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Non-singular graph-manifolds of dimension 4

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Abstract A compact 4-dimensional manifold is a non-singular graph-manifold if it can be obtained by the glueing T^2-bundles over compact surfaces (with boundary) of negative Euler characteristics. If none of glueing diffeomorphisms respect the bundle structures, the graph-structure is called reduced. We prove that any homotopy equivalence of closed oriented 4-manifolds with reduced nonsingular graph-structures is homotopic to a diffeomorphism preserving the structures.

AMS Classification 57M50, 57N35

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Introduction

In the paper [18], Waldhausen introduced a class of orientable 3-manifolds called graph-manifolds which can be obtained by glueing blocks that are Seifert manifolds along homeomorphisms of their boundary tori. These manifolds are not always sufficiently large, but for them one can introduce a notion of reduced graph-structure (i.e. a structure in which no family of neighboring blocks can be replaced by a single block), and then, with a few explicit exceptions, the existence of a homeomorphism between two 3-dimensional graph-manifolds with reduced graph-structures implies the existence of a homeomorphism respecting reduced graph-structures, which leads to a classification of such 3-manifolds.

3-dimensional graph-manifolds are important because they naturally arise as the boundary of resolved isolated complex singularities of polynomial maps $(\mathbb{C}^2,0) \to (\mathbb{C},0)$ [3], as the surfaces of constant energy of integrable hamiltonian systems with two degree of freedom [4], and as 3-manifolds admitting an injective F-structure (a generalization of an injective torus action) [13].

Our goal is to study a class of smooth four-dimensional manifolds generalizing three-dimensional graph-manifolds (with blocks without singular fibers) and having fundamental groups of exponential growth (hence, to which the high-dimensional techniques do not apply).

Definition 1) A (nonsingular) block is a T^2-bundle over a compact surface (with boundary) of negative Euler characteristic.

2) A (nonsingular) graph-manifold structure on a manifold is a decomposition as a union of blocks, glued by diffeomorphisms of the boundary.

Note that the boundary of a block has the structure of a T^2-bundle over a circle.

Definition A graph-manifold structure is *reduced* if none of the glueing maps are isotopic to fiber-preserving maps of T^2-bundle.

Any graph-structure gives rise to a reduced one by forming blocks glued by bundle maps into larger blocks.

Main theorem *Any homotopy equivalence of closed oriented 4-manifolds with reduced nonsingular graph-structures is homotopic to a diffeomorphism preserving the structures.*

The text is organized as follows. Section 1 contains the main technical result. A standard fact about two incompressible surfaces in an orientable irreducible 3-manifold is that one can move one of them by isotopy in such a way that the new intersection becomes π_1-injective. We provide a basis for doing a similar thing for π_1-injective maps of 3-manifolds into a 4-manifold W with $\pi_2(W) = \pi_3(W) = 0$ and for moving by (regular) homotopy. The gain is the same: the intersection of images of 3-manifolds becomes completely visible in $\pi_1(W)$. Section 2 contains a recapitulation of facts about T^2-bundles over aspherical spaces. Section 3 introduces four-dimensional non-singular graph-manifolds and proves the main theorem.

1 3-dimensional π_1-injective submanifolds in 4-manifolds

Manifolds here will be C^∞. Denote the tangent map of f by $df : TM \to TW$. An *immersion* is a smooth map $f : M \to W$ such that at every point of M the derivative df is an injective (linear) map. The set of immersions is open in the $C^\infty(M, W)$-topology ([12], Theorem 3.10). A *regular homotopy* is a homotopy through immersions. Any immersion $F : M \times I \to W$ such that $F|_{M\times\{0\}} = f_0$ and $F|_{M\times\{1\}} = f_1$ gives a regular homotopy between f_0 and f_1. The converse is false: two embedded not concentric circles in $\mathbb{R}^2$ are regularly homotopic, but their embeddings can not be extended into an immersion $S^1 \times I \to \mathbb{R}^2$.

Construction of regular homotopies One particular method to construct a regular homotopy between two immersions $f_0, f_1 : M \to W$ is to immerse into W not $M \times I$, but the image of an isotopy. Precisely, suppose one has a manifold $\mathcal{M}$ which contains the image of an isotopy $\mathcal{I}$ between two embeddings $i_0 : M \hookrightarrow \mathcal{M}$, $i_1 : M \hookrightarrow \mathcal{M}$, i.e. there is a map:

$$\mathcal{I} : M \times I \to \mathcal{M} \quad \text{such that} \quad \mathcal{I}|_{M\times 0} = i_0,\ \mathcal{I}|_{M\times 1} = i_1$$

and $\mathcal{I}|_{M\times\{t\}} \equiv \mathcal{I}_t$ is an embedding. Suppose also that there is an immersion $\mathcal{J} : \mathcal{M} \to W$ such that $\mathcal{J}\mathcal{I}|_{M\times 0} = \mathcal{J}i_0 = f_0$ and $\mathcal{J}\mathcal{I}|_{M\times 1} = \mathcal{J}i_1 = f_1$. Then the map $\mathcal{J}\mathcal{I} : M \times I \to W$ gives a smooth regular homotopy between the immersions f_0 and f_1 (see Figure 1).

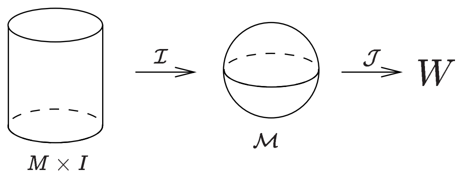

Figure 1: Construction of a regular homotopy

We will refer to the above construction by saying *"push $f_0(M)$ to $f_1(M)$ across $\mathcal{J}\mathcal{I}(M \times I)$"*.

Extension of immersions Let $H : M \times I \to W$ be a map such that $H|_{M\times\{0\}}$ is an immersion and $\mathsf{dim}\, M + 1 < \mathsf{dim}\, W$. The immersion $H|_{M\times\{0\}} : M \to W$ determines a bundle injection

$$A : T(M \times \{0\}) \to (H^*TW)|_{M\times\{0\}} = (H|_{M\times\{0\}})^*TW.$$

Suppose that this bundle injection can be extended to an injection $A_I : T(M \times I) \to H^*TW$. Then, by the Immersion Theorem ([16], [8]) there exists an immersion $\mathcal{H} : M \times I \to W$ which is homotopic to H, inducing the same tangent bundle injection:

$$\begin{tikzcd} T(M \times I) \arrow[r, "A_I"] \arrow[dr, "d\mathcal{H}"'] & H^*TW \arrow[d, equal] \\ & \mathcal{H}^*TW \end{tikzcd}$$

The immersion $\mathcal{H}$ can be chosen in such a way that $\mathcal{H}|_{M\times\{0\}} = H|_{M\times\{0\}}$.

Main technical result The following proposition is a generalization and a detailed proof of the Proposition 2.B.2 of [17], where very few details of the proof are given.

