ISSN 1472-2739 (on-line) 1472-2747 (printed) **71**

Algebraic & Geometric Topology
Volume 5 (2005) 71–106
Published: 3 February 2005

Clover calculus for homology 3–spheres via basic algebraic topology

Emmanuel Auclair
Christine Lescop

Abstract We present an alternative definition for the Goussarov–Habiro filtration of the $\mathbb{Z}$–module freely generated by oriented integral homology 3–spheres, by means of Lagrangian-preserving homology handlebody replacements (LP–surgeries). Garoufalidis, Goussarov and Polyak proved that the graded space $(\mathcal{G}_n)_n$ associated to this filtration is generated by Jacobi diagrams. Here, we express elements associated to LP–surgeries as explicit combinations of these Jacobi diagrams in $(\mathcal{G}_n)_n$. The obtained coefficient in front of a Jacobi diagram is computed like its weight system with respect to a Lie algebra equipped with a non-degenerate invariant bilinear form, where cup products in 3–manifolds play the role of the Lie bracket and the linking number replaces the invariant form. In particular, this article provides an algebraic version of the graphical clover calculus developed by Garoufalidis, Goussarov, Habiro and Polyak. This version induces splitting formulae for all finite type invariants of homology 3–spheres.

AMS Classification 57M27; 57N10

Keywords 3–manifolds, homology spheres, finite type invariants, Jacobi diagrams, Borromeo surgery, clover calculus, clasper calculus, Goussarov–Habiro filtration

1 Introduction

In 1995, in [Oht], Tomotada Ohtsuki introduced a notion of finite type invariants for homology 3–spheres (that are compact oriented 3–manifolds with the same homology with integral coefficients as the standard 3–sphere S^3), following the model of the theory of Vassiliev invariants for knots in the ambient space $\mathbb{R}^3$. He defined a filtration of the real vector space freely generated by homology 3–spheres and began the study of the associated graded space. In [Le], Thang Le finished identifying this graded space to a space of Jacobi diagrams called $\mathcal{A}_{\mathbb{R}}(\emptyset)$. The Jacobi diagrams, precisely defined in Subsection 2.1, are represented by trivalent finite graphs with additional orientation information.

Similar filtrations of the $\mathbb{Z}$–module freely generated by homology 3–spheres and their relationships have been studied by Garoufalidis, Goussarov, Polyak and others. See [GGP] and references therein. Over $\mathbb{Z}[1/2]$, all of them are equivalent to the Ohtsuki filtration [GGP].

Among these filtrations, the most convenient one is the Goussarov–Habiro one where the Matveev Borromeo surgeries [Mat] (defined in Subsection 2.2) play the role of the crossing changes in the knot case. It allowed Garoufalidis, Goussarov and Polyak to define a set of generators $\Psi_n(\Gamma)$ for the degree n part $\mathcal{G}_n$ of the associated Goussarov–Habiro graded $\mathbb{Z}$–module, for Jacobi diagrams Γ with at most n vertices [GGP]. See Subsection 2.3. Garoufalidis, Goussarov and Polyak also gave some graphical rules that allow one to reduce an element to a combination of their generators. This set of rules is the so-called *clover calculus.* Here, these rules are enclosed in two propositions 4.13 and A.1.

Our main theorem 3.5 expresses elements of $\mathcal{G}_n$ associated to the *LP–surgeries* defined in Subsection 3.1, as explicit combinations of the $\Psi_n(\Gamma)$, in terms of intersection forms (or cup products) and linking numbers. Therefore, this article presents a completely algebraic version of the Garoufalidis–Goussarov–Habiro–Polyak clover calculus. Furthermore, it tightens the links between Jacobi diagrams and topology by relating the vertices of the Jacobi diagrams to cup products in 3–manifolds and the diagram edges with linking numbers.

We also give an alternative definition of the Goussarov–Habiro filtration of the $\mathbb{Z}$–module of integral homology 3-spheres, by means of LP–surgeries. See Corollary 3.2.

Let us now give a slightly more specific description of our main theorem 3.5.

A *homology genus g handlebody* is an oriented compact 3–manifold with the same integral homology as the standard genus g handlebody H_g. The boundary ∂A of such a manifold A is then homeomorphic to the genus g surface ∂H_g. The *Lagrangian* $\mathcal{L}_A$ of A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Z})$ to $H_1(A;\mathbb{Z})$. A *Lagrangian-preserving surgery* or *LP–surgery* on a homology sphere M consists in removing the interior of such a homology handlebody ($A \subset M$) and replacing it by another such B whose boundary ∂B is identified to ∂A so that $\mathcal{L}_A = \mathcal{L}_B$.

In our definition of the Goussarov–Habiro filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of the $\mathbb{Z}$–module $\mathcal{F} = \mathcal{F}_0$ freely generated by the oriented homology spheres up to orientation-preserving homeomorphisms, the n^{th} module $\mathcal{F}_n$ is generated by *brackets* $[D]$ of so-called *n–component LP–surgeries* D that are made of n disjoint LP–

surgeries (A_i, B_i) in M. (The A_i are disjoint in M.)

$$[D] = \sum_{J \subset \{1,\dots,n\}} (-1)^{\sharp J} \left((M \setminus \sqcup_{j\in J} \mathrm{Int}(A_j)) \cup_\partial (\sqcup_{j \in J} B_j) \right).$$

Our main result expresses the bracket $[D]$ of an n–component LP–surgery D as

$$[D] = \sum_\Gamma \ell(D;\Gamma)\Psi_n(\Gamma)$$

in $\mathcal{G}_n = \mathcal{F}_n/\mathcal{F}_{n+1}$, where the coefficient $\ell(D;\Gamma)$ of $\Psi_n(\Gamma)$ is an explicit function of the cup products in the manifolds $(A_i \cup -B_i)$, of the linking pairings on $H_1(A_i) \otimes H_1(A_j)$, $i \neq j$, and of variations of the Rohlin invariant when replacing A_i by B_i.

Let us roughly define $\ell(D;\Gamma)$ when n is the number of vertices of Γ and when Γ admits no non-trivial automorphism. The general definition of $\ell(D;\Gamma)$ is given in Subsection 3.2. When a bijection σ from the set of vertices of Γ to $\{1,\dots,n\}$ is given, the algebraic intersection of surfaces (or the cup product) of each $(A_i \cup -B_i)$ is placed at the vertex $\sigma^{-1}(i)$. The cup products are next contracted along the edges with respect to the linking pairing to produce a number $\ell(D;\Gamma;\sigma)$, and $\ell(D;\Gamma) = \sum_\sigma \ell(D;\Gamma;\sigma)$. This construction is similar to the construction of weight systems associated to Lie algebras.

The proof of Theorem 3.5 goes as follows. We first prove that the standard Goussarov, Garoufalidis and Polyak generators have appropriate coefficients in Subsection 4.1. Then we use the similarities between the behaviour of the bracket in $\mathcal{G}_n$ and the behaviour of our coefficients to reduce the proof to this former case.

Though this article is largely inspired by [GGP], it is written in a self-contained way in an attempt to replace all the graphical arguments in [GGP] by more intrinsic arguments of geometric or algebraic topology.

Theorem 3.5 can be used to derive formulae on the behaviour under LP–surgeries of all finite-type invariants of homology spheres in the Goussarov–Habiro sense. For example, it immediately leads to splitting formulae for the restriction to homology spheres of the Kontsevich–Kuperberg–Thurston universal finite-type invariant Z_{KKT} of rational homology spheres. In [L2], the second author proved that these formulae generalise to rational homology spheres and to rational homology handlebody replacements that preserve the rational Lagrangians. These generalized splitting formulae are fairly easy to guess from the Kontsevich–Kuperberg–Thurston construction (but much harder to prove in general), they actually led the second author to the formulae of Theorem 3.5.

These formulae had been previously noticed by G. Kuperberg and D. Thurston in the special case where rational homology handlebodies are reglued by a homeomorphism that induces the identity in homology [KT]. This special case is sufficient to prove that Z_{KKT} is universal among finite-type invariants of homology spheres.

The second author thanks Thang Le, Gregor Masbaum and Dylan Thurston for useful and pleasant conversations.

2 Background

2.1 Jacobi diagrams

In what follows, a *Jacobi diagram* Γ is a trivalent graph without simple loop like . Let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices of Γ and the set of edges of Γ, respectively. A *half-edge* c of Γ is a pair $c = (v(c); e(c))$ where $v(c) \in V(\Gamma)$, $e(c) \in E(\Gamma)$ and $v(c)$ belongs to $e(c)$. The set of half-edges of Γ will be denoted by $H(\Gamma)$ and its two natural projections above onto $V(\Gamma)$ and $E(\Gamma)$ will be denoted by v and e, respectively. An *automorphism* of a Jacobi diagram Γ is a permutation ϕ of $H(\Gamma)$ verifying the two following conditions

$$\begin{array}{ccc} \big(e(c) = e(c')\big) & \Rightarrow & \big(e\big(\phi(c)\big) = e\big(\phi(c')\big)\big) \\ \big(v(c) = v(c')\big) & \Rightarrow & \big(v\big(\phi(c)\big) = v\big(\phi(c')\big)\big) \end{array}$$

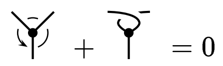
for any $c, c' \in H(\Gamma)$. An automorphism ϕ of a Jacobi diagram Γ *preserves the vertices of* Γ if

$$\forall c \in H(\Gamma), v\big(\phi(c)\big) = v(c).$$

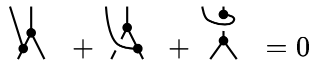
Let $\mathrm{Aut}(\Gamma)$ be the set of automorphisms of Γ. Let $\mathrm{Aut}_V(\Gamma)$ denote the set of automorphims of Γ that preserve the vertices of Γ. Let $\sharp\mathrm{Aut}_V(\Gamma)$ denote the number of automorphisms of Γ that preserve the vertices. A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of each vertex of Γ, that is a cyclic order of the three half-edges that meet at that vertex. Two vertex-orientations of Γ are equivalent if and only if the cardinality of the set of vertices where they differ is even. An *orientation* of Γ is an equivalence class of vertex-orientations. An *oriented* Jacobi diagram is a Jacobi diagram carrying an orientation. A Jacobi diagram Γ is *reversible* if there exists an automorphism ϕ of Γ that reverses an orientation of Γ. For any automorphism ϕ of Γ, set

$$\mathrm{sign}(\phi) = \begin{cases} 1 & \text{if } \phi \text{ preserves the orientation} \\ -1 & \text{if } \phi \text{ reverses the orientation.} \end{cases}$$

Note that, for all $\phi \in \mathrm{Aut}_V(\Gamma)$, $\mathrm{sign}(\phi) = 1$. The *degree* of a Jacobi diagram is half the number of all its vertices. Let $\mathcal{A}_k$ denote the free abelian group generated by the degree k oriented Jacobi diagrams, quotiented out by the following relations AS and IHX.



AS-relation



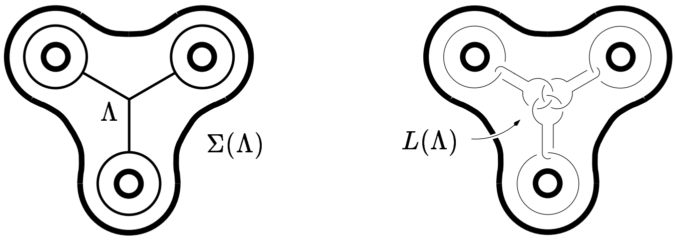
IHX-relation

Each of these relations relate diagrams which can be represented by immersions that are identical outside the part of them represented in the pictures. In the pictures, the cyclic order of the half-edges is represented by the counterclockwise order. For example, AS identifies the sum of two diagrams which only differ by the orientation at one vertex to zero. The space $\mathcal{A}_0$ is equal to $\mathbb{Z}$ generated by the empty diagram. In what follows, if Γ is an oriented Jacobi diagram, then $-\Gamma$ denotes the same Jacobi diagram with the opposite orientation. If Γ is reversible, then $\Gamma = -\Gamma$.

2.2 Y–graphs and the Goussarov–Habiro filtration

Here, we briefly review the *Y–surgery*, or the *surgery along Y–links*, which is presented in [GGP]. The Y–surgery is equivalent to the *Borromeo transformation* in Matveev's work [Mat].

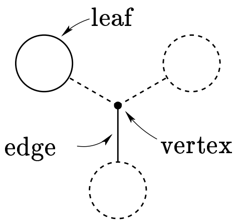
Let Λ be the graph embedded in the surface $\Sigma(\Lambda)$ shown in Figure 1(a). In the 3–handlebody ($N = \Sigma(\Lambda) \times [-1, 1]$), the edges of Λ are framed by a vector field normal to $\Sigma(\Lambda) = \Sigma(\Lambda) \times \{0\}$. $\Sigma(\Lambda)$ is called a *framing surface* for Λ. Let $L(\Lambda) \subset N$ be the link presented in Figure 1(b) with six framed components that inherit their framings from $\Sigma(\Lambda)$.



