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Overtwisted open books from sobering arcs

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Abstract We study open books on three manifolds which are compatible with an overtwisted contact structure. We show that the existence of certain arcs, called sobering arcs, is a sufficient condition for an open book to be overtwisted, and is necessary up to stabilization by positive Hopf-bands. Using these techniques we prove that some open books arising as the boundary of symplectic configurations are overtwisted, answering a question of Gay in [9].

AMS Classification 57R17; 57M99

Keywords Open book, contact structure, overtwisted, sobering arc, symplectic configuration graph

1 Introduction

A contact structure on a three-manifold is a nowhere integrable two-plane field (we consider only oriented, co-orientable, positive contact structures). A fundamental theorem of Eliashberg [3] shows that contact structures fall into two classes: *overtwisted*, and *tight*. The overtwisted contact structures are determined by their homotopy type as plane fields, while the tight ones are more subtle.

An open book is a pair (Σ, ϕ), where Σ is a surface with boundary, and ϕ, the monodromy, is an automorphism of Σ. The mapping torus $\Sigma \times_\phi S^1$ has torus boundary with preferred meridian and longitude, we can thus form a three manifold $M_{(\Sigma,\phi)}$ by gluing solid tori to the boundary. We say that (Σ, ϕ) is an open book on $M_{(\Sigma,\phi)}$, each $\Sigma \subset M_{(\Sigma,\phi)}$ is a page, while the core of the solid tori are the binding.

A contact structure and an open book on the same manifold are called *compatible* if the contact planes can be isotoped arbitrarily close to the tangent planes of the pages, while the binding remains transverse. For example the standard tight contact structure on S^3, written ξ_0, is compatible with the positive Hopf-band, H^+, whose binding is two fibers of the Hopf fibration.

If we have two contact structures compatible with the same open book it is not hard to show (by first pushing the contact planes close to the pages, then using a straight-line homotopy) that they must be isotopic. (We will write ξ_B for a contact structure compatible with an open book B, unique up to isotopy.) Because of this uniqueness the contact structures compatible with a given open book are either all tight or all overtwisted.

Definition 1.1 An open book (Σ, ϕ) is called *overtwisted* if there is an overtwisted contact structure $\xi_{(\Sigma,\phi)}$ compatible with it.

Giroux has shown [11] that there is an open book compatible with any contact structure, so it is natural to study contact structures by studying open books. The first question one would like to answer is: when is an open book overtwisted?

A few open books can be shown to be overtwisted by general considerations or special techniques. For example, every contact structure on the Poincare homology-sphere with reversed orientation is overtwisted [6], so any compatible open book must also be overtwisted. This manifold arises as $+1$-surgery on the right-handed trefoil T^+, so it has an open book which comes from the natural open book structure of T^+ by adding a boundary parallel negative Dehn twist. In section 3 we'll show more directly that this open book is overtwisted.

In this paper we study arcs (which we call *sobering arcs*) which intersect their monodromy images in specific ways. We show that the existence of a sobering arc is a sufficient condition for an open book to be overtwisted. This is not a necessary condition (there are overtwisted open books with no sobering arc), but it is necessary up to stabilization (by Murasugi sum with positive Hopf-bands). The most satisfying statement of our result in this paper is:

Theorem 1.2 *An open book is overtwisted if and only if it is stably equivalent to an open book with a sobering arc.*

This technique gives some interesting examples: we look at the open books that arise from positive surface configurations in symplectic manifolds, studied by Gay in [9], and show that some of these open books are overtwisted, and hence the corresponding symplectic configuration cannot embed in a closed symplectic 4-manifold (answering a question of [9]).

In section 2 we review some necessary background material on contact structures and open books. In section 3 we introduce sobering arcs, prove that an open book with a sobering arc is overtwisted, and give some examples. In section 4 we investigate Hopf-bands and open books which have a Hopf-band as a Murasugi summand. In section 5 we show that every overtwisted open book is stably-equivalent to one with a sobering arc. In section 6 we look at contact manifolds that arise as the boundaries of symplectic configurations and show that some of these are overtwisted.

This results of this paper were contained in the author's Ph.D. dissertation [13].

We would like to thank John Etnyre for a great deal of encouragement and many helpful conversations, and the anonymous reviewer for very useful comments.

2 Background

Throughout this section we will omit proofs that can be easily found in the literature. In particular, there are now many good references on contact geometry, see [5].

2.1 Topological conventions

Unless we explicitly state otherwise all manifolds are three dimensional and oriented, and all surfaces are also oriented.

If $S \subset M$ is a surface with boundary embedded in three-manifold M, and T another surface so that $\partial S \subset T$, then the *framing difference* for ∂S between S and T, written $Fr(\partial S; S, T)$ or just $Fr(S, T)$, is the oriented number of intersections between S and a push-off of ∂S along T.

An automorphism of a surface is a bijective self-homeomorphism, fixing the boundary point-wise. The *mapping torus* of a surface automorphism, $\phi : \Sigma \to \Sigma$, is the three-manifold $\Sigma \times_\phi S^1 = (\Sigma \times I)/(p, 1) \sim (\phi(p), 0)$.

Given a simple closed curve $c \subset \Sigma$ we can define an automorphism $D_c^+ : \Sigma \to \Sigma$, which has support only near c, as follows. Let N be a neighborhood of c which is identified (by oriented coordinate charts) with the annulus $\{a \in \mathbb{C} | 1 \leq ||a|| \leq 2\}$ in $\mathbb{C}$. Then D_c is the map $a \mapsto e^{-i2\pi(||a||-1)}a$ on N, and the identity on $\Sigma \setminus N$. We call D_c^+ a positive Dehn twist; the inverse, written D_c^- is a negative Dehn twist.

2.2 Open books

Open books (Σ_i, ϕ_i), $i = 1, 2$, are *equivalent* if $\Sigma_1 = \Sigma_2$ and ϕ_1 is isotopic to $h^{-1} \circ \phi_2 \circ h$ for some automorphism h.

Associated to each open book there is a three-manifold, $M_{\Sigma,\phi} = (\Sigma \times_\phi S^1) \cup_\partial (D^2 \times \partial\Sigma)$, where the union is taken by gluing boundary tori in such a way that the curve $\{p\} \times S^1$, for $p \in \partial\Sigma$, glues to meridian $\partial(D^2 \times \{p\})$, and $\partial\Sigma \times \{\theta\}$ glues to $\{e^{i\theta}\} \times \partial\Sigma$. The link $\{0\} \times \partial\Sigma \subset D^2 \times \partial\Sigma \subset M_{\Sigma,\phi}$ is called the *binding*, sometimes we'll write it $\partial\Sigma \subset M_{\Sigma,\phi}$. As expected, equivalent open books give homeomorphic $(M_{\Sigma,\phi}, \partial\Sigma)$.

