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Limits of (certain) CAT(0) groups, I: Compactification

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Abstract The purpose of this paper is to investigate torsion-free groups which act properly and cocompactly on CAT(0) metric spaces which have isolated flats, as defined by Hruska [18]. Our approach is to seek results analogous to those of Sela, Kharlampovich and Miasnikov for free groups and to those of Sela (and Rips and Sela) for torsion-free hyperbolic groups.

This paper is the first in a series. In this paper we extract an $\mathbb{R}$-tree from an asymptotic cone of certain CAT(0) spaces. This is analogous to a construction of Paulin, and allows a great deal of algebraic information to be inferred, most of which is left to future work.

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1 Introduction

Using the theory of isometric actions on $\mathbb{R}$-trees as a starting point, Sela has solved the isomorphism problem for hyperbolic groups (at least for torsion-free hyperbolic groups which do not admit a small essential action on an $\mathbb{R}$-tree [28], though he has a proof in the general torsion-free case), has proved that torsion-free hyperbolic groups are Hopfian [31], and recently has classified those groups with the same elementary theory as a given torsion-free hyperbolic group [32, 33, 34]. Kharlampovich and Miasnikov have a similar, but more combinatorial, approach to this last problem for free groups; see [21] and references contained therein.[1]

[1]Neither Sela's nor Kharlampovich and Miasnikov's work on the elementary theory of groups have entirely appeared in refereed journals.

It seems that Sela's methods will not work for non-positively curved groups in general (whatever the phrase 'non-positively curved group' means). For example, Wise [39] constructed a group which acts properly and cocompactly on a CAT(0) metric space, but is non-Hopfian.

The class of groups acting properly and cocompactly on CAT(0) spaces with the isolated flats condition is in many ways an intermediary between hyperbolic groups (which are the 'negatively curved groups' in the context of discrete group) and CAT(0) groups. Sela [35, Question I.8] asked whether such a group is Hopfian, and whether one can construct *Makanin-Razborov diagrams* for these groups. In the second paper in this series, [17], we will provide a positive answer to these questions (under certain extra hypotheses, described below). The purpose of this paper is to develop tools for addressing these questions.

The initial ingredient in many of Sela's arguments is a result of Paulin ([23, 24]; see also [1] and [9]; and see [22] for work preceding Paulin's) which extracts an isometric action on an $\mathbb{R}$-tree from (certain) sequences of actions on δ-hyperbolic spaces. Given two finitely generated groups G and Γ, and a sequence of non-conjugate homomorphisms $\{h_i\colon G \to \Gamma\}$, it is straightforward to construct an action of G on a certain asymptotic cone of Γ with no global fixed point. If Γ acts properly and cocompactly by isometries on a metric space X, then a G-action can be constructed on an asymptotic cone of X (which is bi-Lipschitz homeomorphic, but not necessarily isometric, to the analogous asymptotic cone of G). For δ-hyperbolic groups, this is in essence the above-mentioned result of Paulin. In the case of groups acting on CAT(0) spaces, it is carried out by Kapovich and Leeb in [20], but the general case is hardly more complicated. Of course, for a general finitely generated group Γ, the existence of an action of G on an asymptotic cone of Γ with no global fixed point provides little information about G or Γ. In this paper we place certain restrictions on Γ so that we can find a G-action on an $\mathbb{R}$-tree which provides much the same information as Paulin's result. We study the case where Γ acts properly and cocompactly on a CAT(0) metric space with isolated flats.

We study the asymptotic cone of such a space. Under a further hypothesis (that the stabilisers of maximal flats are free abelian), we construct an $\mathbb{R}$-tree, which allows many of Sela's arguments to be carried out in this context (though we leave most such applications to subsequent work).

The first application of this construction is the following:

Theorem 5.9 *Suppose that Γ is a torsion-free group acting properly and cocompactly on a $CAT(0)$ space X which has isolated flats, so that flat stabilisers in Γ are abelian. Suppose further that $Out(\Gamma)$ is infinite. Then Γ admits a nontrivial splitting over a finitely generated free abelian group.*

This partially answers a question of Swarup (see [3, Q 2.1]). However, Theorem 5.9 is only the first application. Our hope is that much of Sela's program for free groups and torsion-free hyperbolic groups can be carried out for groups Γ as in the statement of Theorem 5.9. In future work, we will consider the automorphism groups of such groups (in analogy with [26, 29]), the Hopf property (in analogy with [31]) and Makanin-Razborov diagrams for these groups (in analogy with [32, 34]). The last of these involves finding a description of $\mathrm{Hom}(G, \Gamma)$, where G is an arbitrary finitely-generated group. A key argument in Sela's solution to all of these problems for torsion-free hyperbolic groups is the *shortening argument*, which we present for these CAT(0) groups with isolated flats in [17].

The outline of this paper is as follows. In Section 2 we recall some basic definitions and results and prove some preliminary results about CAT(0) spaces with isolated flats and groups acting properly and cocompactly on such spaces. In Section 3, we consider a torsion-free group Γ which acts properly and cocompactly on a CAT(0) metric space X with isolated flats. Given a finitely generated group G and a sequence of homomorphisms $\{h_n \colon G \to \Gamma\}$ no two of which differ only by an inner automorphism of Γ, it is straightforward to construct an action of G on the asymptotic cone of X. A key feature of this action is that it has no global fixed point. This construction amounts to a compactification of a certain space of G-actions on X (those actions which factor through a fixed homomorphism $q \colon \Gamma \to \mathrm{Isom}(X)$). In Section 4, we restrict to a torsion-free group Γ which acts properly and cocompactly on a CAT(0) space with isolated flats and has abelian flat stabilisers. Under this additional hypothesis, we are able to extract an isometric action of G on an $\mathbb{R}$-tree T with no global fixed point. The action of G on T largely encodes the same information from the homomorphisms $\{h_n\}$ as Paulin's construction does in the case where Γ is δ-hyperbolic. See, in particular, Theorem 4.4, the main technical result of this paper. Finally, in Section 5 we discuss a few simple relations between our limiting objects, Γ-limit groups, and other definitions of Γ-limit groups, and prove Theorem 5.9.

I would like to thank Jason Manning for several conversations which illustrated my naïveté, and in particular for pointing out an incorrect argument in a previous construction of the limiting tree T in §4.

2 CAT(0) metric spaces with isolated flats and isometric actions upon them

For the definition of $\mathbb{R}$-trees and the basic properties of their isometries, we refer the reader to [36], [37], [12] and [2]. For this paper, we do not need much of this theory.

For the definition and a multitude of results about CAT(0) metric spaces, and isometric actions upon them, we refer the reader to [8]. We recall only a few basic properties and record our notation.

Suppose that X is a geodesic metric space. If $p, q, r \in X$, then $[p, q]$ denotes a geodesic between p and q, and $\Delta(p, q, r)$ denotes the triangle consisting of the geodesics $[p, q], [q, r], [r, p]$. Geodesics (and hence geodesic triangles) need not be unique in geodesic metric spaces, but they are in CAT(0) spaces. If $p, q, r \in X$ then $[p, q, r]$ denotes the path $[p, q] \cup [q, r]$. Expressions such as $[p, q, r, s]$ are defined similarly.

If Γ is a group acting properly and cocompactly by homeomorphisms on a connected simply-connected topological space then Γ is finitely presented (see [8, Theorem I.8.10, pp.135-137]). Obviously, if Γ is torsion-free, then the action is free.

Suppose now that X is a CAT(0) metric space and that Γ acts properly and cocompactly by isometries on X. Then (see [8, II.6.10.(2), p.233]) each element of Γ acts either elliptically (fixing a point) or hyperbolically (there is an invariant axis upon which the element acts by translation). If also Γ is torsion-free then all isometries are hyperbolic.

Recall the following two results.

Lemma 2.1 [8, Proposition II.2.2, p. 176] *Let X be a CAT(0) space. Given any pair of geodesics $c\colon [0,1] \to X$ and $c'\colon [0,1] \to X$ parametrised proportional to arc length, the following inequality holds for all $t \in [0,1]$:*

$$d_X(c(t), c'(t)) \leq (1-t)d_X(c(0), c'(0)) + t(d_X(c(1), c'(1)).$$

Proposition 2.2 [8, Proposition II.2.4, pp. 176–177] *Let X be a CAT(0) space, and let C be a convex subset which is complete in the induced metric. Then,*

(1) *for every $x \in X$, there exists a unique point $\pi_C(x) \in C$ such that $d(x, \pi_C(x)) = d(x, C) := \inf_{y \in C} d(x, y)$;*

(2) *if x' belongs to the geodesic segment $[x, \pi_C(x)]$, then $\pi_C(x') = \pi_C(x)$;*

(3) *the map $x \to \pi_C(x)$ is a retraction of X onto C which does not increase distances.*

2.1 CAT(0) spaces with isolated flats and groups acting on them

Definition 2.3 A *flat* in a CAT(0) space X is an isometric embedding of Euclidean space $\mathbb{E}^k$ into X for some $k \geq 2$.

Note that we do not consider a geodesic line to be a flat.

Definition 2.4 [18, 2.1.2] A CAT(0) metric space X has *isolated flats* if it contains a family $\mathcal{F}_X$ of flats with the following properties:

(1) (Maximal) There exists $B \geq 0$ such that every flat in X is contained in a B-neighbourhood of some flat in $\mathcal{F}_X$;

(2) (Isolated) There is a function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for every pair of distinct flats $E_1, E_2 \in \mathcal{F}_X$ and for every $k \geq 0$, the intersection of the k-neighbourhoods of E_1 and E_2 has diameter less than $\phi(k)$.

This definition is due to C. Hruska [18], but such an idea is implicit in Chapter 11 of [15], and in the work of Wise [40] and of Kapovich and Leeb [20].

Convention 2.5 To simplify constants in the sequel, we assume that $\phi(k) \geq k$ for all $k \geq 0$ and that ϕ is a nondecreasing function. We can certainly make these assumptions, and usually do so without comment.

For the basic properties of CAT(0) metric spaces with isolated flats, for examples of such spaces, and for some properties of isometric actions upon them, we refer the reader to [18].

Hruska also introduced the *relatively thin triangles* property:

Definition 2.6 [18, 3.1.1] A geodesic triangle in a metric space X is *δ-thin relative to the flat E* if each side of the triangle lies in the δ-neighbourhood of the union of E and the other two sides of the triangle (see Figure 1). A metric space X has the *relatively thin triangle property* if there is a constant δ so that each triangle in X is either δ-thin in the usual sense or δ-thin relative to some flat in $\mathcal{F}_X$.

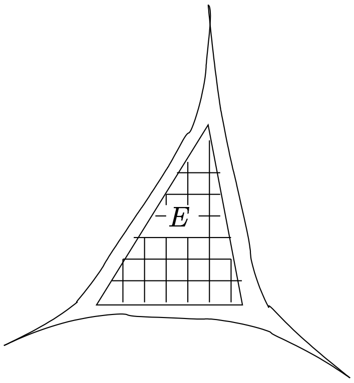

Figure 1: A triangle which is thin relative to the flat E

Using work of Druţu and Sapir [14] on asymptotic cones of relatively hyperbolic groups, Hruska and Kleiner [19] have proved that if X is a CAT(0) space with isolated flats which admits a cocompact isometric group action then X satisfies the relatively thin triangles condition. In this paper, the symbol 'δ' will always refer to the constant from Definition 2.6.

Terminology 2.7 When we refer to a *CAT(0) group with isolated flats* we mean a group which admits a proper, cocompact and isometric action on a CAT(0) space with isolated flats.

We now consider some of the basic properties of CAT(0) spaces with isolated flats, and groups acting properly, cocompactly and isometrically upon them, which are necessary in the sequel.

Proposition 2.8 [18, 2.1.4] *Suppose X is a CAT(0) space with isolated flats. The family $\mathcal{F}_X$ of flats in Definition 2.4 may be assumed to be invariant under all isometries of X.*

Lemma 2.9 [18, 2.1.9] *Suppose that the CAT(0) space X has isolated flats and admits a proper and cocompact action by some group of isometries. Then any maximal flat in X is periodic.*

Lemma 2.10 *Suppose that X is a CAT(0) space with isolated flats, and that $\Delta = \Delta(a,b,c)$ is a geodesic triangle in X. If Δ is not $(\delta + \frac{\phi(\delta)}{2})$-thin then Δ is δ-thin relative to a* unique *flat $E \in \mathcal{F}_X$.*

Proof Let $l_{a,b}$ be that part of the geodesic $[a,b]$ which lies outside of the δ-neighbourhood of $[a,c]\cup[b,c]$, and define $l_{a,c}$ and $l_{b,c}$ similarly.

Suppose that Δ is δ-thin relative to $E,E'\in\mathcal{F}_X$, where $E\neq E'$. Then $l_{a,b},l_{a,c},l_{b,c}$ all lie in the δ-neighbourhood both of E and of E'. The intersection of these δ-neighbourhoods has diameter at most $\phi(\delta)$. Therefore, the length of $l_{a,b}$ is at most $\phi(\delta)$ (since it is a geodesic). Thus, from any point on $l_{a,b}$, the distance to $[a,c]\cup[b,c]$ is at most $\delta+\frac{\phi(\delta)}{2}$.