(a) (b)

Figure 1: Y–graph and associated link

Let M be a 3–manifold. A *Y–graph* in M is an embedding ϕ of N (or $\Sigma(\Lambda)$) into M up to isotopy. Such an isotopy class is determined by the framed image of the framed unoriented graph Λ under ϕ. A *leaf* of a Y–graph ϕ is the image under ϕ of a simple loop of our graph Λ. An *edge* of ϕ is an edge of $\phi(\Lambda)$ that is not a leaf. The *vertex* of ϕ is the unique vertex of $\phi(\Lambda)$ adjacent to the three edges. With this terminology, a Y–graph has one vertex, three edges and three leaves:

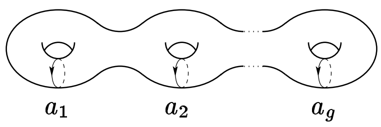


Let $G \subset M$ be a Y–graph. A leaf l of a Y–component of G is *trivial* if l bounds an embedded disc that induces the framing of l, in $M \setminus G$.

The *Y–surgery along the Y–graph* $\phi(\Lambda)$ is the surgery along the framed link $\phi(L(\Lambda))$ (see [Rol, Chapter 9], or [Lic, Chapter 11] for details about surgery on framed knots). The resulting manifold is denoted by $M_{\phi(\Lambda)}$. An *n–component Y–link* $G \subset M$ is an embedding of the disjoint union of n copies of N into M up to isotopy. The Y–surgery along a Y–link G is defined as the surgery along each Y–component of G. The resulting manifold is denoted by M_G.

In this article, the homology coefficients will always be integers. A *$\mathbb{Z}$–sphere* is a compact oriented 3–manifold M such that $H_*(M) = H_*(S^3)$. It is also called a *homology sphere.* A *homology handlebody* or *$\mathbb{Z}$–handlebody* is an oriented, compact 3–manifold A with the same homology (with integral coefficients) as the standard (solid) handlebody H_g below.

H_g



Note that the boundary ∂A of such a $\mathbb{Z}$–handlebody A is homeomorphic to the boundary Σ_g of H_g. For any surface Σ, let $\langle , \rangle_\Sigma$ be the intersection form on $H_1(\Sigma)$. For a $\mathbb{Z}$–handlebody A, $\mathcal{L}_A$ denotes the kernel of the map from $H_1(\partial A)$ to $H_1(A)$ induced by the inclusion. It is a Lagrangian of $(H_1(\partial A); \langle , \rangle_{\partial A})$. It is called the *Lagrangian* of A.

If A is a $\mathbb{Z}$–handlebody and if G is a Y–link in the interior $\mathrm{Int}(A)$ of A, then A_G is still a $\mathbb{Z}$–handlebody whose boundary ∂A is canonically identified with ∂A_G, so that $\mathcal{L}_A = \mathcal{L}_{A_G}$. Similarly, if G is a Y–link in a homology sphere M, then M_G is still a homology sphere.

Let $\mathcal{F}$ be the abelian group freely generated by the oriented $\mathbb{Z}$–spheres up to orientation-preserving diffeomorphisms. Let M be a $\mathbb{Z}$–sphere and let $G \subset M$ be a Y–link with n components indexed by $\{1,\dots,n\}$. For any subset $J \subset \{1,\dots,n\}$, let $G(J)$ be the Y–sublink of G made of the components of G whose indices are in J. Set

$$[M,G] = \sum_{J\subset\{1,\dots,n\}} (-1)^{\sharp J} M_{G(J)} \in \mathcal{F}.$$

Let $\mathcal{F}_n$ denote the subgroup of $\mathcal{F}$ generated by all the elements $[M,G]$, where G is an n–component Y–link in a $\mathbb{Z}$–sphere M. This defines a filtration

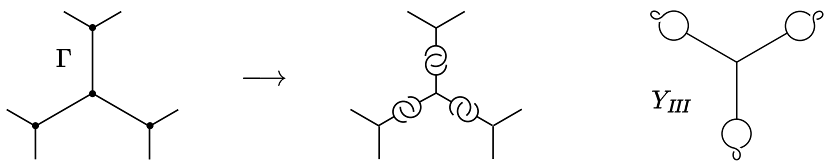
$$\mathcal{F}_0 = \mathcal{F} \supset \mathcal{F}_1 \supset \dots \mathcal{F}_n \supset \dots$$

of $\mathcal{F}$. It is called the Goussarov–Habiro filtration (see [GGP] and [Hbo]). Set

$$\mathcal{G}_n = \mathcal{F}_n/\mathcal{F}_{n+1}.$$

2.3 Linking Jacobi diagrams to the Goussavov–Habiro filtration

Below, following [GGP], we describe a surjective map from $\oplus_{2k\le n}\mathcal{A}_k$ to $\mathcal{G}_n$, whose tensor product by $\mathbb{Z}[1/2]$ is an isomorphism. Let k and n be integers such that $2k \le n$. Let Γ be a degree k oriented Jacobi diagram. Let $\tilde\Gamma$ be an arbitrary framed embedding of Γ in S^3, where the framing is induced by a regular projection of $\tilde\Gamma$ in $\mathbb{R}^2$ that induces the counterclockwise orientation of the trivalent vertices of Γ. Insert a Hopk link on each edge of $\tilde\Gamma$ as illustrated in Figure 2(a). Let $G(\tilde\Gamma)$ denote the resulting Y–link in S^3.



(a) (b)

Figure 2: Turning a Jacobi diagram into a Y–link

Let Y_{III} be the framed Y–graph embedded in S^3 shown in Figure 2(b). Let $\phi_n(\Gamma)$ be the disjoint union of $G(\tilde\Gamma)$ and of $n-2k$ copies of Y_{III}.

Theorem 2.1 [GGP, Theorem 4.13] *The linear map*

$$\begin{array}{rccc} \Psi_n\colon & \bigoplus_{2k\leq n}\mathcal{A}_k & \longrightarrow & \mathcal{G}_n \\ & \Gamma & \longmapsto & \overline{[S^3,\phi_n(\Gamma)]} \end{array}$$

does not depend on the choice of ϕ_n, is well-defined and is surjective. Moreover, in $\mathcal{G}_n$,

$$2\Psi_n(\bigoplus_{2k<n}\mathcal{A}_k) = 0.$$

That Ψ_n is independent of the choice of ϕ_n, factors through AS and satisfies $2\Psi_n(\oplus_{2k<n}\mathcal{A}_k) = 0$ is a consequence of Proposition 4.13 proved below. In this article, the class $\overline{[M,G]} \in \mathcal{G}_n$ of the bracket $[M,G]$ of any n–component Y–link, will be expressed as an explicit combination of the $\Psi_n(\Gamma)$ for oriented Jacobi diagrams Γ with at most n vertices. Therefore, the surjectivity of Ψ_n will be reproved. For the sake of completeness, a proof that Ψ_n factors through IHX is given in the appendix.

3 Statement of the main result

3.1 LP–surgeries

An *n–component LP–surgery* is a 3–tuple

$$D = (M; n; (A_i, B_i)_{i=1,\dots,n})$$

where

- M is a homology sphere, $n \in \mathbb{N}$,
- for any $i = 1, 2, \dots n$, A_i and B_i are $\mathbb{Z}$–handlebodies whose boundaries are identified by implicit diffeomorphisms (we shall write $\partial B_i = \partial A_i$), so that $\mathcal{L}_{B_i} = \mathcal{L}_{A_i}$,
- the disjoint union of the A_i is embedded in M. We shall write

$$\sqcup_{i=1}^{n} A_i \subset M.$$

For such an LP–surgery D, and for any subset $J \subset \{1, \dots, n\}$, set

$$D(J) = \big(M; \sharp J; (A_i, B_i)_{i\in J}\big).$$

Let $M_{D(J)}$ denote the homology sphere obtained by replacing A_i by B_i for any element i of J.

$$M_{D(J)} = \big(M \setminus \sqcup_{i\in J}\mathrm{Int}(A_i)\big) \bigcup_{\sqcup_{i\in J}\partial A_i} \big(\sqcup_{i\in J} B_i\big).$$

Define

$$[D] = \sum_{J \subset \{1,\ldots,n\}} (-1)^{\sharp J} M_{D(J)} \ \in \mathcal{F}.$$

The following proposition will be proved in Subsection 4.2.

Proposition 3.1 *For any n–component LP–surgery D,*

$$[D] \in \mathcal{F}_n.$$

Conversely, any n–component Y–link $G = (G_i)_{i \in \{1,\ldots,n\}}$ in a homology sphere M, induces the n–component LP–surgery

$$(M;G) = \big(M;n;(A_i,B_i)_{i=1,\ldots,n}\big)$$

such that, for any i in $\{1,\ldots,n\}$, A_i is a regular neighbourhood of the Y–component G_i of G, and $B_i = (A_i)_{G_i}$. Then $[(M;G)] = [M,G]$ and $(M;G)$ is called the *LP–surgery induced by* G.

This allows us to give the following alternative definition for the Goussarov–Habiro filtration.

Corollary 3.2 *$\mathcal{F}_n$ is the subspace of $\mathcal{F}$ generated by the elements $[D]$, where D runs among the n–component LP–surgeries.*

In what follows, for any n–component LP–surgery D, $\overline{[D]}$ denotes the class of $[D]$ in $\mathcal{G}_n$. It is called the *bracket* of D.

3.2 The linking number of an LP–surgery with respect to a Jacobi diagram

This subsection is devoted to the definition of the linking number $\ell(D;\Gamma)$ of an n–component LP–surgery D with respect to a degree k Jacobi diagram Γ, with $2k \le n$.

Let Γ be an oriented degree k Jacobi diagram. Define a map

$$h\colon\ H(\Gamma) \longrightarrow \{1,2,3\}$$

such that, for any vertex w of Γ, the map

$$h_w\colon\ v^{-1}(w) \longrightarrow \{1,2,3\}$$

is a bijection. Set

$$\operatorname{sign}(h) = \prod_{w \in V(\Gamma)} \operatorname{sgn}(h_w)$$

where, for any vertex w of Γ, $\operatorname{sgn}(h_w) = 1$ if the orientation of w is induced by the order of the half-edges given by h_w, and $\operatorname{sgn}(h_w) = -1$ otherwise. A *coloration of* Γ is a bijection $\sigma\colon V(\Gamma) \longrightarrow \{1,\dots,2k\}$. Below, σ also denotes the induced map $\sigma \circ v\colon H(\Gamma) \longrightarrow \{1,\dots,2k\}$. Let $D = \big(M; 2k; (A_i, B_i)\big)$ be a $2k$–component LP–surgery. Let us define the *linking number* $\ell(D;\Gamma;\sigma)$ *of* D *with respect to* Γ *and* σ.

The boundary of an oriented manifold is always oriented with the outward normal first convention. The Mayer–Vietoris boundary map

$$\partial_{i,MV}\colon H_2(A_i \cup_{\partial A_i} -B_i) \longrightarrow \mathcal{L}_{A_i},$$

that maps the homology class of an oriented surface to the oriented boundary of its intersection with A_i, is an isomorphism. This isomorphism carries the triple intersection of surfaces in the closed 3–manifold $(A_i \cup_{\partial A_i} -B_i)$ on $\bigotimes^3 H_2(A_i \cup_{\partial A_i} -B_i)$ to a linear form $\mathcal{I}(A_i,B_i)$ on $\bigotimes_{j=1}^3 \mathcal{L}_{A_i}^{(j)}$ which is antisymmetric with respect to the permutation of two factors, where $\mathcal{L}_{A_i}^{(j)}$ denotes the j^{th} copy of $\mathcal{L}_{A_i}$. Then the linear form $\mathcal{I}(A_i,B_i)$ is an element of $\bigotimes_{j=1}^3 \big(\mathcal{L}_{A_i}^{(j)}\big)^*$ where $\big(\mathcal{L}_{A_i}^{(j)}\big)^*$ denotes the dual $\operatorname{Hom}(\mathcal{L}_{A_i}^{(j)};\mathbb{Z})$ of $\mathcal{L}_{A_i}^{(j)}$. Let $c \in H(\Gamma)$. Define

$$X(c) = \left(\mathcal{L}_{A_{\sigma(c)}}^{(h(c))}\right)^*.$$

The linear form $\mathcal{I}(A_i,B_i)$ belongs to

$$\bigotimes_{\{c \in H(\Gamma);\ \sigma(c)=i\}} X(c).$$

Then define

$$T(D;\Gamma;\sigma) = \operatorname{sign}(h) \bigotimes_{w \in V(\Gamma)} \mathcal{I}(A_{\sigma(w)}, B_{\sigma(w)}) \in \bigotimes_{c \in H(\Gamma)} X(c).$$

Note that $T(D;\Gamma;\sigma)$ is independent of h.