Given two open books, and some attaching data, we can get a new open book via the Murasugi sum. Unless stated otherwise our Murasugi sum will be the simplest kind, defined below in terms of rectangles. A more general Murasugi sum, also called generalized plumbing or $*$-product in the literature, can be defined similarly by using polygons with $2m$ sides.

Definition 2.1 Let (Σ_i, ϕ_i), $i = 1, 2$, be open books, $l_i \subset \Sigma_i$ properly embedded arcs, and R_i rectangular neighborhoods of l_i such that $R_i \cap \partial\Sigma_i$ is two arcs of ∂R_i. The surface $\Sigma_1 * \Sigma_2$ is the union $\Sigma_1 \cup_{R_i} \Sigma_2$ with R_1 identified to R_2 in such a way that $\partial R_1 \cap \partial\Sigma_1 = \partial R_2 \setminus \partial\Sigma_2$, and vice versa. (This is the identification which gives a new surface, rather than a branched surface.) The *Murasugi sum* of the (Σ_i, ϕ_i) along l_i is the open book $(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2) = (\Sigma_1 * \Sigma_2, \phi_1 \circ \phi_2)$ (where each ϕ_i is understood to extend to $\Sigma_1 * \Sigma_2$ by the identity on $(\Sigma_1 * \Sigma_2) \setminus \Sigma_i$). We will often suppress mention of the attaching arcs l_i when they are clear or irrelevant.

Theorem 2.2 *The manifold $M_{(\Sigma_1,\phi_1)*(\Sigma_2,\phi_2)}$ is $M_{\Sigma_1,\phi_1} \# M_{\Sigma_2,\phi_2}$.*

Proof See [8]. □

The *Hopf-bands*, $H^\pm$, are the open books with surface an annulus and monodromy a single positive or negative Dehn twist about the center of the annulus.

2.3 Contact structures

A one dimensional submanifold $L \subset (M, \xi)$ is *Legendrian* if $T_pL \subset \xi$ at each $p \in L$, and is *transverse* if $T_pL \not\subset \xi$. Any curve in a contact manifold can be made Legendrian or transverse by a C^0-small perturbation.

Given an embedded surface $S \subset (M,\xi)$ the contact planes restrict to a (singular) line field on S. This singular line field integrates into a singular foliation (since any line field is integrable), which we call the *characteristic foliation*, $\xi|_S$.

For $S \subset M^3$ a surface with ∂S Legendrian, $tw(\partial S, S)$ is the difference in framing of ∂S between ξ and S. (That is the intersection number of S with a push-off of ∂S along ξ.)

A contact structure is *overtwisted* if there is an embedded disk D with Legendrian boundary such that $tw(\partial D, D) = 0$. A contact structure that is not overtwisted is *tight*.

We now state some key theorems about overtwisted contact structures:

Theorem 2.3 (The Bennequin Inequality) *If (M,ξ) is a tight contact manifold, $S \subset M$ is an embedded surface, and ∂S is Legendrian, then:*

$$tw(\partial S, S) \leq -\chi(S) - |r| \leq -\chi(S).$$

In the previous theorem, r is the *rotation number* of the contact structure over the surface. We will use only the courser inequality in this paper.

Theorem 2.4 (Eliashberg [3]) *Overtwisted contact structures are isotopic if, and only if, they are homotopic as plane fields.*

The following theorem is due to Lutz and Martinet (see [3] for a review):

Theorem 2.5 *There is an overtwisted contact structure in each homotopy class of plane fields.*

A surface $S \subset (M,\xi)$ is *convex* if there is a vector field transverse to S whose flow preserves ξ. If v is such a vector field, the curves $\Gamma = \{p \in S | v(p) \in \xi\}$ are called the *dividing curves* (or *divides*) of S. For a convex surface with Legendrian boundary $tw(\partial S, S)$ is negative half the number of intersection points in $\Gamma \cap \partial S$.

In [10] Giroux gives a useful characterization of convex surfaces. A characteristic foliation with isolated singularities is called *Morse-Smale* if all singularities are hyperbolic (in the dynamical systems sense) and there are no saddle-to-saddle connections. (There is another technical requirement in the definition of Morse-Smale, it will be satisfied if there are no closed orbits, and this will be enough for us.) A surface $S \subset (M,\xi)$ is convex if and only if $\xi|_S$ is Morse-Smale. For details see [10].

Lemma 2.6 *Any closed surface $S \subset (M, \xi)$ can be made convex by a C^0-small perturbation. A surface with Legendrian boundary can be made convex by a C^0-small perturbation, relative to ∂S, if $tw(\partial S, S) \leq -1$ for each component of ∂S.*

We'll frequently need the following lemma, called the Legendrian Realization Principal (LeRP), due to Ko Honda [14]:

Lemma 2.7 (LeRP) *Let $c \subset S$ be a set of curves and arcs (possibly with $\partial c \subset \partial S$), on a convex surface S. If each component of $S \setminus c$ intersects the dividing curves of S, then S may be perturbed, by a C^0-small perturbation through convex surfaces, to make c Legendrian.*

We will often use "LeRP" as a verb, meaning "to apply lemma 2.7 to a set of curves".

The folding trick, implicit in [14]:

Lemma 2.8 *Let $c \subset S$ be a non-separating collection of simple closed curves on convex surface S which don't intersect the divides $\Gamma \subset S$, and c' a parallel copy of c. Then S can be perturbed to a convex surface divided by $\Gamma \cup c \cup c'$. This perturbation is C^0-small, but not through convex surfaces.*

2.4 Compatibility

As mentioned above, an open book is compatible with a contact structure, ξ if there is a homotopy ξ_t of plane fields (smooth for $t > 0$) so that $\xi_1 = \xi$, ξ_0 is tangent to the pages, ξ_t is contact for $t > 0$, and the binding is transverse to ξ_t.

Torisu shows in [17] that Murasugi sum corresponds to contact connect sum for the compatible contact structure, that is $\xi_{B*C} = \xi_B \# \xi_C$ (where equality means contact isotopy). Since H^+ is compatible with the standard structure ξ_0, this implies (see [1]) that $\xi_{B*H^+} = \xi_B$. We say that $B * H^+$ is a *stabilization* of B and that two open books related by a series of stabilizations are *stably-equivalent.*

A key theorem of Giroux relates compatible open books ([11], see [13] for an exposition of the proof):

Theorem 2.9 (Giroux) *Two open books compatible with a fixed contact structure are stably-equivalent.*

3 Sobering arcs

Definition 3.1 Let $\alpha, \beta \subset \Sigma$ be properly embedded oriented arcs which intersect transversely, Σ an oriented surface. If $p \in \alpha \cap \beta$ and T_α, T_β are tangent vectors to α and β (respectively) at p, then the intersection number $i_p = +1$ if the ordered basis (T_α, T_β) agrees with the orientation of Σ at p, and $i_p = -1$ otherwise. Let $\tilde{\beta}$ be an arc which minimizes intersections with α over boundary fixing isotopies of β. Sign conventions are illustrated in Figure 1. We define the following intersection numbers:

(1) The *algebraic intersection number*, $i_{alg}(\alpha,\beta) = \sum_{p \in \alpha \cap \beta \cap int(\Sigma)} i_p$, is the oriented sum over intersections on the interior of Σ.