A symmetric argument for $l_{a,c}$ and $l_{b,c}$ finishes the proof. □

2.2 Bieberbach groups and toral actions on CAT(0) spaces with isolated flats

Given a proper and cocompact isometric action of a group Γ on a CAT(0) space X with isolated flats, we are compelled to study the subgroups $\mathrm{Stab}(E)$, where E is a maximal flat in X.[2]

By Lemma 2.9 we have a proper and cocompact action of the group $\mathrm{Stab}_\Gamma(E)$ on $E\cong\mathbb{E}^n$. Recall the following celebrated result of Bieberbach [6, 7]. [3]

Theorem 2.11 (Bieberbach; see for example [38], 4.2.2, p.222)

(a) *A group Γ is isomorphic to a discrete group of isometries of $\mathbb{E}^n$, for some n, if and only if Γ contains a subgroup of finite index that is free abelian of finite rank;*

(b) *An n-dimensional crystallographic group Γ contains a normal subgroup of finite index that is free abelian of rank n and equals its own centraliser. This subgroup is characterised as the unique maximal abelian subgroup of finite index in Γ, or as the translation subgroup of Γ.*

The structure of the subgroups $\mathrm{Stab}_\Gamma(E)$ will be important to us in the sequel. In particular, when there are such groups which are not free abelian the construction in Section 4 does not work. Motivated by this consideration, we make the following

[2] By $\mathrm{Stab}(E)$ we mean $\{g\in\Gamma \mid g.E=E\}$. The point-wise stabiliser is $\mathrm{Fix}(E)=\{g\in\Gamma\mid g.x=x,\ \forall x\in E\}$.

[3] An *n-dimensional crystallographic group* is a cocompact discrete group of isometries of $\mathbb{E}^n$.

Definition 2.12 Suppose that X is a CAT(0) space with isolated flats and that a group Γ acts properly and cocompactly by isometries on X. We say that the action of Γ on X is *toral* if for each maximal flat $E \subseteq X$, the subgroup $\mathrm{Stab}(E) \leq \Gamma$ is free abelian. We say that Γ is a *toral* CAT(0) group with isolated flats if there is a proper, cocompact and toral action of Γ on a CAT(0) space X with isolated flats.

Remark 2.13 We observe in Lemma 2.18 below that if a torsion-free group Γ admits a proper, cocompact and toral action on some CAT(0) space X with isolated flats then *any* proper and cocompact action of Γ on a CAT(0) space with isolated flats is toral. Thus the property of being toral belongs to the group rather than the given action on a CAT(0) space with isolated flats. Also, Hruska and Kleiner have proved [19] that any CAT(0) space X on which a CAT(0) group with isolated flats acts properly and cocompactly by isometries has isolated flats.

2.3 Basic algebraic properties of CAT(0) groups with isolated flats

In this paragraph we consider a few basic algebraic properties of torsion-free CAT(0) groups with isolated flats.

Definition 2.14 A subgroup K of a group G is said to be *malnormal* if for all $g \in G \smallsetminus K$ we have $gKg^{-1} \cap K = \{1\}$.

A group G is said to be *CSA* if any maximal abelian subgroup of G is malnormal.

The following lemma is straightforward and certainly well known, but we record and prove it for later use.

Lemma 2.15 *Suppose that G is a CSA group. Then every soluble subgroup of G is abelian. Also, every virtually abelian subgroup of G is abelian.*

Proof Suppose that S is a nontrivial soluble subgroup of G. Let $S^{(i)}$ be the smallest nontrivial term of the derived series of S. Then $S^{(i)}$ is a normal abelian subgroup of S. However, it is an abelian subgroup of G, so is contained in a maximal abelian subgroup A. If $g \in S$, then g normalises $S^{(i)}$, so $g \in A$, since A is malnormal. Therefore, S is contained in A and S is abelian.

Any virtually abelian subgroup H has a finite index normal abelian subgroup A. By the above argument, the normaliser of A is abelian and contains H, so H is abelian. $\square$

Proposition 2.16 *Suppose that Γ is a torsion-free group which admits a proper and cocompact action on a CAT(0) space X with isolated flats. Then the stabiliser in Γ of any maximal flat in X is malnormal.*

Proof Let $\mathcal{F}_X$ be the collection of flats from Definition 2.4, and let E be a maximal flat in X. Consider $M = \mathrm{Stab}(E)$. Without loss of generality, we may assume that $E \in \mathcal{F}_X$.

Suppose that $g \in \Gamma$ is such that $gMg^{-1} \cap M \neq \{1\}$. We prove that $g \in M$. There exist $a_1, a_2 \in M \smallsetminus \{1\}$ so that $ga_1g^{-1} = a_2$.

Now, $ga_1g^{-1} = a_2$ leaves both E and gE invariant. Therefore, there is an axis for ga_1g^{-1} in each of E and gE, and there is a Euclidean strip, isometric to $[0,k] \times \mathbb{R}$ for some k, joining these axes. However, E and gE are both in $\mathcal{F}_X$ by Proposition 2.8, and we have seen that the k-neighbourhoods of E and gE intersect in an unbounded set, so we must have that $E = gE$, which is to say that $g \in M$. $\square$

Corollary 2.17 *Suppose that Γ is a torsion-free toral CAT(0) group with isolated flats. Then Γ is CSA.*

Proof Let A be a maximal abelian subgroup of Γ and let X be a CAT(0) space with isolated flats with a proper, cocompact and toral action of Γ.

Suppose first that A is noncyclic. Then A stabilises some flat $E \in \mathcal{F}_X$, and hence some maximal flat (by the Isolated Flats condition). Since A is maximal abelian, and the action of Γ on X is toral, $A = \mathrm{Stab}(E)$. In this case the result follows from Proposition 2.16.

Suppose now that A is a cyclic maximal abelian subgroup, and that for some $g \in \Gamma \smallsetminus A$ we have $gAg^{-1} \cap A \neq \{1\}$. Let $A = \langle a \rangle$. Then $ga^pg^{-1} = a^q$ for some p, q. Since A is maximal abelian, we do not have $p, q = 1$. However, Γ is a CAT(0) group, so $|p| = |q|$ (see [8, Theorem III.Γ.1.1(iii)]). Thus g^2 commutes with a^p. Therefore, $\langle a^p \rangle$ is central in $G = \langle g^2, a^p \rangle$. By [8, II.6.12], there is a finite index subgroup H of G so that $H = \langle a^p \rangle \times H_1$, for some group H_1. If H_1 is infinite, then $\langle a^p \rangle$ is contained in a subgroup isomorphic to $\mathbb{Z}^2$. This $\mathbb{Z}^2$ stabilises a flat, and hence a maximal flat, so $\langle a^p \rangle$ is contained in $\mathrm{Stab}(E)$ for some $E \in \mathcal{F}_X$. However, a normalises $\langle a^p \rangle$, so by Proposition

2.16 $a \in \mathrm{Stab}(E)$. This subgroup is abelian since the action of Γ on X is toral, which contradicts A being maximal abelian. Therefore, H_1 is finite, and since Γ is torsion-free, H_1 is trivial. Therefore, G is virtually cyclic, and being torsion-free, is itself infinite cyclic. Hence g^2 commutes with a and so $\langle g^2 \rangle$ is central in $G_1 = \langle g, a \rangle$. Exactly the same argument as above applied to G_1 and $\langle g^2 \rangle$ implies that G_1 is cyclic. Since A is maximal abelian, $g \in A$, a contradiction to the choice of g. Thus A is malnormal, as required. □

Lemma 2.18 *Suppose that Γ is a torsion-free group which admits a proper, cocompact and toral action on a $CAT(0)$ space X with isolated flats. Then any proper and cocompact action of Γ on a $CAT(0)$ space with isolated flats is toral. If Γ is a torsion-free $CAT(0)$ group with isolated flats then Γ is toral if and only if Γ is CSA.*

Proof Let Γ act properly and cocompactly on a $\mathrm{CAT}(0)$ space Y with isolated flats, and let M be the stabiliser of a maximal flat $E \in \mathcal{F}_Y$.

Since Γ admits a proper, cocompact and toral action on a $\mathrm{CAT}(0)$ space X with isolated flats, by Corollary 2.17 any maximal abelian subgroup of Γ is malnormal, and so the normaliser of any abelian group is abelian.

Since M is a Bieberbach group, it has a normal abelian subgroup A of finite index. However, by the above, the normaliser of A is abelian and it certainly contains M, so M is abelian. Therefore the action of Γ on Y is toral. This proves the first claim of the lemma. The second claim follows from the proof of the first and Corollary 2.17. □

Definition 2.19 A group G is said to be *commutative transitive* if for all $u_1, u_2, u_3 \in G \smallsetminus \{1\}$, whenever $[u_1, u_2] = 1$ and $[u_2, u_3] = 1$ we necessarily have $[u_1, u_3] = 1$.

CSA groups are certainly commutative transitive, so we have

Corollary 2.20 *Suppose Γ is torsion-free toral $CAT(0)$ group with isolated flats. Then Γ is commutative transitive. Hence every abelian subgroup in Γ is contained in a unique maximal abelian subgroup.*

2.4 Projecting to flats

Fix X, a $\mathrm{CAT}(0)$ space with isolated flats. Let δ be the constant from Definition 2.6 and ϕ the function from Definition 2.4.

We study the closest-point projection from X onto a flat $E \subset X$.

Lemma 2.21 *Suppose that $E \in \mathcal{F}_X$. Suppose that $x, y \in E$ and $z \in X$. There exist $u \in [x,z]$ and $v \in [y,z]$ so that u and v lie in the 2δ-neighbourhood of E, and*

$$d_X(u,v) \leq \phi(\delta).$$

Proof If z lies in the 2δ-neighbourhood of E then the result is immediate, so we assume that this is not the case. The key (though trivial) observation is that $[x,y]$ lies entirely within E.

Let u_1 be the point on $[x,z]$ which is furthest from x in the δ-neighbourhood of E. The convexity of E and the convexity of the metric on X ensures that u_1 is unique.

We consider the triangle $\Delta = \Delta(x,y,z)$. If Δ is δ-thin, then there is clearly a point v_1 on $[y,z]$ within δ of v_1, and we take $u = u_1, v = v_1$ (this is because if u_1 is not δ-close to $[y,z]$ then neither is a point nearby u_1, but there are points arbitrarily close to u_1 on $[x,z]$ which are not δ-close to E, which would contradict Δ being δ-thin since $[x,y] \subset E$).

Thus suppose that Δ is not δ-thin, so that Δ is δ-thin relative to a flat E'. If $E' = E$, then we have a point $v_1 \in [y,z]$ which is within δ of u_1, by the same reasoning as above. Again, we take $u = u_1, v = v_1$.

Suppose then that $E' \neq E$. Either u_1 is δ-close to $[y,z]$, in which case we proceed as above, or u_1 is δ-close to E'. In this case define v_2 to be the point on $[y,z]$ which is furthest from y but in the δ-neighbourhood of E. Again, v_2 is either δ-close to $[x,z]$ or δ-close to E'. In the first situation, we proceed as above, with $v = v_2$ and u a point on $[x,z]$ which is within δ of v_2. In the second situation, both u_1 and v_2 are within δ of E and of E', and the intersection of the δ-neighbourhoods of E and E' has diameter less than $\phi(\delta)$. Thus in this case, $d_X(u_1,v_2) < \phi(\delta)$ and we may take $u = u_1, v = v_2$.

We have proved that there exist u and v in the 2δ-neighbourhood of E so that $d_X(u,v) \leq \max\{\delta, \phi(\delta)\}$. However, $\delta \leq \phi(\delta)$ by Convention 2.5, so the proof is complete. □

Proposition 2.22 *Suppose that $E \subset \mathcal{F}_X$ is a flat and that $x, y \in X$ are such that $[x,y]$ does not intersect the 4δ-neighbourhood of E. Let $\pi\colon X \to E$ be the closest-point projection map. Then $d_X(\pi(x), \pi(y)) \leq 2\phi(3\delta)$.*

Proof By Lemma 2.21, there are points $w_1 \in [\pi(x), y]$ and $w_2 \in [\pi(y), y]$, both in the 2δ-neighbourhood of E, so that $d_X(w_1, w_2) \leq \phi(\delta)$.

Now consider the triangle $\Delta' = \Delta(\pi(x), x, y)$. By a similar argument to the proof of Lemma 2.21, we find points $u_1 \in [\pi(x), x]$ and $u_2 \in [\pi(x), y]$ which lie outside the 2δ-neighbourhood of E such that $d_X(u_1, u_2) \leq \max\{\delta, \phi(3\delta)\} \leq \phi(3\delta)$. Indeed, let v_1 be the point on $[\pi(x), x]$ furthest from $\pi(x)$ which lies in the 3δ-neighbourhood of E, and let v_2 be the point on $[\pi(x), y]$ furthest from $\pi(y)$ which lies in the 3δ-neighbourhood of E.

If Δ' is δ-thin then there is a point u_2 on $[\pi(x), y]$ within δ of v_1. We may take $v_1 = u_1$. Similarly, if Δ' is δ-thin relative to E then once again there must be such a point u_2.

Therefore, suppose that Δ' is δ-thin relative to $E' \neq E$. Then v_1 does not lie within δ of $[x, y]$ since $[x, y]$ does not intersect the 4δ-neighbourhood of E. Therefore, either v_1 lies within δ of E' or within δ of $[\pi(y), x]$. The second case is unproblematic as usual. Also, v_2 either lies within δ of E' or within δ of $[\pi(x), x]$, and in this second case we proceed as usual.

So suppose that v_1 and v_2 both lie within δ of E'. Then they both lie within the 3δ-neighbourhoods of E and E' and so $d_X(v_1, v_2) \leq \phi(3\delta)$.