Notation 3.3 Let A be a $\mathbb{Z}$–handlebody. Then $H_1(A)$ is canonically isomorphic to $\frac{H_1(\partial A)}{\mathcal{L}_A}$. Furthermore, the intersection form $\langle\ ,\ \rangle_{\partial A}$ induces the map

$$\begin{array}{rccc} \langle\ ,\ .\rangle\colon & H_1(\partial A) & \longrightarrow & \mathcal{L}_A^* \\ & x & \longmapsto & \langle\ .\ ,\ x\rangle \end{array}$$

that in turn induces an isomorphism from $\frac{H_1(\partial A)}{\mathcal{L}_A}$ to $\mathcal{L}^*_A$. Then

$$\varphi_A \colon H_1(A) \longrightarrow \mathcal{L}^*_A$$

will denote the composition of these two isomorphisms.

For $\{i,j\} \subset \{1,2,\ldots,2k\}$, the linking number in M induces a bilinear form on $H_1(A_i) \times H_1(A_j)$ that is viewed as a linear form on $\mathcal{L}^*_{A_i} \otimes \mathcal{L}^*_{A_j}$ via $\varphi^{-1}_{A_i} \otimes \varphi^{-1}_{A_j}$. Therefore, for each edge $f \in E(\Gamma)$ made of two half-edges c and d (such that $e^{-1}(f) = \{c,d\}$), the linking number yields a contraction

$$\ell_f \colon X(c) \otimes X(d) \longrightarrow \mathbb{Z}.$$

Applying all these contractions to the tensor $T(D;\Gamma;\sigma)$ maps $T(D;\Gamma;\sigma)$ to the integral *linking number* $\ell(D;\Gamma;\sigma)$ *of* D *with respect to* Γ *and* σ.

For any automorphism ϕ in $\mathrm{Aut}(\Gamma)$, let

$$\phi_v \colon V(\Gamma) \longrightarrow V(\Gamma)$$

denote the bijection such that $v \circ \phi = \phi_v \circ v$. Let $\mathrm{Bij}(\Gamma)$ denote the set of colorations of Γ. Then $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Bij}(\Gamma)$ by the action

$$\phi \, . \, \sigma = \sigma \circ (\phi_v)^{-1}.$$

Let $\mathrm{Bij}(\Gamma)/\mathrm{Aut}(\Gamma)$ denote the quotient of $\mathrm{Bij}(\Gamma)$ under this action. Note that, for any automorphism ϕ of Γ,

$$\ell(D;\Gamma;\sigma) = \mathrm{sign}(\phi).\ell(D;\Gamma;\phi \, . \, \sigma).$$

The following lemma is proved at the end of the next subsection.

Lemma 3.4 *There exists an integer* $\ell_0(D;\Gamma;\sigma)$ *such that*

$$\ell(D;\Gamma;\sigma) = \sharp\mathrm{Aut}_V(\Gamma) \, . \, \ell_0(D;\Gamma;\sigma).$$

In what follows, for any $\mathbb{Z}$–sphere M, $\mu(M) \in \mathbb{Z}/2\mathbb{Z}$ will denote the *Rohlin invariant* of M that is the reduction mod 2 of the Casson invariant (see [GM, Proposition 1.3, Definition 1.6]).

For any n–component LP–surgery $D = \big(M;n;(A_i,B_i)\big)$ and for any subset $J \subset \{1,\ldots,n\}$, set

$$\mathcal{L}(D(\bar{J})) = \prod_{i \in (\{1,\ldots,n\} \setminus J)} \big(\mu((M \setminus \mathrm{Int}(A_i)) \cup B_i) - \mu(M)\big).$$

Let Γ be an oriented degree k Jacobi diagram. Let $D = \big(M;n;(A_i,B_i)\big)$ be an n–component LP–surgery with $2k \leq n$. Here, we define the *linking number* $\ell(D;\Gamma)$ *of* D *with respect to* Γ.

- If $2k=n$ and if Γ is not reversible, then set
$$\ell(D;\Gamma)=\sum_{\sigma\in \mathrm{Bij}(\Gamma)}\frac{\ell(D;\Gamma;\sigma)}{\sharp \mathrm{Aut}(\Gamma)}\in \mathbb{Z}.$$
Note that
$$\ell(D;\Gamma)=\sum_{\overline{\sigma}\in \mathrm{Bij}(\Gamma)/\mathrm{Aut}(\Gamma)}\ell_0(D;\Gamma;\sigma).$$
- If $2k=n$ and if Γ is reversible, then set
$$\ell(D;\Gamma)=\sum_{\overline{\sigma}\in \mathrm{Bij}(\Gamma)/\mathrm{Aut}(\Gamma)}\overline{\ell_0(D;\Gamma;\sigma)}\in \mathbb{Z}/2\mathbb{Z}$$
where $\overline{\ell_0(D;\Gamma;\sigma)}\in \mathbb{Z}/2\mathbb{Z}$ denotes the mod 2 reduction of $\ell_0(D;\Gamma;\sigma)$.
- If $2k<n$, then set
$$\ell(D;\Gamma)=\sum_{\{J\subset\{1,\dots,n\}\ ;\ \sharp J=2k\}}\ell\big(D(J);\Gamma\big)\ .\ \mathcal{L}\big(D(\bar{J})\big)\in \mathbb{Z}/2\mathbb{Z}.$$

3.3 Expression of brackets of LP–surgeries in terms of Jacobi diagrams

Let $n\in\mathbb{N}$. Let $\mathcal{J}_n$ be a set of oriented Jacobi diagrams of degree at most $n/2$ that contains one Jacobi diagram in each isomorphism class of non-oriented Jacobi diagrams of degree at most $n/2$. The main goal of this paper is to show the following result.

Theorem 3.5 *Let D be an n–component LP–surgery. Then*
$$\overline{[D]}=\sum_{\Gamma\in\mathcal{J}_n}\ell(D;\Gamma).\Psi_n(\Gamma)\in\mathcal{G}_n.$$

Proof of Lemma 3.4 For any $i\in\{1,\dots,2k\}$, let $(a^i_j)_{j\in J_i}$ be a basis of $\mathcal{L}_{A_i}$, where $J_i=\{1,\dots,g_i\}$ and g_i is the genus of ∂A_i. Let $(z^i_j)_{j\in J_i}$ be the basis of $H_1(A_i)$ such that, for any k and l in J_i, $(\varphi_{A_i}(z^i_k))(a^i_l)=\delta_{kl}$. Let c_1, c_2 and c_3 be the three half-edges of Γ such that $\sigma(c_k)=i$ and $h(c_k)=k$. Then
$$\mathcal{I}(A_i,B_i)=\sum_{(j_1,j_2,j_3)\in J_i^3}\mathcal{I}\left(A_i,B_i\right)\left(a^i_{j_1},a^i_{j_2},a^i_{j_3}\right)\,\varphi_{A_i}(z^i_{j_1})\otimes\varphi_{A_i}(z^i_{j_2})\otimes\varphi_{A_i}(z^i_{j_3})$$

$$= \sum_{\substack{(j_1,j_2,j_3)\in J_i^3 \\ j_1<j_2<j_3}} \left(\mathcal{I}(A_i,B_i)(a^i_{j_1},a^i_{j_2},a^i_{j_3}) \sum_{\tau\in\mathcal{S}_3} \left(\mathrm{sgn}(\tau) \bigotimes_{k=1,2,3} \varphi_{A_i}(z^i_{j_{\tau(k)}}) \right)\right)$$

where $\varphi_{A_i}(z^i_{j_{\tau(k)}}) \in X(c_k)$, $\mathcal{S}_3$ denotes the set of the permutations of $\{1,2,3\}$ and $\mathrm{sgn}(\tau)$ denotes the signature of the permutation τ.

Let $\mathcal{H}(\Gamma)$ denote the set of maps $h'\colon H(\Gamma) \longrightarrow \{1,2,3\}$ such that $h'(v^{-1}(w)) = \{1,2,3\}$ for any $w \in V(\Gamma)$. Set

$$J = \left\{ (j^1_1,j^1_2,j^1_3,\ldots,j^{2k}_1,j^{2k}_2,j^{2k}_3) \in \prod_{i=1}^{2k} (J_i)^3 \ ;\forall i\in\{1,\ldots,2k\},\ j^i_1<j^i_2<j^i_3 \right\}.$$

For any $j\in J$, set $\mathcal{J}(j) = \prod_{i=1}^{2k} \mathcal{I}(A_i,B_i)(a^i_{j_1},a^i_{j_2},a^i_{j_3})$. Then

$$T(D;\Gamma;\sigma) = \sum_{j\in J} \mathcal{J}(j) \Bigg(\sum_{h'\in\mathcal{H}(\Gamma)} \mathrm{sign}(h') \bigotimes_{c\in H(\Gamma)} \varphi_{A_{\sigma(c)}} \left(z^{\sigma(c)}_{j^{\sigma(c)}_{h'(c)}} \right) \Bigg).$$

Then $\ell(D;\Gamma;\sigma) = \sum_{h'\in\mathcal{H}(\Gamma)} \ell(D;\Gamma;\sigma;h')$ where

$$\ell(D;\Gamma;\sigma;h') = \mathrm{sign}(h') \sum_{j\in J} \mathcal{J}(j) \Bigg(\prod_{e=(c_1,c_2)\in E(\Gamma)} \ell k\big(z^{\sigma(c_1)}_{j^{\sigma(c_1)}_{h'(c_1)}}, z^{\sigma(c_2)}_{j^{\sigma(c_2)}_{h'(c_2)}} \big) \Bigg).$$

For any automorphism $\zeta \in \mathrm{Aut}_V(\Gamma)$, $\ell(D;\Gamma;\sigma;h'\circ\zeta) = \ell(D;\Gamma;\sigma;h')$. Then $\ell_0(D;\Gamma;\sigma)$ is the sum of the integers $\ell(D;\Gamma;\sigma;h')$ running over all classes $\overline{h'}$ of $\mathcal{H}(\Gamma)/\mathrm{Aut}_V(\Gamma)$. □

4 Proof of the theorem

4.1 Proof of Theorem 3.5 for LP–surgeries induced by Jacobi diagrams

Here we prove Theorem 3.5 when $D = (S^3;\phi_n(\Gamma_Y))$, where the Y–link $\phi_n(\Gamma_Y)$ is the image of a Jacobi diagram Γ_Y under the map ϕ_n of Subsection 2.3. It is a direct corollary of the proposition below (and of Theorem 2.1).

Proposition 4.1 *Let Γ be an oriented degree k Jacobi diagram. Let Γ_Y be an oriented degree k' Jacobi diagram. Let n be an integer such that $n \geq max(2k,2k')$. Then*

$$\ell\big(\phi_n(\Gamma_Y);\Gamma\big) = \begin{cases} 1 & \text{if } \Gamma_Y \cong \Gamma \\ -1 & \text{if } \Gamma_Y \cong -\Gamma \\ 0 & \text{if } \Gamma_Y \not\cong \pm\Gamma \end{cases}$$

where $\Gamma \cong \Gamma'$ iff Γ and Γ' are isomorphic as oriented Jacobi diagrams.

Lemma 4.2 *Let G be a framed Y–graph embedded in the interior of a 3–handlebody A as in Figure 3. Let B be the $\mathbb{Z}$–handlebody obtained by Y–surgery on A along G. Let $(a_1, a_2, a_3) \subset \partial A$ be the oriented curves represented in Figure 3. Then (a_1, a_2, a_3) is a basis of $\mathcal{L}_A = \mathcal{L}_B$ and*

$$|\big(\mathcal{I}(A,B)\big)(a_1 \otimes a_2 \otimes a_3)| = 1.$$

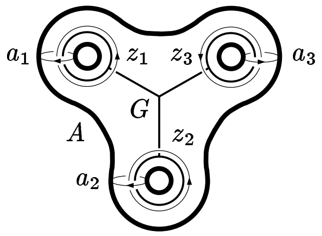


Figure 3

Proof This can be computed directly, or we can use that

$$A \cup_{\partial A} (-B) = S^1 \times S^1 \times S^1$$

$(A \cup (-B)$ is the manifold obtained by surgery on the 0–framed Borromean link in S^3 that is $(S^1)^3$, see [Thu, 13.1.5]). Let S_1, S_2 and S_3 be the three following surfaces in $(S^1)^3$.

$$\begin{array}{rcl} S_1 & = & \{\star\} \times S^1 \times S^1 \\ S_2 & = & S^1 \times \{\star\} \times S^1 \\ S_3 & = & S^1 \times S^1 \times \{\star\}. \end{array}$$

Let $\mathcal{I} \in \left(\bigotimes^3 H_2\big((S^1)^3\big)\right)^*$ be the intersection form of $A \cup_{\partial A} (-B) = (S^1)^3$. Since $S_1 \cap S_2 \cap S_3 = \{\star\} \times \{\star\} \times \{\star\}$ is a single transverse intersection point, then

$$|\mathcal{I}(S_1 \otimes S_2 \otimes S_3)| = 1.$$

By the isomorphism from $H_2\big((S^1)^3\big)$ to $\mathcal{L}_A$ induced by the Mayer–Vietoris boundary map, (a_1, a_2, a_3) can be seen as a basis of $H_2\big((S^1)^3\big)$. Therefore, $\mathcal{I}(S_1 \otimes S_2 \otimes S_3)$ is a multiple of $\big(\mathcal{I}(A,B)\big)(a_1 \otimes a_2 \otimes a_3)$. Then

$$|\big(\mathcal{I}(A,B)\big)(a_1 \otimes a_2 \otimes a_3)| = 1. \qquad \square$$

Lemma 4.3 $\mu(S^3_{Y_{III}}) = 1$

This lemma is a direct consequence of Corollary 4.21 that is proved in Subsection 4.4. It relies on the results of Subsections 4.2 and 4.3 that are logically independent of the proof below that illustrates our formulae.