Proposition 1 *Let W be a compact smooth oriented 4-dimensional manifold with $\pi_2(W) = 0$ and M_1, M_2 be compact oriented 3-manifolds with $\pi_2(M_1) = \pi_2(M_2) = 0$. Let $f_1 : M_1 \to W$ be a π_1-injective map and $f_2 : M_2 \to W$ be a π_1-injective embedding.*

Then

- *f_1 is homotopic to a map $\widetilde{f_1}$ such that each connected component of $\widetilde{f_1}^{-1}(f_2(M_2))$ is π_1-injective in M_1;*
- *if $\pi_3(W) = 0$ and M_1 is irreducible, then all the S^2-components of $\widetilde{f_1}^{-1}(f_2(M_2))$ can be eliminated by homotopy of f_1;*
- *in addition, if f_1 is an immersion, then the homotopies can be made regular.*

Proof Move f_1 by a small (regular if f_1 immersion) homotopy to make it transverse to the submanifold $f_2(M_2)$. Then $F = f_1^{-1}(f_2(M_2))$ is a closed 2-dimensional surface which is embedded into M_1 (and immersed into M_2 if f_1 is immersion):

$$\begin{tikzcd} & M_1 \arrow[dr, "f_1"] & \\ F \arrow[ur] \arrow[dr] & & W \\ & M_2 \arrow[ur, "f_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(M_1) \arrow[dr, "f_{1*}"] & \\ \pi_1(F) \arrow[ur] \arrow[dr] & & \pi_1(W) \\ & \pi_1(M_2) \arrow[ur, "f_{2*}"'] & \end{tikzcd}$$

As $f_2(M_2)$ is closed and M_1 is compact, the surface F has only a finite number of connected components ([1], corollary 17.2(IV)).

Step 1 Construction of a map (resp. immersion) $\alpha : D^2 \times I \to W$ — the image of the future homotopy (resp. regular homotopy)

Suppose $F \subset M_1$ is not π_1-injective, so F is compressible in M_1, i.e. there exists an embedding $\beta : D^2 \to M_1$ such that its boundary loop $\beta(\partial D^2)$ is not contractible in F but $\beta(D^2) \bigcap F = \beta(\partial D^2)$.

Consider the map $f_1\beta : D^2 \to W$. As $F = f^{-1}(f_2(M_2))$ and $\beta(\partial D^2) \subset F$, hence $f_1\beta(\partial D^2) \subset f_2(M_2)$. Thus the map $f_1\beta$ is in fact

$$f_1\beta : (D^2, \partial D^2) \to (W, f_2(M_2)).$$

Note now that $\pi_2(W, f_2(M_2)) = 0$ because for the embedding f_2 the corresponding map induced in the fundamental groups $f_{2*} : \pi_1(M_2) \to \pi_1(W)$ is injective and the homotopy sequence of the pair $(W, f_2(M_2))$

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(W) \arrow[r] \arrow[d, equal] & \pi_2(W, f_2(M_2)) \arrow[r] & \pi_1(M_2) \arrow[r, "f_{2*}"] \arrow[d, equal] & \pi_1(W) \arrow[r] & \cdots \\
& 0 & & \pi_1(f_2(M_2)) & &
\end{tikzcd}$$

is exact. This implies that $f_1\beta$ is homotopic to a map $D^2 \to f_2(M_2)$ which can be written as $f_2\beta_2 : D^2 \to f_2(M_2)$, i.e. there exists a map $H : D^2 \times I \to W$ such that $H|_{D^2\times\{0\}} = f_1\beta$, $H|_{D^2\times\{1\}} = f_2\beta_2$. More, the homotopy can be made in such a way, that $\forall t \;\; H|_{\partial D^2\times\{t\}} = H|_{\partial D^2\times\{0\}} = f_1\beta|_{\partial D^2}$.

Step 1.1 The case of ordinary homotopy In the case when f_1 is just a *map* and we are interested in an *ordinary homotopy*, put $\alpha := H$. As $D^4 = D^3 \times I$ retracts on $D^3 \times \{\frac{1}{2}\} = (D^2 \times I) \times \frac{1}{2}$, we can say that α extends to a map $D^4 \to W^4$:

$$\begin{tikzcd}
D^3 \times \{\frac{1}{2}\} \arrow[dr, "\alpha"] \arrow[dd, shift right, "retraction"'] & \\
& W \\
D^4 = D^3 \times I \arrow[ur] \arrow[uu, dashed, shift right] &
\end{tikzcd}$$

and move to the next step.

Step 1.2 The case of regular homotopy In the case when f_1 is an *immersion* (with trivial normal bundle since everything is orientable) and we are looking for a *regular homotopy*, let us show that this map H can be changed to an immersion.

As $H|_{D^2\times\{0\}} = f_1\beta : D^2 \to W$ is an immersion, its derivative $dH|_{D^2\times\{0\}} = d(f_1\beta)$ is correctly defined and gives a bundle injection

$$T(D^2 \times \{0\}) \longrightarrow \big(H|_{D^2\times\{0\}}\big)^* TW = H^*TW|_{D^2\times\{0\}}.$$

Since $D^2 \times I$ retracts to $D^2 \times \{0\}$, this bundle injection extends to a bundle injection $T(D^2 \times I) \longrightarrow H^*TW$ that on $D^2 \times \{1\}$ restricts to a subbundle of $(H|_{D^2\times I})^*T_{M_2}$. Applying the Immersion Theorem gives an immersion $\alpha: D^2\times I\to W$ such that $\alpha|_{D^2\times\{0\}} = H|_{D^2\times\{0\}} = f_1\beta$ and $\alpha(D^2\times\{1\})\subset f_2(M_2)$. Note that the immersion α is flat ($D^2\times I$ and W being oriented): there exists a map

$$\alpha_\varepsilon : D^4 = (D^2\times I)\times I\to W \quad \text{such that} \quad \alpha_\varepsilon|_{(D^2\times I)\times\{\frac{1}{2}\}}\equiv\alpha.$$

As β is flat (being an embedding), we have in M_1 an embedded 3-disk $\beta_\varepsilon(D^2\times\{0\}\times I)$ which is the normal bundle of $\beta(D^2)$. As $\alpha_\varepsilon|_{(D^2\times I)\times\{\frac{1}{2}\}}\equiv\alpha$ and $\alpha|_{D^2\times\{0\}\times\frac{1}{2}}$ we can write $\alpha_\varepsilon|_{(D^2\times\{0\})\times I} = f_1\beta_\varepsilon$.

We will use the notation $\partial D^3 = S^2_+\cup S^2_-$ with $S^2_+ = D^2\times\{0\}$ and $S^2_- = (D^2\times\{1\})\cup(\partial D^2\times I)$.

Step 2 Homotopy description

As the result of the above construction we have:

- an embedding of 2-disk $\beta: D^2\to M_1$ such that $\beta(D^2)\cap F = \beta(\partial D^2)$,
- a map (respectively, a flat immersion) of 3-disk $\alpha: D^3 = D^2\times I\to W$ such that $\alpha|_{D^2\times\{0\}} = f_1\beta$, $\alpha|_{D^2\times\{1\}}\subset f_2(M_2)$ and $\alpha(\partial(D^2\times\{0\}))) = \alpha(\partial(D^2\times\{1\})))$.

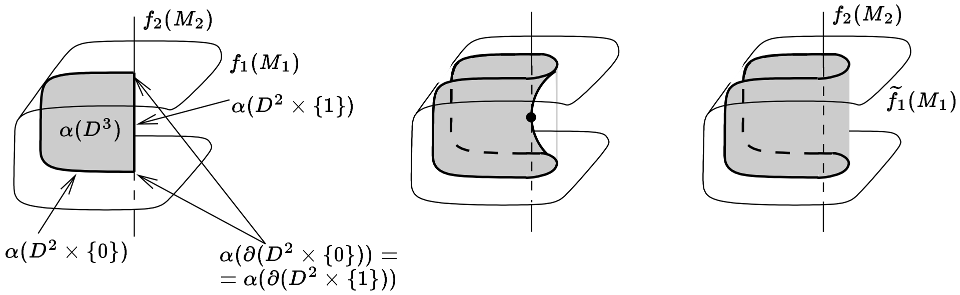

Figure 2: Pushing $f_1|_{\beta_\varepsilon(D^2\times I)}$ across $\alpha_\varepsilon(D^3\times I)$

We will now change the map f_1 firstly by pushing $f_1|_{\beta_\varepsilon(D^2\times I)}$ across $\alpha_\varepsilon(D^3\times I)$ to a map (respectively, an immersion) into W whose image lies in $f_2(M_2)$; secondly we compose it with the pushing along the normal bundle of $f_2(M_2)$ in W in such a way that:

- in a small neighborhood $U\supset\beta_\varepsilon(D^2\times I)$ the map (respectively, the immersion) f_1 is changed by homotopy (resp. regular homotopy) to a map (respectively, an immersion) $\widetilde{f_1}$ such that $\widetilde{f_1}^{-1}(f_2(M_2)) := F'$ is F surgered on the disk $\beta(D^2)$
- and the map f_1 does not change on the complement of U in M_1.