Now, u_2 is closer to y along $[\pi(x), y]$ than w_1, since u_2 lies outside the 2δ-neighbourhood of E, and w_1 lies within. Hence, the convexity of the metric in X ensures that there is a point $u_3 \in [\pi(y), y]$ so that $d_X(u_2, u_3) \leq d_X(w_1, w_2)$.

Now, $u_1 \in [\pi(x), x]$ so $\pi(u_1) = \pi(x)$, and similarly $\pi(y_3) = \pi(y)$. Therefore,

$$\begin{aligned}
d_X(\pi(x), \pi(y)) &= d_X(\pi(u_1), \pi(u_3)) \\
&\leq d_X(\pi(u_1), \pi(u_2)) + d_X(\pi(u_2), \pi(u_3)) \\
&\leq d_X(\pi(u_1), \pi(u_2)) + d_X(\pi(w_1), \pi(w_2)) \\
&\leq \phi(3\delta) + \phi(\delta) \\
&\leq 2\phi(3\delta),
\end{aligned}$$

as required. □

3 Asymptotic cones of CAT(0) spaces with isolated flats

In this section, we construct a limiting action from a sequence of homomorphisms from a fixed finitely generated group G to Γ, a CAT(0) group with isolated flats. The action is of G on the asymptotic cone of X, where X is the CAT(0) space with isolated flats upon which G acts properly and cocompactly.

Asymptotic cones of CAT(0) spaces have been studied in [20] and we use or adapt many of their results. We note that one of the results of this section is that the asymptotic cone of X is a *tree-graded metric space*, in the terminology of [14]. This follows from [19] and [14]. This paper was written before [19] or [14] appeared publicly, and we need more results from this section than follow directly from either [14] or [19]. Thus, we prefer to leave this section unchanged, rather than referring to [14] or [19] for some of the results herein.

Remark 3.1 The construction below could be carried out in a similar way to those found in [23, 24] (see also [1] and [9]) using the equivariant Gromov topology on 'approximate convex hulls' of finite orbits of a basepoint x under the various actions of G on X. For the sake of brevity, however, we use asymptotic cones. However, having used asymptotic cones we use Lemma 3.15 below to pass back to the context of the equivariant Gromov topology.

3.1 Constructing the asymptotic cone

Suppose that X is a CAT(0) space with isolated flats, and $\Gamma \to \mathrm{Isom}(X)$ is a proper, cocompact and isometric action of Γ on X.

Let G be a finitely generated group, and suppose that $\{h_n \colon G \to \Gamma\}$ is a sequence of nontrivial homomorphisms. A homomorphism $h \colon G \to \Gamma$ gives rise to a sequence of proper isometric actions of G on X:

$$\lambda_h \colon G \times X,$$

given by $\lambda_h = \iota \circ h$, where $\iota \colon \Gamma \to \mathrm{Isom}(X)$ is the fixed homomorphism given by the action of Γ on X.

Because the action of Γ on X is proper and cocompact, we have the following:

Lemma 3.2 *For any $y \in X$, $j \geq 1$ and $g \in G$, the function $\iota_{g,j,y} \colon \Gamma \to \mathbb{R}$ defined by*

$$\iota_{g,j,y}(\gamma) = d_X\left(\gamma.y, \lambda_j(g, \gamma.y)\right),$$

achieves its infimum for some $\gamma' \in \Gamma$.

Let $\mathcal{A}$ be a finite generating set for G and let $x \in X$ be arbitrary. For a homomorphism $h \colon G \to \Gamma$ define μ_h and $\gamma_h \in \Gamma$ so that

$$\begin{aligned} \mu_h &= \max_{g \in \mathcal{A}} d_X\left(\gamma_h.x, \lambda_h(g, \gamma_h.x)\right) \\ &= \min_{\gamma \in \Gamma} \max_{g \in \mathcal{A}} d_X\left(\gamma.x, \lambda_h(g, \gamma.x)\right). \end{aligned}$$

For the chosen sequence of homomorphisms $h_n\colon G \to \Gamma$, we write λ_n instead of λ_{h_n}, μ_i instead of μ_{h_i} and γ_i instead of γ_{h_i}.

Now define the pointed metric spaces (X_n, x_n) to be the set X with basepoint $x_n = x$, with the metric $d_{X_n} = \frac{1}{\mu_n} d_X$. Since there is a natural identification between $\mathrm{Isom}(X)$ and $\mathrm{Isom}(X_i)$, we consider λ_i to give an action of G on X_i, as well as on X.

The next lemma follows from the fact that $\Gamma.x$ is discrete, and that G is finitely generated.

Lemma 3.3 *Suppose that for all $j \neq i$ there is no element $\gamma \in \Gamma$ so that $h_i = \tau_\gamma \circ h_j$ where τ_γ is the inner automorphism of Γ induced by γ. Then the sequence $\{\mu_j\}$ does not contain a bounded subsequence.*

We use the homomorphism h_j and the translation minimising element γ_j to define an isometric action $\hat{\lambda}_j\colon G \times X_n \to X_n$ by defining

$$\hat{\lambda}_n(g, y) = \left(\gamma_n^{-1} h_n(g) \gamma_n\right).y.$$

Convention 3.4 For the remainder of the paper, we assume that the homomorphisms h_n were chosen so that $\gamma_n = 1$ for all n. Therefore, $\hat{\lambda}_n(g,x) = \lambda_n(g,x) = h_n(g).x$ for all $n \geq 1$, $g \in G$ and $x \in X$.

Using the spaces (X_n, x_n) and the actions λ_n of G on X_n, we construct an action of G on the asymptotic cone of X, with respect to the basepoints $x_n = x$, scalars μ_n and an arbitrary non-principal ultrafilter ω.

We briefly recall the definition of asymptotic cones. For more details, see [13] and [14], or [20] in the context of CAT(0) spaces.

Definition 3.5 A *non-principal ultra-filter*, ω, is a $\{0,1\}$-valued finitely additive measure on $\mathbb{N}$ defined on all subsets of $\mathbb{N}$ so that any finite set has measure 0.

The existence of non-principal ultrafilters is guaranteed by Zorn's Lemma.

Fix once and for all a non-principal ultrafilter ω.[4] Given any bounded sequence $\{a_n\} \subset \mathbb{R}$ there is a unique number $a \in \mathbb{R}$ so that, for all $\epsilon > 0$,

[4]The choice of ultrafilter will affect the resulting construction, but will not affect our results. Thus we are unconcerned which ultrafilter is chosen.

$\omega(\{a_n \mid |a - a_n| < \epsilon\}) = 1$. We denote a by ω-$\lim\{a_n\}$. This notion of limit exhibits most of the properties of the usual limit (see [13]).

The *asymptotic cone of* X *with respect to* $\{x_n\}$, $\{\mu_n\}$ *and* ω, denoted X_ω is defined as follows. First, define the set $\tilde{X}_\omega$ to consist of all sequences $\{y_n \mid y_n \in X_n\}$ for which $\{d_{X_n}(x_n, y_n)\}$ is a bounded sequence. Define a pseudo-metric $\tilde{d}$ on $\tilde{X}$ by

$$\tilde{d}(\{y_n, z_n\}) = \omega\text{-}\lim\ d_{X_n}(y_n, z_n).$$

The asymptotic cone X_ω is defined to be the metric space induced by the pseudo-metric $\tilde{d}$ on $\tilde{X}_\omega$:

$$X_\omega := \tilde{X}_\omega / \sim,$$

where the equivalence relation '$\sim$' on $\tilde{X}_\omega$ is defined by: $x \sim y$ if and only if $\tilde{d}(x, y) = 0$. The pseudo-metric $\tilde{d}$ on $\tilde{X}_\omega$ naturally descends to a metric d_ω on X_ω.

Lemma 3.6 (see [13] and [20], Proposition 3.6) *(X_ω, d_ω) is a complete, geodesic $CAT(0)$ space.*

We now define an isometric action of G on X_ω. Let $g \in G$ and $\{y_n\} \in \tilde{X}_\omega$. Then define $g.\{y_n\}$ to be $\{\lambda_n(g, y_n)\} \in \tilde{X}_\omega$. This descends to an isometric action of G on X_ω.

Remark 3.7 The action of G defined on the asymptotic cone X_ω is slightly different to the one described in [20, §3.4], but the salient features remain the same.[5]

We assume a familiarity with asymptotic cones, but we essentially use only two properties. The first is that finite sets in $\mathbb{N}$ have ω-measure 0. The second property is the following:

Lemma 3.8 *Suppose that X_ω is constructed using the sequence $\{h_n\colon G \to \Gamma\}$ as above. Suppose that $Q \subset G$ is finite and $S \subset X_\omega$ is finite. For each $s \in S$, let $\{s_n\}$ be a sequence of elements from X_n such that $\{s_n\} \in \tilde{X}_\omega$ is a representative of the equivalence class s. Fix $\epsilon > 0$ and define $I_{\epsilon,Q,S}$ to be the set of $i \in \mathbb{N}$ so that for all $q_1, q_2 \in Q \cup \{1\}$ and all $s, s' \in S$ we have*

$$|d_{X_i}(\hat{\lambda}(q_1, s_i), \hat{\lambda}(q_2, s_i')) - d_{X_\omega}(q_1.s, q_2.s')| < \epsilon.$$

Then $\omega(I_{\epsilon,Q,S}) = 1$.

Given finite subsets Q of G and S of X_ω and $\epsilon > 0$ as above, if $i \in I_{\epsilon,Q,S}$ then the pair (X_i, λ_i) is called an *ϵ-approximation* for Q and S.

[5] The difference comes in the choice of scalars μ_n and the choice of basepoints x_n.

3.2 Properties of X_ω

Lemma 3.9 *The action of G on X_ω by isometries does not have a global fixed point.*

Proof Let $K \subseteq X$ be a compact set so that the basepoint x is in K and $\Gamma.K = X$. Let $D = \mathrm{Diam}(K)$.

Suppose that $y \in X_\omega$ is fixed by all points of G. Choose a large i so that (i) $\mu_i > 4D$; and (ii) X_i is a $\frac{1}{2}$-approximation for $\{y\}$ and $\mathcal{A}$. (Recall that $\mathcal{A}$ is the fixed finite generating set for G.) Thus, if $\{y_n\}$ represents y then for all $g \in \mathcal{A}$

$$d_{X_i}(y_i, \lambda_i(g, y_i)) < \frac{1}{2}.$$

This implies that, for all $g \in \mathcal{A}$,

$$d_X(y_i, h_i(g).y_i) < \frac{\mu_i}{2}.$$

Now, there exists $\gamma \in \Gamma$ so that $d_X(x, \gamma.y_i) \leq D$. Let $g \in S$ be the element which realises the maximum:

$$\mu_i = \min_{\overline{\gamma} \in \Gamma} \max_{g \in S} d_X(x, \overline{\gamma} h_i(g) \overline{\gamma}^{-1}).x).$$

Then we have

$$\begin{aligned}
\mu_i &\leq d_X(x, (\gamma h_i(g)\gamma^{-1}).x) \\
&\leq d_X(x, \gamma.y_i) + d_X(\gamma.y_i, (\gamma h_i(g)\gamma^{-1})\gamma.y_i) \\
&\quad + d_X((\gamma h_i(g)\gamma^{-1})\gamma.y_i, (\gamma h_i(g)\gamma^{-1}).x) \\
&= 2d_X(x, \gamma.y_i) + d_X(y_i, h_i(g).y_i) \\
&< 2D + \frac{\mu_i}{2}.
\end{aligned}$$

Since $\mu_i > 4D$ this is a contradiction. Therefore there is no global fixed point for the action of G on X_∞. ☐

We now prove some results about X_ω which are very similar to those obtained in [20] in the context of asymptotic cones of certain 3-manifolds.

Definition 3.10 (See [20], §2-2) Let X be a CAT(0) space and $x, y, z \in X$. Define x', y', z' by $[x, x'] = [x, y] \cap [x, z]$, $[y, y'] = [y, z] \cap [y, x]$ and $[z, z'] = [z, x] \cap [z, y]$. The triangle $\Delta(x', y', z')$ is called the *open triangle* spanned by x, y, z. The triangle $\Delta(x, y, z)$ is called *open* if $x = x'$, $y = y'$ and $z = z'$. An open triangle $\Delta(x', y', z')$ is *non-degenerate* if the three points x', y', z' are distinct.

Let $\mathcal{F}_X$ be the set of flats in X from Definition 2.4. Let $\mathcal{F}_n$ be the set $\mathcal{F}_X$ considered as subsets of X_n. Denote by $\mathcal{F}_\omega$ the set of all flats in X_ω which arise as limits of flats $\{E_i\}_{i\in\mathcal{N}}$ where $E_i \in \mathcal{F}_i$.

Proposition 3.11 (See [20], Proposition 4.3) *The space X_ω satisfies the following two properties:*

(F1) *Every non-degenerate open triangle in X_ω is contained in a flat $E \in \mathcal{F}_\omega$; and*

(F2) *Any two flats in $\mathcal{F}_\omega$ intersect in at most a point.*

Proof Let $\Delta = \Delta(x, y, z)$ be an open triangle in X_ω. Then Δ can be obtained as a limit of triangles Δ_i, where $\Delta_i = \Delta(x_i, y_i, z_i)$ is a triangle in X_i. The triangle Δ_i may be identified with a triangle Δ_i' in X (since X and X_n are the same set with different metrics).

