(\gamma))^2 = \alpha(\gamma)$ for all $\gamma \in \pi_1(M)$. The representation ρ_α is reducible, i.e. $\rho_\alpha(\pi_1(M))$ has global fixed points in $P^1(\mathbb{C})$.

Question 1.1 When can ρ_α be deformed into irreducible representations (i.e. representations whose images have no fixed points in $P^1(\mathbb{C})$)?

Different versions of this question have been studied in [9], [11] and [12] for SU(2) and [2], [3], [5], [14] and [23] for $\mathrm{SL}_2(\mathbb{C})$.

The answer is related to a twisted Alexander invariant. We first choose an isomorphism:

$$H_1(M;\mathbb{Z}) \cong \operatorname{tors}(H_1(M;\mathbb{Z})) \oplus \mathbb{Z}, \tag{2}$$

which amounts to choosing a projection onto the torsion subgroup $H_1(M;\mathbb{Z}) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$ and to fix a generator ϕ of $H^1(M;\mathbb{Z}) \cong \operatorname{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z})$. So, α induces a homomorphism $\operatorname{tors}(H_1(M;\mathbb{Z})) \oplus \mathbb{Z} \to \mathbb{C}^*$ which will be also denoted by α.

The composition of the projection $\pi_1(M) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$ with the restriction of α gives a representation $\sigma\colon \pi_1(M) \to U(1) \subset \mathbb{C}^*$:

$$\sigma\colon \pi_1(M) \to \operatorname{tors}(H_1(M;\mathbb{Z})) \xrightarrow{\alpha} \mathbb{C}^*.$$

A homomorphism $\phi_\sigma\colon \pi_1(M) \to \mathbb{C}[t^{\pm 1}]^*$ to the units of the ring of Laurent polynomials $\mathbb{C}[t^{\pm 1}]$ is given by $\phi_\sigma(\gamma) = \sigma(\gamma) t^{\phi(\gamma)}$. This allows a definition of the twisted Alexander polynomial $\Delta^{\phi_\sigma}(t) \in \mathbb{C}[t^{\pm 1}]$, whose construction will be recalled in Section 2.

We say that α is *a zero of the Alexander invariant* if $\Delta^{\phi_\sigma}(\alpha(0,1)) = 0$, where $(0,1) \in \operatorname{tors}(H_1(M;\mathbb{Z})) \oplus \mathbb{Z}$. We show in Section 4 that being a zero and the order of the zero does not depend of the choice of the isomorphism (2).

We prove in Lemma 4.8 that if ρ_α can be deformed into irreducible representations, then α is a zero of the Alexander invariant. For a simple zero this condition is also sufficient and we have stronger conclusions, as the next theorem shows. Let $R(M) = \operatorname{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))$ denote the variety of representations of $\pi_1(M)$ in $\mathrm{PSL}_2(\mathbb{C})$.

Theorem 1.2 *If α is a simple zero of the Alexander invariant, then ρ_α is contained in precisely two irreducible components of $R(M)$, one of dimension 4 containing irreducible representations and another of dimension 3 containing only abelian ones. In addition, ρ_α is a smooth point of both varieties and the intersection at the orbit of ρ_α is transverse.*

When the representation α is trivial, then it is not a zero of the Alexander invariant, because the Alexander invariant is the usual untwisted Alexander polynomial Δ, and $\Delta(1) = \pm|\operatorname{tors} H_1(M,\mathbb{Z})| \neq 0$.

Let $X(M) = R(M)/\!/\mathrm{PSL}_2(\mathbb{C})$ denote the algebraic quotient where $\mathrm{PSL}_2(\mathbb{C})$ acts by conjugation on $R(M)$. The character of a representation $\rho \in R(M)$ is a map $\chi_\rho\colon \pi_1(M) \to \mathbb{C}$ given by $\chi_\rho(\gamma) = \mathrm{tr}^2\,\rho(\gamma)$ for $\gamma \in \pi_1(M)$. There is a one-to-one correspondence between $X(M)$ and the set of characters. Hence we call $X(M)$ the variety of characters of $\pi_1(M)$ (see [13] for details).

Let χ_α be the character of ρ_α.

Theorem 1.3 *If α is a simple zero of the Alexander invariant, then χ_α is contained in precisely two irreducible components of $X(M)$, which are curves and are the quotients of the components of $R(M)$ in Theorem 1.2. In addition χ_α is a smooth point of both curves and the intersection at χ_α is transverse.*

This paper generalizes the main results of [14] where we considered only representations $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ which factor through $H_1(M;\mathbb{Z})/\operatorname{tors}(H_1(M;\mathbb{Z}))$ and for which $\alpha^{\frac{1}{2}}$ can be chosen as a homomorphism. These conditions imply that ρ_α and its deformations can be lifted to representations into $\mathrm{SL}_2(\mathbb{C})$. On the other hand, it was shown in [13, Theorem 1.4] that the representation variety $R(M)$ can have many components which do not lift to the $\mathrm{SL}_2(\mathbb{C})$–representation variety. Hence Theorem 1.2 and Theorem 1.3 generalize the main results of [14]. Moreover, we have removed the condition that ρ_α is not *∂-trivial* from [14]. Here a representation $\rho \in R(M)$ is called ∂-trivial if

$$\rho \circ i_\#\colon \pi_1(\partial M) \to \mathrm{PSL}_2(\mathbb{C})$$

is trivial. An example will be given where the results of this paper apply but those of [14] do not.

In [14] we considered the usual Alexander polynomial, but here we need a twisted version. The strategy of the proof of Theorem 1.2 in this paper is similar to the one of Theorem 1.1 of [14]: we construct a metabelian representation $\rho^+\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ which is not abelian and has the same character as ρ_α, and we show that ρ^+ is a smooth point of $R(M)$. This involves quite elaborate cohomology computations. More precisely, due to an observation by André Weil, the Zariski tangent space $T^{\mathrm{Zar}}_\rho(R(M))$ of $R(M)$ at a representation $\rho \in R(M)$ may be viewed as a subspace of the space of group cocycles $Z^1(\pi_1(M), \mathfrak{sl}_2(\mathbb{C})_\rho)$. Here $\mathfrak{sl}_2(\mathbb{C})_\rho$ denotes the $\pi_1(M)$-module $\mathfrak{sl}_2(\mathbb{C})$ via $\mathrm{Ad}\circ\rho$. The approach given here for these cohomological computations and for the analysis of the tangent space is completely self contained and simplifies in several aspects the computations from [14]. In particular, the new approach permits us to remove the assumption that ρ_α is not ∂-trivial.

The paper is organized as follows. In Section 2 we recall the definition of the twisted Alexander polynomial and describe its main properties. In Section 3 we recall some basic facts from group cohomology and Weil's construction which will be used in the sequel. Section 4 relates the vanishing of twisted Alexander invariants to some elementary cohomology and deformations of abelian representations. The next three sections are devoted to prove that the metabelian representation ρ^+ can be deformed into irreducible representations. The cohomology computations are done in Section 5, with a key lemma proved in Section 6. The smoothness of $R(M)$ at ρ^+ is proved in Section 7. Theorems 1.2 and 1.3 are proved in Sections 8 and 9 respectively. Finally, Section 10 is devoted to describe the local structure of the set of real valued characters.

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2 Twisted Alexander polynomial

Let M be a manifold as in the introduction. We fix a projection $p\colon\ H_1(M;\mathbb{Z}) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$ and a generator

$$\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) = \operatorname{Hom}(\pi_1(M),\mathbb{Z})\,,$$

i.e. we fix an isomorphism as in (2)

$$\begin{array}{rcc} H_1(M;\mathbb{Z}) & \cong & \operatorname{tors}(H_1(M;\mathbb{Z}))\oplus\mathbb{Z} \\ z & \mapsto & (p(z),\phi(z))\,. \end{array}$$

For every representation $\sigma\colon\ \operatorname{tors}(H_1(M;\mathbb{Z})) \to \mathrm{U}(1) \subset \mathbb{C}^*$ the composition

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \xrightarrow{p} \operatorname{tors}(H_1(M;\mathbb{Z})) \xrightarrow{\sigma} \mathrm{U}(1)$$

will be denoted by $\sigma(p)\colon\ \pi_1(M) \to \mathrm{U}(1)$. We consider the induced representation

$$\begin{array}{rrcl} \phi_{\sigma(p)}\colon & \pi_1(M) & \to & \mathbb{C}[t^{\pm 1}]^* \\ & \gamma & \mapsto & \sigma(p)(\gamma)t^{\phi(\gamma)}\,. \end{array}$$

In this way $\mathbb{C}[t^{\pm1}]$ is a $\pi_1(M)$-module (or a $H_1(M;\mathbb{Z})$-module since $\operatorname{Im}\phi_{\sigma(p)}$ is abelian).

In the sequel we shall fix a projection $p\colon\ H_1(M;\mathbb{Z}) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$. We shall, by convenient abuse of notation, continue to write σ for $\sigma(p)$. Let

$$H^{\phi_\sigma}_*(M) \quad \text{and} \quad H_{\phi_\sigma}^*(M)$$

denote the homology and cohomology twisted by ϕ_σ. Using singular chains, $H_*^{\phi_\sigma}(M)$ and $H^*_{\phi_\sigma}(M)$ can be defined respectively as the homology and cohomology of the chain and cochain complexes:

$$\mathbb{C}[t^{\pm 1}] \otimes_{\pi_1(M)} C_*(\widetilde{M};\mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}_{\pi_1(M)}(C_*(\widetilde{M};\mathbb{Z}), \mathbb{C}[t^{\pm 1}]),$$

where $\widetilde{M}$ denotes the universal covering of M. Alternatively, since ϕ_σ is abelian, we could take the maximal abelian covering of M instead of the universal one.

In the sequel we shall write $R := \mathbb{C}[t^{\pm 1}]$. Since R is a principal ideal domain and since $H_1^{\phi_\sigma}(M)$ is finitely generated, we have a canonical decomposition

$$H_1^{\phi_\sigma}(M) = R/r_0R \oplus \cdots \oplus R/r_mR,$$

where $r_i \in R$ and $r_{i+1} \mid r_i$.

Definition 2.1 The R-module $H_1^{\phi_\sigma}(M)$ is called the *Alexander module* and

$$\Delta_k^{\phi_\sigma} = r_k\, r_{k+1} \cdots r_m$$

the *k-th twisted Alexander polynomial*, for $k = 0, \dots, m$. The first one is also called the twisted Alexander polynomial: $\Delta^{\phi_\sigma} := \Delta_0^{\phi_\sigma}$.

It is well defined up to units in $R = \mathbb{C}[t^{\pm 1}]$, i.e. up to multiplication with elements $a\,t^n$ with $a \in \mathbb{C}^*$ and $n \in \mathbb{Z}$. We use the natural extension $\Delta_k^{\phi_\sigma} = 1$ for $k > m$. Note that if $A \in M_{m,n}(R)$ is a presentation matrix for $H_1^{\phi_\sigma}(M)$ then $\Delta_k^{\phi_\sigma} \in R$ is the greatest common divisor of the minors of A of order $(n-k)$. Alexander module can be done in a more general context using only an U.F.D. (see [24, IV.3]).

Changing the isomorphism Note that $H_1^{\phi_\sigma}(M)$ and hence $\Delta_k^{\phi_\sigma}$ are not invariants of the pair (M,σ), they depend on the isomorphism (2); equivalently, they depend on the choice of the projection $p\colon H_1(M,\mathbb{Z}) \to \mathrm{tors}(H_1(M;\mathbb{Z}))$ and the generator $\phi\colon H_1(M,\mathbb{Z}) \to \mathbb{Z}$.

Let $p_1, p_2\colon H_1(M;\mathbb{Z}) \to \mathrm{tors}(H_1(M;\mathbb{Z}))$ be two projections. They differ by a morphism $\psi\colon \mathbb{Z} \to \mathrm{tors}(H_1(M;\mathbb{Z}))$. Namely, for all $z \in H_1(M;\mathbb{Z})$,

$$p_2(z) = p_1(z) + \psi(\phi(z)). \tag{3}$$

Therefore, given $\sigma\colon \mathrm{tors}(H_1(M;Z) \to \mathrm{U}(1)$, the induced representations $\sigma_i := \sigma(p_i)\colon \pi_1(M) \to \mathrm{U}(1)$ satisfy:

$$\sigma_2(\gamma) = \sigma_1(\gamma)\sigma(\psi(\phi(\gamma))) \qquad \forall \gamma \in \pi_1(M)\,. \tag{4}$$

Hence,

$$\phi_{\sigma_2}(\gamma) = \sigma_2(\gamma)t^{\phi(\gamma)} = \sigma_1(\gamma)\sigma(\psi(\phi(\gamma)))t^{\phi(\gamma)} = \sigma_1(\gamma)(a\,t)^{\phi(\gamma)}$$

and $\phi_{\sigma_1}(\gamma) = \sigma_1(\gamma)t^{\phi(\gamma)}$ differ by replacing t by $a\,t$, where $a = \sigma(\psi(1)) \in \mathrm{U}(1)$ and 1 denotes the generator of $\mathbb{Z}$. Therefore

$$\Delta_k^{\phi_{\sigma_2}}(t) = \Delta_k^{\phi_{\sigma_1}}(a\,t)\,. \tag{5}$$

The generator $\phi\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}$ is unique up to sign, and replacing ϕ by $-\phi$ implies replacing t by t^{-1} in the twisted polynomial.

