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Extensions of maps to the projective plane

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Abstract It is proved that for a 3-dimensional compact metrizable space X the infinite real projective space $\mathbb{R}P^\infty$ is an absolute extensor of X if and only if the real projective plane $\mathbb{R}P^2$ is an absolute extensor of X (see Theorems 1.2 and 1.5).

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1 Introduction

Let X be a compactum (= separable metric space) and let K be a CW complex. $K \in AE(X)$ (read: K is an *absolute extensor* of X) or $X\tau K$ means that every map $f : A \longrightarrow K$, A closed in X, extends over X. The *extensional dimension* $\mathrm{edim} X$ of X is said to be dominated by a CW-complex K, written $\mathrm{edim} X \leq K$, if $X\tau K$. Thus for the covering dimension $\dim X$ of X the condition $\dim X \leq n$ is equivalent to $\mathrm{edim} X \leq S^n$ where S^n is an n-dimensional sphere and for the cohomological dimension $\dim_G X$ of X with respect to an abelian group G, the condition $\dim_G X \leq n$ is equivalent to $\mathrm{edim} X \leq K(G,n)$ where $K(G,n)$ is an Eilenberg-Mac Lane complex of type (G,n).

Every time the coefficient group in homology is not explicitely stated, we mean it to be integers.

In case of CW complexes K one can often reduce the relation $\mathrm{edim} X \leq K$ to $\mathrm{edim} X \leq K^{(n)}$, where $K^{(n)}$ is the n-skeleton of K.

Proposition 1.1 *Suppose X is a compactum and K is a CW complex. If $\dim X \leq n$, then $\mathrm{edim} X \leq K$ is equivalent to $\mathrm{edim} X \leq K^{(n)}$.*

The proof follows easily using $\mathrm{edim} X \leq n$ to push maps off higher cells.

Dranishnikov [4] proved the following important theorems connecting extensional and cohomological dimensions.

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Theorem 1.2 *Let K be a CW-complex and let a compactum X be such that* $\text{edim}X \leq K$. *Then* $\dim_{H_n(K)} X \leq n$ *for every* $n > 0$.

Theorem 1.3 *Let K be a simply connected CW-complex and let a compactum X be finite dimensional. If* $\dim_{H_n(K)} X \leq n$ *for every* $n > 0$, *then* $\text{edim}X \leq K$.

The requirement in Theorem 1.3 that X is finite dimensional cannot be omitted. To show that take the famous infinite-dimensional compactum X of Dranishnikov with $\dim_{\mathbb{Z}} X = 3$ as in [3]. Then the conclusion of Theorem 1.3 does not hold for $K = S^3$. Let us mention in this connection another result [6]: there is a compactum X satisfying the following conditions:

(a) $\text{edim}X > K$ for every finite CW-complex K with $\tilde{H}_*(K) \neq 0$,

(b) $\dim_G X \leq 2$ for every abelian group G,

(c) $\dim_G X \leq 1$ for every finite abelian group G.

Here $\text{edim}X > K$ means that $\text{edim}X \leq K$ is false.

With no restriction on K, Theorem 1.3 does not hold. Indeed, the conclusion of Theorem 1.3 is not satisfied if K is a non-contractible acyclic CW-complex and X is the 2-dimensional disk. Cencelj and Dranishnikov [2] generalized Theorem 1.3 for nilpotent CW-complexes K (see [1] for the case of K with fundamental group being finitely generated).

The real projective plane $\mathbb{R}P^2$ is the simplest CW-complex not covered by Cencelj-Dranishnikov's result. Thus we arrive at the following well-known open problem in Extension Theory.

Problem 1.4 Let X be a finite dimensional compactum. Does $\dim_{\mathbb{Z}_2} X \leq 1$ imply $\text{edim}X \leq \mathbb{R}P^2$? More generally, does $\dim_{\mathbb{Z}_p} X \leq 1$ imply $\text{edim}X \leq M(\mathbb{Z}_p, 1)$, where $M(\mathbb{Z}_p, 1)$ is a Moore complex of type $(\mathbb{Z}_p, 1)$?