For ω-almost all i, the triangle Δ_i' is not δ-thin, for otherwise the limit would not be a non-degenerate open triangle. Therefore, Δ_i' is δ-thin relative to some flat $E_i \in \mathcal{F}_X$. Consider a point $w \in [x, y] \smallsetminus \{x, y\}$. The point $w \in X_\infty$ corresponds to a sequence of points $\{w_i\}$. Now $d_\omega(w, [y, z]) > 0$ and $d_\omega(w, [x, z])$, so for ω-almost all i the point w_i is not contained in the δ-neighbourhood of $[y_i, z_i]$ or the δ-neighbourhood of $[x_i, z_i]$. Therefore, for ω-almost all i the point w_i is contained in the δ-neighbourhood of E_i. Let u_i be a point in E_i within δ of w_i. It is clear that the sequences $\{w_i\}$ and $\{u_i\}$ have the same limit, namely w (although u_i is only defined for ω-almost all i). Therefore, w is contained in the limit of the flats $\{E_i\}$. This proves Property (F1).

Now suppose that the flats $E, E' \in \mathcal{F}_\infty$ intersect in more than one point. Let $x, y \in \hat{E}_1 \cap \hat{E}_2$ be distinct. By Property (F1), there is a sequence of flats $\{E_i\}$ which approximate E. Let $u, w \in [x, y] \smallsetminus \{x, y\}$ be arbitrary ($u \neq w$) and let $\{x_i\}, \{y_i\}, \{u_i\}, \{w_i\} \subseteq E_i$ be sequences of points representing x, y, u and w, respectively.

Let $z \in E'$ be arbitrary so that $\Delta(x, y, z)$ is a non-degenerate triangle, and let $\{z_i \in X_i\}$ be a sequence of points representing z. Since the triangle $\Delta(x, y, z)$ is an open triangle, there is a sequence of flats $\{E_i'\}$ whose limit contains $\Delta(x, y, z)$.

For ω-almost all i, neither u_i nor w_i is contained in the δ-neighbourhood of $[x_i, z_i] \cup [y_i, z_i]$, and so are contained in the δ-neighbourhood of E_i'. Therefore the points u_i and w_i are each contained in the δ-neighbourhoods of both E_i

and E_i'. However u and w are distinct, so for ω-almost all i the points u_i and w_i are at least $\phi(\delta)$ apart, which implies that $E_i = E_i'$ for ω-almost all i. Therefore, the triangle $\Delta(x,y,z)$ is contained in E. Since z was arbitrary, $E' \subseteq E$, and a symmetric argument shows that the two flats are equal. □

Using only the properties (F1) and (F2) from the conclusion of Proposition 3.11 above, Kapovich and Leeb proved the following two results.

Lemma 3.12 [20, Lemma 4.4] *Let $E \in \mathcal{F}_\omega$ be a flat in X_ω and let $\pi_E\colon X_\omega \to E$ be the closest-point projection map. Let $\gamma\colon [0,1] \to X_\omega \smallsetminus E$ be a curve in the complement of E. Then $\pi_E \circ \gamma\colon [0,1] \to E$ is constant.*

Lemma 3.13 [20, Lemma 4.5] *Every embedded loop in Y is contained in a flat $E \in \mathcal{F}_\omega$.*

3.3 The equivariant Gromov topology

From the sequence of homomorphisms $h_n\colon G \to \Gamma$, we have constructed a space X_ω, a basepoint x_ω and an isometric action of G on X_ω with no global fixed point. Let X_∞ be the convex hull of the set $G.x_\omega$, and let $\mathcal{C}_\infty$ be the union of the geodesics $[x_\omega, g.x_\omega]$, along with the flats $E \in \mathcal{F}_\omega$ which contain some non-degenerate open triangle contained in a triangle $\Delta(g_1.x_\omega, g_2.x_\omega, g_3.x_\omega)$, for $g_1, g_2, g_3 \in G$. Certainly $X_\infty \subseteq \mathcal{C}_\infty$. The set $\mathcal{C}_\infty$, and hence also X_∞, is separable.

Note that $\mathcal{C}_\infty$ is a CAT(0) space and that Proposition 3.11 and Lemmas 3.12 and 3.13 hold for $\mathcal{C}_\infty$ also. The action of G on X_ω leaves $\mathcal{C}_\infty$ invariant, so there is an isometric action of G on $\mathcal{C}_\infty$. Since $C_\infty \subseteq X_\omega$, Lemma 3.9 implies:

Lemma 3.14 *There is no global fixed point for the action of G on $\mathcal{C}_\infty$.*

We have chosen to consider the space $\mathcal{C}_\infty$ rather than X_∞ so that if some flat in X_ω intersects our subspace in a set containing a non-degenerate open triangle then the entire flat containing this triangle is contained in the subspace.

Suppose that $\{(Y_n, \lambda_n)\}_{n=1}^\infty$ and (Y, λ) are pairs consisting of metric spaces, together with actions $\lambda_n\colon G \to \mathrm{Isom}(Y_n)$, $\lambda\colon G \to \mathrm{Isom}(Y)$. Recall (cf. [5, §3.4, p. 16]) that $(Y_n, \lambda_n) \to (Y, \lambda)$ in the *G-equivariant Gromov topology* if and only if: for any finite subset K of Y, any $\epsilon > 0$ and any finite subset P of G, for sufficiently large n, there are subsets K_n of Y_n and bijections $\rho_n\colon K_n \to K$ such that for all $s_n, t_n \in K_n$ and all $g_1, g_2 \in P$ we have

$$|d_Y(\lambda(g_1).\rho_n(s_n), \lambda(g_2).\rho_n(t_n)) - d_{Y_n}(\lambda_n(g_1).s_n, \lambda_n(g_2).t_n)| < \epsilon.$$

To a homomorphism $h\colon G \to \Gamma$, we naturally associate a pair (X_h, λ_h) as follows: let X_h be the convex hull in X of $G.x$ (where x is the basepoint of X), endowed with the metric $\frac{1}{\mu_h} d_X$; and let $\lambda_h = \iota \circ h$, where $\iota\colon \Gamma \to \mathrm{Isom}(X)$ is the fixed homomorphism.

Lemma 3.15 *Let Γ, X, G and $\{h_n\colon G \to \Gamma\}$ be as described above. Let X_ω be the asymptotic cone of X, and $\mathcal{C}_\infty$ be as described above. Let $\lambda_\infty\colon G \to$ Isom$(\mathcal{C}_\infty)$ denote the action of G on $\mathcal{C}_\infty$ and $(\mathcal{C}_\infty, \lambda_\infty)$ the associated pair.*

There exists a subsequence $\{f_i\} \subseteq \{h_i\}$ so that the elements (X_{f_i}, λ_{f_i}) converge to $(\mathcal{C}_\infty, \lambda_\infty)$ in the G-equivariant Gromov topology.

Proof Since $\mathcal{C}_\infty$ is separable, there is a countable dense subset of $\mathcal{C}_\infty$, S say. Let $S_1 \subset S_2 \subset \ldots$ be a collection of finite sets whose union is S.

Let $\{1\} = Q_1 \subset Q_2 \subset \ldots \subset G$ be an exhaustion of G by finite subsets. Define J_i to be the collection of $i \in \mathbb{N}$ so that (X_{h_i}, λ_i) is a $\frac{1}{i}$-approximation for Q_i and S_i. By the definition of asymptotic cone, $\omega(J_i) = 1$, and in particular each J_i is infinite.

Let n_1 be the least element of J_1, and let $f_1 = h_{n_1}$. Inductively, define n_k to be the least element of J_k which is not contained in $\{n_1, \ldots, n_{k-1}\}$, and define $f_k = h_{n_k}$.

It is straightforward to see that the sequence $\{f_i\}$ satisfies the conclusion of the lemma. □

The above result can be interpreted as a compactification of a certain space of metric spaces equipped with G-actions. This is the 'compactification' referred to in the title of this paper.

Convention 3.16 For the remainder of the paper, we will assume that we started with the sequence $\{f_i\colon G \to \Gamma\}$ found in Lemma 3.15 above. This will allow us to speak of 'all but finitely many n' instead of 'ω-almost all n'.

To make the use of Convention 3.16 more transparent, when using this convention we speak of the homomorphisms f_i, rather than h_i. However, we still write λ_i for the action of G on X induced by f_i, and we write μ_i for μ_{f_i} and X_i for X endowed with the metric $d_{X_i} := \frac{1}{\mu_i} d_X = \frac{1}{\mu_{f_i}} d_X$.

Let $\mathcal{F}_\infty$ be the set of flats in $\mathcal{C}_\infty$.

Corollary 3.17 *Under Convention 3.16, for each $E \in \mathcal{F}_\infty$ there is a sequence $\{E_i \subset X_i\}$ so that $E_i \to E$ in the G-equivariant Gromov topology.*

Proof This follows from the proofs of Proposition 3.11 and Lemma 3.15. □

3.4 The action of G on X_ω

Lemma 3.18 *Let $E \in \mathcal{F}_\infty$ be a flat which is a limit of the flats $\{E_i\}$. If $g \in G$ and $g.E = E$ then for all but finitely many j we have $f_j(g).E_j = E_j$.*

Proof Choose a non-degenerate triangle $\Delta(a,b,c)$ in E. Let $\{E_i\}$ be a sequence of flats from X_i approximating E. Let $\{a_i\},\{b_i\}$ and $\{c_i\}$ be sequences of points representing a,b and c, respectively.

By the definition of the action of G on X_ω, the point $g.x$ is represented by the sequence $\{\lambda_i(g,a_i)\}$. The triangle $\Delta(g.a,g.b,g.c)$ is also a non-degenerate triangle in E and at least one of the triangles $\Delta(a,b,g.a)$, $\Delta(a,c,g.a)$, $\Delta(b,c,g.a)$ is non-degenerate in E. The argument from the proof of Proposition 3.11 (along with Corollary 3.17) applied to this non-degenerate triangle shows that for all but finitely many i the point $\lambda_i(g,a_i)$ is δ-close to the flat E_i. Similarly, for all but finitely many i the point $\lambda_i(g,b_i)$ is δ-close to E_i. Since $E_i \in \mathcal{F}_X$, so is the flat $f_i(g).E_i$, by Proposition 2.8. However,

$$d_X(\lambda_i(g,a_i),\lambda_i(g,b_i)) = d_X(a_i,b_i),$$

which is greater than $\phi(\delta)$ for all but finitely many i. Therefore, by the definition of the function ϕ, for all but finitely many i the flats E_i and $f_i(g).E_i$ are the same, as required. □

Lemma 3.19 *Suppose that Γ is a group acting properly and cocompactly on a CAT(0) space X with isolated flats, and suppose that this action is toral. Let G and X_ω be as above. Suppose that $g \in G$ leaves a flat E in $\mathcal{C}_\infty$ invariant as a set. Then g acts by translation on E.*

Proof Suppose that g acts nontrivially on E, but not as a translation. Then there are $y,z \in E$ which are moved different distances by g (suppose that y is moved further than z by g). Let $\{E_i\}$ be a sequence of flats in X_i which converge to E. Since g maps E to itself, for all but finitely many i, we have $f_i(g).E_i = E_i$, by Lemma 3.18.

Let $\{z_i\}, \{y_i\} \subseteq E_i$ be sequences of points in representing z and y, respectively. Suppose that $d_{X_\omega}(y, g.y) - d_{X_\omega}(z, g.z) = \epsilon > 0$. Choose large i so that the points z_i, y_i, $\lambda_i(g, z_i)$ and $\lambda_i(g.y_i)$ satisfy

$$\begin{aligned} |d_{X_n}(\lambda_i(g, z_i), z_i) - d_X(g.z, z)| &< \frac{\epsilon}{3}; \text{ and} \\ |d_{X_n}(\lambda_i(g, y_i), y_i) - d_X(g.y, y)| &< \frac{\epsilon}{3}. \end{aligned}$$

Since Γ is toral, the action of g on X_n via λ_i is by (possibly trivial) translations. Therefore,

$$d_{X_n}(\lambda_i(g, z_i), z_i) = d_{X_n}(\lambda_i(g, y_i), y_i).$$

However, $d_X(g.y, y) - d_X(g.z, z) = \epsilon > 0$ and we have a contradiction. □

Remark 3.20 As we shall see in Example 3.26 below, Lemma 3.19 does not hold when Γ is a non-toral CAT(0) group with isolated flats.

3.5 Algebraic Γ-limit groups

Definition 3.21 (cf. [32], Definition 1.2) Define the normal subgroup K_∞ of G to be the kernel of the action of G on $\mathcal{C}_\infty$:

$$K_\infty = \{g \in G \mid \forall y \in \mathcal{C}_\infty,\ g(y) = y\}.$$

The *strict* Γ*-limit group* is $L_\infty = G/K_\infty$. Let $\eta\colon G \to L_\infty$ be the natural quotient map.

A Γ*-limit group* is a group which is either a strict Γ-limit group or a finitely generated subgroup of Γ.

Recall the following (see [5, Definition 1.5]).

Definition 3.22 Let G and Ξ be finitely generated groups. A sequence $\{f_i\} \subseteq \mathrm{Hom}(G, \Xi)$ is *stable* if, for all $g \in G$, the sequence $\{f_i(g)\}$ is eventually always 1 or eventually never 1.

For any sequence $\{f_i\colon G \to \Xi\}$ of homomorphisms, the *stable kernel* of $\{f_i\}$, denoted $\underrightarrow{\mathrm{Ker}}\,(f_i)$, is

$$\{g \in G \mid f_i(g) = 1 \text{ for all but finitely many } i\}.$$

Definition 3.23 An *algebraic* Γ*-limit group* is the quotient $G/\underrightarrow{\mathrm{Ker}}\,(h_i)$, where $\{h_i\colon G \to \Gamma\}$ is a stable sequence of homomorphisms.