Proof of Proposition 4.1

- First assume that $2k = 2k' = n$.

Let σ be a coloration of Γ. Let $D = \big(S^3; \phi_n(\Gamma_Y)\big) = \big(M; n; (A_i, B_i)\big)$ be the LP–surgery induced by the Y–link $\phi_n(\Gamma_Y)$. Each pair (A_i, B_i) is a copy of the pair (A, B) presented in Lemma 4.2. Let $i \in \{1, \dots, n\}$. Let (a^i_1, a^i_2, a^i_3) be the basis of $\mathcal{L}_{A_i}$ that corresponds to the curves (a_1, a_2, a_3) in Figure 3. Let (z^i_1, z^i_2, z^i_3) be the basis of $H_1(A_i)$ that corresponds to the curves (z_1, z_2, z_3) in Figure 3. Under the (implicit from now on) isomorphism

$$\varphi_{A_i} \colon H_1(A_i) \longrightarrow \mathcal{L}^*_{A_i}$$

presented in Notation 3.3, (z^i_1, z^i_2, z^i_3) is the dual basis to (a^i_1, a^i_2, a^i_3), i.e.

$$\big(\varphi_{A_i}(z^i_k)\big)(a^i_l) = \delta_{kl}.$$

Then

$$\mathcal{I}(A_i, B_i) = \sum_{\tau} \operatorname{sgn}(\tau)\ \mathcal{I}\big(A, B\big)(a_1 \otimes a_2 \otimes a_3)\ z^i_{\tau(1)} \otimes z^i_{\tau(2)} \otimes z^i_{\tau(3)}.$$

Since $\big|\big(\mathcal{I}(A, B)\big)(a_1 \otimes a_2 \otimes a_3)\big| = 1$ by Lemma 4.2,

$$T(D; \Gamma; \sigma) = \operatorname{sign}(h) \sum_{(\tau_i) \in (\mathcal{S}_3)^n} \Big(\big(\prod_{i=1}^{n} \operatorname{sgn}(\tau_i)\big) \bigotimes_{c \in H(\Gamma)} z^{\sigma(c)}_{\tau_{\sigma(c)}(h(c))} \Big)$$

where h is as in Subsection 3.2. For any $\tau = (\tau_i)_{i=1,\dots,n} \in (\mathcal{S}_3)^n$, let $\zeta(\sigma; \tau)$ denote the map

$$\begin{array}{rccl} \zeta(\sigma;\tau)\colon & H(\Gamma) & \longrightarrow & \{1, \dots, n\} \times \{1, 2, 3\} \\ & c & \longmapsto & \big(\sigma(c), \tau_{\sigma(c)}(h(c))\big). \end{array}$$

Let

$$\begin{array}{rccl} \xi\colon & H(\Gamma_Y) & \longrightarrow & \{1, \dots, n\} \times \{1, 2, 3\} \\ & c & \longmapsto & (\xi_1(c), \xi_2(c)) \end{array}$$

be the bijection such that, for any half-edge c of Γ_Y, $z^{\xi_1(c)}_{\xi_2(c)}$ is the core of the leaf corresponding to c. Set $\phi(\sigma; \tau) = \xi^{-1} \circ \zeta(\sigma; \tau)$. Then $\phi(\sigma; \tau)$ is a bijection from $H(\Gamma)$ to $H(\Gamma_Y)$ such that

- for any c, c' in $H(\Gamma)$, $v(\phi(\sigma; \tau)(c)) = v(\phi(\sigma; \tau)(c'))$ if and only if $v(c) = v(c')$

- for any edge $e=(c,c')$ of $H(\Gamma)$,

$$\ell k(z^{\sigma(c)}_{\tau_{\sigma(c)}(h(c))}, z^{\sigma(c')}_{\tau_{\sigma(c')}(h(c'))}) = \begin{cases} 1 & \text{if } \phi(\sigma;\tau)(c) \text{ and } \phi(\sigma;\tau)(c') \\ & \text{belong to the same edge of } \Gamma_Y \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\ell(D;\Gamma;\sigma) = \sum_{\{\tau\in(\mathcal{S}_3)^n;\phi(\sigma;\tau) \text{ is an isomorphism}\}} \operatorname{sign}(\phi(\sigma;\tau))$$

where

$$\operatorname{sign}(\phi(\sigma;\tau)) = \operatorname{sign}(h)\ (\prod_{i=1}^{n} \operatorname{sgn}(\tau_i)).$$

Hence, if $\Gamma \not\cong \pm\Gamma_Y$, then for any coloration σ of Γ, $\ell(D;\Gamma;\sigma)=0$, and $\ell(D;\Gamma)=0$.

Otherwise, there exists a coloration σ of Γ and a map $\tau \in (\mathcal{S}^3)^n$ such that $\phi(\sigma;\tau)$ is an orientation-preserving isomorphism from Γ to $\operatorname{sign}(\phi(\sigma;\tau))\,\Gamma_Y$. For any map $\tau' \in (\mathcal{S}^3)^n$ such that $\phi(\sigma;\tau')$ is an isomorphism, $(\phi(\sigma;\tau'))^{-1}\circ\phi(\sigma;\tau)$ is an automorphism of Γ that preserves the vertices.

Then $\operatorname{sign}\big(\phi(\sigma;\tau)\big) = \operatorname{sign}\big(\phi(\sigma;\tau')\big)$.

Conversely, any automorphism of $\mathrm{Aut}_V(\Gamma)$ provides such a map τ'. Then

$$\ell(D;\Gamma;\sigma) = \operatorname{sign}\big(\phi(\sigma;\tau)\big)\ \sharp\mathrm{Aut}_V(\Gamma).$$

For any other pair $(\sigma';\tau')$ such that $\phi(\sigma';\tau')$ is an isomorphism from Γ to Γ_Y, σ' is obtained from σ by composition by an automorphism of Γ. Then $\ell(D;\Gamma)=\ell_0(D;\Gamma;\sigma)=\operatorname{sign}\big(\phi(\sigma;\tau)\big)$ and Proposition 4.1 is proved in this case.

- If $2k' < 2k = n$, then $\ell(\phi_{2k}(\Gamma_Y);\Gamma)=0$ because when A is the regular neighbourhood of Y_{III}, the elements of $H_1(A)$ do not link any element of the other $H_1(A_i)$'s.

- When $2k<n$, let $J\subset\{1,\dots,n\}$ and let $\overline{J}=\{1,\dots,n\}\setminus J$. Let J_Y be the set of indices of the $2k'$–component Y–link $G(\tilde{\Gamma}_Y)$. See Subsection 2.3. $\sharp J_Y = 2k'$. Set $\overline{J_Y}=\{1,\dots,n\}\setminus J_Y$.

If $J_Y\cap\overline{J}\neq\emptyset$, then $\mathcal{L}(D(\overline{J}))=0$ since $S^3_{Y_0}=S^3$ when Y_0 is a Y–graph in S^3 with a trivial leaf.

If $\overline{J_Y}\cap J\neq\emptyset$, then $\ell(D(J);\Gamma)=0$ like in the previous case. Then

$$\ell(D;\Gamma) = \begin{cases} 0 & \text{if } k\neq k' \\ \ell(D(J_Y);\Gamma)\ .\ \mathcal{L}(D(\overline{J_Y})) & \text{if } k=k'. \end{cases}$$

Then $\mathcal{L}(D(\overline{J_Y}))=1$ by Lemma 4.3 and $\ell(D(J_Y);\Gamma)=\ell(\phi_{2k}(\Gamma_Y);\Gamma)$. Thus the result follows from the first case. $\square$

4.2 Decomposition of LP–surgeries into surgeries on Y–links

In this subsection, we recall known facts and state useful lemmas about the theory of Borromeo surgeries [Mat, GGP]. We shall see all these facts as consequences of the following single lemma 4.4. As an application of the theory of Borromeo surgeries, we shall prove Proposition 3.1.

Lemma 4.4 [GGP, Lemma 2.1] *Let M be an oriented 3–manifold (with possible boundary). Let G be a Y–graph in M with a trivial leaf that bounds a disc D in $M \setminus G$. Then*

- *for any framed graph T_0 in $M \setminus G$ that does not meet D, the pair (M_G, T_0) is diffeomorphic to the pair (M, T_0).*
- *If T is a framed graph in $M \setminus G$ that meets $\mathrm{Int}(D)$ at exactly one point, then the pair (M_G, T) is diffeomorphic to the pair (M, T_G), where T_G is the framed graph in M presented in Figure 4.*

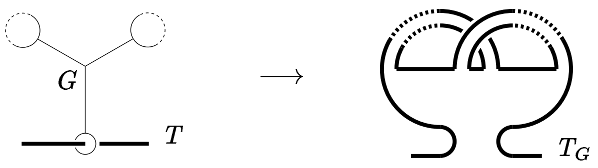


Figure 4

Corollary 4.5 *Let M be an oriented 3–manifold. Let Σ denote a genus 1 surface in M. Let I_1 and I_2 be two intervals such that*

- $\partial\Sigma = I_1 \cup I_2$
- $I_1 \cap I_2 = \partial I_1 = \partial I_2$
- *I_1 and I_2 are framed by a vector field normal to the surface Σ.*

Let T be a framed graph such that $I_1 = T \cap \Sigma$. Then there exists a Y–graph G in $M \setminus T$ with a trivial leaf that is a meridian curve of I_1 such that the pair (M_G, T) is diffeomorphic to the pair $(M, (T \setminus \mathrm{Int}(I_1)) \cup I_2)$

Lemma 4.6 [GGP, Theorem 3.2] *Let Λ be the Y–graph in the 3–handlebody $(N = \Sigma(\Lambda) \times [-1,1])$ presented in Figure 1(a). Then there exists a Y–graph Λ^{-1} in $N \setminus \Lambda$ such that the Y–surgery along $\Lambda \cup \Lambda^{-1}$ does not change N. In particular, if M is a 3–manifold then, for any Y–graph G in M, there exists a Y–graph G^{-1} in a regular neighbourhood of G such that $M_{G \cup G^{-1}} = M$.*

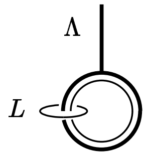


Figure 5: Trivializing a leaf

Proof Let L be the framed link in $N \setminus \Lambda$ made of the two framed knots presented in Figure 5, such that $N_L = N$ and such that Λ is isotopic to a Y–graph Λ_0 with a trivial leaf in N_L. Let L^{-1} denote a framed link in $N \setminus (\Lambda \cup L)$ such that the surgery along $L \cup L^{-1}$ is trivial in $N \setminus \Lambda$. Then L^{-1} corresponds to a framed link L' in N_L. Then

$$N_\Lambda = N_{\Lambda \cup L \cup L^{-1}} = (N_L)_{\Lambda_0 \cup L'}.$$

The Y–surgery along Λ_0 is fully determined by Lemma 4.4. It takes the tube piercing the trivial leaf and makes it describe its complement in the boundary of a genus one surface.

By Corollary 4.5, there exists a Y–graph Λ_0^{-1} in $N_L \setminus (\Lambda_0 \cup L')$ that undoes it.

$$N = \left((N_L)_{\Lambda_0 \cup \Lambda_0^{-1}}\right)_{L'}.$$

After surgery on L', that does not change N since the surgery on L did not change N, Λ_0^{-1} corresponds to a Y–graph Λ^{-1} in $N \setminus \Lambda$ such that the Y–surgery along $\Lambda \cup \Lambda^{-1}$ is trivial. □

Remark 4.7 What is used in the above proof and will be used again is the following principle. Up to surgery along links, one leaf of a Y–graph can be assumed to bound a disk D (pierced by surgery arcs). Then surgery along that Y–graph amounts to move the pack T of framed surgery arcs piercing D as indicated in Lemma 4.4, that therefore fully determines the effect of the surgery along the Y–graph.