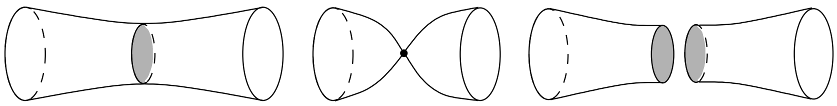

Figure 3: Trace that the pushing of f_1 makes on the intersection of images of f_1 and f_2

The homotopy (resp. regular homotopy) works as follows.

Step 3 Homotopy on a disk

Let us decompose ∂D^4 as union of two 3-discs S^3_+ and S^3_- with $S^3_+ \cap S^3_- = S^2$. Let $\mathcal{I}$ be an isotopy $D^3 \times I \to D^4$ that sends 3-disk S^3_+ on 3-disk S^3_- as shown on Figure 4.

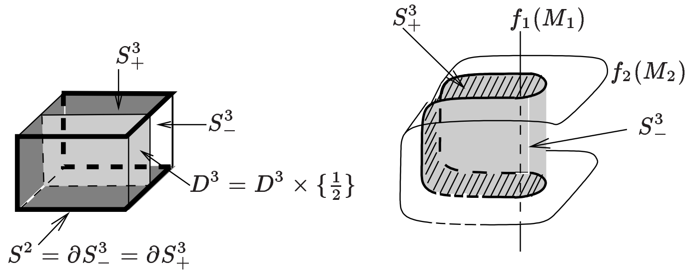

Figure 4: Homotopy sending S^3_+ on S^3_-

Take now the composition $\alpha_\varepsilon \mathcal{I} : D^3 \times I \to D^4$ (α_ε being the flat extension of α): it provides a homotopy (respectively, a regular homotopy) sending $\alpha_\varepsilon|_{(D^2\times\{0\})\times I} \equiv f_1\beta_\varepsilon(D^3)$ into $f_2(M_2)$: we "push $f_1|_{\beta(D^2)}$ across $\alpha_\varepsilon\mathcal{I}(D^4)$". Take then the composition of $\alpha_\varepsilon\mathcal{I}$ with the pushing out along the normal bundle of $f_2(M_2)$ in W^4 (which is trivial, because $f_2(M_2)$ and W^4 are orientable). At this moment the map f_1 will be changed not only on the 3-disc $\beta_\varepsilon(D^2 \times I)$, but on its small neighborhood $U \subset M_1$.

Step 4 Change α to make $\alpha(\mathring{D}^3)$ miss $f_2(M_2)$

Motivation If we want the homotopy described on the previous step to create no new intersections of $f_1(M_1)$ and $f_2(M_2)$, we have to make the image of the interior of the disk $\alpha(\mathring{D}^3)$ disjoint from $f_2(M_2)$. We have $\alpha^{-1}(f_2(M_2)) = S^2_- \cup G$, where G are some closed surfaces.

Denote by Δ the union of G and all components of $D^3 \setminus G$ that do not contain ∂D^3. Note some components of G may be in the interior of Δ. Let $\hat{G} = \partial\Delta$. Since Δ is an open subspace of a manifold, it is a manifold. Let us show that

Δ is aspherical. If we show that $\pi_2(\Delta) = 0$, it will give us the asphericity: take the universal covering $\widetilde{\Delta}$, we have $H_i(\widetilde{\Delta}) = 0,\ i \geq 3$ because it's an open 3-manifold; then, by Whitehead's theorem, $\pi_i(\widetilde{\Delta}) = 0,\ i \geq 3$, and we conclude that $\pi_i(\Delta) = \pi_i(\widetilde{\Delta}) = 0$. Suppose that $\pi_2(\Delta) \neq 0$. Then, by the Sphere Theorem, there exists an embedded $S^2 \hookrightarrow \Delta$ representing a non-trivial element in $\pi_2(\Delta)$. This S^2 bounds a ball in D^3. This ball must be contained in Δ, therefore $\pi_2(\Delta) = 0$.

Note that Δ can be rather complex, for example, be a knot complement.

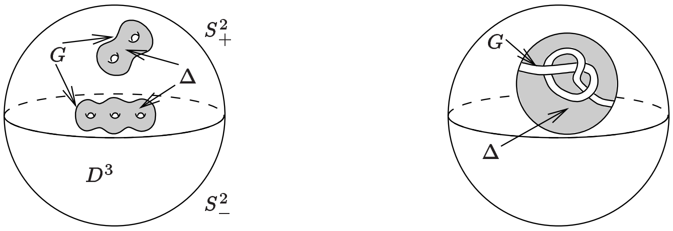

Figure 5: Pre-image of $f_2(M_2)$ by α: the cases of Δ being handlebodies and a knot complement

Extension of $\alpha|_{\hat{G}}$ on Δ and change α Now, let's show that we can always extend the map $\alpha|_{\hat{G}} : \hat{G} \to f_2(M_2)$ to a map $\alpha' : \overline{\Delta} \to f_2(M_2)$. As $\hat{G}$ and $\overline{\Delta}$ are aspherical, it will be enough to extend this map on the fundamental group of each component of Δ. As f_2 is a π_1-injective embedding, we have $\pi_1(M_2) \cong \pi_1(f_2(M_2))$.

$$\begin{tikzcd}
\hat{G} \arrow[r, "\subseteq"] \arrow[d, "\alpha|_{\hat{G}}"'] & \overline{\Delta} \arrow[r, "\subseteq"] \arrow[dl, dashed, "\alpha'"'] \arrow[d, "\alpha|_{\overline{\Delta}}"] & D^3 \arrow[dl, "\alpha"] \\
f_2(M_2) \arrow[r, "\subseteq"'] & W &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(\hat{G}) \arrow[r, "a"] \arrow[d, "b"'] & \pi_1(\overline{\Delta}) \arrow[r, "d_1"] \arrow[dl, dashed, "\alpha'_*"'] \arrow[d, "d"] & 1 \arrow[dl, "d_2"] \\
1 \arrow[r] & \pi_1(M_2) \arrow[r, "c"'] & \pi_1(W) &
\end{tikzcd}$$

We have $cb = d_2 d_1 a$, so that $Im\ cb = 1$. As c is a monomorphism, it follows that $Im\ b = 1$, thus, $b \equiv 1$. We can define the homomorphism $\alpha'_* : \pi_1(\overline{\Delta}) \to \pi_1(M_2) \cong \pi_1(f_2(M_2))$ as being the constant 1, too.

So we can define a new map $\widetilde{\alpha} : D^3 \longrightarrow W$ as follows:

$$\widetilde{\alpha} = \begin{cases} \alpha & \text{on } D^3 \backslash \Delta \\ \alpha' & \text{on } \Delta \end{cases}$$

Now the whole image $\widetilde{\alpha}(\Delta)$ lies in $f_2(M_2)$, hence we can push $\widetilde{\alpha}(D^3)$ off $f_2(M_2)$ across $\alpha'(\Delta) \subset f_2(M_2)$ using the normal bundle of $f_2(M_2)$ in W. We have a new map (which for simplicity we still note by α) $\alpha : D^3 \longrightarrow W$ such that the image of the interiour $\alpha(\mathring{D}^3)$ is disjoint from $f_2(M_2)$.