(2) The *geometric intersection number*, $i_{geom}(\alpha,\beta) = \sum_{p \in \alpha \cap \tilde{\beta} \cap int(\Sigma)} |i_p|$, is the unsigned count of interior intersections, minimized over all boundary fixing isotopies.

(3) The *boundary intersection number*, $i_\partial(\alpha,\beta) = \frac{1}{2} \sum_{p \in \alpha \cap \tilde{\beta} \cap \partial\Sigma} i_p$, is one-half the oriented sum over intersections at the boundaries of the arcs, after the arcs have been isotoped, fixing boundary, to minimize geometric intersection.

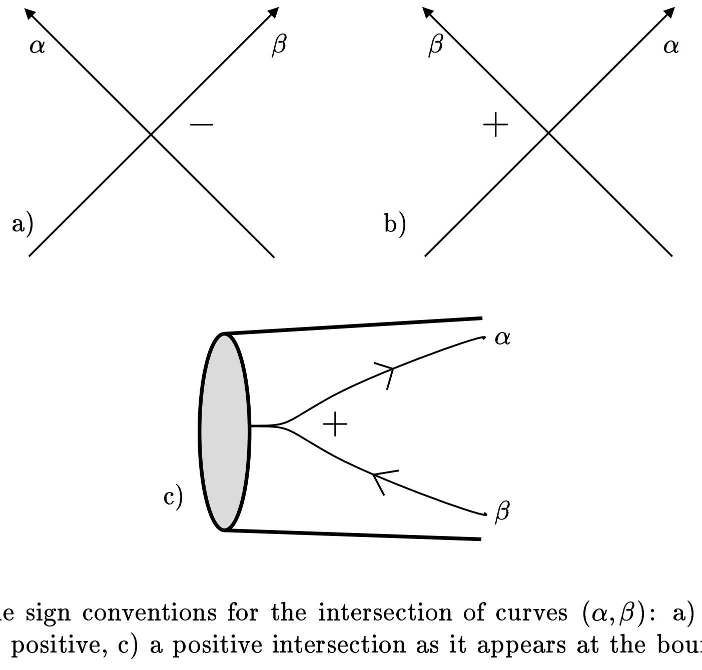

Figure 1: The sign conventions for the intersection of curves (α,β): a) negative intersection, b) positive, c) a positive intersection as it appears at the boundary of the surface

In particular, for an arc $\alpha \subset \Sigma$ in the page of an open book (Σ, ϕ) we may consider the intersection numbers: $i_{alg}(\alpha, \phi(\alpha))$, $i_{geom}(\alpha, \phi(\alpha))$, $i_{\partial}(\alpha, \phi(\alpha))$. We orient $\phi(\alpha)$ by reversing a pushed forward orientation on α. Since reversing the orientation of α will also reverse that on $\phi(\alpha)$, without changing the intersection signs, it is irrelevant which orientation we choose.

Much subtle information about a surface automorphism can be gleaned by comparing these intersection numbers for properly embedded arcs. For instance the algebraic monodromy (the induced map $\phi_* : H_*(\Sigma) \mapsto H_*(\Sigma)$,) is unable to distinguish between (Σ, ϕ) and (Σ, ϕ^{-1}), since ϕ_* is related to ϕ_*^{-1} by conjugation (through the change of basis which reverses the orientation of each basis vector). However, $i_{\partial}(\alpha, \phi(\alpha)) = -i_{\partial}(\alpha, \phi^{-1}(\alpha))$ is a geometrically well-defined quantity which can make this distinction.

Definition 3.2 A properly embedded arc $\alpha \subset \Sigma$ is *sobering*, for a monodromy ϕ, if $i_{alg}(\alpha, \phi(\alpha)) + i_{\partial}(\alpha, \phi(\alpha)) + i_{geom}(\alpha, \phi(\alpha)) \leq 0$, and α is not isotopic to $\phi(\alpha)$.

In particular, since $i_{\partial} \geq -1$ and each positive intersection contributes 2 to the sum of intersection numbers, there can be no interior intersections with positive sign. So we can reinterpret the definition:

Lemma 3.3 *An arc, α, is sobering if and only if, after minimizing geometric intersections, $i_{\partial} \leq 0$, there are no positive (internal) intersections of α with $\phi(\alpha)$, and α isn't isotopic to $\phi(\alpha)$.*

There is a good reason to be interested in such arcs:

Theorem 3.4 *If there is a sobering arc $\alpha \subset \Sigma$ for ϕ, then the open book (Σ, ϕ) is overtwisted.*

To prove this theorem we will construct a surface with Legendrian boundary which violates the Bennequin inequality, theorem 2.3. Since this inequality is a necessary condition for tightness, we'll conclude that $\xi_{\Sigma,\phi}$ is overtwisted. In doing so, however, we will not find an overtwisted disk – such a disk must exist, but will, in general, not be nicely positioned with respect to the open book.

We begin our construction by suspending α in the mapping torus: $\alpha \times I \subset \Sigma \times_{\phi} I$. Let $p \in \partial\Sigma$ be an endpoint of α. Since ϕ fixes the boundary, $p \times I$ is a meridian around the binding. Let D_{α} be the surface made by gluing a meridional disc to $\alpha \times I \subset M_{\Sigma,\phi}$ along $p \times I$ for each $p \in \partial\alpha$ (this extends

$\alpha \times I$ across the binding in $M_{\Sigma,\phi}$). The surface D_α is a disk which has $\partial D_\alpha \subset \Sigma$, and is embedded on its interior, but possibly immersed on its boundary.

We isotope $\phi(\alpha)$, relative to the boundary, to have minimal and transverse intersection with α. This induces an isotopy of D_α, which intersects itself at transverse double points on its boundary. To proceed we need to remove the double points of D_α. However, if we simply separate the edges of the disk meeting at an intersection, the boundary of the disk won't stay on Σ, eliminating any control we might have had over the framing. Instead we will opt to resolve the double points by decreasing the Euler characteristic of the surface:

Definition 3.5 Let F be an orientable surface which is embedded on $Int(F)$ and has isolated double points on ∂F. The *resolution* of F is constructed by thickening each double point into a half-twisted band in such a way that the resulting surface remains oriented. This is done according to the following local model (see Figure 2). In $\mathbb{R}^3$ coordinates, in a small neighborhood of a double point, F looks like $\{(x,y,z) \in \mathbb{R}^3 | z \geq 0, y = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 | z \leq 0, x = 0\}$, oriented by $\hat{x}$ and $\pm\hat{y}$, respectively. We resolve by gluing in the triangles $\{(x,y,z) \in \mathbb{R}^3 | z = 0, x \geq 0, 0 \geq \pm y \geq x - 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 | z = 0, x \leq 0, 0 \leq \pm y \leq x + 1\}$ (that is, a pair of triangles depending on the intersection sign). Note that the boundary of both F and its resolution are on the plane $z = 0$.