Lemma 4.8 *Let ϕ be an embedding of the genus g handlebody H_g into S^3. Let $z_1, \dots, z_g$ denote the curves in ∂H_g presented in Figure 6. If each curve $\phi(z_i)$ bounds an embedded surface in $S^3 \setminus \mathrm{Int}(\phi(H_g))$, then there exists a Y–link G in $S^3 \setminus \phi(H_g)$ such that $S^3_G = S^3$ and the curves $\phi(z_i)$ bound embedded discs in $S^3_G \setminus \mathrm{Int}(\phi(H_g))$.*

Proof Thanks to Corollary 4.5 and to the fact that any orientable surface is a connected sum of genus one surfaces, there exists a Y–link G_1 in $S^3 \setminus \phi(H_g)$

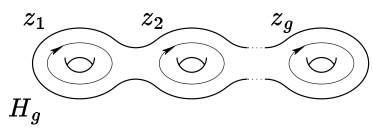


Figure 6

such that $S^3_{G_1} = S^3$ and $\phi(z_1)$ bounds an embedded disc D_1 in $S^3 \setminus \phi(H_g)$ after Y–surgery on G_1. In particular the lemma is true for $g = 1$. Assume that the lemma is true for handlebodies of genus $g-1$. We shall use this induction hypothesis for a regular neighbourhood N of $\phi(H_g) \cup D_1$ equipped with the curves $\phi(z_2), \dots, \phi(z_g)$ that are still homologically trivial in $S^3 \setminus N$. Let $\hat{G}_1 \subset S^3_{G_1}$ be the union of the 3–handlebodies reglued during the Y–surgery on G_1. By induction hypothesis, there exists a Y–link G_2 in $S^3_{G_1} \setminus N$ such that $(S^3_{G_1})_{G_2} = S^3$ and the curves $\phi(z_2), \dots, \phi(z_g)$ bound embedded discs in $S^3 \setminus N$ after Y–surgery on G_2. After a possible isotopy in $S^3_{G_1}$, G_2 avoids $\hat{G}_1$. Then G_2 corresponds to a Y–link G'_2 in $S^3 \setminus \big(\phi(H_g) \cup G_1\big)$ such that $G_1 \cup G'_2$ satisfies the conclusion of the lemma. □

Let M be a $\mathbb{Z}$–sphere. Let ℓk denote the linking number in M. Let $L \subset M$ be a link, and let $L_1, \dots, L_n$ be the components of L. Then L is *algebraically split* if and only if

$$\big(i \neq j\big) \Rightarrow \big(\ell k(L_i, L_j) = 0\big).$$

Then Lemma 4.8 induces the following corollary.

Corollary 4.9 ([Mat, Lemma 2] or [MN, Lemma 1.2]) *Let L be an algebraically split link in S^3. Then there exists a Y–link G in $S^3 \setminus L$ such that $S^3_G = S^3$ and L is trivially embedded in S^3_G.*

Proof Embed H_g in S^3 so that the curves z_i are the components of L. □

Theorem 4.10 [Mat, Theorem 2] *If M and M' are homology spheres, then there exists a Y–link G in M such that $M_G = M'$.*

Proof Since any $\mathbb{Z}$–sphere can be obtained by surgery on S^3 along an algebraically split link framed by ± 1 (see [GM, Lemma 2.1]), and since the surgery on the trivial knot in S^3 framed by ± 1 gives S^3, Theorem 4.10 is an easy corollary of Lemma 4.6 and Corollary 4.9. □

Then we can prove the following useful lemma (see [Hbg, Theorem 2.5], too).

Lemma 4.11 *Let A and B be two $\mathbb{Z}$–handlebodies with the same genus, whose boundaries are identified so that $\mathcal{L}_A = \mathcal{L}_B$. Then there exists a Y–link G embedded in the interior of A such that $A_G = B$, where the identification of ∂A with ∂B is induced by the natural identification of ∂A with ∂A_G.*

Proof Let us first prove the lemma when $B = H_g$ is the standard handlebody of genus g with the boundary of A identified with the boundary Σ_g of H_g so that $\mathcal{L}_A = \mathcal{L}_{H_g}$. Embed H_g trivially in S^3 so that

$$\tilde{H}_g = S^3 \setminus \mathrm{Int}(H_g)$$

is a standard g–handlebody. Let $z_1, \dots, z_g$ be the meridian curves of H_g on Σ_g presented in Figure 7.

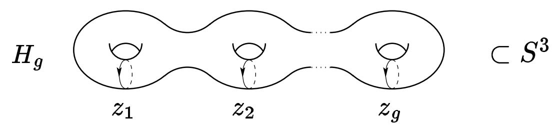


Figure 7

Let $$M = \left(S^3 \setminus \mathrm{Int}(H_g)\right) \cup_{\Sigma_g} A = \tilde{H}_g \cup_{\Sigma_g} A.$$

Then M is a $\mathbb{Z}$–sphere. Thus, by Theorem 4.10, there exists a Y–link $G \subset M$ such that $M_G = S^3$. By isotopy, G can avoid $\tilde{H}_g$. Then

$$S^3 = A_G \cup_{\Sigma_g} \tilde{H}_g$$

Now A_G is the complement in S^3 of a possibly knotted g–handlebody $\tilde{H}_g$. Thanks to Lemma 4.8, there exists a Y–link $G' \subset \mathrm{Int}(A_G)$ such that $S^3_{G'} = S^3$ and $\tilde{H}_g$ is embedded in $S^3_{G'}$ so that the curves z_i bound embedded discs in $A_{G \cup G'}$. Thus $A_{G \cup G'} = H_g$ with the expected boundary identification. The general case follows easily with the help of Lemma 4.6. □

We have the following obvious lemma.

Lemma 4.12 *Let $D = \big(M; n; (A_i, B_i)\big)$ be an n–component LP–surgery. Let A'_1 be a $\mathbb{Z}$–handlebody such that $\partial A'_1$ and ∂A_1 are identified so that $\mathcal{L}_{A'_1} = \mathcal{L}_{A_1}$. Let*

$$M_{A'_1/A_1} = \big(M \setminus \mathrm{Int}(A_1)\big) \cup_{\partial A_1} A'_1$$

denote the manifold obtained by surgery on M along the pair (A_1, A'_1). Set

$$\begin{aligned} D' &= \big(M; n; (A_1, A'_1), (A_2, B_2), \dots, (A_n, B_n)\big) \\ D'' &= \big(M_{A'_1/A_1}; n; (A'_1, B_1), (A_2, B_2), \dots, (A_n, B_n)\big). \end{aligned}$$

Then

$$[D] = [D'] + [D''].$$

Proof of Proposition 3.1 Let $D = \big(M; n; (A_i, B_i)\big)$ be an n–component LP–surgery. Thanks to Lemma 4.11, for any $i \in \{1, \dots, n\}$, there exists a Y–link $G^i \subset \mathrm{Int}(A_i)$ such that $(A_i)_{G^i} = B_i$. Let k_i denote the minimal number of components for such a G_i. Consider the sum $k = \sum_i k_i$.

If there exists $i \in \{1, \dots, n\}$ such that $k_i = 0$, then $[D] = 0 \in \mathcal{F}_n$. If, for all i, $k_i = 1$, then $[D] \in \mathcal{F}_n$ by definition. Therefore $[D] \in \mathcal{F}_n$ if $k \leq n$.

If $k > n$, assume that $k_1 > 1$, without loss of generality. Then there exists a $\mathbb{Z}$–handlebody A_1' verifying the hypotheses of Lemma 4.12 such that A_1' can be obtained from A_1 by Y–surgery along a Y–graph in $\mathrm{Int}(A_1)$, and B_1 can be obtained from A_1' by Y–surgery along a Y–link in $\mathrm{Int}(A_1')$ with $k_1 - 1$ components. Thus, with the notation of Lemma 4.12, $[D'] \in \mathcal{F}_n$ and $[D''] \in \mathcal{F}_n$ by induction on k. Then $[D] \in \mathcal{F}_n$ thanks to Lemma 4.12. The proposition follows. □

4.3 Review of the clover calculus

In this section, we review the clover calculus following [GGP]. However we produce alternative proofs in the spirit of the present paper only based on Lemma 4.4. Furthermore, we summarize what we shall use about the clover calculus in Proposition 4.13.

A Y–graph Λ is *oriented* if its framing surface $\Sigma(\Lambda)$ is equipped with an orientation. Such an orientation provides an orientation for every leaf and (a cyclic order) for the set of leaves of Λ. Figure 8 shows the induced orientations when $\Sigma(\Lambda)$ is given the standard orientation of $\mathbb{R}^2$. Reversing the orientation of $\Sigma(\Lambda)$ reverses these four orientations.

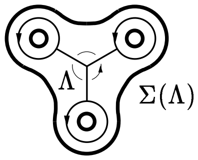


Figure 8: oriented Y–graph

Recall that a *framing* of a knot is a nonzero vector field normal to the knot, up to homotopy, or a parallel to the knot up to isotopy. In a homology sphere these two canonically equivalent notions are represented by the linking number of the knot and its parallel induced by the framing. This linking number is therefore also called the *framing* of the knot.

The goal of this subsection is to prove the following proposition.

Proposition 4.13 *Let G be an oriented n–component Y–link in a $\mathbb{Z}$–sphere M.*

(i) *The bracket $\overline{[(M;G)]}$ (in $\mathcal{G}_n$) is a function independent of M of*

- *the linking numbers $\ell k(l,l')$ where l and l' are leaves in two distinct Y–components of G*
- *the products $\bar{f}(l_1)\bar{f}(l_2)\bar{f}(l_3)$ where l_1, l_2 and l_3 are leaves of a same Y–component, and where $\bar{f}(l)$ is the framing of l in $\mathbb{Z}/2\mathbb{Z}$.*

(ii) *Fix G except for a leaf l in the complement of the other parts of G. Let $[l]$ denote the class of l in $H_1(M \setminus \cup_{l'\neq l} l')$, where the union runs over all leaves l' distinct of l. Then the bracket ($\overline{[(M;G)]} \in \mathcal{G}_n$) of G is a linear map of $([l],\bar{f}(l)) \in H_1(M \setminus \cup_{l'\neq l} l') \times \mathbb{Z}/2\mathbb{Z}$.*

Lemma 4.14 *Let G be an oriented n–component Y–link in a $\mathbb{Z}$–sphere M. The bracket $\overline{[(M;G)]}$ is a function independent of M of*

- *the linking numbers $\ell k(l,l')$ where l and l' are leaves of G*
- *the framings $f(l)$ where l runs over the leaves of G.*

Proof Let Υ be the diagram made of n copies of the diagram Λ connected by an additional edge from the internal vertex of Λ to a common n–valent vertex p. Embed Υ in $\mathbb{R}^3$. Let A be a regular neighbourhood of Υ in $\mathbb{R}^3$. Then A is a union of a ball B with n copies of the genus 3 handlebody N that are glued on ∂B along n disjoint discs. Let $\phi_G\colon A \longrightarrow M$ be an embedding of A in M that extends the embedding G. Set $Z = M \setminus \mathrm{Int}(\phi_G(A))$. Then Z is a genus $3n$ homology handlebody whose Lagrangian $\mathcal{L}_Z \subset H_1(\partial A)$ is fully determined by the framings and by the linking numbers of the leaves of G. Therefore if $G' \subset M'$ is another oriented n–component Y–link with the same linking numbers and framing data, then $Z' = M' \setminus \mathrm{Int}(\phi_{G'}(A))$ is a homology handlebody with the same lagrangian as Z in $H_1(\partial A)$.

By Lemma 4.11, there exists a Y–link $G'' \subset \mathrm{Int}(Z)$ such that $Z_{G''} = Z'$. Then $[(M';G')] = [(M_{G''};G)]$. If G'' is a one-component Y–link, then $[(M;G\cup G'')] = [(M;G)] - [(M_{G''};G)]$, and $\overline{[(M;G)]} = \overline{[(M_{G''};G)]}$. By induction on the number of components of G'', $\overline{[(M;G)]} = \overline{[(M_{G''};G)]}$. Then $\overline{[(M;G)]} = \overline{[(M';G')]}$. □

A framed knot $K_1\sharp_b K_2$ is a *band sum* of two framed oriented knots K_1 and K_2 if there exists an embedding of a 2–hole disk

- that factors the three knot embeddings by the embeddings of the three curves pictured in Figure 9 representing the disk, and
- that induces the three framings.