Symmetry Consider the following involution on $\mathbb{C}[t^{\pm 1}]$:

$$\overline{\sum_i a_i\, t^i} = \sum_i \overline{a_i} t^{-i},$$

where $\overline{a_i}$ denotes the complex conjugate of $a_i \in \mathbb{C}$. An ideal $I \subset \mathbb{C}[t^{\pm 1}]$ is called *symmetric* if $I = \bar{I}$ and an element $\eta \in \mathbb{C}[t^{\pm 1}]$ is called *symmetric* if it generates a symmetric ideal. Hence an element $\eta \in \mathbb{C}[t^{\pm 1}]$ is symmetric if and only if there exists a unit $\epsilon \in \mathbb{C}[t^{\pm 1}]^*$ such that $\bar{\eta} = \epsilon\eta$. Notice that some authors use the expression *weakly symmetric* [24].

Proposition 2.2 *Let M be a 3-manifold such that $\beta_1(M) = 1$ and that ∂M is a torus. For each homomorphism $\sigma\colon \operatorname{tors}(H_1(M;\mathbb{Z})) \to \mathrm{U}(1)$ and for each splitting of (2) we have that $\Delta_k^{\phi_\sigma}$ is symmetric i.e., $\Delta_k^{\phi_\sigma}$ and $\overline{\Delta_k^{\phi_\sigma}}$ are equal up to multiplication by a unit of $\mathbb{C}[t^{\pm 1}]$.*

Proof Given a $R = \mathbb{C}[t^{\pm 1}]$-module N, $\overline{N}$ denotes the R-module with the opposite R-action, i.e. $r\bar{n} := \bar{r}\bar{n}$, for $r \in R$ and $\bar{n} \in \overline{N}$. Using the Blanchfield duality pairing we obtain an isomorphism of R-modules $D\colon H_{3-p}^{\phi_\sigma}(M,\partial M) \to \overline{H^p_{\phi_\sigma}(M)}$. Since R is a P.I.D., we obtain

$$D\colon\ \operatorname{tors}(H_{n-p}^{\phi_\sigma}(M,\partial M)) \cong \overline{\operatorname{tors}(H_{p-1}^{\phi_\sigma}(M))} \tag{6}$$

and

$$\operatorname{rk}_R H_{n-p}^{\phi_\sigma}(M,\partial M) = \operatorname{rk}_R H_p^{\phi_\sigma}(M)$$

(see [6], [19], [17, Sec. 2] and [10, Sect. 7]).

The proposition follows from the duality formula (6) (see the proof of Theorem 7.7.1 in [16, p. 97] for the details). ∎

Remark 2.3 In contrast to the untwisted situation, the Alexander module $H_1^{\phi_\sigma}(M)$ can have nonzero rank. Examples are easily obtained as follows: let M_1 be the complement of knot in a homology sphere and let M_2 be a rational homology sphere. Then $\pi_1(M_1\# M_2)\cong\pi_1(M_1)*\pi_1(M_2)$ and $H_1(M_1\# M_2;\mathbb{Z})\cong H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})$ comes with a canonical splitting.

Since $H_1(M_1;\mathbb{Z})$ is torsion free, we can choose $\phi\colon\ \pi_1(M_1\# M_2)\to\mathbb{Z}$ to be the composition

$$\phi\colon\ \pi_1(M_1\# M_2)\to H_1(M_1\# M_2;\mathbb{Z})\to H_1(M_1;\mathbb{Z})\cong\mathbb{Z}\,.$$

For each nontrivial representation $\sigma\colon\ H_1(M_2;\mathbb{Z})\to \mathrm{U}(1)$ we obtain $\phi_\sigma(h_1+h_2)=\sigma(h_2)t^{\phi(h_1)}\in\mathbb{C}[t^{\pm1}]^*$ for $h_i\in H_1(M_i;\mathbb{Z})$ and hence $\sigma=\phi_\sigma\,|_{H_1(M_2;\mathbb{Z})}$.

Since σ is nontrivial it follows that $H_0^{\phi_\sigma}(M_2)=H_0^{\phi_\sigma}(M_1\# M_2)=0$. Moreover, we obtain that $H_0^{\phi_\sigma}(S^2)\cong\mathbb{C}[t^{\pm1}]$ and hence the Mayer-Vietoris sequence gives a short exact sequence:

$$0\to H_1^{\phi_\sigma}(M_1)\oplus H_1^{\phi_\sigma}(M_2)\to H_1^{\phi_\sigma}(M_1\# M_2)\to\operatorname{Ker} j\to 0$$

where $j\colon\ H_0^{\phi_\sigma}(S^2)\to H_0^{\phi_\sigma}(M_1)$ is surjective. Since $H_0^{\phi_\sigma}(M_1)$ is torsion it follows that $\operatorname{Ker} j$ is a free $\mathbb{C}[t^{\pm1}]$-module of rank one and hence, $H_1^{\phi_\sigma}(M_1\# M_2)$ has nonzero rank.

3 Group cohomology: Fox calculus and products

Fox calculus will be used to compute the twisted Alexander polynomial. Since this is a tool in group cohomology, we first need the following lemma, that will also be used later. Details and conventions in group cohomology can be found in [7] and [25].

Lemma 3.1 *Let A be a $\pi_1(M)$-module and let X be any CW-complex with $\pi_1(X)\cong\pi_1(M)$. Then there are natural morphisms $H_i(X;A)\to H_i(\pi_1(M);A)$ which are isomorphisms for $i=0,1$ and a surjection for $i=2$. In cohomology there are natural morphisms $H^i(\pi_1(M);A)\to H^i(X;A)$ which are isomorphisms for $i=0,1$ and an injection for $i=2$.*

Proof It is possible to construct an Eilenberg-MacLane space K of type $(\pi_1(M),1)$ from X by attaching k-cells, $k\geq 3$. In this way we obtain a CW-pair (K,X) and it follows that $H_j(K,X;A)=0$ and $H^j(K,X;A)=0$ for $j=1,2$ (this a direct application of Theorems (4.4) and (4.4*) of [27, VI.4]). Hence the exact sequences of the pair (K,X) give the result. $\square$

Fox calculus Let $\sigma\colon \operatorname{tors}(H_1(M;\mathbb{Z}) \to \mathrm{U}(1)$ be a representation and fix a splitting of $H_1(M;\mathbb{Z})$ as in (2). We can use Fox calculus to compute $\Delta_k^{\phi_\sigma}$: choose a cell decomposition of M with only one zero cell x_0. Since every presentation of $\pi_1(M)$ obtained from a cell decomposition of M has deficiency one, we have:

$$\pi_1(M) = \langle S_1, \dots, S_n \mid R_1, \dots, R_{n-1}\rangle\,.$$

Denote by $\pi\colon F_n \to \pi_1(M)$ the canonical projection where $F_n = F(S_1, \dots, S_n)$ is the free group generated by n elements and by $\partial/\partial S_i\colon \mathbb{Z}F_n \to \mathbb{Z}F_n$ the partial derivations of the group ring of the free group.

The *Jacobian* $J^{\phi_\sigma} := (J^{\phi_\sigma}_{ji}) \in M_{n-1,n}(\mathbb{C}[t^{\pm1}])$ is defined by $J^{\phi_\sigma}_{ji} := \phi_\sigma \circ \pi(\partial R_j/\partial S_i) \in \mathbb{C}[t^{\pm1}]$. Analogous to [8, Chapter 9], one can show that J^{ϕ_σ} is a presentation matrix for $H_1^{\phi_\sigma}(M, x_0)$. The exact sequence for the pair (M, x_0) yields $H_1^{\phi_\sigma}(M, x_0) \cong H_1^{\phi_\sigma}(M) \oplus \mathbb{C}[t^{\pm1}]$ (see [24, pp. 61-62]). Hence, $\Delta_k^{\phi_\sigma}(M) = \Delta_{k+1}^{\phi_\sigma}(M, x_0)$ and $\Delta_k^{\phi_\sigma}(M)$ is the greatest common divisor of the $(n-k-1)$-minors of the Jacobian $J^{\phi_\sigma} \in M_{n-1,n}(\mathbb{C}[t^{\pm1}])$.

Example 3.2 Let M be the punctured torus bundle over S^1 whose action of the monodromy on $H_1(\dot T^2, \mathbb{Z})$ is given by the matrix $\left(\begin{smallmatrix}1&2\\2&5\end{smallmatrix}\right)$. The fundamental group $\pi_1(\dot T^2)$ is a free group of rank 2 generated by α and β. A presentation of $\pi_1(M) \cong \mathbb{Z} \ltimes \pi_1(\dot T^2)$ is given by

$$\pi_1(M) = \langle \alpha, \beta, \mu \mid \mu\alpha\mu^{-1} = \alpha\beta^2, \mu\beta\mu^{-1} = \beta(\alpha\beta^2)^2\rangle\,.$$

Moreover, $H_1(M;Z) \cong (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}$ comes with a canonical splitting $s(1) = \mu$ i.e., $p(\mu) = 0$, $p(\alpha) = \alpha$ and $p(\beta) = \beta$. A generator $\phi \in H^1(M;\mathbb{Z}) \cong \operatorname{Hom}(\pi_1(M);\mathbb{Z})$ is given by $\phi(\mu) = 1$ and $\phi(\alpha) = \phi(\beta) = 0$.

There are exactly four representations $\sigma_i\colon (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \to \mathrm{U}(1)$ which give rise to the following homomorphisms $\phi_{\sigma_i}\colon \pi_1(M) \to \mathbb{C}[t^{\pm1}]^*$:

$$\phi_{\sigma_1}\colon \begin{cases} \mu & \mapsto t \\ \alpha & \mapsto 1 \\ \beta & \mapsto 1 \end{cases}, \qquad \phi_{\sigma_2}\colon \begin{cases} \mu & \mapsto t \\ \alpha & \mapsto 1 \\ \beta & \mapsto -1 \end{cases},$$

$$\phi_{\sigma_3}\colon \begin{cases} \mu & \mapsto t \\ \alpha & \mapsto -1 \\ \beta & \mapsto 1 \end{cases}, \qquad \phi_{\sigma_4}\colon \begin{cases} \mu & \mapsto t \\ \alpha & \mapsto -1 \\ \beta & \mapsto -1 \end{cases}.$$

Now, a direct calculation gives:

$$J^{\phi_{\sigma_1}} = \begin{pmatrix} 0 & t-1 & -2 \\ 0 & -2 & t-5 \end{pmatrix}, \qquad J^{\phi_{\sigma_2}} = \begin{pmatrix} 0 & t-1 & 0 \\ 2 & 2 & t-1 \end{pmatrix}$$

$$J^{\phi_{\sigma_3}} = \begin{pmatrix} 2 & t-1 & 2 \\ 0 & 0 & t-1 \end{pmatrix}, \qquad J^{\phi_{\sigma_4}} = \begin{pmatrix} 2 & t-1 & 0 \\ 2 & 0 & t-1 \end{pmatrix}.$$

Hence, $\Delta^{\phi_{\sigma_1}} = t^2 - 6t + 1$ and $\Delta^{\phi_{\sigma_i}} = t - 1$ for $i = 2, 3, 4$.

Products in cohomology Let Γ be a group and let A be a Γ-module. We denote by $(C^*(\Gamma; A), d)$ the normalized cochain complex. The coboundaries (respectively cocycles, cohomology) of Γ with coefficients in A are denoted by $B^*(\Gamma; A)$ (respectively $Z^*(\Gamma; A)$, $H^*(\Gamma; A)$).

Let A_1, A_2 and A_3 be Γ-modules. The cup product of two cochains $u \in C^p(\Gamma; A_1)$ and $v \in C^q(\Gamma; A_2)$ is the cochain $u \cup v \in C^{p+q}(\Gamma; A_1 \otimes A_2)$ defined by

$$u \cup v(\gamma_1, \dots, \gamma_{p+q}) := u(\gamma_1, \dots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p \circ v(\gamma_{p+1}, \dots, \gamma_{p+q}) . \tag{7}$$

Here $A_1 \otimes A_2$ is a Γ-module via the diagonal action.