In the case that Γ is a free group (acting on its Cayley graph), Bestvina and Feighn [5] define *limit groups* to be those groups of the form $G/\underrightarrow{\mathrm{Ker}}\, f_i$, where $\{f_i\}$ is a stable sequence in $\mathrm{Hom}(G, \Gamma)$. When Γ is a free group, this leads to the same class of groups as the geometric definition analogous to Definition 3.21 above (see [32]; this is also true when Γ is a torsion-free hyperbolic group, see [34]). When Γ is a torsion-free $\mathrm{CAT}(0)$ group with isolated flats we may have torsion in G/K_∞, but $G/\underrightarrow{\mathrm{Ker}}$ is always torsion-free. However, for any stable sequence $\{f_i\}$, we always have $\underrightarrow{\mathrm{Ker}}\, f_i \subseteq K_\infty$. Torsion in G/K_∞ can only occur when $\mathcal{C}_\infty$ is a single flat, in which case $f_i(G)$ is virtually abelian for almost all i.

In Section 4 below, when Γ is a torsion-free toral $\mathrm{CAT}(0)$ group with isolated flats we use the action of G on $\mathcal{C}_\infty$ to construct an action of G on an $\mathbb{R}$-tree T. In this case, the class of Γ-limit groups and algebraic Γ-limit groups coincides. It will be this fact that allows us in [17] to prove many results about torsion-free toral $\mathrm{CAT}(0)$ groups with isolated flats in analogy to Sela's results about free groups and torsion-free hyperbolic groups.

The following are elementary.

Lemma 3.24 *Suppose that $\{f_n\colon G \to \Gamma\}$ gives rise to the action of G on $\mathcal{C}_\infty$ as in the previous section. Then $\underrightarrow{\mathrm{Ker}}\,(f_n) \subseteq K_\infty$.*

Lemma 3.25 *Let L be a Γ-limit group. Then L is finitely generated.*

3.6 A non-toral example

In this paragraph we consider an example of the above construction in the case that Γ is a torsion-free *non*-toral $\mathrm{CAT}(0)$ group with isolated flats.

For any torsion-free group acting an properly and cocompactly on a $\mathrm{CAT}(0)$ space X with isolated flats, and for any maximal flat $E \in \mathcal{F}_\omega$, we know that $H := \mathrm{Stab}(E)$ is a torsion-free, proper cocompact lattice in $\mathbb{R}^n$, for some n. Hence, by Bieberbach's theorem, H has a free abelian group of finite index. We have an exact sequence

$$1 \to \mathbb{Z}^n \to H \to A \to 1,$$

where A is a finite subgroup of $O(n)$. Each element $g \in H$ acts on $\mathbb{R}^n$ as

$$g(v) = r_g(v) + t_g,$$

where $r_g \in A \subset O(n)$ and $t_v \in \mathbb{R}^n$. The homomorphism $H \to A$ is given by $g \to r_g$.

Example 3.26 Let G_1 be a non-abelian torsion-free crystallographic group as above, with the exact sequence

$$1 \to \mathbb{Z}^n \to G_1 \to A \to 1,$$

where A is a nontrivial finite group and let $\Gamma = G_1 \times \mathbb{Z}$. Let w be the generator of the $\mathbb{Z}$ factor of Γ. Clearly the group Γ acts properly and cocompactly by isometries on a CAT(0) space with isolated flats.

Let Γ be generated by $\{g_1, \ldots, g_k, w\}$, where $\{g_1, \ldots, g_k\}$ is a generating set for G_1 and let F_{k+1} be the free group of rank $k+1$ with basis $\{x_1, \ldots, x_{k+1}\}$. For $n \geq 1$, define the homomorphism $\phi_n \colon F_{k+1} \to G_3$ by $\phi_n(x_i) = g_i$ for $1 \leq i \leq k$, and $\phi_n(x_{k+1}) = w^n$. All of the kernels of ϕ_n are identical, so the algebraic Γ-limit group is $F/\underrightarrow{\mathrm{Ker}}\,(\phi_n) \cong \Gamma$.

In this case $\mathcal{C}_\infty = X_\infty = X_\omega = \mathbb{R}^{n+1}$. In the geometric Γ-limit group, the $\mathbb{Z}^n$ in G_1 acts trivially, but the elements not in $\mathbb{Z}^n$ act like the corresponding element of A. The element w acts nontrivially by translation, and the Γ-limit group is isomorphic to $A \times \mathbb{Z}$, which is not torsion-free.

4 The $\mathbb{R}$-tree T

For the remainder of the paper, we suppose that Γ is a torsion-free *toral* CAT(0) group with isolated flats. In this section we extract an $\mathbb{R}$-tree T from the space X_ω and an isometric action of G on T. The idea is to remove the flats in X_ω in order to obtain an $\mathbb{R}$-tree. We replace the flats with lines.

4.1 Constructing the $\mathbb{R}$-tree

Suppose that Γ is a torsion-free toral CAT(0) group with isolated flats acting on the space X, and that the sequence of homomorphisms $h_n \colon G \to \Gamma$ gives rise to the limiting space X_ω, as in the previous section. Let $\mathcal{C}_\infty$ be the collection of geodesics and flats as described in Subsection 3.4, and let $\mathcal{F}_\infty$ be the set of flats in $\mathcal{C}_\infty$.

Suppose that $E \in \mathcal{F}_\infty$. By Proposition 3.11, for any $g \in G$, exactly one of the following holds: (i) $g.E = E$; (ii) $|g.E \cap E| = 1$; or (iii) $g.E \cap E = \emptyset$. By Lemmas 3.18 and 3.19, the action of $\mathrm{Stab}(E)$ on E is as a finitely generated free abelian group, acting by translations on E.

Let $\mathcal{D}_E$ be the set of directions of the translations of E by elements of $\mathrm{Stab}(E)$.

For each element $g \in G \smallsetminus \mathrm{Stab}(E)$, let $l_g(E)$ be the (unique) point where any geodesic from a point in E to a point in $g.E$ leaves E, and let $\mathcal{L}_E$ be the set of all $l_g(E) \subset E$. Note that if $g.E \cap E$ is nonempty (and $g \not\in \mathrm{Stab}(E)$) then $g.E \cap E = \{l_g(E)\}$.

Since G is finitely generated, and hence countable, both sets $\mathcal{D}_E$ and $\mathcal{L}_E$ are countable. Given a (straight) line $p \subset E$, let χ_E^p be the projection from E to p. Since $\mathcal{L}_E$ is countable, there are only countably many points in $\chi_E^p(\mathcal{L}_E)$. Therefore, there is a line $p_E \subseteq E$ such that

(1) the direction of p_E is not orthogonal to a direction in $\mathcal{D}_E$;

(2) if x and y are distinct points in $\mathcal{L}_E$, then $\chi_E^{p_E}(x) \neq \chi_E^{p_E}(y)$;

Project E onto p_E using $\chi_E^{p_E}$. The action of $\mathrm{Stab}(E)$ on p_E is defined in the obvious way (using projection) – this is an action since the action of $\mathrm{Stab}(E)$ on E is by translations. Connect $\mathcal{C}_\infty \smallsetminus E$ to p_E in the obvious way – this uses the following observation which follows immediately from Lemma 3.12.

Observation 4.1 Suppose S is a component of $\mathcal{C}_\infty \smallsetminus E$. Then there is a point $x_S \in E$ so that S is a component of $\mathcal{C}_\infty \smallsetminus \{x_E\}$.

Glue such a component S to p_E at the point $\chi_E^{p_E}(x_S)$.

Perform this projecting and gluing construction in an equivariant way for all flats $E \subseteq \mathcal{C}_\infty$ – so that for all $E \subseteq \mathcal{C}_\infty$ and all $g \in G$ the lines $p_{g.E}$ and $g.p_E$ have the same direction (this is possible since the action of $\mathrm{Stab}(E)$ on E is by translations, so doesn't change directions).

Having done this for all flats $E \subseteq \mathcal{C}_\infty$, we arrive at a space T, which is endowed with the (obvious) path metric.

The action of G on T is defined in the obvious way from the action of G on X_ω. This action is clearly by isometries.

The space T has a distinguished set of geodesic lines, namely those of the form $\chi_E^{p_E}(E)$, for $E \in F_\infty$. Denote the set of such geodesic lines by $\mathbb{P}$.

Lemma 4.2 *T is an $\mathbb{R}$-tree and there is an action of G on T by isometries without global fixed points.*

Proof That T is an $\mathbb{R}$-tree is obvious, since there are no embedded loops. We have already noted that there is an isometric action of G on T.

Finally, suppose that there is a fixed point y for the action of G on T. If y is not contained in some geodesic line in $\mathbb{P}$, then y would correspond to a fixed point for the action of G on X_ω, and there are no such fixed points, by Lemma 3.14.

Thus y is contained in some geodesic line $p_E \in \mathbb{P}$, corresponding to the flat $E \in \mathcal{F}_\infty$. Let $g \in G$. If g does not fix p_E then it takes p_E to some line $p_{E'}$, and g takes E to E' in X_ω, fixing the point of intersection. Suppose that g_1 and g_2 are elements of G which fix y but not p_E. Then let $\alpha \in X_\omega$ be the point of intersection of E and $g_1.E$ and let β be the point of intersection of E and $g_2.E$. Then α and β are both in $\mathcal{L}_E$ and $\chi_E^{p_E}(\alpha) = \chi_E^{p_E}(\beta)$, so by the choice of p_E we must have $\alpha = \beta$. Therefore, there is a point $\alpha \in E$ so that all elements $g \in G$ which do not fix $p_E \subseteq T$ fix $\alpha \in X_\omega$.

If g does leave p_E invariant then it fixes E as a set, and so acts by translations on E, and hence by translations on p_E. Therefore g fixes p_E pointwise, and the direction of translation of g on E is orthogonal to the direction of p_E. By the choice of the direction of p_E above, this means that g acts trivially on E, and in particular fixes the point α found above. Thus α is a global fixed point for the action of G on $\mathcal{C}_\infty$, contradicting Lemma 3.14. □

Remark 4.3 Since $K_\infty \leq G$ acts trivially on $\mathcal{C}_\infty$, it also acts trivially on T and the action of G on T induces an isometric action of L_∞ on T.

4.2 The actions of G and L_∞ on T

The following theorem is the main technical result of this paper, and the remainder of this section is devoted to its proof.

Let G be a finitely generated group, Γ a torsion-free toral CAT(0) group with isolated flats and $\{h_i\colon G \to \Gamma\}$ a sequence of homomorphisms, no two of which differ only by conjugation in Γ. Let X_ω, $\mathcal{C}_\infty$ and T be as in Section 3 and Subsection 4.1 above. Let $\{f_i\colon G \to \Gamma\}$ be the subsequence of $\{h_i\}$ as in the conclusion of Lemma 3.15. Let K_∞ be the kernel of the action of G on $\mathcal{C}_\infty$ and let $L_\infty = G/K_\infty$ be the associated strict Γ-limit group.

Theorem 4.4 (Compare [32], Lemma 1.3) *In the above situation, the following properties hold.*

(1) *Suppose that $[A,B]$ is a non-degenerate segment in T. Then $\mathrm{Fix}_{L_\infty}[A,B]$ is an abelian subgroup of L_∞;*

(2) *If T is isometric to a real line then for all sufficiently large n the group $f_n(G)$ is free abelian. Furthermore in this case L_∞ is free abelian.*

(3) *If $g \in G$ fixes a tripod in T pointwise then $g \in \underrightarrow{\mathrm{Ker}}(f_i)$;*

(4) *Let $[y_1, y_2] \subset [y_3, y_4]$ be a pair of non-degenerate segments of T and assume that the stabiliser Fix$([y_3, y_4])$ of $[y_3, y_4]$ in L_∞ is non-trivial. Then*

$$\mathit{Fix}([y_1, y_2]) = \mathit{Fix}([y_3, y_4]).$$

In particular, the action of L_∞ on the $\mathbb{R}$-tree T is stable.

(5) *Let $g \in G$ be an element which does not belong to K_∞. Then for all but finitely many n we have $g \notin \ker(f_n)$;*

(6) *L_∞ is torsion-free;*

(7) *If T is not isometric to a real line then $\{f_i\}$ is a stable sequence of homomorphisms.*

We prove Theorem 4.4 in a number of steps.

First, we prove 4.4(1). Suppose that $[A, B] \subseteq T$ is a non-degenerate segment with a nontrivial stabiliser. If there is a line $p_E \in \mathbb{P}$ such that $[A, B] \cap p_E$ contains more than one point, then any elements $g_1, g_2 \in \mathrm{Fix}([A, B])$ fix p_E and hence fix $E \in \mathcal{F}_\infty$. Therefore, by Lemma 3.18 for all but finitely many i the elements g_1 and g_2 fix the flat $E_i \in X_i$, where $\{E_i\} \to E$. The stabiliser of E_i is free abelian, so $[g_1, g_2] \in \ker(f_i)$. Thus $[g_1, g_2] \in \underrightarrow{\mathrm{Ker}}(f_i)$. By Lemma 3.24, $\underrightarrow{\mathrm{Ker}}(f_i) \subseteq K_\infty$, so $[g_1, g_2] \in K_\infty$. Hence, in this case the stabiliser in L_∞ of $[A, B]$ is abelian.