Note that

$$f(K_1\sharp_b K_2) = f(K_1) + f(K_2) + 2\ell k(K_1, K_2).$$

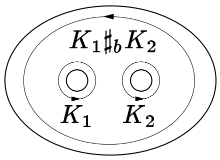


Figure 9: Band sum of two knots

Lemma 4.15 [GGP, Theorem 3.1] *Let G be an oriented framed Y–graph with leaves K_1, K_2, K_3 in a $\mathbb{Z}$–sphere M. Assume that K_3 is a band sum of two framed knots K_3^2 and K_3^3. For $k = 1$ and 2, let K_k^2 and K_k^3 be two parallels of K_k equipped with the framing $f(K_k)$ of K_k, and such that $\ell k(K_k^2, K_k^3) = f(K_k)$. Then*

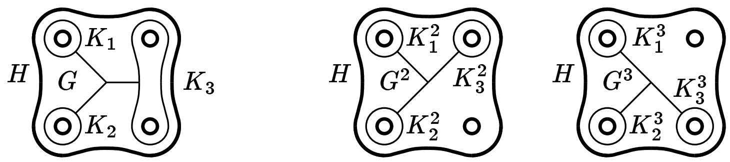


Figure 10: Splitting a leaf

(i) *There exist two oriented disjoint framed Y–graphs G^2 and G^3 in M whose framed leaves are K_1^2, K_2^2, K_3^2 and K_1^3, K_2^3, K_3^3, respectively, such that the surgery along G is equivalent to the surgery along $G^2 \cup G^3$.*

(ii) *For any $(n-1)$–component Y–link L in the complement in M of the embedded neighbourhood H of G represented in Figure 10,*

$$\overline{[(M; L\cup G)]} = \overline{[(M; L\cup G^2)]} + \overline{[(M; L\cup G^3)]}.$$

Proof The surgery operation on G is thought of as the move of two packs P_2 and P_3 of arcs of surgery components that go through the two holes on the right hand-side of H in Figure 10 as in Remark 4.7. Then Lemma 4.4 says that the surgery along G moves these two framed packs of arcs by adding the boundary of a genus 1 surface Σ. This operation can be made in two steps. Move P_3 first, which means do the surgery along a Y–graph G^3 whose leaves are K_1^3, K_2^3 and K_3^3. Then move P_2 so that it is parallel to $\partial\Sigma$ inside Σ. It can be done by a surgery along a Y–graph $G^2 \subset H \setminus G^3$ whose leaves are K_1^2, K_2^2 and K_3^2. Then $M_{G^2\cup G^3} = M_G$. Therefore

$$[(M;L\cup G^2\cup G^3)] = -[(M;L\cup G)] + [(M;L\cup G^2)] + [(M;L\cup G^3)]$$

and

$$\overline{[(M;L\cup G)]} = \overline{[(M;L\cup G^2)]} + \overline{[(M;L\cup G^3)]}. \qquad \square$$

Lemma 4.16 [GGP, Lemma 4.8] *Let $G \subset M$ be an n–component Y–link. Suppose that a Y–component of G contains a 2–framed leaf l that bounds an embedded disc in $M \setminus G$. Then $\overline{[M,G]} = 0$.*

Proof If l is a 2–framed leaf that bounds an embedded disc in $M \setminus G$, then l is a band sum of two knots K^2 and K^3 that form a trivial Hopf link (see Figure 11) in $M \setminus G$. Thanks to Lemma 4.15, there exist two n–components Y–links G^2 and G^3 with a trivial leaf such that

$$\overline{[(M;G)]} = \overline{[(M;G^2)]} + \overline{[(M;G^3)]} = 0. \qquad \square$$

Proof of part (ii) of Proposition 4.13 Consider the bracket of G as a function of a leaf l of G by fixing $G\setminus l$. According to Lemma 4.14, the bracket of G only depends on $[l] \in H_1(M\setminus \cup_{l'\neq l} l')$ and on $f(l)$. Applying Lemmas 4.15 and 4.16 when adding a disjoint 2–framed trivial knot to l shows that this function of l only depends on $[l]$ and on $f(l)$ mod 2. Then Lemma 4.15 implies (ii). $\square$

Lemma 4.17 [GGP, Lemma 2.3] *Let $G \subset M$ be an n–component Y–link. Suppose that G contains a Y–component with two leaves l and l' that form the trivial Hopf link of Figure 11. Then $\overline{[(M;G)]} = 0$.*

Proof The surgery along a Y–graph with this trivial Hopf link is trivial: Think of this surgery as the move of surgery arcs along the boundary of the surface corresponding to these two leaves as in Lemma 4.4. It implies that the bracket of G vanishes. $\square$

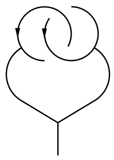


Figure 11: Two leaves that form the trivial Hopf link

Definition 4.18 Let c be a curve in a surface Σ that is the image of $S^1 \times \{\pi\}$ under an orientation-preserving embedding $\phi\colon\ S^1 \times [0, 2\pi] \longrightarrow \Sigma$. A *left-handed Dehn twist* of Σ along c is the homeomorphism of Σ that is the identity outside $\phi(S^1 \times]0, 2\pi[)$ and that maps $\phi(z,t)$ to $\phi(ze^{-it}, t)$.

Lemma 4.19 *[GGP, Theorem 3.1] Let H be an oriented Y–graph in a $\mathbb{Z}$–sphere M. Let l^- and l' be two oriented leaves of H, and let Σ be the genus one surface presented in Figure 12. Let l be an oriented parallel of l^- in Σ equipped with the framing induced by Σ. Let l'' be obtained from l' by a left-handed Dehn twist along l, and equipped with the framing induced by the surface Σ, that is $f(l'') = f(l) + f(l') + 2\ell k(l^-, l') - 1$. Let H' be the Y–graph obtained from H by changing l' into l''. Then $M_H = M_{H'}$ and, for any Y–link*

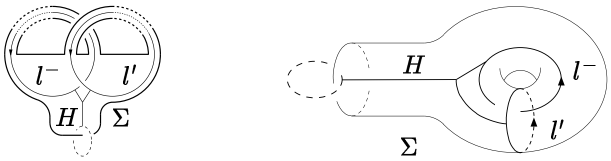


Figure 12

L in the complement in M of a neighbourhood of H,

$$[(M; L \cup H')] = [(M; L \cup H)].$$

Proof Thanks to Lemma 4.4, the surgery on H is uniquely determined by Σ that is unchanged by a Dehn twist of Σ. It implies that $M_H = M_{H'}$ and that, for any sublink $L(J)$ of L, $M_{L(J) \cup H} = M_{L(J) \cup H'}$. The equality of the brackets follows. □

End of proof of Proposition 4.13

- Let l be a leaf of an oriented Y–component H of G. Let l' be the next leaf of H (with respect to the cyclic order). We prove that increasing $\ell k(l, l')$ does not change $\overline{[(M; G)]}$. By Lemma 4.15, adding a 0–framed meridian

$m_0(l')$ of l' to l adds $\overline{[(M;G(m_0(l')/l))]}$ to $\overline{[(M;G)]}$, where $G(m_0(l')/l)$ is obtained from G by changing l into $m_0(l')$. Now, $l' = m' + l'_0$, where l'_0 does not intersect a disk bounded by $m_0(l')$, and m' is a meridian of $m_0(l')$. Then

$$\begin{aligned}\overline{[(M;G(m_0(l')/l))]} &= \overline{[(M;G(m_0(l'),m'/l,l'))]} \\ &\quad +\overline{[(M;G(m_0(l'),l'_0/l,l'))]} \\ &= 0\end{aligned}$$

since $G(m_0(l'),m'/l,l')$ is a Y–link with a trivial Hopf link and since $G(m_0(l'),l'_0/l,l')$ has a trivial leaf.

- To conclude, it is enough to show that if l^- is a 0–framed leaf of G, if l' is the previous leaf in the component of l^- in G (w.r.t. the cyclic order), and if l' is 1–framed, then changing the framing of l' into 0 does not change $\overline{[(M;G)]}$. By linearity, we may assume that l' is a trivial knot, and then it is enough to show that $\overline{[(M;G)]} = 0$. By linearity on l^-, we can assume that l^- is a 0–framed meridian of some leaf l_0 in another Y–component of G. Let G' be the Y–link obtained from G by changing l' into the twisted curve $l'' = l + l'$ as in Lemma 4.19 so that $\bar{f}(l'') = \bar{f}(l') + 1 = 0$. Then l^- and l'' are 0–framed meridians of l_0. By linearity with respect to l_0, $\overline{[(M;G')]}$ is the sum of brackets of two Y–links with a trivial leaf. Then $\overline{[(M;G')]} = \overline{[(M;G)]} = 0$. □

4.4 Proof of Theorem 3.5 for LP–surgeries induced by Y–links

Here we prove Theorem 3.5 when $D = (M;G)$ is an LP–surgery induced by a Y–link G.

Lemma 4.20 *Let A and A' be two $\mathbb{Z}$–handlebodies whose boundaries are identified so that $\mathcal{L}_A = \mathcal{L}_{A'}$. Let Z and Z' be two other $\mathbb{Z}$–handlebodies whose boundaries are identified so that $\mathcal{L}_Z = \mathcal{L}_{Z'}$. Assume that ∂A and $(-\partial Z)$ are identified so that $A \cup_{\partial A} Z$ is a $\mathbb{Z}$–sphere. Then*

$$\mu(A \cup_{\partial A} Z) - \mu(A' \cup_{\partial A} Z) = \mu(A \cup_{\partial A} Z') - \mu(A' \cup_{\partial A} Z').$$

Proof For any $\mathbb{Z}$–sphere M, let $\lambda(M)$ be the *Casson invariant* of M. Then $\mu(M) \equiv \lambda(M) \bmod 2$. Thanks to [Les, Theorem 1.3],

$$\lambda(A \cup_{\partial A} Z) - \lambda(A' \cup_{\partial A} Z) - (\lambda(A \cup_{\partial A} Z') - \lambda(A' \cup_{\partial A} Z'))$$

is an even number. It implies the result. □

Together with Proposition 4.13, it implies

Corollary 4.21 *Let H be a Y–graph in a $\mathbb{Z}$–sphere M. Let*

$$p = \bar{f}(l_1)\bar{f}(l_2)\bar{f}(l_3)$$

denote the product of the framings of the three leaves of H in $\mathbb{Z}/2\mathbb{Z}$. Then

$$\mu(M_H) - \mu(M) = p.$$

Proof First, we prove that μ vanishes on $\mathcal{F}_2$. Let $G = G_1 \cup G_2$ be a 2–component Y–link in a $\mathbb{Z}$–sphere M. Let A be a regular neighbourhood of G_1. Let Z be the complement of $\mathrm{Int}(A)$ in M. Set $A' = A_{G_1}$ and set $Z' = Z_{G_2}$. Then A, A', Z and Z' satisfy the assumptions of Lemma 4.20 and

$$\begin{aligned}&\mu(M_G) - \mu(M_{G_1}) - \mu(M_{G_2}) + \mu(M)\\ &= \mu(A' \cup Z') - \mu(A' \cup Z) - \mu(A \cup Z') + \mu(A \cup Z) = 0.\end{aligned}$$

Then $\mu(M_H) - \mu(M)$ only depends on $\overline{[(M;H)]} \in \mathcal{G}_1$, thus it only depends on the product p thanks to Proposition 4.13, and it vanishes when $p = 0$.

Then $\mu(M_H) - \mu(M) = p$ since μ is a non-trivial invariant on $\mathcal{F}$. □

Lemma 4.22 *Let $\tilde{D} = (M; 2k-1; (A_i, B_i)_{i=2,\dots,2k})$ be a $(2k-1)$–component LP–surgery. Let N be a $\mathbb{Z}$–handlebody in $M \setminus (\cup_{i=2}^{2k} A_i)$. Let (A_1, B_1) be a pair of $\mathbb{Z}$–handlebodies such that $A_1 \subset \mathrm{Int}(N)$ and ∂A_1 and ∂B_1 are identified so that $\mathcal{L}_{A_1} = \mathcal{L}_{B_1}$. Set $D(A_1, B_1) = \big(M; 2k; (A_i, B_i)\big)$. Let*

$$i_*\colon\ H_1(A_1) \longrightarrow H_1(N)$$

denote the homomorphism induced by the inclusion map of A_1 into N. Set

$$\mathcal{J}_1(A_1, B_1) = \big(\otimes^3 (i_* \circ \varphi_{A_1}^{-1})\big)\big(\mathcal{I}(A_1, B_1)\big) \in \bigotimes_{i=1}^{3} H_1(N)^{(i)}$$

where φ_{A_1} has been defined in Notation 3.3. Then, for any oriented degree k Jacobi diagram Γ, there exists a linear form $\ell_N(\tilde{D};\Gamma)$ in $(\bigotimes^3 H_1(N))^*$ such that, for any pair (A_1, B_1) as above,

$$\ell\big(D(A_1, B_1);\Gamma\big) = \langle \mathcal{J}_1(A_1, B_1)\ ,\ \ell_N(\tilde{D};\Gamma)\rangle.$$

Proof Let σ be a coloration of Γ. Set

$$T'(\tilde{D};\Gamma;\sigma) = \mathrm{sign}(h) \bigotimes_{i=2}^{2k} \mathcal{I}(A_i, B_i) \in \otimes_{\{c \in H(\Gamma); \sigma\circ v(c) > 1\}} X(c).$$