Step 5 The number of disks in M_1, on which the homotopy of f_1 must be done, is finite

Suppose we made the homotopy of the map f_1 on one disk. Suppose that the obtained surface F (whose topological type has changed) is still not π_1-injective in M_1. After the surgery the new surface is still oriented, hence again there is an embedded compressing disk in M_1, on which again one can do the surgery by homotopy of f_1 etc. After each surgery the topological type of the surface F changes as follows: either the genus of one component of F decrements, or one component splits into two components, the sum of genera of which is not greater than the genus of the original component. As F is compact, the genera of all its components are finite, and as it was pointed out before Step 1, the number of components of F is finite, hence, *the process will terminate after a finite number of steps.* This is the advantage that we get from replacing the homotopic information ($Ker\, g_* \neq 0$) by the geometric information (there exists an embedded loop which is trivialized by an embedded disk): the infinite kernel is killed in a finite number of steps.

As the result we obtain a surface that is π_1-injective in M_1, but which could contain spheres among its components.

Step 6 Elimination of S^2-components provided $\pi_3(W) = 0$ and M_1 is irreducible

If the obtained surface F contains S^2-components, then, as M_1 is irreducible and $\pi_2(M_2) = 0$, every such S^2-component bounds an embedded 3-disk in M_1 and a homotopy 3-disk in M_2: there exist an embedding $\gamma_1 : D^3 \to M_1$ and a map $\gamma_2 : D^3 \to M_2$ such that $f_1\gamma_1(\partial D^3) = f_2\gamma_2(\partial D^3)$. Denote $f_1\gamma_1(D^3) = S^3_+$ and $f_2\gamma_2(D^3) = S^3_-$.

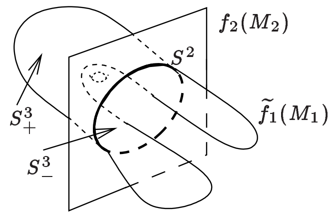

Figure 6: Elimination of S^2-components

If $\pi_3(W) = 0$, then the map of 3-sphere, whose image is $S^3_+ \bigcup S^3_-$, bounds a homotopy 4-disk: there exists $\lambda : D^3 \times I \to W$ such that $\lambda(D^2 \times \{0\}) = S^3_+$, $\lambda(D^2 \times \{1\} = S^3_-$ and $\lambda(\partial D^3 \times I) = S^3_+ \cap S^3_-$, and which, in addition, can be made an immersion on each $D^3 \times \{t\}$. In order to eliminate a chosen S^2-component of F, push $\widetilde{f}_1(\gamma_1(D^3))$ along $\lambda(D^3 \times I) \subset W$ to make S^3_+ coincide with S^3_- (see Figure 6), then, push it off $f_2(M_2)$ along the normal bundle of $f_2(M_2)$ in W, then glue with the map on $M_1 \setminus S^3_+$. □

2 Torus bundles

A *torus bundle* here will be a fiber bundle $f : M \to B$ with fibers diffeomorphic to T^2, smooth if the base is a smooth manifold. The *monodromy* is the action of $\pi_1(B)$ on H_1 of the fiber:

$$\pi_1(B, b) \to Aut\,(H_1(f^{-1}(b)); \mathbb{Z}).$$

Choosing an identification of the fiber with T^2 (equivalently, a basis for $H_1(T^2)$) identifies the automorphism group as $GL(2, \mathbb{Z})$. The classifying map for a torus bundle is $B \to B_{Diff(T^2)}$. There is a 2-stage Postnikov decomposition

$$K(\mathbb{Z} \oplus \mathbb{Z}, 2) \to B_{Diff(T^2)} \to B_{GL(2,\mathbb{Z})}$$

([6], ch.4, p.51). If B is a surface with non-empty boundary, this implies that a bundle is determined up to isomorphism by the conjugacy class of its monodromy. However the nontrivial π_2 in the classifying space shows bundles on surfaces are not defined *rel boundary* by the monodromy. A *fiber map* is a pair of maps, one on the total spaces and one on the bases so that the diagram

$$\begin{tikzcd} E_1 \arrow[r, dashed] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r, dashed] & B_2 \end{tikzcd}$$

commutes. *Bundle map* is a fiber map so that on coordinate charts it is given by function into the structure group. As the inclusion $Diff(T^2) \hookrightarrow G(T^2)$ (the monoid of self-homotopy equivalences of torus) is homotopy equivalence [5], the existence of a bundle map between T^2-bundles is equivalent to the existence of a fiber map inducing homotopy equivalence on the fibers.

A *fiber covering map* of bundles here will be a fiber map, which is finite covering on fibers. The degree of the covering on different fibers is clearly the same. If B is aspherical, there exists a fiber covering map of T^2-bundles with monodromies φ_1, φ_2 if and only if there exists a monomorphism $\alpha : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ with $\alpha\,\varphi_1(\gamma) = \varphi_2(\gamma)\,\alpha$ for all $\gamma \in \pi_1(B)$.

Lemma 1 *Let $f : E_1 \to E_2$ be π_1-injective map of T^2-bundles over aspherical spaces. Then f is homotopic to a fiber covering map if and only if the induced map on π_1 sends the fiber subgroup of $\pi_1(E_1)$ into the fiber subgroup of $\pi_1(E_2)$.*

Proof The π_1 condition is well-defined (independent of basepoints) because the fiber defines a normal subgroup of π_1. A fiber covering map clearly verifies the condition.

Now suppose f sends the subgroup of fiber of E_1 into the subgroup of fiber of E_2. Then it induces an homomorphism on quotient groups. This gives a π_1-injective map of the base spaces and a commutative up to homotopy diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d] & E_2 \arrow[d] \\ G_1 \arrow[r, "g"] & G_2 \end{tikzcd}$$

Let E_2^* be the pullback of E_2 to G_1. Then f factors as $f^* : E_1 \to E_2^*$ and a map $E_2^* \to E_2$ which is isomorphic on fibers, so it is sufficient to show that f^* is homotopic to a fiber covering map, which follows from the commutative diagram of short exact sequences. □

Proposition 2 *Suppose E is homotopy equivalent to a T^2-fibration over a graph G. Then this structure is unique up to homotopy unless $G \cong S^1$ and the monodromy is conjugate to $\left(\begin{smallmatrix} 1 & n \\ 0 & m \end{smallmatrix}\right)$.*

Proof According to the lemma 1 it is sufficient to show that there is a unique normal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and with free quotient except when $G \cong S^1$ and the monodromy has the specified form.

Case 0 G contractible, so $E \cong T^2$, and $\pi_1(E) \cong \mathbb{Z} \oplus \mathbb{Z}$, then $\pi_1(E)$ is the only such subgroup.

Case 1 $G \cong S^1$. Then the fundamental group of the fiber is the unique normal $(\mathbb{Z} \oplus \mathbb{Z})$-subgroup of $\pi_1(E)$ unless the monodromy has an eigenvector with eigenvalue 1. This shows the monodromy is conjugate to $\left(\begin{smallmatrix} 1 & n \\ 0 & m \end{smallmatrix}\right)$. In this case either $\pi_1(E) \cong \mathbb{Z}^3$ or the commutator subgroup is $\mathbb{Z}$, with quotient $\mathbb{Z}^2$. This means E is homotopic to an S^1-bundle over T^2. Taking non-homotopic fibering $T^2 \to S^1$ induces non-homotopic T^2-bundle structure on E.