Let S be the resolution of D_α, with additionally ∂D_α pushed into $Int(\Sigma)$ at the points $\partial\alpha$ (see Figure 3). Figure 2b) shows, on Σ, how the boundary ∂D_α changes into $\partial S \subset \Sigma$ at a resolution.

Lemma 3.6 $\chi(S) = 1 - i_{geom}(\alpha, \phi(\alpha))$.

Proof The construction of S starts with the disk D_α which has $i_{geom}(\alpha, \phi(\alpha))$ double points on its boundary. At each double point we glue a band from the disk to itself. The counting follows since a disk has $\chi = 1$, and adding a band reduces the Euler characteristic by 1. □

Lemma 3.7 *The difference in framings of ∂S with respect to S and Σ is: $Fr(\partial S; S, \Sigma) = -i_{alg}(\alpha, \phi(\alpha)) - i_\partial(\alpha, \phi(\alpha))$.*

Proof The relative framing can be computed as the oriented intersection number of S with a push-off of ∂S along Σ. To choose the push-off first choose

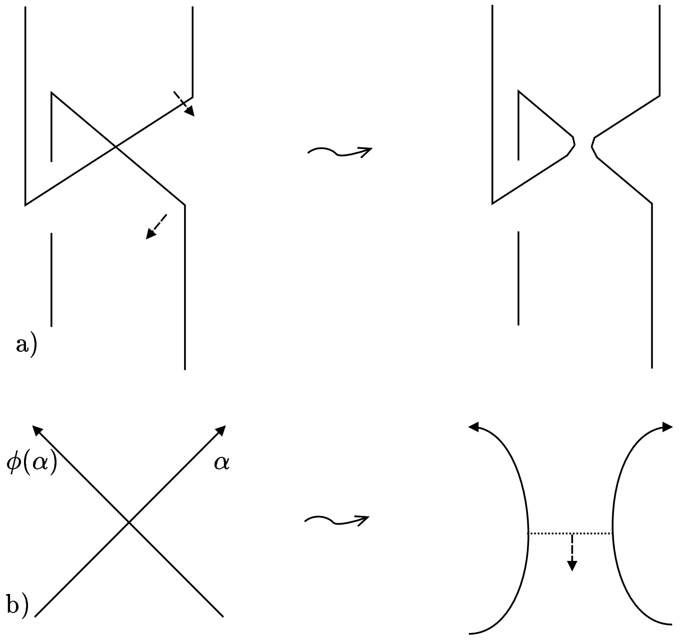

Figure 2: The local model for resolutions of immersed points on the boundary of a surface: a) the surface, with dashed arrows indicating orientations, b) the resolution of $\alpha \cap \phi(\alpha)$, drawn on Σ, for the case of D_α (the dotted line represents $Int(D_\alpha) \cap \Sigma$)

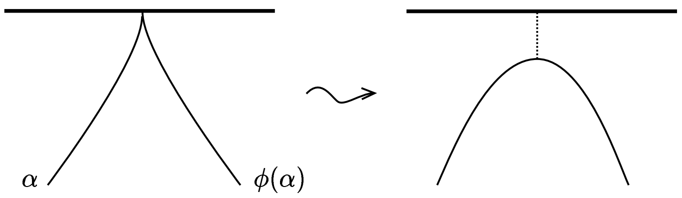

Figure 3: Smoothing the boundary of the suspension disk, D_α: The dashed line is $D_\alpha \cap \Sigma$.

an orientation for S, and let s be a tangent vector to ∂S which gives the boundary orientation. Choose vector $t \in T\Sigma$ so that (s,t) provide an oriented basis agreeing with the orientation of Σ. Let L be the push-off of ∂S in the t direction.

Now, $L \subset \Sigma$ can intersect S only on $Int(S) \cap \Sigma$. This set consists of arcs where intersections were resolved in the construction of S (including the boundary intersections, which are smoothed, as in Figure 3). For each of the internal intersections of α with $\phi(\alpha)$, L intersects S once with sign opposite that of the intersection, as shown in Figure 4.

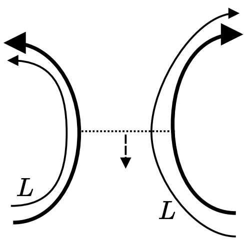

Figure 4: Framing at the resolution of a positive intersection: The dashed line is $S \cap \Sigma$, while the arrow is the orientation of S. L is a push-off of ∂S.

It remains to account for the boundary intersections. Since ϕ fixes $\partial\Sigma$ (as for any open book), there are exactly two intersections points in $\partial\alpha \cap \partial\phi(\alpha) = \partial\alpha$ (so $i_\partial(\alpha, \phi(\alpha)) \in \{+1, -1, 0\}$). There are four possibilities for the intersection of L with S at these resolutions, according to the signs of the intersections at the beginning and end of α. One can compute each of these possibilities, as in the example of Figure 5, the result is shown in table 1. There will be one intersection near the boundary, with sign ∓ 1, if $i_\partial(\alpha, \phi(\alpha)) = \pm 1$, and either two with opposite signs, or none, if $i_\partial(\alpha, \phi(\alpha)) = 0$. Therefore the boundary intersections contribute $-i_\partial(\alpha, \phi(\alpha))$ to the framing, $Fr(S, \Sigma)$.

Finally, in the above analysis we chose an orientation on α (which then orients $\phi(\alpha)$, D_α and S). Reversing this orientation is equivalent to reversing the orientation of ∂S, and the effect on the framing computation is to take the push-off of ∂S along Σ but to the other side of S. However, the relative framing $Fr(S, \Sigma)$ is well-defined independent of which push-off is used, so our result doesn't depend on the chosen orientation. (The independence can also be seen directly by repeating the calculation using the other push-off.) $\square$

	start α	end α
pos.	-1	0
neg.	0	+1

Table 1: The framing intersections contributed by a boundary intersection of the given sign at the beginning or end of α. Computations were made as in Figure 5.

We are now ready to prove the main theorem of this section.

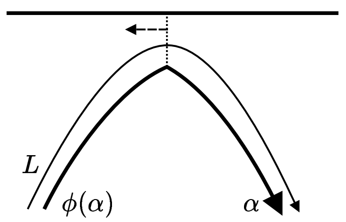

Figure 5: Framing near boundary: an example of computing the intersection of L with S, on Σ. Here a positive boundary intersection at the start of α contributes -1 to the framing. The dotted line is $S \cap \Sigma$, while the dashed arrow is the orientation of S.