Suppose therefore that there is no $p_E \in \mathbb{P}$ which intersects $[A, B]$ in more than a single point. In particular, A and B are not both contained in p_E for any $p_E \in \mathbb{P}$.

In fact, we need something stronger than this. First we prove the following:

Lemma 4.5 *Suppose that $\alpha, \beta \in X_n$ and $g \in G$ are such that there is a segment of length at least $6\phi(4\delta) + 4\max\{d_X(g.\alpha, \alpha), d_X(g.\beta, \beta)\}$ in $[\alpha, \beta]$ which is within δ of a flat $E \in \mathcal{F}_X$. Then $g \in \mathit{Fix}(E)$.*

Proof In this lemma, all distances are measured with the metric d_X. Let $L = \max\{d_X(g.\alpha, \alpha), d_X(g.\beta, \beta)\}$.

Let $[\alpha_1, \beta_1]$ be the segment in $[\alpha, \beta]$ of length at least $6\phi(4\delta) + 4L$ which is in the δ-neighbourhood of E.

Consider first the triangle $\Delta_1 = \Delta(\alpha, \beta, g.\beta)$.

If Δ_1 is δ-thin, then there is a segment $[\alpha_2, \beta_2] \in [\alpha, g.\beta]$ which has length at least $6\phi(4\delta) + 3L - 2\delta$ and is within δ of $[\alpha_1, \beta_1]$. Hence $[\alpha_2, \beta_2]$ is in the 2δ-neighbourhood of E. Also, when $[\alpha, \beta]$ and $[\alpha, g.\beta]$ are both parametrised by arc length, there is an interval of time of length at least $6\phi(4\delta) + 3L - 2\delta$ when $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ both occur.

Suppose then that Δ_1 is δ-thin relative to a flat $E' \neq E$. Then a subsegment $[\alpha_1', \beta'] \subseteq [\alpha_1, \beta_1]$ of length at least $6\phi(4\delta) + 3L - \phi(\delta)$ does not intersect the δ-neighbourhood of E', and so must be contained in the δ-neighbourhood of $[\alpha, g.\beta] \cup [\beta, g.\beta]$. However, $d_X(\alpha, \beta) \geq 6\phi(4\delta) + 4L$, so there is an interval $[\alpha_2, \beta_2]$ in $[\alpha, g.\beta]$ of length at least $6\phi(4\delta) + 3L - \phi(\delta)$ which is within δ of $[\alpha_1, \beta_1]$ and hence in the 2δ-neighbourhood of E. Once again, when $[\alpha, \beta]$ and $[\alpha, g.\beta]$ are parametrised by arc length there is an interval of time of length at least $6\phi(4\delta) + 3L - \phi(\delta)$ when both $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ occur.

Finally suppose that Δ_1 is δ-thin relative to E. Then there is certainly a segment $[\alpha_2, \beta_2] \subseteq [\alpha, g.\beta]$ of length at least $6\phi(4\delta) + 3L - 2\delta$ in the δ-neighbourhood of E. Since

$$\begin{aligned} d_X(\alpha, g.\beta) &\leq d_X(\alpha, \beta) + d_X(\beta, g.\beta) \\ &\leq d_X(\alpha, \beta) + L, \end{aligned}$$

then just as above there are subsegments $[\alpha_1', \beta_1']$ and $[\alpha_2', \beta_2']$ (contained in $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$,respectively) which occur at the same time for an interval of at least $6\phi(4\delta) + 2L - 2\delta$.

In any of these cases, denote by $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ the intervals in $[\alpha, \beta]$ and $[\alpha, g.\beta]$ of length at least $6\phi(4\delta) + 2L - 2\phi(\delta)$ which occur at the same time when $[\alpha, \beta]$ and $[\alpha, g.\beta]$ are parametrised by arc length and such that $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ are both contained in the 2δ-neighbourhood of E. We now consider $\Delta_2 = \Delta(g.\alpha, \beta, g.\beta)$.

Using the same arguments as those which found $[\alpha_2, \beta_2]$ as above, it is not difficult to find an interval $[\alpha_3, \beta_3] \subseteq [g.\alpha, g.\beta]$ which is of length at least $6\phi(4\delta) - 4\phi(\delta)$ and occurs at the same time as $[\alpha_2, \beta_2]$ when $[\alpha, g.\beta]$ and $[g.\alpha, g.\beta]$ are parametrised by arc length. The only wrinkle in this argument occurs when Δ_2 is δ-thin relative to E and we may have to change $[\alpha_2, \beta_2]$ as in the third case above. However, in this case we can find an appropriate $[\alpha_1, \beta_1]$.

Now, $[\alpha_3, \beta_3]$ is contained in the 4δ-neighbourhood of E. Also, the time at which it occurs overlaps the time at which $[\alpha_1, \beta_1]$ occurs by at least $6\phi(4\delta) -$

$4\phi(\delta)$. Note that $6\phi(4\delta)-4\phi(\delta) > \phi(4\delta)+1$. Since $[\alpha_1,\beta_1] \subseteq [\alpha,\beta]$ is contained in the δ-neighbourhood of E, and since $[\alpha_3,\beta_3] \subseteq [g.\alpha, g.\beta]$ occurs at the same time as $[\alpha_1,\beta_1]$, the interval $[\alpha_3,\beta_3]$ is contained in the δ-neighbourhood of $g.E$. Therefore, the 4δ-neighbourhoods of E and $g.E$ intersect in a geodesic segment of length at least $\phi(4\delta)+1$, which implies that $E = g.E$, so $g \in \mathrm{Fix}(E)$, as required. □

Fix a finite subset $Q \in \mathrm{Fix}_G([A,B])$. Let the sequences $\{A_k\}$ and $\{B_k\}$ converge to A and B, respectively.

Suppose that, for some $\epsilon > 0$, for all but finitely many i there is a segment $[\alpha_i,\beta_i] \subseteq [A_i,B_i]$ for which (i) $d_{X_i}(\alpha_i,\beta_i) \geq \epsilon$; and (ii) there is a flat E_i so that $[\alpha_i,\beta_i]$ is in the δ-neighbourhood of E_i. In this case, for all but finitely many i, the conditions of Lemma 4.5 are satisfied for each $g \in Q$ and the segment $[\alpha_i,\beta_i]$. Therefore, $f_i(Q) \subseteq \mathrm{Stab}_\Gamma(E_i)$, so $\langle f_i(Q)\rangle$ is abelian for all but finitely many i. In this case $\langle qK_\infty \mid q \in Q\rangle$ is certainly abelian. Thus we can suppose that for all $\epsilon > 0$ there is no such segment $[\alpha_i,\beta_i]$.

Let $Q = \{g_1,\ldots,g_s\}$. Let $[\rho,\sigma] \in [A,B]$ be the middle third, and let the sequences $\{\rho_k\}$, $\{\sigma_k\}$, $\{\lambda_k(g_i,\rho_k)\}$ and $\{\lambda_k(g_i,\sigma_k)\}$ converge to ρ, σ, $g_i.\rho$, and $g_i.\sigma$. Consider the triangles $\Delta(\rho_k,\sigma_k,\lambda(g_i,\sigma_k))$ and $\Delta(\lambda(g_i,\rho_k),\sigma_k,\lambda(g_i,\sigma_k))$. By the argument in Lemma 4.5, the argument in the above paragraph and the argument in [24, Proposition 2.4], for all but finitely many k there is a segment $[\tau_k,\upsilon_k] \subseteq [\rho_k,\sigma_k]$ whose length (measured in d_{X_k}) is at least $\frac{1}{3}d_{X_k}(\rho_k,\sigma_k)$ and such that for each $g_i \in Q$, the image $g_i.[\tau_k,\upsilon_k]$ lies in the 4δ-neighbourhood of $[\rho_k,\sigma_k]$. Note that $d_{X_k}(\tau_k,\upsilon_k) \geq \frac{1}{9}(A_k,B_k)$.

Now, since translations on a line commute, each element of the form $[g,g']$, $g,g' \in Q$ moves the midpoint of $[\tau_k,\upsilon_k]$ at most 16δ (see [24, Proposition 2.4]).

There is an absolute bound on the size of the ball of radius 16δ around any point in X. Let this bound be D_1. Since the action of Γ is free, we have $|\{[f_n(g),f_n(g')] \mid g,g' \in Q\}| \leq D_1$ irrespective of the size of Q.

Let $x_1,x_2 \in \mathrm{Fix}_G([A,B])$. Either for all but finitely many n there is a flat E_n so that $x_1,x_2 \in \mathrm{Fix}(E_n)$ or the above argument bounding the size of the set of commutators holds. In the first case, $\langle f_n(x_1),f_n(x_2)\rangle$ is abelian for all but finitely many n.

Suppose then that the second case holds. By the above argument, with $Q = \{x_2,x_1,x_1^2,\ldots,x_1^{D_1+1}\}$, for all but finitely many n we can find $1 \leq s_1 < s_2 \leq D_1+1$ for which $f_n([x_1^{s_1},x_2]) = f_n([x_1^{s_2},x_2])$. Therefore $f_n(x_1^{s_1}x_2x_1^{-s_1}) = f_n(x_1^{s_2}x_2x_1^{-s_2})$, which implies that $f_n([x_1^{s_2-s_1},x_2]) = 1$. Hence $f_n(x_1)^{s_2-s_1}$

commutes with $f_n(x_2)$. Also, $\langle f_n(x_1)^{s_2-s_1}\rangle \subseteq \langle f_n(x_1)^{s_2-s_1}, f_n(x_2)\rangle^{f_n(x_1)}$, so by Proposition 2.16, $f_n(x_1)$ is contained in the same maximal abelian subgroup as $\langle f_n(x_1)^{s_2-s_1}, f_n(x_2)\rangle$, which is to say that $f_n(x_1)$ commutes with $f_n(x_2)$.

Therefore, in any case if $x_1, x_2 \in \mathrm{Fix}_G([A,B])$ then for all but finitely many n $f_n(x_1)$ commutes with $f_n(x_2)$. Therefore x_1K_∞ commutes with x_2K_∞. Since x_1 and x_2 were arbitrary, we have proved that the group $\mathrm{Fix}_{L_\infty}([A,B])$ is abelian. This finishes the proof of 4.4(1).

We now prove 4.4(2). Suppose that T is isometric to a real line, so L_∞ is a subgroup of $\mathrm{Isom}(\mathbb{R})$.

Suppose first that $\mathcal{C}_\infty$ is not a single flat.

Suppose that $k_1, k_2 \in G$ are arbitrary. Let $H = \langle k_1, k_2\rangle$ and $\overline{H}$ be the image of $\langle k_1, k_2\rangle$ in L_∞. Then $\overline{H}$ is a 2-generator subgroup of $\mathrm{Isom}(\mathbb{R})$ and so is one of the following (i) cyclic; (ii) infinite dihedral; or (iii) free abelian of rank 2. We first prove that $\overline{H}$ cannot be infinite dihedral, so that it is abelian.

Suppose that $k \in G$ reverses the orientation of T (which we are assuming is isometric to $\mathbb{R}$). Let $a, b \in T$ be distinct points so that $k(a) = b$ and $k(b) = a$. Approximate the segment $[a,b]$ by a segment $[a_i, b_i] \subset X_i$ for large i, and let $[c_i, d_i]$ be the middle third of $[a_i, b_i]$. Since $\mathcal{C}_\infty$ is not a single flat, the segment $k.[c_i, d_i]$ lies within 2δ of $[a_i, b_i]$, with orientation reversed (distances are being measured in d_X). It is not hard to see that there must be a point $e_i \in [c_i, d_i]$ which is moved at most 2δ by k. Therefore $d_X(k^2.e_i, e_i) \leq 6\delta$. In turn, this implies that k^2 moves each point on $[c_i, d_i]$ distance at most 10δ. Repeating this argument (with a larger i) with the elements $k, k^3, \ldots, k^{2D_1+1}$ we find $1 \leq i_1 < i_2 \leq D_1$ so that $k^{2(2i_1+1)}.e_i = k^{2(2i_2+1)}.e_i$, which implies that $k^{2(2i_1+1)}k^{-2(2i_2+1)} \in \ker(f_i)$ for large enough i. This in turn implies, since Γ is torsion-free, that $k \in \ker(f_i)$ for all but finitely many i. Therefore k acts trivially on T, and so cannot reverse the orientation. This proves that $\overline{H}$ is abelian, and being an orientation preserving subgroup of $\mathrm{Isom}(\mathbb{R})$, it is free abelian.

Suppose then that k_1 and k_2 act as translations on T, with translations lengths τ_1, τ_2, say. Choose $\kappa > 100D_1(\max\{\tau_1, \tau_2\} + 1)$, and choose $a, b \in T$ distance κ from each other. For large enough i, the approximation $[a_i, b_i] \subset X_i$ of $[a,b]$ is such that for all $j \in \{1, \ldots, D_1\}$, the elements k_1^j and k_2 move the middle third $[c_i, d_i]$ of $[a_i, b_i]$ entirely within the 4δ neighbourhood of $[a_i, b_i]$. As in the proof of 4.4(1) above, the commutator $[k_1^j, k_2]$ moves the midpoint of $[c_i, d_i]$ a distance at most 16δ. Therefore, there is $1 \leq j_1 < j_2 \leq D_1$ so that

$f_i([k_1^{j_1}, k_2]) = f_i([k_1^{j_2}, k_2])$, which as above implies that $f_i([k_1, k_2]) = 1$ for large enough i.