Apply all the contractions corresponding to the edges that do not contain any half-edge c in $(\sigma \circ v)^{-1}(1)$. These are all the edges except the three edges $\{c, j(c)\}$ where $c \in (\sigma \circ v)^{-1}(1)$ and $j(c)$ is the other half-edge of $e(c)$, and the obtained tensor is in $\otimes_{\{c\in H(\Gamma);\sigma\circ v(c)=1\}} X(j(c))$. Now apply $\bigotimes_{\{c\in H(\Gamma);\sigma\circ v(c)=1\}} \varphi^{-1}_{A_{\sigma\circ v(j(c))}}$ in order to obtain the tensor

$$\ell'(\tilde{D};\Gamma;\sigma) \in \bigotimes_{\{c\in H(\Gamma);\sigma\circ v(c)=1\}} H_1(A_{\sigma\circ v(j(c))}).$$

The linking number maps $H_1(A_{\sigma\circ v(j(c))})$ to $\left(H_1(N)\right)^*$ and therefore maps $\ell'(\tilde{D};\Gamma;\sigma)$ to an element

$$\ell_N(\tilde{D};\Gamma;\sigma) \in \bigotimes_{i=1}^{3} \left(H_1(N)^*\right)^{(i)}.$$

By definition, $\ell(\tilde{D};\Gamma;\sigma)$ is the contraction of $\ell_N(\tilde{D};\Gamma;\sigma) \otimes \mathcal{J}_1(A_1, B_1)$. Then

$$\ell(D;\Gamma) = \langle \mathcal{J}_1(A_1, B_1)\ ,\ \sum_{\sigma\in \mathrm{Bij}(\Gamma)/\mathrm{Aut}(\Gamma)} \frac{\ell_N(\tilde{D};\Gamma;\sigma)}{\sharp \mathrm{Aut}_V(\Gamma)} \rangle. \qquad \square$$

Proposition 4.23 *Let $n \in \mathbb{N}$. Let M be a $\mathbb{Z}$–sphere. Let Γ be an oriented degree k Jacobi diagram with $2k \leq n$. Consider $\ell((M;G);\Gamma)$ as a function of oriented n–component Y–links G in M. Then*

- *The linking number $\ell((M;G);\Gamma)$ only depends on*
  - *the linking numbers $\ell k(l, l')$, where l and l' are leaves in two distinct Y–components of G*
  - *the products $\bar{f}(l_1)\bar{f}(l_2)\bar{f}(l_3)$, where l_1, l_2 and l_3 are leaves in a same Y–component.*
- *Considered as a map of a leaf l of G, $\ell((M;G);\Gamma)$ is a linear map in $([l], \bar{f}(l)) \in H_1(M \setminus \cup_{l'\neq l} l') \times \mathbb{Z}/2\mathbb{Z}$.*

Proof Assume $n = 2k$. Lemma 4.22 and the expression of $\mathcal{I}(A_1, B_1)$ given in Subsection 3.3 show that $\ell((M;G);\Gamma)$ does not depend on the framing of the leaves of G and that $\ell((M;G);\Gamma)$, seen as a map of a leaf l of a component H of G, linearly depends on the homology class of l in $M \setminus (G \setminus H)$. It implies the result. If $2k < n$, the result follows from Corollary 4.21 and from the previous case. $\square$

Proposition 4.24 *Let G be an n–component Y–link in a $\mathbb{Z}$–sphere M. Then Theorem 3.5 is true when $D = (M;G)$.*

Proof The simultaneous multilinearities of the bracket in Proposition 4.13 and of the linking number of LP–surgeries induced by Y–links in Proposition 4.23 allow us to cut the leaves of G and to reduce the proof in the case where

- The non-zero-framed leaves are ± 1–framed and bound discs disjoint from G.
- Any 0–framed leaf is a meridian of another leaf.

If a ± 1–framed leaf is in a component with a 0–framed leaf, its framing can be changed without changing either side of the equality. Then we can assume than the only ± 1–framed leaves are parts of components like Y_{III}. Similarly, we can assume that any 0–framed leaf is a meridian of one leaf in another component of G. Then, up to orientation changes of leaves, we can assume that G is a Y–link induced by a Jacobi diagram. Since Theorem 3.5 is true for LP–surgeries induced by Jacobi diagrams, the result follows. □

4.5 Proof of Theorem 3.5 in the general case

Let (A, B) be a pair of $\mathbb{Z}$–handlebodies whose boundaries ∂A and ∂B are identified so that $\mathcal{L}_A = \mathcal{L}_B$. In what follows, $\mathcal{I}(A, B)$ denotes the linear form on $\bigotimes^3 \mathcal{L}_A$ induced by the intersection form on $\bigotimes^3 H_2\big(A \cup (-B)\big)$, and

$$\varphi_A\colon\, H_1(A) \longrightarrow \mathcal{L}_A^*$$

denotes the isomorphism presented in Notation 3.3.

Lemma 4.25 *Let A, B and C be three $\mathbb{Z}$–handlebodies with the same genus. Assume that ∂A, ∂B and ∂C are identified so that $\mathcal{L}_A = \mathcal{L}_B = \mathcal{L}_C$. Then*

$$\mathcal{I}(A, B) = \mathcal{I}(A, C) + \mathcal{I}(C, B).$$

Proof Let a_1, a_2 and a_3 be three oriented curves in ∂A that represent elements of $\mathcal{L}_A$ still denoted by a_1, a_2 and a_3 such that the curves a_i do not intersect each other. Let $M = A \cup (-B)$. For any $i \in \{1, 2, 3\}$, let S_A^i and S_B^i be oriented surfaces in A and in B, respectively, such that a_i bounds S_A^i in A and a_i bounds S_B^i in B. Assume that all the surfaces are transverse to each other and to ∂A. Set

$$\Sigma_M^i = S_A^i \cup_{a_i} (-S_B^i) \subset M.$$

The orientation of Σ_M^i and the orientation of M induce a positive normal vector field n_i on Σ_M^i. The algebraic intersection $\langle \Sigma_M^1, \Sigma_M^2, \Sigma_M^3 \rangle_M$ is the sum of the signs of the intersection points, where the sign is defined as follows. For any intersection point x, the sign is $+1$ if $\big(n_1(x), n_2(x), n_3(x)\big)$ is a direct basis of T_xM according to the orientation of M, and -1 otherwise. By definition,

$$\big(\mathcal{I}(A,B)\big)\big(a_1\otimes a_2\otimes a_3\big) \;=\; \langle \Sigma^1_M, \Sigma^2_M, \Sigma^3_M\rangle_M.$$

Then

$$\begin{aligned}\big(\mathcal{I}(A,B)\big)\big(a_1\otimes a_2\otimes a_3\big) &= \langle \Sigma^1_M, \Sigma^2_M, \Sigma^3_M\rangle_M\\ &= \langle S^1_A, S^2_A, S^3_A\rangle_M + \langle(-S^1_B),(-S^2_B),(-S^3_B)\rangle_M\\ &= \langle S^1_A, S^2_A, S^3_A\rangle_A + \langle(-S^1_B),(-S^2_B),(-S^3_B)\rangle_{(-B)}.\end{aligned}$$

Note that the normal vector field n^i_B to S^i_B induced by the orientation of S^i_B and by the orientation of B is equal to the normal vector field to $(-S^i_B)$ induced by the orientation of $(-S^i_B)$ and by the orientation of $(-B)$. Now, for each point of $S^1_B\cap S^2_B\cap S^3_B$, (n^1_B, n^2_B, n^3_B) is direct according to the orientation of B if and only if it is not direct according to the orientation of $(-B)$. It implies that

$$\big(\mathcal{I}(A,B)\big)\big(a_1\otimes a_2\otimes a_3\big) \;=\; \langle S^1_A, S^2_A, S^3_A\rangle_A - \langle S^1_B, S^2_B, S^3_B\rangle_B$$

and the lemma follows. □

Lemma 4.26 *Under the hypotheses of Lemma 4.12, for any Jacobi diagram* Γ,

$$\ell(D;\Gamma) = \ell(D';\Gamma) + \ell(D'';\Gamma).$$

Proof The result follows from the equality $\mathcal{I}(A_1,B_1) = \mathcal{I}(A_1,A_1') + \mathcal{I}(A_1',B_1)$ given by Lemma 4.25 and from the equality

$$\mathcal{L}(D(\{1\})) = \mathcal{L}(D'(\{1\})) + \mathcal{L}(D''(\{1\})).$$ □

Lemma 4.27 *Consider an* n*-component LP–surgery*

$$D = (M; n; (A_1,B_1), (A_2,B_2), \dots (A_n,B_n)).$$

Let A_1' *and* B_1' *be two* $\mathbb{Z}$*–handlebodies such that*

- $A_1' \subset \mathrm{Int}(A_1)$
- $\partial A_1'$ *and* $\partial B_1'$ *are identified so that* $\mathcal{L}_{A_1'} = \mathcal{L}_{B_1'}$
- $B_1 = (A_1)_{B_1'/A_1'}$ *is the* $\mathbb{Z}$*–handlebody obtained from* A_1 *by replacing* A_1' *by* B_1'.

Set $D' = \big(M; n; (A_1',B_1'), (A_2,B_2), \dots, (A_n,B_n)\big)$. *Then* $[D'] = [D]$ *while, for any Jacobi diagram* Γ,

$$\ell(D';\Gamma) = \ell(D;\Gamma).$$

Sublemma 4.28 *Under the hypotheses of Lemma 4.27, let*

$$\begin{array}{rcl} \partial_{A_1}: & H_2(A_1, \partial A_1) & \to & \mathcal{L}_{A_1} \\ \partial_{A_1'}: & H_2(A_1', \partial A_1') & \to & \mathcal{L}_{A_1'} \end{array}$$

denote the isomorphisms induced by the long exact homology sequences. Let

$$\begin{array}{rcl} i_{A_1}: & H_2(A_1, \partial A_1) & \to & H_2\big(A_1, A_1 \setminus \mathrm{Int}(A_1')\big) \\ i_{A_1'}: & H_2(A_1', \partial A_1') & \to & H_2\big(A_1, A_1 \setminus \mathrm{Int}(A_1')\big) \end{array}$$

be the homomorphisms induced by the inclusion maps. Then $i_{A_1'}$ is an isomorphism by the excision axiom. Set

$$\Phi = \partial_{A_1'} \circ i_{A_1'}^{-1} \circ i_{A_1} \circ \partial_{A_1}^{-1} \colon \mathcal{L}_{A_1} \to \mathcal{L}_{A_1'}.$$

Then

$$\big(\mathcal{I}(A_1', B_1')\big) \circ (\otimes^3 \Phi) = \mathcal{I}(A_1, B_1).$$

Proof of Lemma 4.27 assuming Sublemma 4.28 The assertion $[D'] = [D]$ is obvious. Since $\mu(M_{B_1/A_1}) = \mu(M_{B_1'/A_1'})$, it suffices to prove that

$$\ell(D;\Gamma) = \ell(D';\Gamma)$$

when $n = 2k$ is even and when Γ is a degree k Jacobi diagram. Set $\tilde{D} = \big(M; 2k-1; (A_i, B_i)_{i=2,\dots,2k}\big)$. Let

$$i_*\colon H_1(A_1') \longrightarrow H_1(A_1)$$

be the map induced by the inclusion map of A_1' into A_1. By Lemma 4.22, there exists a linear form

$$\ell_{A_1}(\tilde{D};\Gamma) \in \Big(\bigotimes^3 H_1(A_1)\Big)^*$$

such that

$$\begin{array}{rcl} \ell(D;\Gamma) & = & \langle \big(\otimes^3 \varphi_{A_1}^{-1}\big)\big(\mathcal{I}(A_1, B_1)\big)\ ,\ \ell_{A_1}(\tilde{D};\Gamma)\rangle \\ \ell(D';\Gamma) & = & \langle \big(\otimes^3 (i_* \circ \varphi_{A_1'}^{-1})\big)\big(\mathcal{I}(A_1', B_1')\big), \ell_{A_1}(\tilde{D};\Gamma)\rangle. \end{array}$$

The following diagram is commutative

$$\begin{array}{ccc} H_1(A_1') & \overset{\varphi_{A_1'}}{\longrightarrow} & \mathcal{L}^*_{A_1'} \\ i_* \downarrow & & \downarrow \Phi^* \\ H_1(A_1) & \overset{\varphi_{A_1}}{\longrightarrow} & \mathcal{L}^*_{A_1}. \end{array}$$

Indeed both compositions, seen as elements of $\left(H_1(A_1')\otimes\mathcal{L}_{A_1}\right)^*$, map $([b],[a]) \in H_1(A_1') \times \mathcal{L}_{A_1}$ to the algebraic intersection in A_1 of a surface bounded by a and of the curve b. Thus,

$$\left(\otimes^3 (i_* \circ \varphi_{A_1'}^{-1})\right)\left(\mathcal{I}(A_1', B_1')\right) = \left(\otimes^3 \varphi_{A_1}^{-1}\right)\left(\mathcal{I}(A_1', B_1') \circ (\otimes^3 \Phi)\right).$$