It is not difficult to see that this problem can be answered affirmatively if $\dim X \leq 2$ (use 1.1). Sharing a belief that Problem 1.4 has a negative answer in higher dimensions the authors made a few unsuccessful attempts to construct a counterexample in the first non-trivial case $\dim X = 3$ and were surprised to discover the following result.

Theorem 1.5 *Let X be a compactum of dimension at most three. If* $\dim_{\mathbb{Z}_2} X \leq 1$, *then* $\text{edim}X \leq \mathbb{R}P^2$.

Notice (see [5]) that there exist compacta X of dimension 3 such that $\dim_{\mathbb{Z}_2} X \leq 1$, so Theorem 1.5 is not vacuous.

This paper is devoted to proving of Theorem 1.5. Theorem 1.5 can be formulated in a slightly different form. Let X be a compactum. Take $\mathbb{R}P^\infty$ as an Eilenberg-Mac Lane complex $K(\mathbb{Z}_2, 1)$. Then $\dim X \leq 3$ and $\dim_{\mathbb{Z}_2} X \leq 1$ imply $\text{edim} X \leq \mathbb{R}P^3$. On the other hand by Theorem 1.2 the condition $\text{edim} X \leq \mathbb{R}P^3$ implies that $\dim_{\mathbb{Z}_2} X = \dim_{H_1(\mathbb{R}P^3)} X \leq 1$, $\dim_{\mathbb{Z}} X = \dim_{H_3(\mathbb{R}P^3)} X \leq 3$ and by Alexandroff's theorem $\dim X \leq 3$ if X is finite dimensional (note that $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ if $1 \leq k < n$ is odd, $H_k(\mathbb{R}P^n) = 0$ if $1 < k \leq n$ is even, and $H_n(\mathbb{R}P^n) = \mathbb{Z}$ if n is odd - see p.89 of [7]). Thus Theorem 1.5 is equivalent to the following, more general result.

Theorem 1.6 *Let X be a compactum of finite dimension. If* $\text{edim} X \leq \mathbb{R}P^3$, *then* $\text{edim} X \leq \mathbb{R}P^2$.

We end this section with two questions related to Theorems 1.5 and 1.6.

Question 1.7 Let X be a compactum of dimension at most three. Does $\dim_{\mathbb{Z}_p} X \leq 1$ imply $\text{edim} X \leq M(\mathbb{Z}_p, 1)$?

Question 1.8 Does $\text{edim} X \leq \mathbb{R}P^3$ imply $\text{edim} X \leq \mathbb{R}P^2$ for all, perhaps infinite-dimensional, compacta X?

2 Preliminaries

Maps on projective spaces

Recall that the real projective n-space $\mathbb{R}P^n$ is obtained from the n-sphere S^n by identifying points x and $-x$. The resulting map $p_n : S^n \longrightarrow \mathbb{R}P^n$ is a covering projection and $\mathbb{R}P^1$ is homeomorphic to S^1. By $q_n : B^n \longrightarrow \mathbb{R}P^n$ we denote the quotient map of the unit n-ball B^n obtained by identifying B^n with the upper hemisphere of S^n. We consider all spheres to be subsets of the infinite-dimensional sphere S^∞. Similarly, we consider all projective spaces $\mathbb{R}P^n$ to be subsets of the infinite projective space $\mathbb{R}P^\infty$. Clearly, there is a universal covering projection $p : S^\infty \longrightarrow \mathbb{R}P^\infty$. It is known that $\mathbb{R}P^\infty$ has a structure of a CW complex making it an Eilenberg-MacLane complex of type $K(\mathbb{Z}_2, 1)$ as S^∞ is contractible.