In this argument k_1 and k_2 were arbitrary, so letting k_1 and k_2 run over all pairs in a finite generating set $\{g_1, \ldots, g_k\}$ for G we see that for all $i, j \in \{1, \ldots, k\}$ the elements $f_n(g_i)$ and $f_n(g_j)$ commute for all but finitely many n. Thus, $f_n(G)$ is abelian for all but finitely many n.

We have also proved that L_∞ is an orientation preserving subgroup of $\mathrm{Isom}(\mathbb{R})$, which is free abelian, as required.

Now suppose that $\mathcal{C}_\infty$ is a single flat. By Lemma 3.18, for any $g \in G$ for all but finitely many i the element g fixes a flat $E_i \subseteq X_i$. Take a finite generating set for G and note that for all but finitely many i each of the elements in this set fix the flat E_i. Therefore, for all but finitely many i, $f_i(G) \subseteq \mathrm{Fix}(E_i)$, which is free abelian. This proves also that L_∞ is abelian, and again the only abelian subgroups of $\mathrm{Isom}(\mathbb{R})$ which are not free abelian have a global fixed point. This proves 4.4(2).

We now prove 4.4(3). Let $T(A,B,C)$ be a tripod in T and let N be the valence three vertex in $T(A,B,C)$. Suppose that $g \in G \smallsetminus \{1\}$ stabilises A, B and C and therefore also N. We prove that $g \in \ker(f_n)$ for all but finitely many n.

Let K_0 be the maximum number of elements of any orbit $\Gamma.y$ in any ball of radius $170\phi(16\phi(\phi(3\delta)))$ in X (with metric d_X). Such a K_0 exists because the action of Γ on X is proper and cocompact.

Suppose that some point $\alpha \in T(A,B,C)$ is contained in a line $p_E \in \mathbb{P}$. Certainly not all of $T(A,B,C)$ is contained in p_E, so let $y \in T(A,B,C) \smallsetminus p_E$. Let $\alpha \in \mathcal{C}_\infty$ be a point corresponding to $y \in T$ and $\beta = \pi_E(\alpha)$. Let $\{E_i\}$ be a sequence of flats converging to E (such a sequence exists by Corollary 3.17). By Lemma 3.18 $f_i(g) \in \mathrm{Fix}(E_i)$ for all but finitely many i.

By fixing a sufficiently small ϵ and finding an X_i which is an ϵ approximation for $Q = \{1, g, \ldots, g^{K_0+1}\}$ and $\{\alpha, \beta\}$ we can ensure that for all $q \in Q$ the geodesic $[\alpha_i, \lambda_i(q, \alpha_i)]$ does not intersect the 4δ-neighbourhood of E_i, where E is the limit of the flats $\{E_i\}$. Hence by Proposition 2.22 $d_X(\pi_{E_i}(\alpha_i), \pi_{E_i}(\lambda_i(q, \alpha_i))) \leq 2\phi(3\delta)$, for all $q \in Q$. However, $|Q| > K_0$ and there are no more than K_0 elements of $\Gamma.y$ in a ball of radius $2\phi(3\delta)$ in X, by the choice of K_0. Note also that if $f_i(g)$ leaves E_i invariant then $\pi_{E_i}(\lambda_i(q, \alpha_i)) = \lambda_i(q, \pi_{E_i}(\alpha_i))$. Therefore, there exist $1 \leq s_1 < s_2 \leq K_0 + 1$ so that $\lambda_i(g^{s_1}, \pi_{E_i}(\alpha_i)) = \lambda_i(g^{s_2}, \pi_{E_i}(\alpha_i))$, which implies that $f_i(g^{s_2 - s_1})$ fixes $\pi_{E_i}(\alpha_i)$. Therefore, $f_i(g^{s_2-s_1}) = 1$, and since Γ is torsion-free $g \in \ker(f_i)$ for all but finitely many i, as required.

Therefore, we assume for the moment that no point in the tripod $T(A,B,C)$ is contained in any line $p_E \in \mathbb{P}$. In this case, A,B and C correspond to points $\overline{A},\overline{B}$ and $\overline{C}$ in $\mathcal{C}_\infty$ for which $\Delta(\overline{A},\overline{B},\overline{C})$ is a tripod in $\mathcal{C}_\infty$. Let $\overline{N}$ be the valence three vertex in the tripod $T(\overline{A},\overline{B},\overline{C})$.

Let $\overline{A}',\overline{B}',\overline{C}'$ be the midpoints of $[\overline{A},\overline{N}],[\overline{B},\overline{N}]$ and $[\overline{C},\overline{N}]$ respectively and let $S=\{\overline{A},\overline{B},\overline{C},\overline{N},\overline{A}',\overline{B}',\overline{C}'\}$. We also define the set $Q=\{1,g,g^2,\ldots,g^{K_0+1}\}$. For varying ϵ, we will consider those X_i which are ϵ approximations for Q and S. Consider the triangle $\Delta=\Delta(\overline{A}_k,\overline{B}_k,\overline{C}_k)$ in X_k, an ϵ-approximation for Q and S. Suppose that Δ is δ-thin relative to a flat E. If ϵ is small enough, then necessarily $\overline{A}_k$ is at least distance $\frac{1}{3}d_{X_i}(\overline{A}_k,\overline{A}_k')$ from E. See Figure 2.

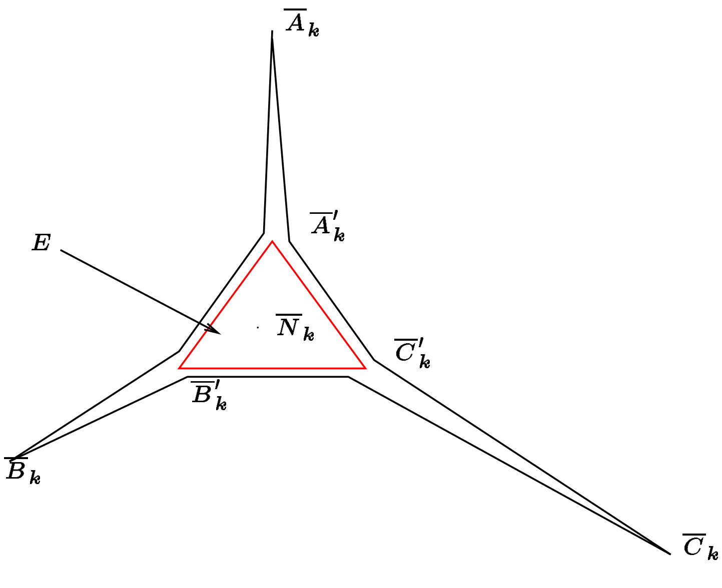

Figure 2: Ensuring $\overline{A},\overline{B}$ and $\overline{C}$ do not lie close to the flat E

Note that since $\overline{A}_k$, $\overline{B}_k$ and $\overline{C}_k$ are not moved far by $q\in Q$, compared to the distances $d_{X_i}(\overline{A}_k,\overline{A}_k')$, etc., the same property is true for triangles such as $\Delta(\overline{A}_k,\overline{B}_k,q.\overline{C}_k)$.

Fix X_i, an ϵ-approximation for Q and S so that ϵ is 'small enough' in the sense of the previous two paragraphs, and also $\epsilon<\frac{1}{100}$.

Consider the triangle $\Delta=\Delta(\overline{A}_i,\overline{B}_i,\overline{C}_i)$ in X_i. Define the constant $\delta' = 17\phi(16\phi(\phi(3\delta)))$.

Suppose that Δ is not δ'-thin, so it is δ-thin relative to a unique flat $E\subset X_i$. In this case $d_X(\pi_E(\overline{A}_i),\pi_E(\overline{B}_i)) \geq \delta'-2\delta$. Now, since $\epsilon<\frac{1}{100}$ and

$d_X(\overline{A}_i, \pi_E(\overline{A}_i)) \ge \frac{1}{3} d_{X_i}(\overline{A}_k, \overline{A}_k')$, for large enough i the geodesic $[\overline{A}_i, \lambda_i(q, \overline{A}_i)]$ avoids the 4δ-neighbourhood of E for all $q \in Q$. Therefore, for all $q \in Q$, $d_X(\pi_E(\overline{A}_i), \pi_E(\lambda_i(q, \overline{A}_i))) \le 2\phi(3\delta)$. Fix $q \in Q$. We now prove that $f_i(q)$ leaves E invariant, and then as above we argue that $g \in \ker(f_i)$.

To prove that $f_i(q)$ leaves E invariant, we consider the triangle

$$\Delta' = \Delta(\lambda_i(q, \overline{A}_i), \lambda_i(q, \overline{B}_i), \lambda_i(q, \overline{C}_i)),$$

and prove that it is $16\phi(\phi(3\delta))$-thin relative to E. Since it is also δ-thin relative to $f_i(q).E$ and since it is not δ'-thin, we must have that $E = f_i(q).E$, by an argument similar to that which proved Lemma 2.10.

Let $\alpha_1 = \lambda_i(q, \overline{A}_i), \alpha_2 = \pi_E(\alpha_1), \beta_1 = \lambda_i(q, \overline{B}_i)$ and $\beta_2 = \pi_E(\beta_1)$. Now,

$$\begin{aligned} d_X(\alpha_2, \pi_E(\overline{A}_i)), d_X(\beta_2), \pi_E(\overline{B}_i)) &\le 2\phi(3\delta), \text{ and} \\ d_X(\pi_E(\overline{A}_i), \pi_E(\overline{B}_i)) &\ge \delta' - 2\delta, \end{aligned}$$

so we have

$$d_X(\alpha_2, \beta_2) \ge 17\phi(16\phi(\phi(3\delta))) - 4\phi(3\delta) - 2\delta.$$

Now, by Lemma 2.21 there exists u, v in the 2δ-neighbourhood of E so that $u \in [\alpha_1, \alpha_2]$ and $v \in [\alpha_1, \beta_2]$ and $d_X(u, v) \le \phi(\delta)$. Now, since $\alpha_2 = \pi_E(\alpha_1)$, $d_X(u, \alpha_2) \le 2\delta$, and so $d_X(v, \alpha_2) \le \phi(\delta) + 2\delta)$. Therefore, $[\alpha_2, \beta_2]$ and $[v, \beta_2]$ $3\phi(\delta)$-fellow travel and $\Delta(\alpha_1, \alpha_2, \beta_2)$ is $3\phi(\delta)$-thin. Also $d_X(v, \beta_2) \ge \delta' - 7\phi(3\delta) - 2\delta$.

Now, consider the triangle $\Delta_1 = \Delta(\alpha_1, \beta_1, \beta_2)$. Suppose it is δ-thin relative to a flat $E' \ne E$. Let w_1 be the point on $[\beta_2, \beta_1]$ which lies in the $4\phi(\delta)$-neighbourhood of E furthest from β_2. Then w_1 is not in the δ-neighbourhood of $[v, \beta_2]$, and so is not in the δ-neighbourhood of $[\alpha_1, \beta_2]$. If w_1 is in the δ-neighbourhood of $[\beta_1, \alpha_1]$ then Δ_1 is $5\phi(\delta)$-thin. Thus suppose that w_1 is in the δ-neighbourhood of E'.

Now, because (i) $\beta_2 = \pi_E(\beta_1)$; (ii) $d_X(v, \beta_2) \ge \delta' - 7\phi(3\delta) - 2\delta$; and (iii) $[v, \beta_2]$ is contained in the $3\phi(\delta)$-neighbourhood of E, either Δ_1 is $(\phi(3\phi(\delta)) + 3\phi(\delta) + \delta)$-thin or there is a segment in $[v, \beta_2]$ of length at least $\phi(\phi(3\delta))$ which lie in the δ-neighbourhood of E'. However, this segment also lies in the $\phi(3\delta)$-neighbourhood of E, which is a contradiction. Therefore, in any case either Δ_1 is $5\phi(\phi(3\delta))$-thin or Δ_1 is δ-thin relative to E.

Suppose that Δ_1 is δ-thin relative to E. The geodesic $[\beta_2, \beta_1]$ intersects the δ-neighbourhood of E in a segment of length at most δ and so in this case Δ_1 is 2δ-thin.

We have proved that $\Delta_1 = \Delta(\alpha_1, \beta_1, \beta_2)$ is $5\phi(\phi(3\delta))$-thin, and also that $\Delta(\alpha_1, \alpha_2, \beta_2)$ is $3\phi(\delta)$-thin. Therefore, the geodesic $[\alpha_1, \beta_1]$ must $8\phi(\phi(3\delta))$-fellow travel the path $[\alpha_1, \alpha_2, \beta_2, \beta_1]$.

Similar arguments applied to the geodesic segments $[\lambda_i(q, \overline{A}_i), \lambda_i(q, \overline{C}_i)]$ and $[\lambda_i(q, \overline{B}_i), \lambda_i(q, \overline{C}_i)]$ show that the triangle $\Delta' = \Delta(\lambda_i(q, \overline{A}_i), \lambda_i(q, \overline{B}_i), \lambda_i(q, \overline{C}_i))$ is $16\phi(\phi(3\delta))$-thin relative to E. Since Δ' is not δ'-thin, the argument from Lemma 2.10 implies that Δ' is $16\phi(\phi(3\delta))$-thin relative to a unique flat. Since Δ' is certainly $16(\phi(3\delta))$-thin relative to $f_i(q).E$ we must have that $f_i(q).E = E$, as required. Now we know for all $q \in Q$ that $d_X(\lambda_i(q, \pi_E(\overline{A}_i)), \pi_E(\overline{A}_i)) \le 2\phi(3\delta)$, which implies as above that $g \in \ker(f_i)$.