Proof of Theorem 3.4 As we will later wish to LeRP ∂S, we must first rule out any components of ∂S which bound disks on Σ.

Assume there is a disk $D \subset \Sigma$ such that $\partial D \subset \partial S$. The intersection signs at the corners of a disk (where resolutions have occurred on ∂S) must alternate: see Figure 6. However, by the sobering condition, there is at most one positive intersection, which occurs at the boundary. This implies that the only possible disk is a bi-gon with a corner on the boundary, but in this case either the intersection $\alpha \cap \phi(\alpha)$ isn't minimal, or the other corner of the bi-gon is also on the boundary. The former contradicts the minimal set-up we've arranged, the latter contradicts the definition (α isn't isotopic to $\phi(\alpha)$), so there can be no such disk.

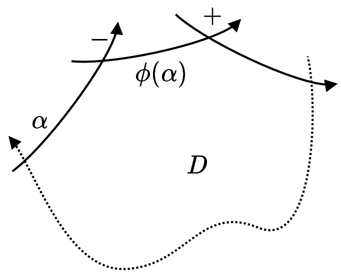

Figure 6: If disk $D \subset \Sigma$ was created by resolving $\alpha \cap \phi(\alpha)$, the signs at the intersections would have to alternate.

In order to use the machinery of convex surfaces, we must make the leaves convex. We can do this by an isotopy of the contact structure.

Lemma 3.8 *Let (Σ, ϕ) be an open book, $\pi : M_{\Sigma,\phi} \setminus \partial\Sigma \to S^1$ the fibration of the complement to the binding. The closure of $\pi^{-1}(0) \cup \pi^{-1}(0.5)$ is convex and* *divided by the binding, $\partial\Sigma = \partial(\pi^{-1}(0))$, for some compatible contact structure.*

Proof Let $\tilde{\Sigma}$ be the closure of $\pi^{-1}(0) \cup \pi^{-1}(0.5)$ (oriented by $\pi^*(ds)$ on $\pi^{-1}(0)$ and $\pi^*(-ds)$ on $\pi^{-1}(0.5)$), we take a compatible contact structure built as in [16] from a one-form λ (see [13]). We may choose the homotopy extending λ to the mapping torus in such a way that the characteristic foliation on $\tilde{\Sigma}$ looks like $ker(\lambda + \phi^*(\lambda))$ on $\pi^{-1}(0.5)$ (and like $ker(\lambda)$ on $\pi^{-1}(0)$). By a small perturbation of this homotopy (eg. by perturbing ϕ) we can assure that the saddles on $\pi^{-1}(0)$ connect to the sources/sinks on $\pi^{-1}(0.5)$, and vice versa, hence the characteristic foliation is Morse-Smale. By the results of [10], a surface with such a characteristic foliation is convex. The dividing curves separate the regions where the contact one-form is positive on a positive normal vector from those where it is negative. Since the contact structure is compatible, $\pi^{-1}(0)$ is positive while $\pi^{-1}(0.5)$ is negative (with the natural orientation: $\pi^*(ds)$, on each), hence the dividing curve is exactly the binding. □

Because the compatible contact structure to an open book is unique up to isotopy, we can perform an isotopy to assure that our contact structure is that of the lemma. Now, with convex pages, we want to apply the LeRP (lemma 2.7) to the curves ∂S, but we must insure that there is a component of the dividing set in each component of $\tilde{\Sigma} \setminus \partial S$. To achieve this, let Γ contain a simple closed curve in each component of $\tilde{\Sigma} \setminus \partial S$. Using the folding lemma 2.8 along Γ we perturb $\tilde{\Sigma}$, changing the dividing set to $\partial\Sigma \cup \Gamma \cup \Gamma'$ (Γ' a parallel copy of Γ). Now we can apply the LeRP, perturbing $\tilde{\Sigma}$ again to make the curves ∂S Legendrian.

By the definition of convex surfaces there is a contact vector field v normal to $\tilde{\Sigma}$. Since ∂S doesn't cross the dividing set, $v|_{\partial S} \notin \xi_{\Sigma,\phi}$, so the framing of S with respect to $\xi_{\Sigma,\phi}$ (the twisting number) may be computed by using a push-off of ∂S in the v direction. But v is normal to $\tilde{\Sigma}$ so a push-off along $\tilde{\Sigma}$ is equivalent, for framing purposes, to the push off along v. Hence, the framing coming from $\xi_{\Sigma,\phi}$ is the same as that from Σ: $tw(\partial S, S) = Fr(\partial S; S, \Sigma)$.

We computed above that $Fr(S,\Sigma) = -i_{alg}(\alpha,\phi(\alpha)) - i_\partial(\alpha,\phi(\alpha))$. So,

$$tw(\partial S, S) + \chi(S) \geq -i_{alg}(\alpha,\phi(\alpha)) - i_\partial(\alpha,\phi(\alpha) + 1 - i_{geom}(\alpha,\phi(\alpha)).$$

The sobering condition $-i_{alg}(\alpha,\phi(\alpha)) - i_\partial(\alpha,\phi(\alpha) - i_{geom}(\alpha,\phi(\alpha)) \geq 0$ then implies:

$$tb(\partial S, S) + \chi(S) > 0,$$

violating the Bennequin inequality (theorem 2.3), which must hold if $\xi_{\Sigma,\phi}$ is tight. Hence, (Σ,ϕ) is overtwisted. □

Remark A word about the use of arcs instead of closed curves inside Σ. Our construction, above, would fail for these curves for simple Euler characteristic reasons, or we may rule out such considerations more directly: in the complement of the binding $\xi_{\Sigma,\phi}$ is a perturbation of the taught foliation given by the leaves of the open book. From the results of Eliashberg and Thurston [4] such a contact structure is symplectically fillable, and hence tight. Thus, no surface entirely in the complement of the binding, such as the suspension of a closed curve on Σ, can violate the Bennequin inequality.

We can slightly extend the class of monodromies for which sobering arcs are useful by a little cleverness:

Theorem 3.9 *If an arc $\alpha \subset \Sigma$ has:*

$$i_{alg}(\alpha,\phi(\alpha)) + i_{\partial}(\alpha,\phi(\alpha)) + i_{geom}(\alpha,\phi(\alpha)) = -1,$$

then each open book (Σ, ϕ^n), for $n > 0$, is overtwisted.

Proof The open book (Σ, ϕ^n) can be built by gluing n-copies of $\Sigma \times I$ to each other by ϕ (including the bottom of the first to the top of the last) to get the mapping torus $\Sigma \times_{\phi^n} S^1$, then adding the binding as usual. In each $\Sigma \times I$ we have a disk $\alpha \times I$. The top of one disk intersects the bottom of the next as $\alpha \cap \phi(\alpha)$. If we resolve these intersections, as in the previous proofs, we get a surface with $\chi = n - ni_{geom}(\alpha,\phi(\alpha))$ and framing $ni_{alg}(\alpha,\phi(\alpha))$. Next we add in the binding and cap off the surface with disks $D^2 \times \partial\alpha$, to get a surface S with $\chi(S) = 2 - n - ni_{geom}(\alpha,\phi(\alpha))$ and framing $ni_{\partial}(\alpha,\phi(\alpha)) + ni_{alg}(\alpha,\phi(\alpha))$.