It is possible to combine the cup product with any bilinear map (compatible with the Γ action) $b\colon A_1 \otimes A_2 \to A_3$. So we obtain a cup product

$$\overset{b}{\cup}\colon C^p(\Gamma; A_1) \otimes C^q(\Gamma; A_2) \to C^{p+q}(\Gamma; A_3) .$$

For details see [7, V.3]. If $A = \mathfrak{g}$ is a Lie algebra, then we obtain the *cup-bracket* of two cochains, which will be denoted by $[u \cup v]$. Note that the cup-bracket is not associative on the cochain level. If A is an algebra then the cup product will be simply denoted by $\dot{\cup}$. This cup product is associative on the cochain level if the multiplication in A is associative.

Let $b\colon A_1 \otimes A_2 \to A_3$ be bilinear and let $z_i \in Z^1(\Gamma; A_i)$, $i = 1, 2$, be cocycles. We define $f\colon \Gamma \to A_3$ by $f(\gamma) := b(z_1(\gamma) \otimes z_2(\gamma))$. A direct calculation gives: $d\,f(\gamma_1, \gamma_2) + b(z_1(\gamma_1) \otimes \gamma_1 \circ z_2(\gamma_2)) + b(\gamma_1 \circ z_1(\gamma_2) \otimes z_2(\gamma_1)) = 0$. This shows that:

$$d\,f + z_1 \overset{b}{\cup} z_2 + z_2 \overset{b\circ\tau}{\cup} z_1 = 0 \tag{8}$$

where $\tau\colon A_1 \otimes A_2 \to A_2 \otimes A_1$ is the twist operator.

Group cohomology and representation varieties Let Γ be a group and let $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be a representation. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ turns into a Γ-module via $\mathrm{Ad} \circ \rho$. We shall denote this Γ-module by $\mathfrak{sl}_2(\mathbb{C})_\rho$. A cocycle $d \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is a map $d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})_\rho$ satisfying

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)}\, d(\gamma_2), \ \forall \gamma_1, \gamma_2 \in \pi_1(M) .$$

It was observed by André Weil [26] that there is a natural inclusion of the Zariski tangent space $T^{\mathrm{Zar}}_\rho(R(\Gamma)) \hookrightarrow Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho)$. Informally speaking, given a smooth curve ρ_ϵ of representations through $\rho_0 = \rho$ one gets a 1-cocycle $d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})_\rho$ by defining

$$d(\gamma) := \left.\frac{d\,\rho_\epsilon(\gamma)}{d\,\epsilon}\right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma\,.$$

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries $B^1(\Gamma;\mathfrak{sl}_2(\mathbb{C})_\rho)$ (see for instance [15, Section 4.5]). Here, $b\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})_\rho$ is a coboundary if there exists $x \in \mathfrak{sl}_2(\mathbb{C})$ such that $b(\gamma) = \mathrm{Ad}_{\rho(\gamma)}\, x - x$. A detailed account can be found in [18, Thm. 2.6] or [21, Ch. VI].

Let $\dim_\rho R(\Gamma)$ be the *local dimension* of $R(\Gamma)$ at ρ (i.e. the maximal dimension of the irreducible components of $R(\Gamma)$ containing ρ, cf. [22, Ch. II, §1.4]). In the sequel we shall use the following lemma:

Lemma 3.3 *Let $\rho \in R(\Gamma)$ be given. If $\dim_\rho R(\Gamma) = \dim Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{C})_\rho)$ then ρ is a smooth point of the representation variety $R(\Gamma)$ and ρ is contained in a unique component of $R(\Gamma)$ of dimension $\dim Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{C})_\rho)$.*

Proof For every $\rho \in R(\Gamma)$ we have

$$\dim_\rho R(\Gamma) \le \dim T_\rho^{\mathrm{Zar}}(R(\Gamma)) \le \dim Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{C})_\rho)\,.$$

The lemma follows from the fact that the equality $\dim_\rho R(\Gamma) = \dim T_\rho^{\mathrm{Zar}}(R(\Gamma))$ is the condition in algebraic geometry that guarantees that ρ belongs to a single irreducible component of $R(\Gamma)$ and it is a smooth point (for more details see [22, Ch. II]). □

4 Abelian representations and the twisted Alexander invariant

Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a homomorphism. In what follows, the induced homomorphism $H_1(M;Z) \to \mathbb{C}^*$ will be also denoted by α and we denote by σ the restriction of α to the torsion subgroup, i.e.

$$\sigma := \alpha\,|_{\mathrm{tors}(H_1(M;\mathbb{Z}))}\colon\ \mathrm{tors}(H_1(M;\mathbb{Z})) \to \mathrm{U}(1)\,.$$

Let us fix an isomorphism $H_1(M;\mathbb{Z}) \cong \mathrm{tors}(H_1(M;\mathbb{Z})) \oplus \mathbb{Z}$ as in (2), i.e. we fix an projection $p\colon H_1(M;\mathbb{Z}) \to \mathrm{tors}(H_1(M;\mathbb{Z}))$ and a generator $\phi \in H^1(M;\mathbb{Z})$. The induced section $s_p\colon \mathbb{Z} \to H_1(M;\mathbb{Z})$ satisfies

$$s_p \circ \phi = \mathrm{Id} - p.$$

Definition 4.1 We say that α is a *zero of the k-th Alexander invariant of order r* if $a := \alpha(s_p(1)) \in \mathbb{C}^*$ is a zero of $\Delta_k^{\phi_\sigma}(t)$ of order r.

Lemma 4.2 *This definition does not depend on the isomorphism (2).*

Proof The independence of the generator $\phi \in H^1(M;\mathbb{Z})$ is clear: if we replace ϕ by $-\phi$ we have to replace $s_p(1)$ by $s_p(-1) = -s_p(1)$; hence, $a = \alpha(s_p(1))$ has to be replaced by a^{-1}. Moreover, we have to replace $\Delta_k^{\phi_\sigma}(t)$ by $\Delta_k^{\phi_\sigma}(t^{-1})$ and the claim follows.

Suppose now that we have two projections $p_1, p_2\colon H_1(M;\mathbb{Z}) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$. Then there is morphism $\psi\colon \mathbb{Z} \to \operatorname{tors}(H_1(M;\mathbb{Z}))$ as in (3) such that

$$\psi \circ \phi = p_2 - p_1 = s_1 \circ \phi - s_2 \circ \phi,$$

because $s_i \circ \phi = \operatorname{Id} - p_i$. In particular

$$\psi = s_1 - s_2.$$

Let $\sigma_i\colon \pi_1(M) \to \mathrm{U}(1)$, $i = 1, 2$, be given by $\sigma_i := \sigma \circ p_i$ and denote $a_i := \alpha(s_{p_i}(1))$, so that $\alpha(\psi(1)) = a_1 a_2^{-1}$. By (5) we get:

$$\Delta_k^{\phi_{\sigma_2}}(t) = \Delta_k^{\phi_{\sigma_1}}(\alpha(\psi(1))\, t) = \Delta_k^{\phi_{\sigma_1}}(a_1 a_2^{-1}\, t),$$

Putting $t = a_2\, s$, we get $\Delta_k^{\phi_{\sigma_2}}(a_2\, s) = \Delta_k^{\phi_{\sigma_1}}(a_1\, s)$ Hence the order of vanishing of $\Delta_k^{\phi_{\sigma_i}}$ at a_i is independent of i. □

Definition 4.3 Following 4.1, we define $\beta_\alpha\colon \mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ to be the evaluation map at $\alpha(s_p(1)) \in \mathbb{C}$, i.e. $\beta_\alpha(\eta(t)) = \eta(s_p(1)) \in \mathbb{C}\ \forall \eta(t) \in \mathbb{C}[t^{\pm 1}]$.

The previous lemma says that the evaluation and the order of vanishing of $\Delta_k^{\phi_\sigma}$ at β_α is independent of the splitting of the first homology group. In addition, we have

$$\alpha = \beta_\alpha \circ \phi_\sigma\,, \tag{9}$$

i.e. $\alpha(\gamma) = \beta_\alpha(\sigma(\gamma) t^{\phi(\gamma)}) = \sigma(\gamma)\alpha(s_p(1))^{\phi(\gamma)}\ \forall \gamma \in \pi_1(M)$.

Example 4.4 Let M be the torus bundle given in Example 3.2. For every $\lambda \in \mathbb{C}^*$ there are representations $\alpha_i\colon \pi_1(M) \to \mathbb{C}^*$, $i = 1, \ldots, 4$, given by $\alpha_i(\mu) = \lambda$ and $\alpha_i|_{\mathbb{Z}/2\oplus\mathbb{Z}/2} = \sigma_i$. Now, α_i is a root of the Alexander invariant if and only if $\Delta^{\phi_{\sigma_i}}(\lambda) = 0$. More precisely, α_1 is a root of the Alexander invariant if and only if $\lambda = 3 \pm \sqrt{8}$ and, for $i = 2, 3, 4$, α_i is a root of the Alexander invariant if and only if $\lambda = 1$. Note that in each case the root is a simple root.

Definition 4.5 We define $\mathbb{C}_\alpha$ to be the $\pi_1(M)$-module $\mathbb{C}$ with the action induced by α, i.e. $\gamma \cdot x = \alpha(\gamma)x\ \forall x \in \mathbb{C}$ and $\forall \gamma \in \pi_1(M)$.

We explain the motivation of this definition. Let $\rho_\alpha\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be the representation in (1). The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ turns into a $\pi_1(M)$-module via $\mathrm{Ad} \circ \rho_\alpha$ which will simply be denoted by $\mathfrak{sl}_2(\mathbb{C})_\alpha$. The $\pi_1(M)$-module $\mathfrak{sl}_2(\mathbb{C})_\alpha$ decomposes as $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathbb{C}_+ \oplus \mathbb{C}_0 \oplus \mathbb{C}_-$, where

$$\mathbb{C}_+ = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbb{C}_0 = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbb{C}_- = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

Here $\mathbb{C}_0$ is a trivial $\pi_1(M)$-module and $\mathbb{C}_\pm := \mathbb{C}_{\alpha^\pm}$, where $\alpha^+ = \alpha$ and α^- is the morphism that maps every element $\gamma \in \pi_1(M)$ to $\alpha(\gamma^{-1})$.

Computing $\mathbf{H^1(\Gamma;\mathbb{C}_\alpha)}$ Let $\Gamma = \langle S_1, \dots, S_n | R_1, \dots, R_m \rangle$ be a finitely presented group and let $\alpha\colon \Gamma \to \mathbb{C}^*$ be a representation. In order to compute $H^1(\Gamma;\mathbb{C}_\alpha)$ we can use the canonical 2-complex with one 0-cell associated to the presentation of Γ (see Lemma 3.1). More precisely, we can identify $Z^1(\Gamma;\mathbb{C}_\alpha)$ with the kernel of the linear map $\mathbb{C}^n \to \mathbb{C}^m$ given by $\mathbf{a} \mapsto \mathbf{A}\,\mathbf{a}$ where $\mathbf{A} = (a_{ji}) \in M_{m,n}(\mathbb{C})$ is given by $a_{ji} = \alpha(\partial R_j / \partial S_i)$.