By Sublemma 4.28, the lemma follows. □

Proof of Sublemma 4.28 Let a_1, a_2 and a_3 be three oriented curves in ∂A_1 that represent elements of $\mathcal{L}_{A_1}$ still denoted by a_1, a_2 and a_3 such that the curves a_i do not intersect each other. Let a_1', a_2', a_3' be oriented curves in $\partial A_1'$ that represent the elements $\Phi(a_i)$ of $\mathcal{L}_{A_1'}$ such that the curves a_i' do not intersect each other. For any $i \in \{1,2,3\}$, the curves a_i and a_i' cobound an oriented surface Σ^i in $A_1 \setminus \mathrm{Int}(A_1')$. The curve a_i' bounds an oriented surface $\sigma^i_{A_1'}$ in A_1' and bounds an oriented surface $\sigma^i_{B_1'}$ in B_1'. Set

$$\begin{array}{rccl} S_i = & \left(\sigma^i_{A_1'} \cup \Sigma^i\right) \cup \left(- (\Sigma^i \cup \sigma^i_{B_1'})\right) & \subset & A_1 \cup (-B_1) \\ S_i' = & \sigma^i_{A_1'} \cup (-\sigma^i_{B_1'}) & \subset & A_1' \cup (-B_1'). \end{array}$$

Set

$$\begin{array}{rcl} \mathcal{J}_{A_1'B_1'} & = & \left(\mathcal{I}(A_1', B_1')\right)\left(\Phi(a_1) \otimes \Phi(a_2) \otimes \Phi(a_3)\right) \\ \mathcal{J}_{A_1B_1} & = & \left(\mathcal{I}(A_1, B_1)\right)\left(a_1 \otimes a_2 \otimes a_3\right). \end{array}$$

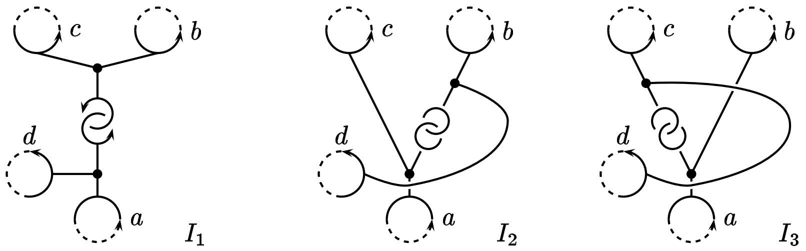
By definition, $\mathcal{J}_{A_1B_1}$ is the intersection in $A_1 \cup (-B_1)$ of the oriented surfaces S_i and $\mathcal{J}_{A_1'B_1'}$ is the intersection in $A_1' \cup (-B_1')$ of the oriented surfaces S_i'. Then $\left(\mathcal{J}_{A_1B_1} - \mathcal{J}_{A_1'B_1'}\right)$ is the contribution of the intersection of the surfaces $\Sigma^i \cup (-\Sigma^i)$. This contribution vanishes when $A_1' = B_1'$ because $\mathcal{I}(A_1, A_1) = \mathcal{I}(A_1', A_1') = 0$ by Lemma 4.25. Hence it always vanishes. □

Proof of Theorem 3.5 Lemmas 4.11, 4.12 and 4.26 allow us to reduce the proof to the case of an LP–surgery $D = \left(M; n; (A_i, B_i)\right)$ where B_i is obtained from A_i by a surgery on a Y–graph embedded in A_i. By Lemma 4.27, D can next be considered as an LP–surgery induced by an n–component Y–link in M. Then Theorem 3.5 follows from Proposition 4.24. □

A The IHX relation

For self-containedness, we finish the proof of Theorem 2.1 by proving that Ψ_n factors through the IHX relation. This is a consequence of the following proposition.

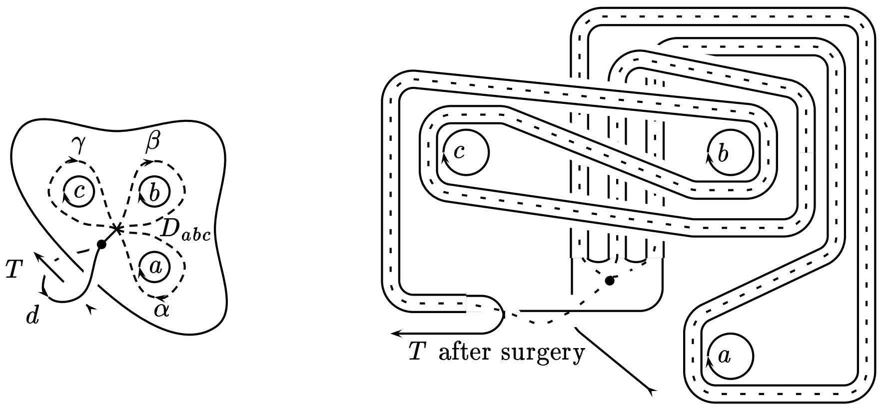
Proposition A.1 *Let G_1 be an oriented Y–link in a $\mathbb{Z}$–sphere M that admits the following two-component sublink I_1. For $i = 2, 3$, let G_i be obtained from G_1 by changing I_1 into I_i. The four leaves a, b, c and d are identical.*



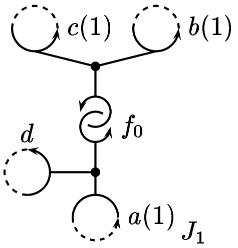
Then

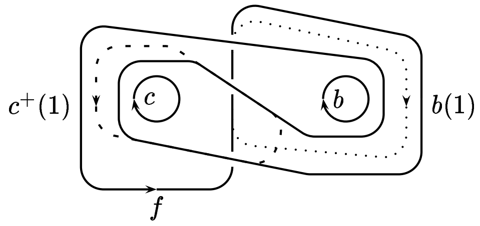
$$\overline{[(M;G_1)]} + \overline{[(M;G_2)]} + \overline{[(M;G_3)]} = 0.$$

Proof First recall that Proposition 4.13 implies that the actual embeddings of the Y–graph edges do not affect the Y–link brackets. Therefore these embeddings will not be specified in the proof below. Consider an embedded product $D_{abc} \times [0,1]$ of the three-hole disk D_{abc} whose three inner boundary components are a, b and c, and the three generators of $\pi_1(D_{abc})$, α, β and γ. We first prove that there exists some two-component Y–link J_1 in $(M \setminus (G_1 \setminus I_1))$ that is obtained from I_1 by changing (the edge adjacent to d and) the leaves a, b and c into leaves $a(1)$, $b(1)$ and $c(1)$ that are homologous to a, b and $(-c)$ in $(M \setminus (G_1 \setminus I_1))$, respectively, such that surgery on J_1 makes a pack T of surgery arcs in a surgered disk bounded by d describe the element $[\beta\alpha\beta^{-1}, [\gamma, \beta]]$ of $\pi_1(D_{abc})$ in $D_{abc} \times [0,1]$ in an ascending way with respect to the height of $[0,1]$. Indeed the second part of the following picture shows such a path that is ascending (after sliding two tongues), and that cobounds a genus one surface with the initial shown portion of T.



Therefore by Lemma 4.4, this path is obtained from T by surgery on a Y–graph with leaves d, $a(1)$ and f, where $a(1)$ and f are the two dashed handle cores of the genus one surface, $a(1)$ is homotopic to $\beta\alpha\beta^{-1}$, and f is homotopic to $[\beta,\gamma]$. Thus, f is obtained from a trivial leaf f_0 by surgery on a Y–graph, with one trivial leaf that makes a Hopf link together with f_0, and two other leaves $b(1)$ and $c(1)$ that do not link $a(1)$ and that are homotopic to β and γ^{-1}, respectively, as Lemma 4.4 and the next picture show.





Similarly, for $i = 2$ or 3, there exists a two-component Y–link J_i, that is obtained from I_i by changing (the edge adjacent to d and) the leaves a, b and c into leaves $a(i)$, $b(i)$ and $c(i)$ that are homologous to $\alpha(i)a$, $\beta(i)b$ and c in $(M \setminus (G_1 \setminus I_1))$, respectively, where $\alpha(i), \beta(i) \in \{-1, 1\}$, and $\alpha(i)\beta(i) = -1$, such that surgery on J_i makes a pack T of surgery arcs in a surgered disk bounded by d describe the element of $\pi_1(D_{abc})$, $[\gamma\beta\gamma^{-1}, [\alpha,\gamma]]$ for $i = 2$, or $[\alpha\gamma\alpha^{-1}, [\beta,\alpha]]$ for $i = 3$, in $D_{abc} \times [0,1]$ in an ascending way with respect to the height of $[0,1]$.

In particular, the following identity in the free group generated by α, β and γ -whose verification is straightforward-

$$[\beta\alpha\beta^{-1}, [\gamma,\beta]][\gamma\beta\gamma^{-1}, [\alpha,\gamma]][\alpha\gamma\alpha^{-1}, [\beta,\alpha]] = 1$$

ensures that the surgery on $J_1 \cup J_2 \cup J_3$ is trivial.

Therefore, if H_i is obtained from G_i by changing I_i into J_i,

$$\begin{aligned}&[M_{J_1\cup J_2}, H_3] + [M_{J_1}, H_2] + [M, H_1]\\ &= \textstyle\sum_{J\subset (G_1\setminus I_1)} (-1)^{\sharp J} \begin{pmatrix} M_{J\cup J_1\cup J_2\cup J_3} - M_{J\cup J_1\cup J_2} \\ +M_{J\cup J_1\cup J_2} - M_{J\cup J_1} \\ +M_{J\cup J_1} - M_J \end{pmatrix} = 0.\end{aligned}$$

Furthermore, Proposition 4.13 ensures that

$$\overline{[(M;G_i)]} = -\overline{[(M;H_i)]} = -\overline{[(M_{\cup_{j<i}J_j};H_i)]},$$

and allows us to conclude the proof. □

This proof shows how the Jacobi IHX relation comes from the Lie algebra structure on the graded space associated to the lower central series of a free group. See [MKS].

References

[GGP] **S Garoufalidis**, **M Goussarov**, **M Polyak**, *Calculus of clovers and finite type invariants of 3–manifolds*, Geom. Topol. 5 (2001) 75–108

[GM] **L Guillou**, **A Marin**, *Notes sur l'invariant de Casson des sphères d'homologie de dimension trois*, Enseign. Math. 38 (1992) 233–290

[Hbg] **N Habegger**, *Milnor, Johnson, and Tree Level Perturbative Invariants*, preprint (2000) `http://www.math.sciences.univ-nantes.fr/~habegger`

[Hbo] **K Habiro**, *Claspers and finite type invariants of links*, Geom. Topol. 4 (2000) 1–83

[KT] **G Kuperberg**, **D P Thurston**, *Perturbative 3–manifold invariants by cut-and-paste topology*, `arXiv:math.GT/9912167`

[Le] **T T Q Le**, *An invariant of integral homology* 3*–spheres which is universal for all finite type invariants*, from: "Solitons, geometry, and topology: on the crossroad", Amer. Math. Soc. Transl. 179 (1997) 75–100

[Les] **C Lescop**, *A sum formula for the Casson–Walker invariant*, Invent. Math. 133 (1998) 613–681

[L2] **C Lescop**, *Splitting formulae for the Kontsevich–Kuperberg–Thurston invariant of rational homology* 3*–spheres*, `arXiv:math.GT/0411431`

[Lic] **W B R Lickorish**, *An Introduction to Knot Theory*, Graduate Texts in Mathematics 175, Springer–Verlag, New York (1997)

[MKS] **W Magnus**, **A Karrass**, **D Solitar**, *Combinatorial group theory. Presentations of groups in terms of generators and relations*, Second revised edition, Dover Publications, Inc. New York (1976)

[Mat] **S V Matveev**, *Generalized surgery of three-dimensional manifolds and representations of homology spheres*, Mat. Zametki 42 (1987) 268–278, 345 (Russian); English translation: Math. Notes 42 (1987) 651–656

[MN] **H Murakami**, **Y Nakanishi**, *On a certain move generating link-homology*, Math. Ann. 284 (1989) 75–89

[Oht] **T Ohtsuki**, *Finite type invariants of integral homology* 3*–spheres*, J. Knot Theory Ramifications 5 (1996) 101–115

[Rol] **D Rolfsen**, *Knots and Links,* Mathematics Lecture Series, No.7. Publish or Perish, Inc. Berkeley, Calif. (1976)

[Thu] **W P Thurston**, *The Geometry and Topology of* 3*–manifolds, Chapter* 13, Princeton University (1978)
`http://www.msri.org/publications/books/gt3m/`

Institut Fourier (UMR 5582 du CNRS), B.P. 74
38402 Saint-Martin d'Hères cedex, France

Email: `auclaire@ujf-grenoble.fr` and `lescop@ujf-grenoble.fr`

URL: `http://www-fourier.ujf-grenoble.fr/~lescop`

Received: 9 February 2004