Case 2 $\pi_1(G)$ is non-abelian free group. Consider the exact homotopy sequences of two T^2-bundles on E. In the sequence of first bundle

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\alpha"] & \pi_1(E) \arrow[r, "\beta"] & \pi_1(G) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[ur, "j"'] & & &
\end{tikzcd}$$

we have $Im\, j \subset Im\, \alpha = Ker\, \beta$, because $\beta(Im\, j) \subset \pi_1(G)$ is an abelian normal subgroup, hence trivial (theorem 2.10 of [10]). Similarly, from the sequence of the second bundle we have $Im\, \alpha \subset Im\, j$, hence these subgroups coincide in $\pi_1(E)$. □

Corollary 1 *If a 4-manifold is a T^2-bundle over a surface with boundary different from annulus and Möbius band, then this structure is unique up to bundle homotopy.*

Proof A surface with boundary has the homotopy type of a graph. □

Proposition 3 *Let $f : E_1 \to E_2$ be a π_1-injective map between T^2-bundles over aspherical surfaces. Then f is homotopic to a fiber covering map unless E_1 either comes from S^1-bundle over 3-dimensional S^1-bundle over aspherical surface (whose π_1 contains a normal $\mathbb{Z}$) or is T^4 or $T^2 \times K^2$.*

Proof Denote the projection $p : E_2 \to B$ and $G := Im\,(p_* f_*)$. Let $B_{2,G}$ be a covering of B_2 corresponding to G, $p_G : E_{2,G} \to B_{2,G}$ be the pullback of $p : E_2 \to B_2$ by $B_{2,G} \to B_2$, and $\hat{f} : E_1 \to E_{2,G}$ be a map covering f.

$$\begin{tikzcd}
 & E_1 \arrow[dl, "\hat{f}"'] \arrow[d, "f"] \arrow[ddl, dashed] \\
E_{2,G} \arrow[r] \arrow[d, "p_G"'] & E_2 \arrow[d, "p"] \\
B_{2,G} \arrow[r] & B_2
\end{tikzcd}$$

Denote the kernel of $\pi_1(E_1) \to \pi_1(B_1)$ by K_1 and the kernel of $(p_G\hat{f})_* : \pi_1(E_1) \to \pi_1(B_{2,G})$ by K_2.

$$\begin{tikzcd}
K_1 \arrow[r] & \pi_1(E_1) \arrow[r] \arrow[dr, "(p_G\hat{f})_*"'] & \pi_1(B_1) \\
K_2 \arrow[ur] & & \pi_1(B_{2,G})
\end{tikzcd}$$

Both kernels are isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, and lemma 1 implies that f is homotopic to a fiber covering map if and only if $K_1 = K_2$.

Case 1 $K_1 \cap K_2 \equiv \mathbb{Z}\oplus\mathbb{Z}$.

As π_1 of aspherical surface has no torsion, it means $K_1 = K_2$. It gives a map $B_1 \to B_{2,G}$ such that up to homotopy all the squares of the diagram commute

$$\begin{tikzcd}
\pi_1(E_1) \arrow[r, "\hat{f}_*"] \arrow[d] \arrow[dr, "(p_G\hat{f})_*"] \arrow[rr, bend left, "f_*"] & \pi_1(E_{2,G}) \arrow[r] \arrow[d, "(p_G)_*"] & \pi_1(E_2) \arrow[d, "p_*"] \\
\pi_1(B_1) \arrow[r, dashed] & \pi_1(B_{2,G}) \arrow[r] & \pi_1(B_2)
\end{tikzcd}$$

and hence a map $B_1 \to B_2$ which with f gives a fiber covering map.

Case 2 $K_1 \cap K_2 \equiv \mathbb{Z}$.

In this case $\pi_1(B_1)$ contains $\mathbb{Z}$ as normal subgroup, hence B_1 is T^2, Klein bottle K^2, $S^1 \times I$ or Möbius band. As $(K_1 \cap K_2) \lhd \pi_1(E_1)$ and the monodromy acts by conjugation, in this case the monodromy of $E_1 \to B_1$ preserves a curve in the fiber. This curve is embeded because there is no torsion in $\pi_1(B_1)$ (and hence $K_1/(K_1 \cap K_2) \cong \mathbb{Z}$). So that, E_1 is a S^1-bundle over 3-dimensional manifold W

$$K_1 \cap K_2 \longrightarrow \pi_1(E_1) \longrightarrow \pi_1(W)$$

and W itself is S^1-bundle over B_1. Denote $K_3 \lhd \pi_1(W)$ the corresponding fiber subgroup. The subgroup $K_2/(K_1 \cap K_2) \cong \mathbb{Z}$ is normal in $\pi_1(W)$ with quotient isomorphic to $\pi_1(B)$. The subgroups K_3 and $K_2/(K_1 \cap K_2)$ coinside if and only if f is fiber covering.

If $K_3 \neq K_2/(K_1 \cap K_2)$, refiber W by S^1 with fiber subgroup $K_2/(K_1 \cap K_2)$. Together with $E_1 \to W$ it will give another T^2-fibration of E_1, in which f will be fiber-covering.

Case 3 $K_1 \cap K_2 \equiv 1$.

In this case $\pi_1(B_1)$ contains a normal $\mathbb{Z}\oplus\mathbb{Z}$, hence B_1 is T^2 or a Klein bottle, and $\pi_1(E_1)$ injects into $\pi_1(B_1) \times \pi_1(B_{2,G})$. The monodromy of $E_1 \to B_1$ is trivial, because in the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
K_1 \arrow[r] \arrow[d] & \pi_1(E_1) \arrow[r] \arrow[d] & \pi_1(B_1) \arrow[d, equal] \\
\pi_1(B_{2,G}) \arrow[r] & \pi_1(B_1) \times \pi_1(B_{2,G}) \arrow[r] & \pi_1(B_1)
\end{tikzcd}$$

the monodromy of the second line is trivial, the morphisms between the lines are injective and the diagram commutes. There are obvious different T^2-fibrations of T^4. For $T^2 \times K^2$, diferent T^2 fibrations can be seen by taking in

$$\begin{tikzcd}
T^2 \times K^2 \arrow[d] & \\
K^2 \times K^2 \arrow[r] \arrow[d] & K^2 \\
K^2 &
\end{tikzcd}$$

the projections of $K^2 \times K^2$ onto different factors. □

Corollary 2 *Any π_1-injective map $f : (E_1, \partial E_1) \to (E_2, \partial E_2)$ of torus bundles over surfaces with non-empty π_1-injective boundary is homotopic rel boundary to a fiber-covering map.*

Proof The condition on base implies that we are in the Case 1 of Proposition 3. Hence f is homotopic to a fiber-covering map. By the lemma 1 it means that induced map on π_1's send the fiber subgroup of E_1 into the fiber subgroup of E_2. From where $f|_{\partial E_1}$ is homotopic to a fiber-covering map too, because ∂E_i is a subbundle of E_i.

Denote the corresponding homotopies by $\{f_t\} : E_1 \times I \to E_2$ and $\{g_t\} : \partial E_1 \times I \to \partial E_2$. For each t, the step maps f_t and g_t are both homotopic to $f|_{\partial E_1 \times \{0\}}$. Presenting E_1 as $(\partial E_1 \times [0;t]) \cup (\partial E_1 \times [t;1]) \cup E_1$, one can define a new homotopy $\{H_t\} : E_1 \times I \to E_2$ of f as follows. On $\partial E_1 \times [0;t]$, the step map H_t will be the (reparametrized) homotopy between g_t and $f|_{\partial E_1 \times \{0\}}$ followed by the reparametrized homotopy between $f|_{\partial E_1 \times \{0\}}$ and f_t. On $(\partial E_1 \times [t;1]) \cup E_1 = E_1$, the map H_t will be f_t.