For the remainder of this section we shall fix a projection $p\colon H_1(M;\mathbb{Z}) \to \mathrm{tors}(H_1(M;\mathbb{Z}))$ and a generator $\phi \in H^1(M;\mathbb{Z})$. Moreover, let σ denote the restriction of α to the torsion subgroup $\mathrm{tors}(H_1(M;\mathbb{Z}))$. Using (9) we obtain $a_{ji} = \beta_\alpha(J^{\phi_\sigma}_{ji})$ where $J^{\phi_\sigma} = (J^{\phi_\sigma}_{ji}) \in M_{n-1,n}(\mathbb{C}[t^{\pm 1}])$ is the Jacobian, i.e. $J^{\phi_\sigma}_{ji} := \phi_\sigma \circ \pi(\partial R_j / \partial S_i)$. We have $\dim B^1(\Gamma;\mathbb{C}_\alpha) = 1$ if α is nontrivial, and $\dim B^1(\Gamma;\mathbb{C}_\alpha) = 0$ if α is trivial. Hence for any nontrivial $\alpha\colon \Gamma \to \mathbb{C}^*$, we have

$$\dim H^1(\Gamma;\mathbb{C}_\alpha) = n - \mathrm{rk}\,\mathbf{A} - 1. \tag{11}$$

Lemma 4.6 *Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a nontrivial homomorphism. Then $\dim H^1(\pi_1(M);\mathbb{C}_\alpha) = k$ if and only if α is a zero of the k-th Alexander invariant and not a zero of the $(k+1)$-th Alexander invariant.*

Proof By (11) we have $k = \dim H^1(\pi_1(M);\mathbb{C}_\alpha) = n - \mathrm{rk}\,\mathbf{A} - 1$, where $\mathbf{A} = (\alpha(\partial R_j/\partial S_i))$. Now $\alpha = \beta_\alpha \circ \phi_\sigma$ and hence $\beta_\alpha(\Delta^{\phi_\sigma}_l(M)) = 0$ if $l < k$ and $\beta_\alpha(\Delta^{\phi_\sigma}_l(M)) \neq 0$ if $l \geq k$. □

Recall that $\alpha^\pm\colon \pi_1(M) \to \mathbb{C}^*$ denotes the homomorphisms given by

$$\alpha^\pm(\gamma) := (\alpha(\gamma))^{\pm 1} \qquad \forall \gamma \in \pi_1(M). \tag{12}$$

It follows from the symmetry of the Alexander invariants that α^+ and α^- are zeros of the same order of the k-th Alexander invariant. More precisely, we have:

Proposition 4.7 *Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a nontrivial homomorphism. Then α^+ and α^- are zeros of the same order of the k-th Alexander invariant. In particular, we have*

$$\dim H^1(\pi_1(M); \mathbb{C}_{\alpha^+}) = \dim H^1(\pi_1(M); \mathbb{C}_{\alpha^-}).$$

Proof Since $\alpha^-(s_p(1))$ is the inverse of $\alpha(s_p(1))$, it suffices to check that $\Delta_k^{\phi_{\sigma^{-1}}}(t^{-1}) = \epsilon\, \Delta_k^{\phi_\sigma}(t)$ where ϵ is a unit in $\mathbb{C}[t^{\pm 1}]$. To verify this, notice that the image of σ is contained in $\mathrm{U}(1)$, so $\sigma^{-1}(\gamma) = \overline{\sigma(\gamma)}$, $\forall \gamma \in \pi_1(M)$. Hence, if $\Delta_k^{\phi_\sigma}(t) = \sum_i a_i t^i$, then $\Delta_k^{\phi_{\sigma^{-1}}}(t) = \sum_i \overline{a_i} t^i$. By Proposition 2.2, $\sum_i a_i t^i$ differs from $\sum_i \overline{a_i} t^{-i}$ by a unit, hence the proposition follows. $\square$

The space of abelian representations Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a representation and let $\varphi\colon \mathbb{Z} \to \mathbb{C}^*$ be a homomorphism. Using multiplication, we obtain a homomorphism $\alpha\varphi\colon \pi_1(M) \to \mathbb{C}^*$ given by $\alpha\varphi(\gamma) = \alpha(\gamma)\varphi(\phi(\gamma))$ for $\gamma \in \pi_1(M)$, where $\phi \in H^1(M;\mathbb{Z})$ is a generator. There is a one dimensional irreducible algebraic set $V_\alpha \subset R(M)$ given by

$$V_\alpha := \{\rho_{\alpha\varphi} \mid \varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*)\} \subset R(M).$$

Moreover, the $\mathrm{PSL}_2(\mathbb{C})$ orbit of $\rho_{\alpha\varphi}$ is two dimensional if $\alpha\varphi$ is nontrivial and hence V_α is contained in an at least three dimensional component. We denote by $S_\alpha(M) \subset R(M)$ the closure of the $\mathrm{PSL}_2(\mathbb{C})$–orbit of V_α. Notice that $\rho_\alpha \in S_\alpha(M)$ and $\dim_{\mathbb{C}} S_\alpha(M) \geq 3$.

Lemma 4.8 *Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a representation. If α is not a zero of the Alexander invariant then there exists a neighborhood of ρ_α in $R(M)$ consisting entirely of points of the component $S_\alpha(M)$. Moreover, $\rho_\alpha \in S_\alpha(M)$ is a smooth point and $S_\alpha(M)$ is the unique component through ρ_α and $\dim S_\alpha(M) = 3$.*

Proof We have $3 \leq \dim S_\alpha(M) \leq \dim_{\rho_\alpha} R(M) \leq \dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$. Therefore Lemma 3.3 implies the result if we can show that

$$\dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = 3.$$

If α is trivial then $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) \cong H^1(M;\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$ and $\dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = 3$ follows.

If α is nontrivial, then the $\pi_1(M)$-module $\mathfrak{sl}_2(\mathbb{C})_{\rho_\alpha}$ splits as $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}_0$. Hence $H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha) \cong H^1(\pi_1(M);\mathbb{C}_+) \oplus H^1(\pi_1(M);\mathbb{C}_0) \oplus H^1(\pi_1(M);\mathbb{C}_-)$ and by Lemma 4.6 and Proposition 4.7 we get

$$\dim H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha) = 1\,.$$

This implies that $\dim Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha) = 3$ and $\rho_\alpha \in S_\alpha(M) \subset R(M)$ is a smooth point. ◻

5 Cohomology of metabelian representations

The aim of the following three sections is to prove that, when α is a simple zero of the Alexander invariant, certain reducible metabelian representations are smooth points of the representation variety $R(M)$. First we construct these reducible representations and then, before proving their smoothness in Section 7, we shall do some cohomological computations in this section and the following one.

Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a homomorphism and let $d\colon \pi_1(M) \to \mathbb{C}$ be a map. The map $\rho_\alpha^d\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ given by

$$\rho_\alpha^d(\gamma) = \begin{pmatrix} 1 & d(\gamma) \\ 0 & 1 \end{pmatrix} \rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha^{\frac{1}{2}}(\gamma) & \alpha^{-\frac{1}{2}}(\gamma) d(\gamma) \\ 0 & \alpha^{-\frac{1}{2}}(\gamma) \end{pmatrix}$$

is a homomorphism if and only if $d \in Z^1(\pi_1(M);\mathbb{C}_\alpha)$. Moreover, ρ_α^d is non-abelian if and only if d is not a coboundary.

Corollary 5.1 (Burde, de Rham) *Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a representation and define $\rho_\alpha\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ as in (1). Then there exists a reducible, non-abelian representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ such that $\chi_\rho = \chi_{\rho_\alpha}$ if and only if α is a zero of the Alexander invariant.*

Proof By Lemma 4.6 we have that $\dim H^1(\pi_1(M);\mathbb{C}_\alpha) > 0$ if and only if α is a zero of the Alexander invariant of M. ◻

If α is a simple zero of the Alexander invariant, then $\alpha^\pm$ defined by (12) is a zero of the first Alexander invariant, but it is not a zero of the second. By Lemma 4.6 we have $H^1(\pi_1(M);\mathbb{C}_\pm) \cong \mathbb{C}$.

Let $d_\pm \in Z^1(\pi_1(M);\mathbb{C}_\pm)$ be a cocycle which represents a generator of the first cohomology group $H^1(\pi_1(M);\mathbb{C}_\pm)$. We denote by $\rho^\pm$ the metabelian representations into the upper/lower triangular group given by Corollary 5.1, i.e.

$$\rho^+(\gamma) = \begin{pmatrix} 1 & d_+(\gamma) \\ 0 & 1 \end{pmatrix} \rho_\alpha(\gamma) \qquad \text{and} \qquad \rho^-(\gamma) = \begin{pmatrix} 1 & 0 \\ d_-(\gamma) & 1 \end{pmatrix} \rho_\alpha(\gamma).$$

If we replace $d_\pm$ by $d'_\pm = c\,d_\pm + b_\pm$ where $c \in \mathbb{C}^*$ and $b_\pm \in B^1(\pi_1(M);\mathbb{C}_\pm)$ then $\rho^\pm$ changes by conjugation by an upper/lower triangular matrix. Let $\mathfrak{b}_+ \subset \mathfrak{sl}_2(\mathbb{C})$ denote the Borel subalgebra of upper triangular matrices. It is a $\pi_1(M)$-module via $\mathrm{Ad}\circ\rho^+$. The short exact sequence of $\pi_1(M)$-modules

$$0 \to \mathbb{C}_+ \to \mathfrak{b}_+ \to \mathbb{C}_0 \to 0$$

gives a long exact sequence in cohomology:

$$\begin{aligned} 0 \to H^0(M;\mathbb{C}_0) \xrightarrow{\delta^1} H^1(M;\mathbb{C}_+) \to H^1(M;\mathfrak{b}_+) \to \\ H^1(M;\mathbb{C}_0) \xrightarrow{\delta^2} H^2(M;\mathbb{C}_+) \to H^2(M;\mathfrak{b}_+) \to 0 \end{aligned} \tag{13}$$

Lemma 5.2 *We have that $H^1(M;\mathfrak{b}_+) = 0$ if and only if $\delta^2\colon H^1(M;\mathbb{C}_0) \to H^2(M;\mathbb{C}_+)$ is an isomorphism.*

Proof The Euler characteristic $\chi(M)$ vanishes. Hence, $H^1(M;\mathfrak{b}_+) = 0$ implies $H^2(M;\mathfrak{b}_+) = 0$ and the sequence (13) gives that $\delta^2\colon H^1(M;\mathbb{C}_0) \to H^2(M;\mathbb{C}_+)$ is an isomorphism.

Suppose that $\delta^2\colon H^1(M;\mathbb{C}_0) \to H^2(M;\mathbb{C}_+)$ is an isomorphism. Then the sequence (13) gives $H^2(M;\mathfrak{b}_+) = 0$ and the vanishing of the Euler characteristic implies $H^1(M;\mathfrak{b}_+) = 0$. □

A cocycle $d_0\colon \pi_1(M) \to \mathbb{C}_0$ is nothing but a homomorphism $d_0\colon \pi_1(M) \to (\mathbb{C},+)$. A direct calculation gives

$$\delta^2(d_0)(\gamma_1,\gamma_2) = -2d_+(\gamma_1)d_0(\gamma_2).$$

The 2-cocycle $\delta^2(d_0)$ is a cup product. In our situation we have the multiplication $\mathbb{C}_0 \otimes \mathbb{C}_+ \to \mathbb{C}_+$. This gives us a cup product

$$\dot\cup\colon H^1(\pi_1(M);\mathbb{C}_0) \otimes H^1(\pi_1(M);\mathbb{C}_+) \to H^2(\pi_1(M);\mathbb{C}_+)$$

and

$$\delta^2(d_0) = -2(d_+ \mathbin{\dot\cup} d_0) \tag{14}$$

(see Equation (7)). Hence we have that δ^2 is an isomorphism if and only if, for each nontrivial homomorphism $d_0\colon \Gamma \to \mathbb{C}$, the cocycle $d_+ \mathbin{\dot\cup} d_0$ represents a nontrivial cohomology class.

The next lemma will be proved in Section 6:

Lemma 5.3 *Let $d_0\colon \pi_1(M) \to \mathbb{C}_0$ be a nontrivial homomorphism and let $d_+\colon \pi_1(M) \to \mathbb{C}_+$ be a cocycle representing a nontrivial cohomology class. If α is a simple zero of the Alexander invariant, then the 2-cocycle $d_+ \mathbin{\dot\cup} d_0 \in Z^2(\pi_1(M);\mathbb{C}_+)$ represents a nontrivial cohomology class.*

Corollary 5.4 *Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a nontrivial homomorphism. If α is a simple zero of the Alexander invariant then $H^1(\pi_1(M);\mathfrak{b}_+) = 0$ and the projection to the quotient $\mathfrak{sl}_2(\mathbb{C})_{\rho^+} \to \mathfrak{sl}_2(\mathbb{C})_{\rho^+}/\mathfrak{b}_+ \cong \mathbb{C}_-$ induces an isomorphism*

$$H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong H^1(\pi_1(M);\mathbb{C}_-) \cong \mathbb{C}.$$

Proof Lemmas 5.3 and 5.2 and equation (14) imply that $H^1(\pi_1(M);\mathfrak{b}_+) = 0$. The isomorphism follows then from the long exact sequence in cohomology corresponding to

$$0 \to \mathfrak{b}_+ \to \mathfrak{sl}_2(\mathbb{C})_{\rho^+} \to \mathbb{C}_- \to 0\,. \qquad \square$$

6 Fox calculus and 2-cocycles

The aim of this section is to prove Lemma 5.3. Let Γ be a finitely presented group and let A be a $\mathbb{C}\Gamma$-module. In the sequel we have to decide when a given 2-*cocycle* $c\colon \Gamma\times\Gamma \to A$ is a coboundary.