Proposition 2.1 *Any map $f : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$ extends to a map $f' : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$.*

Proof It is obvious if f is null-homotopic. Assume that f is not homotopic to a constant map. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $\mathbb{R}P^1$ generates $\pi_1(\mathbb{R}P^2)$, f is homotopic to the inclusion map of $\mathbb{R}P^1$ to $\mathbb{R}P^2$. Obviously, that inclusion extends to the identity map of $\mathbb{R}P^2$, so f extends over $\mathbb{R}P^2$. □

Proposition 2.2 *If $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$ induces the zero homomorphism of the fundamental groups, then f extends to a map $f' : \mathbb{R}P^3 \longrightarrow \mathbb{R}P^2$.*

Proof Since f induces the zero homomorphism of the fundamental group, f can be lifted to $\beta : \mathbb{R}P^2 \longrightarrow S^2$. Since $H_2(\mathbb{R}P^2) = 0$, the map $\gamma = \beta \circ q_3|_{\partial B^3} : \partial B^3 \longrightarrow S^2$ induces the zero homomorphism $\gamma_* : H_2(\partial B^3) \longrightarrow H_2(S^2)$ and hence γ is null-homotopic. Thus γ can be extended over B^3 and this extension induces the corresponding extension of f over $\mathbb{R}P^3$. □

Proposition 2.3 *Let Y be a topological space. A map $f : S^1 \times \mathbb{R}P^1 \longrightarrow Y$ extends over $S^1 \times \mathbb{R}P^2$ if and only if the composition $S^1 \times S^1 \xrightarrow{id \times p_1} S^1 \times \mathbb{R}P^1 \xrightarrow{f} Y$ extends over the solid torus $S^1 \times B^2$.*

Proof Consider the induced map $f' : \mathbb{R}P^1 \longrightarrow Map(S^1, Y)$ to the mapping space defined by $f'(x)(z) = f(z, x)$ for $x \in \mathbb{R}P^1$ and $z \in S^1$. f extends over $S^1 \times \mathbb{R}P^2$ if and only if f' extends over $\mathbb{R}P^2$. Notice that f' extends over $\mathbb{R}P^2$ if and only if $S^1 \xrightarrow{p_1} \mathbb{R}P^1 \xrightarrow{f'} Map(S^1, Y)$ extends over B^2 which is the same as to say that $S^1 \times S^1 \xrightarrow{id \times p_1} S^1 \times \mathbb{R}P^1 \xrightarrow{f} Y$ extends over the solid torus $S^1 \times B^2$. □

Proposition 2.4 *Suppose $(a, b) \in S^1 \times \mathbb{R}P^1$. If $f : S^1 \times \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$ is a map such that $f|\{a\} \times \mathbb{R}P^1$ is null-homotopic and $f|S^1 \times \{b\}$ is not null-homotopic, then f extends over $S^1 \times \mathbb{R}P^2$.*

Proof Assume $f|\{a\} \times \mathbb{R}P^1$ is constant. In view of 2.3 we need to show that the composition $S^1 \times S^1 \xrightarrow{id \times p_1} S^1 \times \mathbb{R}P^1 \xrightarrow{f} \mathbb{R}P^2$ extends over the solid torus $S^1 \times B^2$. Let D be a disk with boundary equal to $\mathbb{R}P^1$. Pick $e : I \longrightarrow S^1$ identifying 0 and 1 with $a \in S^1$. The homotopy $f \circ (e \times id) : I \times \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$ has a lift $H : I \times \mathbb{R}P^1 \longrightarrow S^2$ such that $\{0\} \times \mathbb{R}P^1$ and $\{1\} \times \mathbb{R}P^1$ are each mapped to a point and those points are antipodal as $f|S^1 \times \{b\}$ is not null-homotopic. Therefore H can be extended to $G : \partial(I \times D) \longrightarrow S^2$ so that