The end map $H_1 : (\partial E_1 \times [0;1]) \cup E_1 \to E_2$ is fiber covering on E_1 and on $(\partial E_1 \times \{0\})$. Denote by γ_0 and γ_1 loops in B_2, subbundles over which are covered by $H_1(\partial E_1 \times \{0\})$ and $H_1(\partial E_1 \times \{1\})$. As γ_0 and γ_1 are homotopic in B_2, the homotopy between $H_1|_{\partial E_1\times\{0\}}$ and $H_1|_{\partial E_1\times\{1\}}$ can made fiber covering, and this new homotopy is homotopy to the old one. Then $\{H_t\}$ followed by the new homotopy gives a homotopy *relatively to the boundary* between f and a fiber covering map. □

Corollary 3 *Any homotopy equivalence rel boundary of torus bundles over surfaces with non-empty π_1-injective boundary is homotopic rel boundary to a diffeomorphism.*

Proof According to Corollary 2, both maps of the homotopy equivalence can be made fiber covering maps by homotopy rel boundary. As both of them are of degree ± 1, they are isomorphisms on the fibers. This and the commutative diagram of fundamental groups imply that the monodromies are conjugate, hence the bundles are isomorphic. As the obtained diffeomorphism of aspherical total spaces induces the same preserving peripheral structure isomorphism of π_1's as the initial map, they are homotopic rel boundary. □

Recall that a subgroup A is said to be square root closed in G if for every element $g \in G$ such that $g^2 \in A$ one has $g \in A$, too.

Proposition 4 *Let B surface with boundary, S is a component of ∂B, $E \to B$ a T^2-bundle over B. Then the image $\pi_1(E|_S) \to \pi_1(E)$ is square root closed if and only if B is not a Möbius band.*

Proof If $B = D^2$, the homomorphism $\pi_1(E|_S) \to \pi_1(E)$ is onto and the statement is obvious.

If B is different from the disc and Möbius band, $\pi_1(E|_S) \to \pi_1(E)$ is injective. As the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_1(E|_S) & \longrightarrow & \pi_1(E) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1(B) \end{array}$$

commutes, $\pi_1(E|_S) \subset \pi_1(E)$ is square root closed if and only if $\pi_1(S) \subset \pi_1(B)$ does. Suppose $\pi_1(S) \subset \pi_1(B)$ is not square root closed. Choose $a \notin \pi_1(S)$ with $a^2 \in \pi_1(S)$. As $\pi_1(B)$ is free, and square roots are unique in free groups, so a^2 must be an odd power of the generator of $\pi_1(S)$.

Next observe that there is a Möbius band $(M, \partial M) \to (B, S)$ with $\pi_1(M) \to a$, $\pi_1(\partial M) \to a^2$. Attach disks to M and B to get a map of $\mathbb{R}P^2 = M \cup D^2 \to B \cup_S D^2$. This induces

$$\mathbb{Z}_2 \cong H_1(\mathbb{R}P^2; \mathbb{Z}_2) \to H_2(B \cup_S D^2; \mathbb{Z}_2) \to H_2(D^2, S; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

The composition is the same as boundary map $H_1(\partial M; \mathbb{Z}_2) \to H_1(S; \mathbb{Z}_2)$ which is an isomorphism because a is an odd power of the generator. Therefore we conclude $H_2(B \cup_S D^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\mathbb{R}P^2 \to B \cup_S D^2$ is an isomorphism on H_2 with $\mathbb{Z}_2$ coefficients. It follows that $B \cup_S D^2$ is closed and $\pi_1(\mathbb{R}P^2) \to \pi_1(B \cup_S D^2)$ has finite odd index. But $\mathbb{R}P^2$ is the only closed surface with finite π_1, so $B \cup_S D^2 \cong \mathbb{R}P^2$, and B is a Möbius band. □

3 Graph-manifolds

We use the term *non-singular block* for the total space of a T^2-bundle over a compact surface (with non-empty boundary) different from a 2-disc, an annulus and a Möbius band (hence, a surface with free non-abelian fundamental group). Boundary components of blocks are T^2-bundles over S^1 and are π_1-injective in blocks.

Definition 1 A 4-dimensional closed connected compact oriented manifold is a *non-singular graph-manifold* if it can be obtained by gluing several blocks by diffeomorphisms of their boundaries.

Fot simplicity we will say "blocks" instead of "non-singular blocks" and "graph-manifolds" instead of "non-singular graph-manifolds".

Example The simplest examples of 4-dimensional graph-manifolds are T^2-bundles over closed hyperbolic surfaces (all the glueing diffeomorphisms being trivial). A more interesting examples can be constructed by taking oriented S^1-bundles over some 3-dimensional graph-manifolds: for instance, such that all their blocks have π_1-injective boundary components (for exemple, lens spaces are not good as bases) and all the blocks being locally trivial S^1-bundles (i.e. no exceptional Seifert fibers).

Any decomposition as a union of blocks will be called a *graph-structure*. Topologically, a graph-structure is determined by a system of embedded π_1-injective T^2-bundles over circles, called *decomposing manifolds*. A graph-structure is *reduced* if all the glueing diffeomorphisms are not fiber-preserving, or, equivalently, if the induced isomorphism of π_1's does not preserve the fiber subgroup.

As the fiber subgroup is unique in π_1 of the block, the notion of reduced structure is well defined.

Immediate properties Graph-manifolds are aspherical: since inclusions of boundary components into blocks are π_1-injective, they are graphs of aspherical spaces, and the universal covering of a graph of aspherical spaces is contractible ([14], prop. 3.6 p.156). The definition also implies that the Euler characteristic of graph-manifolds is 0, because the Euler characteristic of block is 0, and gluings are made along 3-manifolds. Finally, graph-manifolds can be smoothed: the blocks are smooth, and gluings are made by diffeomorphisms of 3-manifolds. More, a given graph-structure determines the smoothing in a unique way, because the smooth structure on a 3-manifold is unique and homotopic diffeomorphisms of torus bundles over S^1 are isotopic [19].

Proposition 5 *The signature of a closed oriented graph-manifold W^4 with reduced graph-structure all the blocks of which have orientable bases is $\sigma(W^4) = 0$.*

Proof The blocks of an orientable graph manifold are orientable, and the signature of the graph-manifold induces the orientations on the blocks. One can assume that all the orientations of blocks are such that glueing diffeomorphisms reverse the induced orientation of boundaries. The orientation on a block comes from the orientation of its fiber plus the orientation of its base. Hence we can speak about presentation of boundaries of blocks as some $M_{\varphi_i} = (T^2 \times I)/(x;0) \sim (\varphi_i(x);1)$, $\varphi_i \in SL(2,\mathbb{Z})$.

Determine first the signatures of blocks. In a reduced graph-structure, the boundaries of all the blocks have many non-isotopic T^2-bundle structures. Hence the monodromies of all decomposing manifolds must be conjugate to $\left(\begin{smallmatrix} 1 & n_i \\ 0 & 1 \end{smallmatrix}\right)$ (Proposition 2). Thus by Meyer's Theorem [11], the signature of a block M^4 of such a manifold is

$$\sigma(M^4) = \frac{1}{3}\sum_{i=1}^{k} n_i.$$

Now apply the Novikov's additivity to get $\sigma(W^4)$ by adding the signatures of the blocks. When one glues the blocks, M_φ can be glued either with $M_{a\varphi a^{-1}}$ or with $M_{a\varphi^{-1}a^{-1}}$ for some $a \in SL(2,\mathbb{Z})$. As Meyer's characteristic function $\Psi : SL(2,\mathbb{Z}) \to \mathbb{Z}$ is invariant under conjugation in $SL(2,\mathbb{Z})$, for the signature calculation one can assume that M_φ can be glued either with M_φ or with $M_{\varphi^{-1}}$. There exists an orientation reversing diffeomorphism $M_\varphi \to M_\varphi$ if and only if the Euler number of S^1-bundle of M_φ with a triangular φ is 0 [15], i.e. if $\varphi = \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$. But the boundary component with such monodromy gives contribution 0 into the signature. There always exists an orientation reversing diffeomorphisms $M_\varphi \to M_{\varphi^{-1}}$, but $\Psi\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) + \Psi\left(\begin{smallmatrix}1 & -n\\ 0 & 1\end{smallmatrix}\right) = n + (-n) = 0$ and $\Psi\left(\begin{smallmatrix}-1 & n\\ 0 & -1\end{smallmatrix}\right) + \Psi\left(\begin{smallmatrix}-1 & -n\\ 0 & -1\end{smallmatrix}\right) = n + (-n) = 0$, hence every such pair also gives the contribution 0 in the signature. Hence after adding the signatures of all the blocks we will obtain $\sigma(W^4) = 0$. □

Lemma 2 *Let M be a T^2-bundle over surface with non-empty π_1-injective boundary. Then any non-fiber-covering π_1-injective map*

$$f : (T^2 \times I; \partial(T^2 \times I)) \to (M; \partial M)$$

sending $\partial(T^2 \times I)$ into the same boundary component is homotopic rel boundary to a map into ∂M.