A normalized 2-cochain is a map $c\colon \Gamma\times\Gamma \to A$ where the normalization condition is $c(1,\gamma) = c(\gamma,1) = 0$ for all $\gamma \in \Gamma$. We shall extend c linearly on the first component, i.e. for $\eta = \sum_{\gamma\in\Gamma} n_\gamma \gamma \in \mathbb{C}\Gamma$, $n_\gamma \in \mathbb{C}$, we define

$$c(\eta,\gamma_0) := \sum_{\gamma\in\Gamma} n_\gamma\, c(\gamma,\gamma_0)\,.$$

Proposition 6.1 *Let $\Gamma = \langle S_1,\dots,S_n | R_1,\dots,R_m\rangle$ be a finitely presented group and let $c\colon \Gamma\times\Gamma \to A$ be normalized 2-cocycle.*

Then $c\colon \Gamma\times\Gamma \to A$ is a coboundary if and only if there exists $a_i \in A$, $i = 1,\dots,n$, such that for all $j = 1,\dots,m$ the equation

$$\sum_{i=1}^{n} \pi\Big(\frac{\partial R_j}{\partial S_i}\Big) \circ a_i + \sum_{i=1}^{n} c\Big(\pi\Big(\frac{\partial R_j}{\partial S_i}\Big), \pi(S_i)\Big) = 0 \tag{15}$$

holds.

Here $\pi\colon \langle S_1,\dots,S_n\rangle \to \Gamma$ denotes the natural projection from the free group to Γ.

Proof We start by recalling some well known constructions used in the proof (cf. [7]). Let X be the canonical 2-complex with one 0-cell associated to the presentation of $\Gamma = \langle S_1, \dots, S_n | R_1, \dots, R_m \rangle$ i.e. $X = X^0 \cup X^1 \cup X^2$ where $X^0 = \{e^0\}$, $X^1 = \{e^1_1, \dots, e^1_n\}$ and $X^2 = \{e^2_1, \dots, e^2_m\}$. The universal covering $p\colon \tilde{X} \to X$ gives us a free chain complex $C_k := C_k(\tilde{X})$ of Γ-modules. A basis for C_k is given by choosing exactly one cell $\tilde{e}^k_j \in p^{-1}(e^k_j)$. With respect to this basis, ∂_2 is given by the Fox calculus, i.e.

$$\partial_2(\tilde{e}^2_j) = \sum_{i=1}^{n} \frac{\partial R_j}{\partial S_i} \tilde{e}^1_i \quad \text{and} \quad \partial_1(\tilde{e}^1_i) = (S_i - 1)\tilde{e}^0 \,. \tag{16}$$

Notice that $\partial_1 \circ \partial_2 = 0$ corresponds to the fundamental formula of the Fox calculus (see [8]).

The *normalized* bar resolution for Γ is denoted by $B_* := B_*(\Gamma)$. More precisely, let $B_n := B_n(\Gamma)$ be the free Γ-module with generators $[x_1|\dots|x_n]$, where $x_i \in \Gamma \backslash \{1\}$. In order to give meaning to every symbol $[x_1|\dots|x_n]$ set $[x_1|\dots|x_n] = 0$ if $x_i = 1$ for any i. This is called the *normalization* condition. Note that $B_0 \cong \mathbb{Z}\Gamma$ is the free Γ-module on one generator and the augmentation $\varepsilon\colon B_0 \to \mathbb{Z}$ maps $[\,]$ to 1. In low dimensions the boundary operators are given by

$$\partial[x|y] = x[y] - [xy] + [x], \quad \partial[x] = (x-1)[\,] \,.$$

Moreover, homomorphisms $s_{-1}\colon \mathbb{Z} \to B_0$, $s_n\colon B_n \to B_{n+1}$ of abelian groups are defined by

$$s_{-1}(1) = [\,] \text{ and } s_n(x[x_1|\dots|x_n]) = [x|x_1|\dots|x_n] \,.$$

It turns out that (s_n) is a contracting homotopy for the underlying augmented chain complex $B_* \xrightarrow{\varepsilon} \mathbb{Z}$ of abelian groups.

$$\begin{tikzcd}
0 \arrow[r] & C_2(\tilde{X}) \arrow[r, "\partial_2"] \arrow[d, "f_2"] & C_1(\tilde{X}) \arrow[r, "\partial_0"] \arrow[d, "f_1"] & C_0(\tilde{X}) \arrow[r, "\epsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[d, "="] \\
\cdots \arrow[r] & B_2(\Gamma) \arrow[r, shift left, "\partial"] & B_1(\Gamma) \arrow[r, shift left, "\partial"] \arrow[l, shift left, "s_1"] & B_0(\Gamma) \arrow[r, shift left, "\epsilon"] \arrow[l, shift left, "s_0"] & \mathbb{Z} \arrow[l, shift left, "s_{-1}"]
\end{tikzcd}$$

By using the fact that C_* is a free Γ-complex and that B_* is contractile we obtain a chain map $f_*\colon C_* \to B_*$ which is augmentation preserving i.e. $\varepsilon \circ f_0 = \varepsilon$. Moreover, f is unique up to chain homotopy (see [7, I.7.4]). The contracting homotopy $s_n\colon B_n \to B_{n+1}$ and the basis of C_* determine f_* inductively (see [7, p.24] for details). Hence the maps $f_i\colon C_i \to B_i$, $i = 0, 1, 2$, are given by

$$f_0(\tilde{e}^0) = [\,], \quad f_1(\tilde{e}^1_i) = [S_i] \text{ and } f_2(\tilde{e}^2_j) = \sum_{i=1}^{n} [\frac{\partial R_j}{\partial S_i} | S_i] \,. \tag{17}$$

Here we have used the following convention: if $\eta = \sum_{\gamma\in\Gamma} n_\gamma \gamma \in \mathbb{Z}\Gamma$, then $[\eta,\gamma_0] := \sum_{\gamma\in\Gamma} n_\gamma [\gamma,\gamma_0] \in B_2$.

It follows that the induced cochain map $f_2^*\colon \operatorname{Hom}_\Gamma(B_2;A) \to \operatorname{Hom}_\Gamma(C_2;A)$ is given by

$$f_2^*(c)(\tilde{e}_j^2) = \sum_{i=1}^n c\Big(\frac{\partial R_j}{\partial S_i}, S_i\Big), \text{ where } c \in \operatorname{Hom}_\Gamma(B_2;A). \tag{18}$$

By Lemma 3.1, the map f_2^* induces an injection $f^*\colon H^2(\Gamma;A) \to H^2(X;A)$. Hence, $f_2^*(c)$ is a coboundary if and only if there exists a cochain

$$b \in \operatorname{Hom}_\Gamma(B_1;A) \text{ such that } f_2^*(c)(\tilde{e}_j^2) + b(f_1(\partial_2 \tilde{e}_j^2)) = 0 \tag{19}$$

for all $j = 1, \dots, m$. The proposition follows from Equation (19) by using (16), (17) and (18). □

Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a nontrivial representation. Note that α induces a homomorphism $\alpha\colon \operatorname{tors}(H_1(M;\mathbb{Z})) \oplus \mathbb{Z} \to \mathbb{C}^*$, and let σ denote the restriction of α to the torsion subgroup $\operatorname{tors}(H_1(M;\mathbb{Z}))$. As before we denote by $\mathbb{C}_+$ the $\pi_1(M)$-module $\mathbb{C}_\alpha$, i.e. $\gamma\cdot z = \alpha(\gamma)z$. For the remainder of the section, fix a projection $p\colon H_1(M;\mathbb{Z}) \to \operatorname{tors}(H_1(M;\mathbb{Z}))$ and a generator $\phi \in H^1(M;\mathbb{Z})$.

Let $h\colon \pi_1(M) \to (\mathbb{C},+)$ be a homomorphism and let $d_+\colon \pi_1(M) \to \mathbb{C}_+$ be a cocycle. Since ϕ is a generator of the first cohomology group, there exists $a \in \mathbb{C}$ such that $h = a\,\phi$.

By Proposition 6.1 we have that $h \mathbin{\dot\cup} d_+$ is a coboundary if and only if there exist $x_1, \dots, x_n \in \mathbb{C}_+$ such that

$$\sum_{i=1}^n \alpha\Big(\frac{\partial R_j}{\partial S_i}\Big)\, x_i + h \mathbin{\dot\cup} d_+\Big(\frac{\partial R_j}{\partial S_i}, S_i\Big) = 0 \tag{20}$$

for all $j = 1, \dots, n-1$.

We have $h\dot\cup d_+(\gamma_1,\gamma_2) = h(\gamma_1)\alpha(\gamma_1)d_+(\gamma_2)$ and hence for $\eta = \sum c_\gamma \gamma \in \mathbb{C}\pi_1(M)$ we get from (9):

$$\begin{aligned} h \mathbin{\dot\cup} d_+(\eta,\gamma_0) &= \sum c_\gamma\, (h \mathbin{\dot\cup} d_+)(\gamma,\gamma_0) \\ &= \sum c_\gamma\, h(\gamma)\alpha(\gamma) d_+(\gamma_0) \\ &= a \sum c_\gamma\, \phi(\gamma)\sigma(\gamma)\alpha(s_p(1))^{\phi(\gamma)} d_+(\gamma_0)\,. \end{aligned} \tag{21}$$

Let $D\colon \mathbb{C}[t^{\pm 1}] \to \mathbb{C}[t^{\pm 1}]$ be the following differential operator:

$$D\Big(\sum_{i\in\mathbb{Z}} c_i t^i\Big) = \sum_{i\in\mathbb{Z}} c_i i t^i\,.$$

The operator D satisfies the following rules:

$$\begin{aligned} D(c_1\eta_1 + c_2\eta_2) &= c_1 D(\eta_1) + c_2 D(\eta_2) \qquad && \text{for } c_i \in \mathbb{C} \text{ and } \eta_i \in \mathbb{C}[t^{\pm 1}], \\ D(\eta_1\eta_2) &= D(\eta_1)\eta_2 + \eta_1 D(\eta_2) && \text{for } \eta_i \in \mathbb{C}[t^{\pm 1}]. \end{aligned}$$

It follows from these rules that $D(c) = 0$ for $c \in \mathbb{C}$.

For given $z \in \mathbb{C}^*$ we define $\operatorname{ord}_z \colon \mathbb{C}[t^{\pm 1}] \to \mathbb{N} \cup \{\infty\}$ to be the order of $\eta(t) \in \mathbb{C}[t^{\pm 1}]$ at z, i.e. $\operatorname{ord}_z(\eta) = \infty \Leftrightarrow \eta(t) \equiv 0$ and

$$\operatorname{ord}_z(\eta) = k \in \mathbb{N} \Leftrightarrow \exists \eta' \in \mathbb{C}[t^{\pm 1}] : \eta'(z) \neq 0 \text{ and } \eta(t) = (t-z)^k \eta'(t)\,.$$

It is easy to see that if $\eta(z) = 0$, then $\operatorname{ord}_z(\eta) = \operatorname{ord}_z(D(\eta)) + 1$.

For a fixed $z \in \mathbb{C}^*$ the evaluation map $\mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ which maps $\eta(t)$ to $\eta(z)$ turns $\mathbb{C}$ into a $\mathbb{C}[t^{\pm 1}]$-module which will be denoted by $\mathbb{C}_z$. The kernel of the evaluation map $\mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ is exactly the maximal ideal generated by $(t-z)$.

We choose a splitting of $H_1(M;\mathbb{Z})$ as in (2) and we write $\alpha = \beta_\alpha \circ \phi_\sigma$ as in (9). Recall that β_α is nothing but the evaluation map at $z := \alpha(s_p(1))$. Hence (21) gives:

$$h \mathbin{\dot\cup} d_+(\eta, \gamma_0) = a\, \beta_\alpha(D(\phi_\sigma(\eta))) d_+(\gamma_0), \tag{22}$$

where $\eta \in \mathbb{C}\pi_1(M)$, $\phi_\sigma(\eta) \in \mathbb{C}[t^{\pm 1}]$ and $\gamma_0 \in \pi_1(M)$.

A $\mathbb{C}[t^{\pm 1}]$-module homomorphism $f \colon (\mathbb{C}[t^{\pm 1}])^n \to (\mathbb{C}[t^{\pm 1}])^m$ induces a $\mathbb{C}[t^{\pm 1}]$-morphism $f^z \colon \mathbb{C}_z^n \to \mathbb{C}_z^m$. This follows simply from $f(\operatorname{Ker}(\beta_\alpha)^n) \subset \operatorname{Ker}(\beta_\alpha)^m$.

$$\begin{tikzcd} (\mathbb{C}[t^{\pm 1}])^n \arrow[r, "f"] \arrow[d, "\beta_\alpha^n"] & (\mathbb{C}[t^{\pm 1}])^m \arrow[d, "\beta_\alpha^m"] \\ \mathbb{C}_z^n \arrow[r, "f^z"] & \mathbb{C}_z^m\,. \end{tikzcd}$$

It is easy to see that $D(f) \colon (\mathbb{C}[t^{\pm 1}])^n \to (\mathbb{C}[t^{\pm 1}])^m$ given by $D(f) := D^m \circ f - f \circ D^n$ is a $\mathbb{C}[t^{\pm 1}]$-module morphism. If $\mathbf{A}$ is the matrix of f with respect to the canonical basis, then $\beta_\alpha(\mathbf{A})$ is the matrix of f^z with respect to the canonical basis and $D\mathbf{A}$ is the matrix of $D(f)$ with respect to the canonical basis. Here, β_α and D applied to a matrix means simply applying it to each entry.