To finish the proof we proceed exactly as in the proof of theorem 3.4: rule out components of ∂S which bound disks on a page, then fold and LeRP to make ∂S Legendrian (we'll have to do this along several pages, but they won't interfere since folding and LeRPing can be done in small neighborhoods). Finally we get a surface with $tb(\partial S, S) + \chi(S) = -n(i_{alg}(\alpha,\phi(\alpha)) + i_{\partial}(\alpha,\phi(\alpha))) + 2 - n$. Then under the condition $i_{alg}(\alpha,\phi(\alpha)) + i_{\partial}(\alpha,\phi(\alpha)) + i_{geom}(\alpha,\phi(\alpha)) = -1$ we get: $tb(\partial S, S) + \chi(S) = n + 2 - n = 2 > 0$, so the open book is overtwisted. □

We can apply this theorem to get some interesting examples:

Corollary 3.10 *Let Σ_g be a surface with genus $g > 0$ and one boundary component, and let $\delta \subset \Sigma_g$ be parallel to the boundary. The open book $(\Sigma_g, D_{\delta}^{-n})$ is overtwisted ($n > 0$).*

Proof Let $\{a_i, b_i\}_{i=1,\dots,g}$ be simple closed curves on Σ_g such that a_i intersects b_i in one point, b_i intersects a_{i+1} in one point, and the curves are otherwise disjoint. One can show, as in [15], that $D_\delta = (\prod_i D_{a_i} D_{b_i})^{4g+2}$.

Then $D_\delta^{-n} = (D_{b_g}^- D_{a_g}^- \cdots D_{b_1}^- D_{a_1}^-)^{n(4g+2)}$. An arc which intersects only b_g will be sobering for $\phi = D_{b_g}^- D_{a_g}^- \cdots D_{b_1}^- D_{a_1}^-$. □

On the other hand, since the open books (Σ_g, D_δ^n) are positive, they are Stein fillable, and hence tight. (The connection between fillable contact structures and positive monodromy was found by many people, see [7] for a review.)

We return now to the Poincare homology-sphere with reversed orientation, discussed in the introduction. This manifold arises as $+1$-surgery on the trefoil $T^+ \subset S^3$, so it supports an open book which comes from the natural fibering of $S^3 \setminus T^+$ by adding a boundary parallel negative Dehn twist. (To see this, push the surgery curve onto a page, then verify directly that an arc passing the surgery experiences a negative Dehn twist). Explicitly: let Σ be the punctured torus, δ a boundary parallel curve, and a, b curves which intersect in a point. Then the open book $(\Sigma, D_\delta^{-1} D_b^{+1} D_a^{+1})$ represents the Poincare homology-sphere with reversed orientation. The technique of the corollary shows that the monodromy can be written $\phi = (D_b^- D_a^-)^6 (D_b^+ D_a^+) = (D_b^- D_a^-)^5$, and that this is overtwisted.

4 A special case

Let us consider the simplest open books. In a disk all arcs are boundary parallel, so we turn our attention to annuli, and especially to the Hopf-bands, $H^\pm$, which are annuli with monodromy a single right or left-handed Dehn twist along, c, the center of the annulus. If we let α be an arc which crosses the annulus, we see from Figure 7 that $i_{geom}(\alpha, D_c^\pm(\alpha)) = i_{alg}(\alpha, D_c^\pm(\alpha)) = 0$, while $i_\partial(\alpha, D_c^\pm(\alpha)) = \pm 1$ for $H^\pm$, respectively. Since $D_c^\pm(\alpha)$ isn't isotopic to α, α is sobering for H^- (but not for H^+!).

Lemma 4.1 *The Murasugi sum $(\Sigma, \phi) * H^-$, for any open book (Σ, ϕ), along any attaching curve, is overtwisted.*

Proof Take an arc across H^-, outside the attaching region. It is unmoved by (the extension of) ϕ, so will have the same intersection properties relative to $D_c^- \circ \phi$ as the arc α, above, had relative to D_c^-. So this arc will be sobering in the sum. □

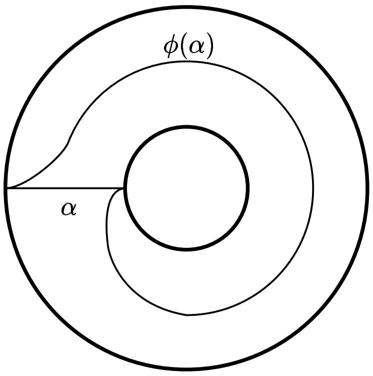

Figure 7: A negative Hopf-band, H^-, with the transverse arc, α, and its image

How special are the intersection properties of the arcs we've just considered? It turns out that they uniquely specify the geometric situation of having a Hopf-band Murasugi summand.

Theorem 4.2 *If $i_{geom}(\alpha, \phi(\alpha)) = 0$ and $i_\partial(\alpha, \phi(\alpha)) = \pm 1$, then α is the transverse arc to a positive (respectively negative) Hopf-band. That is $(\Sigma, \phi) = (\Sigma', \phi') * H^\pm$ where α is parallel to the attaching curve in $H^\pm$.*

Proof Let $c \subset \Sigma$ be the curve which comes from smoothing $\alpha \cup \phi(\alpha)$. Since $i_{geom}(\alpha, \phi(\alpha)) = 0$, c will be a simple closed curve. Note that $\phi(\alpha) = D_c^\pm(\alpha)$ up to isotopy, where the sign of the Dehn-twist agrees with $i_\partial(\alpha, \phi(\alpha))$.

Let $\phi' = D_c^\mp \circ \phi$ on the surface $\Sigma' = \Sigma \setminus \alpha$. This makes sense since $\phi'(\alpha) = D_c^\mp \circ \phi(\alpha) = D_c^\mp \circ D_c^\pm(\alpha) = \alpha$ (up to isotopy, so for some representative of the isotopy class of ϕ').

If we plumb a Hopf-band onto (Σ', ϕ') along $c \setminus \alpha \subset \Sigma'$, we get the map $D_c^\pm \circ \phi' = \phi$ on Σ. Thus $(\Sigma, \phi) = (\Sigma', \phi') * H^\pm$, and α crosses the one-handle, as required. $\square$

Of course if $(\Sigma, \phi) = (\Sigma', \phi') * H^\pm$, then there is such an arc α, described above, so we in fact have a necessary and sufficient condition for the existence of a Hopf-band summand in an open book. This criterion can be used to show that certain fibered links are not Hopf plumbings, and it is often helpful in finding stabilizations in otherwise difficult examples, as in section 6.