Proof Denote the component of ∂M containing $f(T^2 \times \partial I)$ by M_φ. Fix a point $t \in T^2$. As $T^2 \times I$ is aspherical, we have to show that $f|_{t\times I} : (I, \partial I) \to (M, M_\varphi)$ is homotopic to a map into M_φ.

Denote the projections $p : M \to B^2$, $p_b : M_\varphi \to S^1$ and natural inclusions $l : M_\varphi \to M$, $l_p : p(M_\varphi) \to B^2$. Take a path in M_φ that joins $f(t \times \{0\})$ and $f(t \times \{1\})$; the union of this path with $f(t \times I)$ is an element of $\pi_1(M; b)$ where $b = f(t \times \{0\})$.

In the diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "f_*"] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, equal] & \pi_1(M_\varphi) \arrow[r, "p_{b*}"] \arrow[d, "l_*"] & \mathbb{Z} \arrow[r] \arrow[d, "l_{p*}"] & 0 \\
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \pi_1(M) \arrow[r, "p_*"] & \pi_1(B^2) \arrow[r] & 0
\end{tikzcd}$$

denote $G = Im\, f_*$; $p_{b*}(G)$ is non-trivial (by assumption). We have $\gamma l_* f_*(G)\gamma^{-1} \subset Im\, l_*$, hence

$$p_*(\gamma)\, l_{p*} p_{b*}(G)\, p_*(\gamma^{-1}) \subset Im\,(l_{p*} p_{b*}) \cong \mathbb{Z}.$$

As $l_{p*} p_{b*}(G)$ is abelian and non-trivial, and $Im\,(l_{p*} p_{b*})$ is generated by primitive element of $\pi_1(B^2)$, we conclude that $p_*(\gamma) \in Im\,(l_{p*})$ [9], hence $\gamma \in Im\, l_*$. □

Proposition 6 *Let $W = \cup W_i$ and $W' = \cup W'_k$ be non-singular graph-manifolds with reduced graph-structures. Then any π_1-injective map $f : W \to W'$ is homotopic to $\bigcup f_i$, where each $f_i : (W_i, \partial W_i) \to (W'_j, \partial W'_j)$ is fiber covering map.*

Proof

Step 1 *Any π_1-injective map of torus bundle over circle $f : M_\varphi \to W'$ is homotopic to a fiber covering map into one block.*

By Main Technical Result, one can move f by homotopy such that the inverse image by f of decomposing submanifolds becomes disjoint union of π_1-injective 2-tori, embedded in M_φ. Then M_φ cut along them is either $\bigcup_i (T^2 \times I)_i$ or $\bigcup_i (T^2 \times I)_i \cup (K^2 \widetilde{\times} I) \cup (K^2 \widetilde{\times} I)$, each summand lying in a block ($K^2 \widetilde{\times} I$ is twisted oriented I-bundle over Klein bottle).

Case 1 M_φ cut along preimages of decomposing submanifolds is $\bigcup_i (T^2 \times I)$.

First observe that if f sends $T^2 \times \{0\}$ and $T^2 \times \{1\}$ into different boundary components of a block, then f is homotopic to a fiber-covering map. Indeed, denote the block $p : M \to B$ and choose a base point $b \in f(T^2 \times \{0\})$. Then $p_* f_*(\mathbb{Z} \oplus \mathbb{Z})$ lies in the subgroup of $\pi_1(B, p(b))$ corresponding to the component of ∂B containing $pf(T^2 \times \{0\})$, and in the conjugation class of the subgroup corresponding to the component of ∂B containing $pf(T^2 \times \{1\})$. But conjugate classes of different boundary components can intersect only if B is an annulus, because the conjugation defines a map $S^1 \times I \to B$ of non-zero degree. Hence $p_* f_*(\mathbb{Z} \oplus \mathbb{Z}) = 1$ which means that f sends the fiber subgroup of $T^2 \times I$ into the fiber subgroup of the block. Hence f is homotopic to fiber-covering map.

Remark 1 This observation implies that in a reduced graph-structure the fibers of different blocks are not homotopic. Indeed, if a graph-manifold has just 2 blocks, then the claim comes from the definition of the reduced graph-structure. If there are more blocks, take one of them, its fiber satisfies the conditions of the previous observation in all the neighboring blocks. Hence, in every neighboring block this fiber is not homotopic to any torus in the remaining boundary components. But in these components lie in particular the fibers of the next neighboring blocks etc.

As the graph-structure is reduced, in all the neighboring blocks f is not homotopic to a fiber-covering map and hence $f(\partial(T^2 \times I))$ lie in the same boundary component. Hence one can apply Lemma 2 to the neighboring $(T^2 \times I)$'s and move them into the block where the fiber-covering $f(T^2 \times I)$ lies.

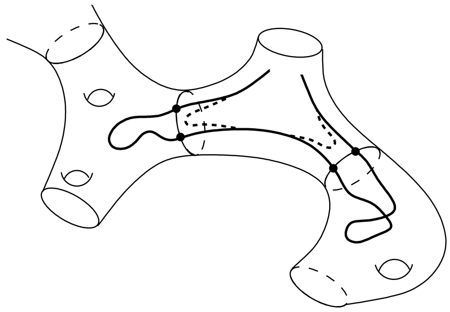

Figure 7: Shrinking in one block

Case 2 M_φ cut along preimages of decomposing submanifolds is $\bigcup_i (T^2 \times I) \cup (K^2 \widetilde{\times} I) \cup (K^2 \widetilde{\times} I)$.

Take the first copy of $K^2 \widetilde{\times} I$, $I = [-1;1]$, denote the decomposing manifold, in which the image under f of its boundary lies, by M_{φ_1}. Its boundary torus is a two-fold covering of the Klein bottle in the base and if $\pi_1(K^2 \widetilde{\times} I) = \pi_1(K^2) = \langle a, b | aba^{-1} = b^{-1} \rangle$, then the boundary torus corresponds to the subgroup $\langle a, b^2 \rangle$. As the subgroups of boundary components are square root closed in the fondamental groups of the blocks (Proposition 4), the subgroup $f_*(\pi_1(K^2 \widetilde{\times} I), x)$ must lie in the subgroup corresponding to M_{φ_1}, because $f_*(\pi_1(\partial(K^2 \widetilde{\times} I)), x)$ lies there. As $K^2 \widetilde{\times} I$ is aspherical, $f|_{K^2 \widetilde{\times} I}$ can be moved by homotopy in M_{φ_1} and, hence, out of its original block in the neighboring one. Repeat the previous reasonnings for the union of $(K^2 \widetilde{\times} I)$ with the next $T^2 \times I$ gives a new $(K^2 \widetilde{\times} I)$. In the end it will be two copies of $(K^2 \widetilde{\times} I)$, and the image of each of them under f can be moved into the same decomposing manifold. Hence, in this case f is homotopic to a map into a decomposing manifold.