Proof of Lemma 5.3 Recall that we made the assumption that α is a simple zero of the Alexander invariant of M.

Let $h \in Z^1(\pi_1(M);\mathbb{C}_0)$ and $d_+ \in Z^1(\pi_1(M);\mathbb{C}_+)$ be cocycles representing nontrivial cohomology classes. It follows from Equation (8) that

$$h \mathbin{\dot\cup} d_+ + d_+ \mathbin{\dot\cup} h \sim 0,$$

and hence $h \dot{\cup} d_+$ is a coboundary if and only if $d_+ \dot{\cup} h$ is a coboundary.

Let $a \in \mathbb{C}^*$ such that $h = a\phi$ and set $z := \alpha(s_p(1))$. Then $\beta_\alpha(\eta(t))$ is simply $\eta(z)$. By writing equations (20) in matrix form, we obtain from (22) that $h \dot{\cup} d_+$ is a coboundary if and only if the system

$$J^{\phi_\sigma}(z)\,\mathbf{x} + a\,(DJ^{\phi_\sigma})(z)\begin{pmatrix} d_+(S_1) \\ \vdots \\ d_+(S_n)\end{pmatrix} = 0 \tag{23}$$

has a solution $\mathbf{x} \in \mathbb{C}^n$. Here, for each matrix $A \in M_{m,n}(\mathbb{C}[t^{\pm 1}])$ we denote by $A(z) \in M_{m,n}(\mathbb{C})$ the matrix obtained by applying the evaluation map to its entries.

From the canonical 2-complex associated to the presentation we obtain the following resolutions:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & (\mathbb{C}[t^{\pm 1}])^{n-1} & \xleftarrow{d_2} & (\mathbb{C}[t^{\pm 1}])^{n} & \xleftarrow{d_1} & \mathbb{C}[t^{\pm 1}] & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_\alpha} & & \downarrow{\scriptstyle \beta_\alpha} & & \downarrow{\scriptstyle \beta_\alpha} & & \\
0 & \longleftarrow & \mathbb{C}_z^{n-1} & \xleftarrow{d_2^z} & \mathbb{C}_z^{n} & \xleftarrow{d_1^z} & \mathbb{C}_z & \longleftarrow & 0\,.
\end{array}$$

The matrix of d_2 (respectively d_2^z) with respect to the canonical basis is J^{ϕ_σ} (respectively $J^{\phi_\sigma}(z)$) and the matrix of $D(d_2)$ with respect to the canonical basis is DJ^{ϕ_σ}.

It follows that $h \dot{\cup} d_+$ is a coboundary if and only if

$$(DJ^{\phi_\sigma})(z)\begin{pmatrix} d_+(S_1) \\ \vdots \\ d_+(S_n)\end{pmatrix} \in \operatorname{Im}(d_2^z)\,.$$

Note that J^{ϕ_σ} is a presentation matrix of $H_1^{\phi_\sigma}(M, x_0) \cong H_1^{\phi_\sigma}(M) \oplus \mathbb{C}[t^{\pm 1}]$. The assumption that α is a simple zero of the Alexander invariant implies that $H_1^{\phi_\sigma}(M)$ is torsion. Hence, there exist a basis $\mathcal{B} = (b_0, \dots, b_{n-1})$ of $(\mathbb{C}[t^{\pm 1}])^n$ and a basis $\mathcal{C} = (c_1, \dots, c_{n-1})$ of $(\mathbb{C}[t^{\pm 1}])^{n-1}$ such that $d_2(b_0) = 0$ and $d_2(b_i) = r_i(t)c_i$, $1 \le i \le n-1$, where $r_i(t) \in \mathbb{C}[t^{\pm 1}]$ are nonzero and $r_{i+1}(t) \mid r_i(t)$. Moreover, we have that $\operatorname{ord}_z(\Delta^{\phi_\sigma}) = 1$ and hence $r_1(z) = 0$ and $(Dr_1)(z) \neq 0$. In particular, $r_j(z) \neq 0$ for $j \geq 2$.

Now, $d_2 \circ d_1 = 0$ gives that $\operatorname{Im} d_1 \subset \mathbb{C}[t^{\pm 1}] \cdot b_0$. Therefore, there exists a $r(t) \in \mathbb{C}[t^{\pm 1}]$ such that $d_1(1) = r(t) b_0$. Since $H^0(M, \mathbb{C}_+) = 0$ we obtain $r(z) \neq 0$.

We define a basis $(b_0^z, \dots, b_{n-1}^z)$ of $\mathbb{C}_z^n$ by $b_0^z = r(z)\beta_\alpha(b_0) = r(z)b_0(z)$ and $b_i^z = \beta_\alpha(b_i) = b_i(z)$, $1 \le i \le n-1$. Analogously, a basis $(c_1^z, \dots, c_{n-1}^z)$ of $\mathbb{C}^{n-1}$ is given by $c_i^z = \beta_\alpha(c_i) = c_i(z)$, $1 \le i \le n-1$.

We have

$$\operatorname{Im} d_2^z = \operatorname{span}(c_2^z, \dots, c_{n-1}^z) \text{ and } \operatorname{Ker} d_2^z = \operatorname{span}(b_0^z, b_1^z)\,.$$

Note that $\operatorname{Ker} d_2^z = \operatorname{span}(b_0^z, b_1^z)$ can be identified with $Z^1(M;\mathbb{C}_+)$ and that the coboundaries correspond to the multiples of b_0^z.

Now a direct calculation gives

$$(Dd_2)(z)\,(b_0^z) = \beta_\alpha(D(d_2)\,(r(t)b_0)) \in \operatorname{Im} d_2^z, \text{ (using } d_2(b_0) = 0),$$

and $(Dd_2)(z)\,(b_1^z) = \beta_\alpha(D(d_2)(b_1)) \in (Dr_1)(z)\,c_1^z + \operatorname{Im} d_2^z$, (using $r_1(z) = 0$).

Moreover, $(Dr_1)(z) \neq 0$ and each element of $\operatorname{Ker} d_2^z$ representing a nonzero cohomology class does not map into $\operatorname{Im} d_2^z$ under $(Dd_2)(z)$. Hence, for each cocycle $d_+\colon \pi_1(M) \to \mathbb{C}_+$ which represents a generator of $H^1(M;\mathbb{C}_+)$ and each nontrivial homomorphism $h\colon \pi_1(M) \to (\mathbb{C},+)$ the system (23) has no solution, i.e. $h \mathbin{\dot\cup} d_+$ is not a coboundary. □

7 Deforming metabelian representations

We suppose in the sequel that α is a simple zero of the Alexander invariant of M. Let $\rho^+ \in R(M)$ denote the metabelian representation defined in Section 5. In this section we use the results of the previous two sections in order to show that ρ^+ is a smooth point of $R(M)$ with local dimension 4.

Let $i\colon \partial M \to M$ be the inclusion.

Lemma 7.1 *The representation $\rho^+ \circ i_\#\colon \pi_1(\partial M) \to \mathrm{PSL}_2(\mathbb{C})$ is nontrivial.*

Proof By Corollary 5.4, $H^1(M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong \mathbb{C}$ and by duality, $H^2(M;\partial M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong H^1(M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong \mathbb{C}$. Thus, by the exact sequence of the pair

$$H^1(M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \to H^1(\partial M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \to H^2(M;\partial M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}),$$

we obtain $\dim H^1(\partial M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \le 2$.

We prove the lemma by contradiction: if ρ^+ restricted to $i_\#(\pi_1(\partial M))$ was trivial, then $\mathfrak{sl}_2(\mathbb{C})_{\rho^+}$ would be a trivial $\pi_1(\partial M)$-module, and therefore

$$H^1(\partial M;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong H^1(\partial M;\mathbb{C}) \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$$

would have dimension six, contradicting the previous upper bound for the dimension. □

Definition 7.2 A non-cyclic abelian subgroup of $\mathrm{PSL}_2(\mathbb{C})$ with four elements is called *Klein's 4-group*. Such a group is realized by rotations about three orthogonal geodesics and it is conjugate to the one generated by $\pm\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix}i & 0\\ 0 & -i\end{smallmatrix}\right)$.

Remark 7.3 The image $\rho^+(i_\#(\pi_1(\partial M)))$ cannot be the Klein group, because the image of ρ^+ is reducible (i.e. the action on $P^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ has a fixed point, ∞), but the Klein group has no fixed point in $P^1(\mathbb{C})$.

Recall that by Weil's construction $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$ contains the Zariski tangent space of $R(M)$ at ρ^+ (cf. Section 3). To prove the smoothness, we show that all cocycles in $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})\cong\mathbb{C}$ are integrable. To do this, we prove that all obstructions vanish, by using the fact that the obstructions vanish on the boundary.

Lemma 7.4 *The variety $R(\mathbb{Z}\oplus\mathbb{Z})$ has exactly two irreducible components. One is four dimensional and smooth except at the trivial representation. The other component is three dimensional and smooth; it is exactly the orbit of a representation onto the Klein group.*

Proof Let $\mathbb{Z}\oplus\mathbb{Z} = \langle x,y | [x,y]=1\rangle$ and let $\rho\colon \mathbb{Z}\oplus\mathbb{Z}\to\mathrm{PSL}_2(\mathbb{C})$ be a representation given by $\rho(x)=\pm A_x$ and $\rho(y)=\pm A_y$. Then $\mathrm{tr}[A_x,A_y]\in\{\pm 2\}$ and ρ lifts to $\mathrm{SL}_2(\mathbb{C})$ if and only if $\mathrm{tr}[A_x,A_y]=2$. Moreover, ρ is a representation onto a Klein group if and only if $\mathrm{tr}[A_x,A_y]=-2$.

Thus $R(\mathbb{Z}\oplus\mathbb{Z})$ has two components, one of dimension four and one of dimension three, which is the orbit of a representation onto the Klein group.

Given a representation $\varrho\in R(\mathbb{Z}\oplus\mathbb{Z})$ which is nontrivial and different from the Klein group, then $H^0(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)\cong\mathfrak{sl}_2(\mathbb{C})^{\varrho(\mathbb{Z}\oplus\mathbb{Z})}\cong\mathbb{C}$. Thus, by duality and Euler characteristic, $H^1(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)\cong\mathbb{C}^2$ and $Z^1(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)\cong\mathbb{C}^4$. This computation shows that the dimension of the Zariski tangent space at this representation is at most four. Since the representation lies in a four dimensional component, it is a smooth point of $R(\mathbb{Z}\oplus\mathbb{Z})$. Note that it follows from the proof of Lemma 7.1 that the trivial representation is a singular point of $R(\mathbb{Z}\oplus\mathbb{Z})$.

If ϱ is a representation onto the Klein group then $H^0(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)=0$ and hence $H^1(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)=0$ by the same Euler characteristic argument. Hence $\dim Z^1(\mathbb{Z}\oplus\mathbb{Z};\mathfrak{sl}_2(\mathbb{C})_\varrho)=3$. Since the orbit of the representation is three dimensional and closed, the lemma is proved. □

Given a cocycle $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$ the first obstruction to integration is the cup product with itself. In general when the n-th obstruction vanishes, the obstruction of order $n+1$ is defined, it lives in $H^2(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$.

Let Γ be a finitely presented group and let $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be a representation. A *formal deformation* of ρ is a homomorphism $\rho_\infty\colon \Gamma \to \mathrm{PSL}_2(\mathbb{C}[[t]])$

$$\rho_\infty(\gamma) = \pm\exp(\sum_{i=1}^{\infty} t^i u_i(\gamma))\rho(\gamma)$$

where $u_i\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})$ are elements of $C^1(\Gamma,\mathfrak{sl}_2(\mathbb{C})_\rho)$ such that $p_0 \circ \rho_\infty = \rho$. Here $p_0\colon \mathrm{PSL}_2(\mathbb{C}[[t]]) \to \mathrm{PSL}_2(\mathbb{C})$ is the evaluation homomorphism at $t = 0$ and $\mathbb{C}[[t]]$ denotes the ring of formal power series. We shall say that ρ_∞ is a *formal deformation up to order k* of ρ if ρ_∞ is a homomorphism modulo t^{k+1}.