Therefore, we may assume that Δ is δ'-thin. Similar arguments to those above allow us to infer that for all $r_1, r_2, r_3 \in \{1, q\}$, the triangle

$$\Delta(f_i(r_1).\overline{A}_i, f_i(r_2).\overline{B}_i, f_i(r_3).\overline{C}_i),$$

is δ'-thin. Now, by the Claim in the proof of [26, Lemma 4.1], the point $\overline{N}_i$ is moved by $f_i(q)$ at most $170\phi(16\phi(\phi(3\delta)))$. Again in this case, we find $1 \le s_1 < s_2 \le K_0+1$ so that $f_i(g^{s_2-s_1})$ fixes $\overline{N}_i$, which implies that $g \in \ker(f_i)$.

We have proved that if $g \in G$ stabilises a tripod in T then for all but finitely many i $g \in \ker(h_i)$. This proves 4.4(3).

The proof of 4.4(4) is identical to the of [26, Proposition 4.2], except that segment stabilisers are abelian, rather than cyclic. However, all that is used in this proof is that segment stabilisers are abelian.

We now prove 4.4(5). Suppose that $g \notin K_\infty$. Then certainly $g \notin \ker(f_k)$ for all but finitely many k, by the choice of the sequence $\{f_i\}$ in Lemma 3.15.

We now prove 4.4(6). If T is isometric to a real line then by 4.4(2) L_∞ is finitely generated free abelian, and is certainly torsion-free.

Therefore suppose that T is not isometric to a real line, that $g \in G$ and that $g^p \in K_\infty$. Since T is not isometric to a real line, g^p stabilises a tripod, so $g^p \in \ker(f_k)$ for all but finitely many k. However Γ is torsion-free, so $g \in \ker(f_k)$ for all but finitely many k and, by 4.4(5), $g \in K_\infty$, as required. This proves 4.4(6).

Finally, we prove 4.4(7). To see that $\{f_i\}$ is a stable sequence of homomorphisms when T is not isometric to a real line, suppose that $g \in G$. If $g \notin K_\infty$ then by 4.4.(5) we have $g \notin \ker(f_i)$ for all but finitely many i. If $g \in K_\infty$ then g stabilises a tripod in T and so by 4.4.(3) $g \in \ker(f_i)$ for all but finitely many i. This proves that $\{f_i\}$ is a stable sequence of homomorphisms.

This finally completes the proof of Theorem 4.4. $\square$

5 Γ-limit groups and concluding musings

5.1 Various kinds of Γ-limit groups

Theorem 5.1 *Suppose that Γ is a torsion-free toral $CAT(0)$ group with isolated flats. The class of Γ-limit groups coincides with the class of algebraic Γ-limit groups.*

Proof The class of abelian Γ-limit groups is exactly class of finitely generated free abelian groups. It is easy to see that these are also the abelian algebraic Γ-limit groups.

Clearly a finitely generated subgroup of Γ is an algebraic Γ-limit group.

Suppose then that $\{h_i : G \to \Gamma\}$ is a sequence of homomorphisms and $\{f_i\}$ is the subsequence obtained from Lemma 3.15. If the limiting tree T is isometric to a real line then the associated Γ-limit group is abelian. We have already covered this case, so we may assume T is not isometric to a real line. Therefore, by 4.4.(3), 4.4.(5) and 4.4.(7), $\{f_i\}$ is a stable sequence and $K_\infty = \underrightarrow{\mathrm{Ker}}\,(f_i)$. Hence the limit group L_∞ is an algebraic Γ-limit group.

Conversely, suppose that $\{h_i : G \to \Gamma\}$ is a stable sequence of homomorphisms. If the associated sequence of stretching factors $\{\mu_j\}$ contains a bounded subsequence, then there is a subsequence $\{h_{i'}\}$ of $\{h_i\}$ so that $h_{i'_1}(G) \cong h_{i'_2}(G)$ for all $i'_1, i'_2 \in \{h_{i'}\}$. In this case, since $\{h_i\}$ is a stable sequence, the associated algebraic Γ-limit group is isomorphic to a finitely generated subgroup of Γ.

Thus suppose that there is not a bounded subsequence of the $\{\mu_j\}$. In this case we can construct a limiting spaces X_ω and $\mathcal{C}_\infty$ and the associated $\mathbb{R}$-tree T. If T is isometric to a real line, we are done. Otherwise since passing to a subsequence of a stable sequence does not change the stable kernel, we see again that $K_\infty = \underrightarrow{\mathrm{Ker}}\,(h_i)$, so that the algebraic Γ-limit group is a Γ-limit group. □

We now recall a topology on the set of finitely generated groups from [11] (see also [16, 10]).

Definition 5.2 A *marked group* $(G, \mathcal{A})$ consists of a finitely generated group G with an ordered generating set $\mathcal{A} = (a_1, \ldots, a_n)$. Two marked groups $(G, \mathcal{A})$ and $(G', \mathcal{A}')$ are *isomorphic* if the bijection taking a_i to a'_i for each i induces an isomorphism between G and G'.

For a fixed n, the set $\mathcal{G}_n$ consists of those marked groups $(G, \mathcal{A})$ where $|\mathcal{A}| = n$.

We now introduce a metric on $\mathcal{G}_n$. First, we introduce the following abuse of notation.

Convention 5.3 Marked groups are always considered up to isomorphism of marked groups. Thus for any marked groups $(G, \mathcal{A})$ and $(G', \mathcal{A}')$ in $\mathcal{G}_n$, we identify an $\mathcal{A}$-word with the corresponding $\mathcal{A}'$-word under the canonical bijection induced by $a_i \to a'_i$, $i = 1, \ldots, n$.

Definition 5.4 A *relation* in a marked group $(G, \mathcal{A})$ is an $\mathcal{A}$-word representing the identity in G. Two marked groups $(G, \mathcal{A})$ and $(G', \mathcal{A}')$ in $\mathcal{G}_n$ are at distance e^{-d} from each other if they have the exactly the same relations of length at most d, but there is a relation of length $d+1$ which holds in one marked group but not the other.

The following result is implicit in [11].

Proposition 5.5 *Let G be a finitely generated group and Ξ a finitely presented group. Suppose that $\{h_i \colon G \to \Xi\}$ is a stable sequence, and $\{g_1, \ldots, g_k\}$ is a generating set for G. Then the marked group*

$$\left(G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}, \left\{g_1 \underrightarrow{\mathrm{Ker}}\,\{h_i\}, \ldots, g_k \underrightarrow{\mathrm{Ker}}\,\{h_i\}\right\}\right),$$

is a limit of marked groups $(G_i, \mathcal{A}_i)$ where each G_i is a finitely generated subgroup of Ξ.

Conversely, if the marked group $(G, \mathcal{A})$ is a limit of finitely generated subgroups of Ξ, then $(G, \mathcal{A})$ is an algebraic Ξ-limit group.

Proof Suppose that $\{h_i \colon G \to \Xi\}$ is a stable sequence, and $\{g_1, \ldots, g_k\}$ is a generating set for G. Consider the marked group

$$\left(G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}, \left\{g_1 \underrightarrow{\mathrm{Ker}}\,\{h_i\}, \ldots, g_k \underrightarrow{\mathrm{Ker}}\,\{h_i\}\right\}\right).$$

For each n, let $H_n = \langle h_n(g_1), \ldots, h_n(g_k) \rangle \leq \Xi$. We consider the marked groups $(H_n, \{h_n(g_1), \ldots, h_n(g_k)\})$, and prove that they converge to $G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}$ with the above marking.

Let $j \geq 1$ be arbitrary and let W_j be the set of all words of length at most j in the alphabet $\{x_1^{\pm 1}, \ldots, x_k^{\pm 1}\}$. Following Convention 5.3, we interpret W_j as words in the various generating sets without changing notation. The set W_j admits a decomposition into $T_j \cup N_j$, where T_j are the words of length at most j which are in $\underrightarrow{\mathrm{Ker}}\,\{h_i\}$ and N_j are the remaining words of length at most j.

Since $\{h_i\}$ is a stable sequence, for each element $w \in T_j$, the element $h_n(w)$ is trivial for all but finitely many n, and for each $w \in N_j$ the element $h_n(w)$ is nontrivial for all but finitely many n.

Thus, for all but finitely many n, the relations in H_n of length at most j are exactly the same as the relations of length at most j in $G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}$. Thus for all but finitely many n, the group H_n with the given marking is at distance at most e^{-j} from $G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}$ with the given marking. This implies that the sequence $\{H_n\}$ (with markings) converges to $G/\underrightarrow{\mathrm{Ker}}\,\{h_i\}$ (with marking).

For the converse, suppose that $(G, \mathcal{A})$ is a limit of a (convergent) sequence of marked finitely generated subgroups of Ξ. Denote these subgroups by $(H_i, \mathcal{A}_i)$. Note that $|\mathcal{A}_i|$ is fixed. Let $\mathcal{A} = \{a_1, \ldots, a_k\}$, let $\mathcal{A}_i = \{b_{i,1}, \ldots, b_{i,k}\}$ and let F be the free group on the set $\mathcal{A}$. Define homomorphisms $h_i \colon F \to \Xi$ by $h_i(a_j) = b_{i,j}$. It is not difficult to see that $G \cong F/\underrightarrow{\mathrm{Ker}}\,\{h_i\}$. □

For a group H, let $T_\forall(H)$ be the *universal theory* of H – the set of all universal sentences which are true in H (see [11] or [25] for the definition, we are interested only in its consequences). The results in [11] now imply

Corollary 5.6 *Let Ξ be a finitely presented group and suppose that L is an algebraic Ξ-limit group. Then $T_\forall(\Xi) \subseteq T_\forall(L)$.*

Corollary 5.7 *Suppose that Γ is a torsion-free toral CAT(0) group with isolated flats and that L is a Γ-limit group. Then*

(1) *any finitely generated subgroup of L is a Γ-limit group;*

(2) *L is torsion-free;*

(3) *L is commutative transitive; and*

(4) *L is CSA.*

Lemma 2.15 now implies the following:

Corollary 5.8 *Let Γ be a torsion-free toral CAT(0) group with isolated flats and let L be a Γ-limit group. Every solvable subgroup of L is abelian.*

5.2 The Main Theorem and conclusions

Finally, we have:

Theorem 5.9 *Suppose that* Γ *is a torsion-free toral* $CAT(0)$ *group with isolated flats such that* $Out(\Gamma)$ *is infinite. Then* Γ *admits a nontrivial splitting over a finitely generated free abelian group.*

Proof Suppose that $\{\phi_i\}$ is an infinite set of automorphisms of Γ which belong to distinct conjugacy classes in $\mathrm{Out}(\Gamma)$. Then the construction from Sections 3 and 4 allows us to find an isometric action of Γ on an $\mathbb{R}$-tree T without global fixed points. Pass to the subsequence $\{f_i\}$ of $\{\phi_i\}$ as in Lemma 3.15.

Suppose first that T is isometric to a real line. Then by Theorem 4.4.(2) the group $f_i(\Gamma)$ is free abelian for all but finitely many i. But $f_i(\Gamma) = \Gamma$, so Γ must be free abelian in this case. The theorem certainly holds for finitely generated free abelian groups.

Therefore, we may suppose that T is not isometric to a real line. In this case, since $K_\infty = \underrightarrow{\mathrm{Ker}}$ is trivial, the Γ-limit group L_∞ is Γ itself. Then by Theorem 9.5 of [4], the group Γ splits over a group of the form E-by-cyclic, where E fixes a non-degenerate segment of T. The stabilisers in Γ of non-degenerate segments are free abelian, by Lemma 4.4.(4). Hence the group of the form E-by-cyclic is soluble, and hence free abelian by Lemma 2.15. Note that free abelian subgroups of Γ are finitely generated. This finishes the proof of the theorem. □

Suppose that Γ is a $\mathrm{CAT}(0)$ group and that $\mathrm{Out}(\Gamma)$ is infinite. Swarup asked (see [3], Q2.1) whether Γ necessarily admits a Dehn twist of infinite order. The above result shows that this is the case for the class of torsion-free toral $\mathrm{CAT}(0)$ groups with isolated flats. Swarup also asked whether there is an analog of the theorem of Rips and Sela that $\mathrm{Out}(\Gamma)$ is virtually generated by Dehn twists.

In the subsequent work [17], we will prove if Γ is a torsion-free toral $\mathrm{CAT}(0)$ group with isolated flats then $\mathrm{Out}(\Gamma)$ is virtually generated by generalised Dehn twists (which take into account the existence of noncyclic abelian groups).

It seems that the techniques developed here will be of little help in answering Swarup's question in the case of a general $\mathrm{CAT}(0)$ group.

It is also clear from the above construction that if L is a non-abelian freely indecomposable strict Γ-limit group then L splits over an abelian group. However, there is no reason to conclude that the edge group in this splitting is finitely generated.

It is straightforward to construct the canonical abelian JSJ decomposition of a strict Γ-limit group L_∞, using acylindrical accessibility [30].[6] However, for example, we do not yet know that the edge groups in the abelian JSJ decomposition of L_∞ are finitely generated. To prove that this is the case if L_∞ is freely indecomposable and nonabelian involves the shortening argument of Sela, which we present for torsion-free toral CAT(0) groups with isolated flats in future work. The shortening argument allows us to prove that torsion-free toral CAT(0) groups with isolated flats are Hopfian, and to construct Makanin-Razborov diagrams for these groups, thus partially answering a question of Sela (see [35, I.8.(i), (ii), (iii)]). This work will be undertaken in [17].

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