Once $f(M_\varphi)$ is shrinked in one block, look at the homomorphism that f induces on π_1's. The image of the subgroup of the fiber of M_φ vanishes when projecting on π_1 of the base of M, because it is abelian normal subgroup of non-abelian free group. Hence, by lemma 1 the map is homotopic to a fiber covering one.

Step 2 *Any π_1-injective map of a block $f: M \to W'$ is homotopic to a map into one block of W'.*

Any block retracts on a torus bundle over a wedge of circles. Torus bundle over a wedge of circles can be obtained from a torus bundle over circle (with the monodromy equal to the product of the monodromies of the petals) by identifying some fibers. Change the map of this “big” single torus-bundle by homotopy given by Step 1. The images of fibers that are identified are two by two homotopic by two kinds of homotopy. The first are given by petals and now lies in one block M'. The second comes from identification and still lies in the whole W'. In order to identify the images, one has to shrink these homotopies into M'.

Apply the Main Technical Result to each of these homotopies, this will make their intersections with decomposing submanifolds W' π_1-injective.

If the tori that must be identified are fiber-covering in M', the corresponding homotopies can lie only in the neighboring blocks, where they are not-fiber-covering, hence by Lemma 2 can be shrinked in M'.

If the tori that must be identified are not fiber-covering in M', then the homotopies lie in the union of M' with its neighboring blocks, because if the part of the homotopy in the neighboring block does fiber-covering, then in the following blocks it does not and Lemma 2 does apply. More, all this homotopies must lie in just one neighboring block of M', because elsewhere we would have a π_1-injective map $(T^2\times I, \partial(T^2\times I)) \to (M', \partial M'))$ which would be non-fiber-covering but sending $\partial(T^2\times I)$ into different components of $\partial M'$. Then the remaining parts of homotopies and the images of all the petals can be shrinked in this neighboring block.

Step 3 For all i, change $f|_{W_i}$ according to Step 2, denote it by f_i. Let $W_i, W_k \subset W$ be neighboring blocks. For each component of $W_i\cap W_k$, by Step 1, the block in which it lies is unique. As $f_i|_{W_i\cap W_k}$ and $f_k|_{W_i\cap W_k}$ are homotopic, one conclude that the corresponding W_i' and W_k' are neighboring and $f_i|_{W_i\cap W_k}, f_k|_{W_i\cap W_k}$ are homotopic to a map into $W_i'\cap W_k'$. Use the corresponding homotopy inside W_i' to define the new $f_i : W_i = W_i\bigcup((W_i\cap W_k)\times I) \to W_i'$ being the homotopy on $(W_i\cap W_k)\times I$ part and the old f_i on W_i part; then make the same for f_k. By doing it on all pairs of neighboring blocks, one obtain the map $f=\bigcup f_i$ with $f_i : (W_i, \partial W_i) \to (W_i', \partial W_i')$. Apply Corollary 2 to every f_i to make it fiber covering. It remains to bind the new $f_i|_{W_i\cap W_k}$ and $f_k|_{W_i\cap W_k}$ by a fiber covering homotopy inside the correponding $W_i'\cap W_k'$, which is possible because they covers the same subbundles of $W_i'\cap W_k'$ with the same degree. We obtain a map $f=\bigcup f_i : W\to W'$ such that every $f_i : (W_i,\partial W_i)\to(W_i',\partial W_i')$ is fiber covering. □

Theorem 1 *Any homotopy equivalence between non-singular graph-manifolds with reduced graph-structures is homotopic to a diffeomorphism.*

Proof Take the collars of decomposing manifolds in the blocks, the blocks without this collars remain the blocks. For each decomposing manifold M_φ, the union of its collars on both side is $M_\varphi \times I$, call it "double collar of M_φ" in the graph-manifold.

According to Proposition 6, both maps of the homotopy equivalence $f : W \to W', g : W' \to W$ can be moved by homotopy such that their restrictions on blocks without collars are rel boundary fiber covering maps. As after homotopies we still have $gf \sim id_W$, the fiber of a block (without collars) M is homotopic to its image by gf. Hence, by Remark 1, $gf(M) \subset M$ for every block of W, i.e. the restrictions of f and g gives the homotopy equivalences rel boundary of blocks without collars. According to Corollary 3, these resrictions are homotopic rel boundary to diffeomorphisms. One has to bind the obtained block's diffeomorphisms on the double collars of the decomposing manifolds. For this note that the diffeomorphisms, that are the restricions of f and g on the boundaries of blocks without collars, are homotopic. Hence restrictions of f and g on double collars of decomposing manifolds are homotopy equivalences rel boundary that are diffeomorphism on the boundaries. As decomposing manifolds are sufficiently large, these homotopy equivalences are homotopic rel boundary to diffeomorphisms, by homotopies that are constant on the boundaries. [19]. $\square$

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References

[1] **R Abraham**, **J Robbin**, *Transversal mappings and flows*, An appendix by Al Kelley, W. A. Benjamin, Inc. New York-Amsterdam (1967)

[2] **M Culler**, *Using surfaces to solve equations in free groups*, Topology 20 (1981) 133–145

[3] **D Eisenbud**, **W Neumann**, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies 110, Princeton University Press, Princeton, NJ (1985)

[4] **A T Fomenko**, **H Zieschang**, *A topological invariant and a criterion for the equivalence of integrable Hamiltonian systems with two degrees of freedom*, Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990) 546–575 (Russian); translation in Math. USSR-Izv. 36 (1991) 567–596

[5] **C J Earle**, **J Eells**, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry 3 (1969) 19–43

[6] **J A Hillman**, *The algebraic characterization of geometric 4-manifolds*, London Mathematical Society Lecture Note Series 198, Cambridge University Press, Cambridge (1994)

[7] **J A Hillman**, *Four-manifolds, geometries and knots*, Geometry and Topology Monographs Volume 5

[8] **M W Hirsch**, *Immersions of manifolds*, Trans. Amer. Math. Soc. 93 (1959) 242–276

[9] **R C Lyndon**, **M P Schützenberger**, *The equation $a^M = b^N c^P$ in a free group*, Michigan Math. J. 9 (1962) 289–298

[10] **W Magnus**, **A Karrass**, **D Solitar**, *Combinatorial group theory*, Dover Publications Inc. New York (1976)

[11] **W Meyer**, *Die Signatur von Flächenbündeln*, Math. Ann. 201 (1973) 239–264

[12] **J R Munkres**, *Elementary differential topology*, Lectures given at Massachusetts Institute of Technology, Fall 1961, Princeton University Press, Princeton, N.J. (1966)

[13] **X Rong**, *The limiting eta invariants of collapsed three-manifolds*, J. Differential Geom. 37 (1993) 535–568

[14] **P Scott**, **T Wall**, *Topological methods in group theory*, from: "Homological group theory (Proc. Sympos. Durham, 1977)", London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press, Cambridge (1979) 137–203

[15] **H Seifert**, *Topology of 3-dimensional fibered spaces*, from: H Seifert, W.Threlfall, "A textbook of topology", Pure and applied Mathematics, vol. 89, Academic Press Inc. (1980)

[16] **S Smale**, *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. (2) 69 (1959) 327–344

[17] **J Stallings**, *Group theory and three-dimensional manifolds*, Yale University Press, New Haven, Conn. (1971)

[18] **F Waldhausen**, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. 3 (1967), 308–333; ibid. 4 (1967) 87–117

[19] **F Waldhausen**, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) 87 (1968) 56–88

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