An easy calculation gives that ρ_∞ is a homomorphism up to first order if and only if $u_1 \in Z^1(\Gamma,\mathfrak{sl}_2(\mathbb{C})_\rho)$ is a cocycle. We call a cocycle $u_1 \in Z^1(\Gamma,\mathfrak{sl}_2(\mathbb{C})_\rho)$ *formally integrable* if there is a formal deformation of ρ with leading term u_1.

Let $u_1,\dots,u_k \in C^1(\Gamma,\mathfrak{sl}_2(\mathbb{C})_\rho)$ such that

$$\rho_k(\gamma) = \exp(\sum_{i=1}^{k} t^i u_i(\gamma))\rho(\gamma)$$

is a homomorphism into $\mathrm{PSL}_2(\mathbb{C}[[t]])$ modulo t^{k+1}. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1,\dots,u_k)} \in H^2(\Gamma;\mathfrak{sl}_2(\mathbb{C})_\rho)$ with the following properties (see [14, Sec. 3]):

(i) There is a cochain $u_{k+1}\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})$ such that

$$\rho_{k+1}(\gamma) = \exp(\sum_{i=1}^{k+1} t^i u_i(\gamma))\rho(\gamma)$$

is a homomorphism modulo t^{k+2} if and only if $\zeta_{k+1} = 0$.

(ii) The obstruction ζ_{k+1} is natural, i.e. if $f\colon \Gamma' \to \Gamma$ is a homomorphism then $f^*\rho_k := \rho_k \circ f$ is also a homomorphism modulo t^{k+1} and $f^*(\zeta_{k+1}^{(u_1,\dots,u_k)}) = \zeta_{k+1}^{(f^*u_1,\dots,f^*u_k)}$.

Lemma 7.5 *Let $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be a reducible, nonabelian representation such that $\dim Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\rho) = 4$.*

If $\rho \circ i_\#\colon \pi_1(\partial M) \to \mathrm{PSL}_2(\mathbb{C})$ is neither trivial nor a representation onto a Klein group, then every cocycle in $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\rho)$ is integrable.

Proof We first show that $i^*\colon H^2(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\rho) \to H^2(\pi_1(\partial M);\mathfrak{sl}_2(\mathbb{C})_\rho)$ is injective. To that extent we use the following commutative diagram:

$$\begin{tikzcd} H^2(M;\mathfrak{sl}_2(\mathbb{C})_\rho) \arrow[r, "\cong"] & H^2(\partial M;\mathfrak{sl}_2(\mathbb{C})_\rho) \\ H^2(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\rho) \arrow[r] \arrow[u] & H^2(\pi_1(\partial M);\mathfrak{sl}_2(\mathbb{C})_\rho) \arrow[u, "\cong"'] \end{tikzcd}$$

The horizontal isomorphism on the top of the diagram comes from the exact sequence of the pair $(M,\partial M)$ and the dimension computation in the proof of Lemma 7.1. The vertical isomorphism on the right is a consequence of asphericity of ∂M. In addition, the vertical map on the left is an injection (see Lemma 3.1).

We shall now prove that every element of $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\rho)$ is integrable. Let $u_1,\dots,u_k\colon \pi_1(M)\to\mathfrak{sl}_2(\mathbb{C})$ be given such that $\rho_k(\gamma)=\exp(\sum_{i=1}^k t^i u_i(\gamma))\rho(\gamma)$ is a homomorphism modulo t^{k+1}. Then the restriction $\rho_k\circ i_\#\colon \pi_1(\partial M)\to \mathrm{SL}_2(\mathbb{C}[[t]])$ is also a formal deformation of order k. On the other hand, it follows from Lemma 7.4 that the restriction $\rho_k\circ i_\#$ is a smooth point of the representation variety $R(\partial M)$. Hence, the formal implicit function theorem gives that $i^*\rho_k$ extends to a formal deformation of order $k+1$ (see [14, Lemma 3.7]). Therefore, we have that

$$0=\zeta_{k+1}^{(i^*u_1,\dots,i^*u_k)}=i^*\zeta_{k+1}^{(u_1,\dots,u_k)}\,.$$

Now, i^* is injective and the obstruction vanishes. □

Proposition 7.6 *The representation ρ^+ is a smooth point of $R(M)$ with local dimension four.*

Proof It follows from Corollary 5.4 that $\dim H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})=1$. Thus $\dim Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) = 1+\dim\mathfrak{sl}_2(\mathbb{C}) = 4$. Moreover, it follows from Lemma 7.1 and Remark 7.3 that the representation ρ^+ verifies the hypothesis of Lemma 7.5. Hence all cocycles in $Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$ are integrable. By applying Artin's theorem [1] we obtain from a formal deformation of ρ^+ a convergent deformation (see [14, Lemma 3.3]). Thus ρ^+ is a smooth point of $R(M)$ with local dimension equal to $4=\dim Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$. □

8 The quadratic cone at the representation ρ_α

Let $\alpha\colon \pi_1(M)\to\mathbb{C}^*$ be a nontrivial representation. We shall suppose in the sequel that α is a simple zero of the Alexander invariant.

We want to show that ρ_α is contained in precisely two components and that their intersection at the orbit of ρ_α is transverse. For this we study the quadratic cone. The Zariski tangent space can be viewed as the space of germs of analytic paths which are contained in $R(M)$ at the first order. The quadratic cone is the analogue at the second order. Since the defining polynomials for the union of two varieties are the products of defining polynomials for each component, the Zariski tangent space of each component (at points of the intersection) can only be detected by the quadratic cone.

Let $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be a representation. The quadratic cone $Q(\rho)$ is defined by

$$Q(\rho) := \{u \in Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\rho) \mid [u \cup u] \sim 0\}.$$

Recall that given two cocycles $u, v \in Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_{\rho_\alpha})$ the cup product $[u \cup v] \in Z^2(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_{\rho_\alpha})$ is the cocycle given by

$$[u \cup v](\gamma_1, \gamma_2) = [u(\gamma_1), \mathrm{Ad}_{\rho_\alpha(\gamma_1)}(u(\gamma_2))] \in \mathfrak{sl}_2(\mathbb{C}), \qquad \forall \gamma_1 \gamma_2 \in \pi_1(M);$$

where $[\,,\,]$ denotes the Lie bracket (see (7)). Since the Lie bracket is antisymmetric, one easily checks that the cup product is symmetric, i.e. the cocycles $[u \cup v]$ and $[v \cup u]$ represent the same cohomology class, by (8).

To compute the quadratic cone $Q(\rho_\alpha)$ we use the decomposition $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathbb{C}_0 \oplus \mathbb{C}_+ \oplus \mathbb{C}_-$ of $\pi_1(M)$-modules, see (10). Let pr_0 and $\mathrm{pr}_\pm$ denote the projections of $\mathfrak{sl}_2(\mathbb{C})_\alpha$ to the respective one dimensional modules. We can easily check the relation

$$\mathrm{pr}_\pm([u \cup v]) = \mathrm{pr}_0(u) \mathbin{\dot\cup} \mathrm{pr}_\pm(v) - \mathrm{pr}_\pm(u) \mathbin{\dot\cup} \mathrm{pr}_0(v). \tag{24}$$

Here we use the products of $\pi_1(M)$-modules $\mathbb{C} \times \mathbb{C}_\pm \to \mathbb{C}_\pm$, and $\mathbb{C}_\pm \times \mathbb{C} \to \mathbb{C}_\pm$, which are nothing but the usual product of complex numbers. Notice that these cup products of cohomology classes of cocycles valued in $\mathbb{C}_\pm$ and $\mathbb{C}$ are antisymmetric (for the same reason that the cup product valued in $\mathfrak{sl}_2(\mathbb{C})$ is symmetric). Therefore (24) in induces $\forall z \in Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$:

$$\mathrm{pr}_\pm([z \cup z]) = 2\,\mathrm{pr}_0(z) \mathbin{\dot\cup} \mathrm{pr}_\pm(z) \quad \text{up to coboundary.} \tag{25}$$

The splitting (10) induces a splitting in cohomology. We recall that

$$\dim H^1(\pi_1(M); \mathbb{C}_\pm) = \dim H^1(\pi_1(M); \mathbb{C}) = 1$$

and we have chosen cocycles $d_\pm$ and d_0 whose cohomology classes generate $H^1(\pi_1(M); \mathbb{C}_\pm)$ and $H^1(\pi_1(M); \mathbb{C})$ respectively (see Section 5). Thus, the cocycle $z \in Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$ is cohomologous to

$$z \sim a_0\, d_0 + a_-\, d_- + a_+\, d_+ \qquad \text{with } a_0, a_+, a_- \in \mathbb{C}. \tag{26}$$

Therefore (25) becomes

$$\mathrm{pr}_{\pm}([z \cup z]) = 2a_0 a_{\pm}\, d_0 \,\dot{\cup}\, d_{\pm} \in Z^1(\pi_1(M); \mathbb{C}_{\pm}) \quad \text{up to coboundary.} \tag{27}$$

Lemma 8.1 *The quadratic cone $Q(\rho_\alpha) \subset Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$ is the union of two subspaces, one of dimension 4 and another one of dimension 3. These subspaces are precisely the kernels of the projections*

$$\mathrm{pr}_0 \colon Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) \to Z^1(\pi_1(M); \mathbb{C}) = H^1(\pi_1(M); \mathbb{C})$$

and

$$Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) \xrightarrow{\mathrm{pr}_+ \oplus \mathrm{pr}_-} Z^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-) \to H^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$$

respectively. In particular, the intersection of these subspaces is the space of coboundaries $B^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$.

Proof Notice that the space

$$Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = Z^1(\pi_1(M); \mathbb{C}_0) \oplus Z^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$$

is five dimensional and that

$$B^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = B^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-).$$

Every cocycle $z \in Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$ can be uniquely written as $z = a_0 d_0 + a_+ d_+ + a_- d_- + b$ where $b \in B^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$. Combining Lemma 5.3, Equations (26) and (27), the condition for the quadratic cone $[z \cup z] \sim 0$ reduces to:

$$a_0\, a_+ = a_0\, a_- = 0\,.$$

This is of course a necessary condition for integrability. In particular we deduce that $z \in Q(\rho_\alpha)$ if and only if $z \in Z^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$ or $z \in Z^1(\pi_1(M); \mathbb{C}_0) \oplus B^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$ and the lemma follows. □

Proof of Theorem 1.2 By Lemma 8.1 it suffices to show that ρ_α is contained in two irreducible components, one of dimension four containing irreducible representations and another of dimension three containing only abelian ones. This will show that the Zariski tangent space of each component has the right dimension which implies that the point of the intersection is a smooth point of each component. Moreover, the intersection is transverse.

The component of dimension 4 is provided by Proposition 7.6. In fact ρ_α is the adherence of the orbit of ρ^+, thus it is contained in the same irreducible component. For the other component, notice that ρ_α is contained in a subvariety of abelian representations $S_\alpha(M)$, with $\dim S_\alpha(M) \geq 3$ (see Lemma 4.8).

Representations in $S_\alpha(M)$ are conjugate to diagonal representations, thus the tangent space to this deformation is clearly contained in the kernel of the projection to $H^1(\pi_1(M); \mathbb{C}_+ \oplus \mathbb{C}_-)$, and this gives an irreducible component of dimension at most three. Thus ρ_α it is a smooth point of the three-dimensional component $S_\alpha(M) \subset R(M)$. In addition, the orbits by conjugation in this component must be two-dimensional and therefore all representations must be abelian. $\square$

Example 8.2 Let M be the torus bundle given in Example 3.2. Following Example 4.4 we choose $\lambda_i \in \mathbb{C}^*$ such that $\alpha_i \colon \pi_1(M) \to \mathbb{C}^*$, $i = 1, \ldots, 4$, given by $\alpha_i(\mu) = \lambda_i$ and $\alpha_i|_{\mathbb{Z}/2\oplus\mathbb{Z}/2} = \sigma_i$ is a simple root of the Alexander invariant, i.e. $\lambda_1 = 3 \pm \sqrt{8}$ and $\lambda_i = 1$ for $i = 2, 3, 4$. Therefore, in each case the representation ρ_{α_i} is the limit of irreducible representations. The deformation for ρ_{α_4} was already observed in [13, Section 4.2]. This deformation was simply obtained as a pullback of a component of the representation variety $R(\mathbb{Z}/2 * \mathbb{Z}/2)$ under the surjection:

$$\pi_1(M) \to \pi_1(M)/\langle \mu = 1 \rangle \cong \langle \alpha, \beta | \alpha^2, \beta^2 \rangle\,.$$

Note that ρ_{α_i} is ∂-trivial for $i = 2, 3, 4$. On the other hand, $\rho^+_{\alpha_i} \circ i_\# \colon \pi_1(\partial M) \to \mathrm{PSL}_2(\mathbb{C})$ is parabolic but nontrivial (cf. Lemma 7.1) and Lemma 7.5 applies. The results of [14] do not apply.