5 A criterion for overtwisting

Theorem 5.1 *An open book is overtwisted if and only if it is H^+-stably equivalent to an open book with a sobering arc.*

Proof *If:* The final open book of the equivalence has a sobering arc so its compatible contact structure is overtwisted. Neither summing nor de-summing H^+ changes the contact structure, so the original open book is overtwisted.

Only If: For any open book B the homotopy classes of plane fields on M_B have a natural $\mathbb{Z}$–action, and we arrive in the homotopy class of ξ_{B*H^-} by acting on the homotopy class of ξ_B with $+1$ (see [12]).

Now let (Σ, ϕ) be an overtwisted open book, let ξ be a contact structure in the homotopy class of $\xi_{(\Sigma,\phi)}$ acted on by -1 (there is a contact structure in any homotopy class), and let B be an open book compatible with ξ. The contact structure ξ_{B*H^-} then is homotopic to $\xi_{(\Sigma,\phi)}$, and both are overtwisted, hence they are isotopic (lemma 2.4). So, $B*H^-$ is H^+-stably equivalent to (Σ,ϕ) by theorem 2.9, and $B*H^-$ has a sobering arc (across the Hopf band, see lemma 4.1). □

6 Boundaries of symplectic configurations

Following [9] we define a *symplectic configuration graph* to be a labeled graph G with no edges from a vertex to itself and with each vertex v_i labeled with a triple (g_i, m_i, a_i), where $g_i \in \{0, 1, 2, \dots\}$, $m_i \in \mathbb{Z}$ and $a_i \in (0, \infty)$. (a_i doesn't enter into our purely three-dimensional concerns, so we'll often omit it.) Let d_i denote the degree of vertex v_i. A configuration graph is called *positive* if $m_i + d_i > 0$ for every vertex v_i.

Given a positive configuration graph G we define an open book $(\Sigma(G), \phi(G))$ as follows: For each vertex v_i let F_i be a surface of genus g_i with $m_i + d_i$ boundary components. $\Sigma(G)$ is the surface obtained by connect sum of the F_i, with one connect sum between F_i and F_j for each edge connecting v_i to v_j. See Figure 8 for an example. For each edge in G there is a circle e_{ij} in $\Sigma(G)$. Let $\sigma(G) = \prod D^+_{e_{ij}}$, a right-handed Dehn twist at each connect sum. Let $\delta(G)$ be the product of one right-handed Dehn twist around each circle of $\partial\Sigma(G)$. Finally, $\phi(G) = \sigma(G)^{-1} \circ \delta(G)$.

In [9] Gay shows that the open books $(\Sigma(G), \phi(G))$ arise as the concave boundary of certain symplectic manifolds. In that paper he asks the question:

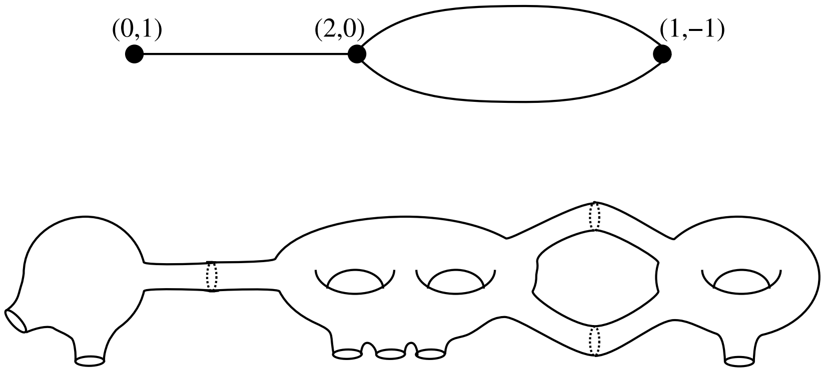

Figure 8: A positive configuration graph (the a_i have been suppressed), and the surface $\Sigma(G)$: The dotted circles are the e_{ij}.

Question (Gay [9]) Are there any positive configuration graphs G for which we can show that $(\Sigma(G), \phi(G))$ is overtwisted (and hence conclude that a symplectic configuration with graph G cannot embed in a closed symplectic 4-manifold)?

We can use the techniques of the previous sections to give such examples.

First, note that if l is a properly embedded arc in $\Sigma(G)$ which crosses from one F_i to an adjacent one, so crosses one negative Dehn twist and two positive, then $i_{geom}(l, \phi(l)) = 0$ and $i_\partial(l, \phi(l)) = +1$ (see Figure 9). From theorem 4.2 we know that there is an H^+ summand. To remove it we'll first re-write the monodromy using the lantern relation (this is a standard fact of mapping class groups, see [2]):

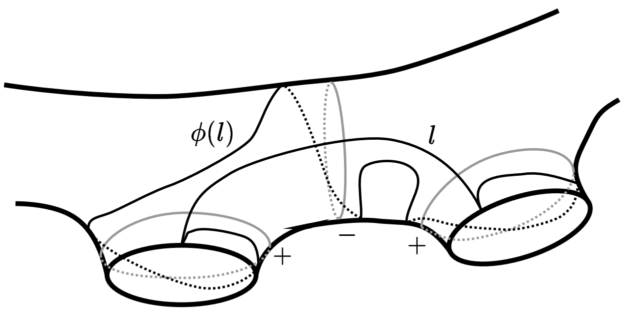

Figure 9: The arc $l \subset \Sigma(G)$, its image under $\phi(G)$, and the twisting curves it crosses

Lemma 6.1 *The following relation holds in the mapping class group of a four-times punctured sphere (fixing the boundary point-wise), referring to Figure 10(a) and (b):*

$$D_a^+ \circ D_b^+ \circ D_c^+ \circ D_d^+ = D_\alpha^+ \circ D_\beta^+ \circ D_\gamma^+.$$

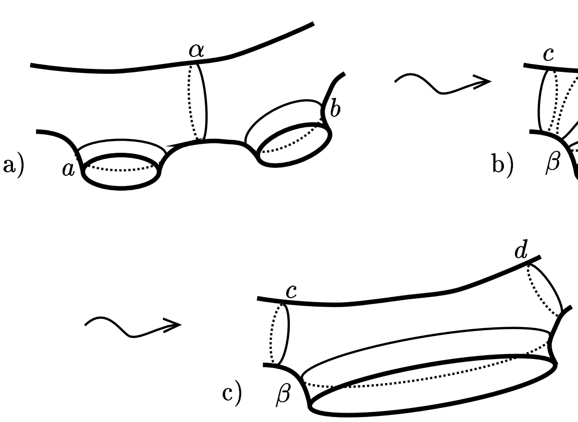

Figure 10: De-plumbing an H^+ from $(\Sigma(G), \phi(G))$. The first step is an application of the lantern lemma to go from $D_a^+ \circ D_b^+ \circ D_\alpha^-$ to $D_c^- \circ D_d^- \circ D_\beta^+ \circ D_\gamma^+$, the second is the removal of a Hopf-band along γ.