9 The variety of characters near χ_α

Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ a representation such that α is a simple zero of the Alexander invariant. Let $\chi_\alpha \in X(M)$ denote the character of ρ_α.

Proof of Theorem 1.3 Notice that there are at least two components of $X(M)$ containing χ_α, which are precisely the quotients of the components of $R(M)$ containing ρ_α.

To study the geometry of $X(M)$ near χ_α we construct a slice as in [4]. Let $\gamma_0 \in \Gamma$ be an element such that $\rho_\alpha(\gamma_0) \neq \pm I$. We define the slice as:

$$\mathcal{S} = \{\rho \in R(M) \mid \rho(\gamma_0) \text{ is a diagonal matrix } \}.$$

By [4], $\mathcal{S}$ is a slice étale and we shall give here some of its properties: firstly, $\mathcal{S}$ is transverse to the orbit by conjugation at ρ_α. More precisely, if for some neighborhood $\mathcal{U}$ of ρ_α, $F\colon \mathcal{U} \subset R(M) \to \mathbb{C}^2$ maps a representation

ρ to the non-diagonal entries of $\rho(\gamma_0)$, then $\mathcal{S}\cap\mathcal{U}=F^{-1}(0,0)$. The tangent map of F restricted to the coboundary space defines an isomorphism $B^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha)\cong\mathbb{C}^2$. Thus

$$T^{\mathrm{Zar}}_{\rho_\alpha}(\mathcal{S})\oplus B^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha)=Z^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha),$$

and $H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{C})_\alpha)$ can be viewed as the Zariski tangent space to $\mathcal{S}$ at ρ_α. Secondly, the projection $\mathcal{S}\to X(M)$ is surjective at least for characters χ with $\chi(\gamma_0)\neq 4$. It is therefore sufficient for our purpose to study the slice $\mathcal{S}$ and its quotient by the stabilizer of ρ_α.

Let $d_0\in Z^1(\pi_1(M);\mathbb{C}_0)$ and $d_\pm\in Z^1(\pi_1(M);\mathbb{C}_\pm)$ denote the cocycles of the previous section. Up to adding a coboundary, we may assume that $d_\pm(\gamma_0)=0$. Thus the tangent space to $\mathcal{S}$ at ρ_α is three dimensional and generated by d_0, d_+ and d_-. The analysis of Section 8 allows to say that $\mathcal{S}$ has two components around ρ_α: one curve tangent to d_0 consisting of diagonal representation and a surface tangent to the space generated by d_+ and d_-, containing irreducible representations. Notice that since $\mathcal{S}$ is transverse to the boundary space, it intersects the orbit of ρ_α by conjugation in a single point.

To analyze $X(M)$, we take the quotient of $\mathcal{S}$ by the stabilizer of ρ_α, which is precisely the group of diagonal matrices:

$$D=\left\{\pm\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\mid \lambda\in\mathbb{C}^*\right\}\subset \mathrm{PSL}_2(\mathbb{C}).$$

Since the curve in $\mathcal{S}$ of diagonal representations commutes with D, it projects to a curve of abelian characters in $X(M)$. To understand the action on the other component, we analyze the action on the Zariski tangent space: a matrix $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ acts by mapping $d_\pm$ to $\lambda^{\pm 2}d_\pm$. In other words, we must understand the *algebraic* quotient $\mathbb{C}^2/\!/\mathbb{C}^*$ where $t\in\mathbb{C}^*$ maps $(x,y)\in\mathbb{C}^2$ to $(t\,x,t^{-1}\,y)$. The quotient $\mathbb{C}^2/\!/\mathbb{C}^*$ is the line $\mathbb{C}$, where the algebra of invariant functions is generated by xy. Thus the quotient of the four dimensional component containing ρ_α is also a smooth curve.

To show that the intersection is transverse, notice that the Zariski tangent space to $X(M)$ at χ_α is $H^1(\pi_1(M);\mathbb{C})\oplus H^1(\pi_1(M);\mathbb{C}_+\oplus\mathbb{C}_-)/\!/\mathbb{C}^*\cong\mathbb{C}^2$. The first factor $H^1(\pi_1(M);\mathbb{C})\oplus 0\cong\mathbb{C}$ is tangent to the curve of abelian characters, and $0\oplus H^1(\pi_1(M);\mathbb{C}_+\oplus\mathbb{C}_-)/\!/\mathbb{C}^*\cong\mathbb{C}$ is tangent to the other curve, thus the intersection is transverse. $\square$

10 Real valued characters

Let $\alpha\colon \pi_1(M) \to \mathbb{C}^*$ be a representation such that α is a simple zero of the Alexander invariant. Moreover, we shall suppose in this section that the character $\chi_\alpha\colon \pi_1(M) \to \mathbb{R}$ is real valued.

We recall from [13] that the character χ_ρ of a representation $\rho \in R(M)$ maps $\gamma \in \pi_1(M)$ to $\chi_\rho(\gamma) = \operatorname{trace}^2(\rho(\gamma))$.

Lemma 10.1 *Let Γ be a finitely generated group. If $\chi \in X(\Gamma)$ is real valued, then there exists a representation $\rho \in R(\Gamma)$ with character $\chi_\rho = \chi$ and such that the image of ρ is contained either in $\mathrm{PSU}(2)$ or in $\mathrm{PGL}_2(\mathbb{R})$.*

Notice that $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$ has two components, the identity component is $\mathrm{PSL}_2(\mathbb{R})$, the other component consists of matrices with determinant -1 (which in $\mathrm{PSL}_2(\mathbb{C})$ are represented by matrices with entries in $\mathbb{C}$ with zero real part).

Looking at the action of $\mathrm{PSL}_2(\mathbb{C})$ on hyperbolic space $\mathbb{H}^3$ by orientation preserving isometries, the group $\mathrm{PGL}_2(\mathbb{R})$ is the stabilizer of a totally geodesic plane in $\mathbb{H}^3$, and the two components of the group are determined by whether their elements preserve or reverse the orientation of the plane. The group $\mathrm{PSU}(2)$ is the stabilizer of a point and it is connected (isomorphic to $\mathrm{SO}(3)$).

Proof Let F_n be a free group with a surjection $F_n \twoheadrightarrow \Gamma$. Following [13], we consider F_n^2, the subgroup of F_n generated by all squared elements γ^2, with $\gamma \in F_n$. This group is a normal subgroup of F_n and gives the following exact sequence

$$1 \to F_n^2 \to F_n \to H_1(F_n, C_2) \to 1,$$

where C_2 is the cyclic group with two elements.

Let $\rho \in R(M)$ be a representation with real valued character $\chi_\rho = \chi$. If ρ is reducible, we may assume that it is diagonal. Therefore we have two cases, either ρ is irreducible or ρ is diagonal.

The composition of ρ with $F_n \twoheadrightarrow \Gamma$ lifts to a representation $\rho'\colon F_n \to \mathrm{SL}_2(\mathbb{C})$. By Propositions 2.2 and 2.4 of [13] the restriction of ρ' to F_n^2 has real trace. Hence, we can apply the known results about these representations.

If the restriction $\rho'|_{F_n^2}$ is irreducible, then the image of $\rho'|_{F_n^2}$ is contained in either $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}(2)$ [20, Prop. III.1.1]. Looking at the action on $\mathbb{H}^3$, this means that $\rho'|_{F_n^2}$ preserves either a totally geodesic plane or a point in $\mathbb{H}^3$. In particular $\rho'|_{F_n^2}$ has either a unique invariant circle in $\partial\mathbb{H}^3 \cong P^1(\mathbb{C})$ or a unique fixed point in $\mathbb{H}^3$ (uniqueness follows from irreducibility). Since F_n^2 is normal in F_n, $\rho'(F_n)$ leaves invariant the same circle or the same point, respectively. This proves the lemma in this case.

If the restriction $\rho'|_{F_n^2}$ is trivial, then the image of ρ is abelian and finite, and therefore it fixes a point in $\mathbb{H}^3$. This means that, up to conjugation, $\rho(\Gamma) \subset \mathrm{PSU}(2)$.

If the restriction $\rho'|_{F_n^2}$ is reducible but non-trivial, then it fixes either a single point or two in $\partial\mathbb{H}^3$. When it fixes two points, using normality, these points must be fixed or permuted by every element of $\rho'(F_n)$, and therefore ρ is either diagonal or the Klein group and the lemma is shown in this case. Finally, if $\rho'|_{F_n^2}$ fixes a single point, it is also fixed by ρ, but this contradicts the fact that ρ is diagonal. □

As χ_α is real valued, $\alpha(\gamma) + \frac{1}{\alpha(\gamma)} \in \mathbb{R}$ $\forall \gamma \in \pi_1(M)$. Thus the image of α is either contained in the reals $\mathbb{R}^*$ or in the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Since we already know which cocycles of $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha)$ are integrable, we can easily describe the subsets of representations into these groups. We distinguish two cases:

Case 1 Assume that there is $\gamma \in \pi_1(M)$ such that $|\alpha(\gamma)| \neq 1$.

In this case the image of α is contained in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ but not in $\{\pm 1\}$. Hence, $\mathrm{Im}(\rho_\alpha)$ is contained in $\mathrm{PGL}_2(\mathbb{R})$. Since $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of $\mathfrak{sl}_2(\mathbb{R})$, we have the corresponding isomorphism of cohomology groups:

$$H^*(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = H^*(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\alpha) \otimes_{\mathbb{R}} \mathbb{C}.$$

In particular, we may assume that the cocycles $d_\pm$ and d_0 are valued in $\mathbb{R}$. Using the complexification of the second cohomology group, we realize that the computation of obstructions can be carried out in the real setting, thus we get:

Proposition 10.2 *Assume that the image of α is contained in $\mathbb{R}^*$ but not in $\{\pm 1\}$. Then the character χ_α is contained in precisely two real curves of characters in $\mathrm{PGL}_2(\mathbb{R})$, one of them with abelian representations and the other one with irreducible ones. In addition χ_α is a smooth point of both curves that intersect transversely at χ_α.*

When the image of α is contained in the positive reals, then the representations in Proposition 10.2 are contained in $\mathrm{PSL}_2(\mathbb{R})$, by connectedness. The case when the image of α is contained in $\{\pm 1\}$, is treated in next case.

Case 2 Assume that the image of α is contained in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Now $\mathrm{Im}(\rho_\alpha)$ is contained in the intersection $\mathrm{PSU}(2) \cap \mathrm{PSU}(1,1)$, where

$$\mathrm{PSU}(2) = \pm \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\}$$

and

$$\mathrm{PSU}(1,1) = \pm \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 - |b|^2 = 1 \right\}.$$

Geometrically, $\mathrm{PSU}(2)$ is the stabilizer of a point in hyperbolic space and $\mathrm{PSU}(1,1)$ is the connected component of the stabilizer of the unit circle in $\mathbb{C}$. Thus $\mathrm{PSU}(1,1)$ is the connected component of the stabilizer of a plane in hyperbolic space, and therefore it is conjugate to $\mathrm{PSL}_2(\mathbb{R})$.

In this case, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of both $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. Thus an argument similar to the previous case will apply. Note that the cocycles d_+ and d_- are not valuated in those Lie algebras. However, since $\overline{\rho_\alpha} = \rho_{1/\alpha}$ we can assume that d_- is the complex conjugate transpose to d_+, thus $d_+ - d_-$ is valuated in $\mathfrak{su}(2)$ and $d_+ + d_-$ is valuated in $\mathfrak{su}(1,1)$.

The tangent directions $d_+ - d_-$ and $d_+ + d_-$ are different in the slice of the variety of representations, but they project to the same direction (with opposite sense) in the variety of characters (see the description of the quotient in the previous section). Notice that this gives a curve of real valued characters. In addition, the set of real valued characters in a smooth complex curve can be at most one dimensional.

Proposition 10.3 *Assume that the image of α is contained in $S^1 \subset \mathbb{C}^*$. Then the character χ_α is contained in precisely two real curves of characters, one of them abelian (contained in S^1) and the other one with irreducible representations, contained in $\mathrm{PSU}(2)$ in one side and in $\mathrm{PSU}(1,1) \cong \mathrm{PSL}_2(\mathbb{R})$ on the other side. In addition, χ_α is a smooth point of both curves that intersect transversely at χ_α.*

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