Applying this relation to the region of Figure 9 around arc l, we can rewrite the monodromy as in Figure 10 (a) and (b): $D_a^+ \circ D_b^+ \circ D_\alpha^- = D_c^- \circ D_d^- \circ D_\beta^+ \circ D_\gamma^+$ (Dehn twists about a, b, c, d commute with each other and with α, β, and γ, since these pairs don't intersect). It is then clear, since l crosses only the final Dehn twist D_γ^+, how to remove a Hopf-band: remove D_γ^+ and cut along l, Figure 10(c). We arrive at a surface with one fewer boundary component, and $D_c^- \circ D_d^- \circ D_\beta^+$ replacing $D_a^+ \circ D_b^+ \circ D_\alpha^-$ in the monodromy. We'll use this move repeatedly to prove the next lemma. Note that this move doesn't necessarily return an open book $(\Sigma(G'), \phi(G'))$, since the resulting open book may have several negative Dehn twists around the same curve.

Theorem 6.2 *Let G be a positive configuration graph with a vertex v_1 such that $d_1 = 1$ and $g_1 = 0$, but not a graph with:*

(1) *one edge, $m_1 = 0$,*

(2) *one edge, $g_2 = m_2 = 0$,*

(3) *one edge, $g_2 = 0$, $m_2 = m_1 = 1$, or,*

(4) *two edges,* $g_3 = 0$, $m_1 = m_3 = 0$.

Then $(\Sigma(G), \phi(G))$ *is overtwisted.*

Proof First we apply the move illustrated in Figure 10 repeatedly, removing positive Hopf-bands until all of the boundary components of F_1 are gone, leaving a boundary component with one positive Dehn twist and $m_1 + 1$ negative twists. These, of course, are equivalent to m_1 negative Dehn twists. We are left with a surface, G', in which F_1 and F_2 have been replaced with a surface of genus g_2, and $m_2 + d_2$ boundary components, which has m_1 negative Dehn twists about one of these punctures (the remaining monodromy twists are unchanged). See Figure 11.

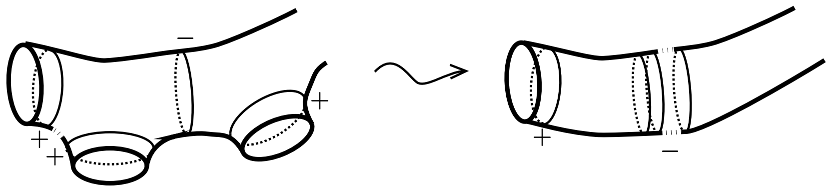

Figure 11: Removing almost all boundary components by applying Figure 10 to remove H^+

The surface G' will have $\sum_{i>1} m_i + d_i$ boundary components and, since $m_i + d_i > 0$, will have a single boundary component only when $e = 1$ and $m_2 = 0$ (where e is the number of edges). In this case we have a surface with one puncture and monodromy consisting of m_1 negative twists about the boundary; by corollary 3.10 the open book will be overtwisted if $g_2 > 0$ and $m_1 > 0$. The 'bad' cases of $e = 1$, $m_2 = 0$, and either $g_2 = 0$ or $m_1 = 0$ are excluded by hypothesis.

Now if G' has several boundary components, let α be an arc from the boundary component with m_1 negative Dehn twists to another boundary component, chosen so that it crosses as many of the e_{ij} (non-boundary parallel negative twists of $\phi(G)$) as possible, but crosses each at most once. The arc α will cross the m_1 negative Dehn twists, k additional negative twists (one for each e_{ij} crossed), and finally a positive Dehn twist (see Figure 12). (Note that if $k = 0$ then $e = 1$, and if $k = 1$ then $e = 2$: if there were more edges then α could be extended.)

Before any isotopy there are only negative internal intersections of α with $\phi(\alpha)$, thus after an isotopy which minimizes intersections there will still be only negative internal intersections. The boundary intersection number will initially be 0 (since α crosses only the one positive twist as in Figure 12), and can only be reduced as internal intersections are minimized (since all internal intersections are negative). So, α will be sobering if it is not isotopic to $\phi(\alpha)$.

If $m_1 + k > 1$, that is α crosses more than one negative twist, the algebraic intersection number (including boundary intersections) is less than 0. This is an isotopy invariant, so α is not isotopic to $\phi(\alpha)$.

We can rule out the case $m_1 + k = 0$ since we must have $e = 1$ and $m_1 = 0$. What if $m_1 + k = 1$? Then α crosses one negative and one positive Dehn twist, call them D^+_β and D^-_γ. Up to homotopy (relative to the boundary) $[\phi(\alpha)] = [\alpha] + [\beta] + [\gamma]$ (as β and γ can't cross). If α is isotopic to $\phi(\alpha)$ then $[\beta] = [\gamma]$, up to sign.

There are two cases, first consider $m_1 = 1$ and $k = 0$. In this case $e = 1$, so both β and γ are boundary parallel – in order for them to be homotopic G' must be a cylinder, so $g_2 = 0$ and $m_2 = m_1 = 1$ (a case excluded by hypothesis).

In the other case $m_1 = 0$ and $e = 2$, now β is again boundary parallel (near the boundary of F_3), but γ is the curve separating F_3 from F_2. Their homotopy equivalence implies that F_3 is a cylinder, so $g_3 = 0 = m_3$ (another excluded case).

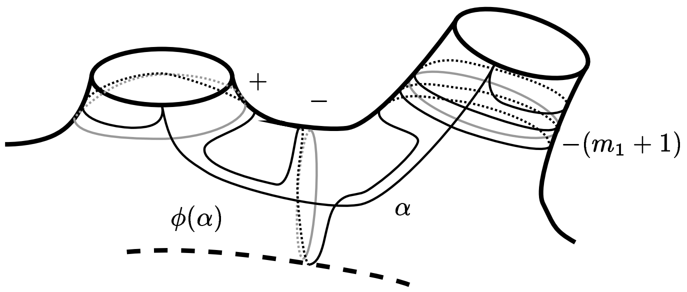

Figure 12: Schematic of the sobering curve

We have shown that α will be sobering in the allowed cases. By theorem 3.4, the presence of a sobering arc implies that the open book is overtwisted. When stabilizing, to get back the open book $(\Sigma(G), \phi(G))$, the compatible contact structure remains overtwisted (in fact, the same up to isotopy), so $(\Sigma(G), \phi(G))$ is overtwisted. □

Remark Since they are overtwisted these contact manifolds can't be the convex boundary of a symplectic four-manifold. Corollary 2.1 of [9] also shows this for some (but not all) of these graphs. The techniques used there are four-dimensional – a clever application of the adjunction inequality.

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