$G|\{0\}\times D$ and $G|\{1\}\times D$ are constant. Fix the orientation of $\partial(I\times D)$ and let c be the degree of G. Define F on $I\times S^1$ as the composition $G\circ(id\times p_1)$ and use the orientation on $\partial(I\times B^2)$ induced by that on $\partial(I\times D)$. Define F on $D_1=\{1\}\times B^2$ as a map with the same value on ∂D_1 as $G(\partial(\{1\}\times D))$ so that the induced map from $D_1/(\partial D_1)\longrightarrow S^2$ is of degree $-c$ (the orientation on $D_1/(\partial D_1)$ is induced by the orientation of $\partial(I\times B^2)$). The new map is called F. Define F on $D_0=\{0\}\times B^2$ as the map with the same value on ∂D_0 as $G(\partial(\{0\}\times D))$ so that $F(0,x)=-F(1,x)$ for all $x\in B^2$. The cumulative map $F:\partial(I\times B^2)\longrightarrow S^2$ is of degree 0, so it extends to $F':I\times B^2\longrightarrow S^2$. Notice that $J=p_2\circ F':I\times B^2\longrightarrow \mathbb{R}P^2$ has the property that $J(0,x)=J(1,x)$ for all $x\in B^2$. Therefore it induces an extension $S^1\times B^2\longrightarrow\mathbb{R}P^2$ of the composition $S^1\times S^1\xrightarrow{id\times p_1}S^1\times\mathbb{R}P^1\xrightarrow{f}\mathbb{R}P^2$. □

The first modification M_1 of $\mathbb{R}P^3$

Let $B^3\subset\mathbb{R}^3$ be the unit ball and let D be the 2-dimensional disk of radius $1/3$ lying in the yz-coordinate plane and centered at the point $(0,1/2,0)$. Denote by L the solid torus obtained by rotating D about the z-axis. We consider L with the structure of cartesian product $L=S^1\times D$ such that the rotations of L about the z-axis correspond to the rotations of S^1. Think of S^1 as the circle $x^2+y^2=1/4, z=0$ (the circle traced by the center of D). Since L is untouched under the quotient map $q_3:B^3\longrightarrow\mathbb{R}P^3$, we may assume $L\subset\mathbb{R}P^3$. The *first modification* M_1 of $\mathbb{R}P^3$ is obtained by removing the interior of L from $\mathbb{R}P^3$ and attaching $S^1\times\mathbb{R}P^2$ via the map $S^1\times\mathbb{R}P^1\to\partial L$, where $\mathbb{R}P^1$ is identified with ∂D. Notice that $\mathbb{R}P^2=q_3(\partial B^3)\subset M_1$.

Proposition 2.5 *There is a retraction $r:M_1\longrightarrow\mathbb{R}P^2$ of the first modification M_1 of $\mathbb{R}P^3$ to the projective plane.*

Proof We use the notation that we introduced above defining the first modification of $\mathbb{R}P^3$. Let I be the interval of the points of B^3 lying on the z-axis and let $M=\partial B^3\cup I\subset B^3$. Denote $K=B^3\setminus(L\setminus\partial L)$. Consider the group Γ of rotations of $\mathbb{R}^3$ around the z-axis. Note that L, M and K are invariant under rotations in Γ and every such rotation induces the corresponding homeomorphism of $\mathbb{R}P^2$ which will be called the corresponding rotation of $\mathbb{R}P^2$. Recall that L is represented as the product $L=S^1\times D$ in such a way that the rotations of L are induced by the rotations of S^1. Let $\alpha:K\longrightarrow M$ be a retraction which commutes with the rotations in Γ (this means that for every rotation $\rho\in\Gamma$ of $\mathbb{R}^3$ and $x\in K$, $\alpha(\rho(x))=\rho(\alpha(x))$). Let $\beta:M\longrightarrow\mathbb{R}P^2$ be the extension of q_3 restricted to ∂B^3 sending the interval I to the point

$q_3(\partial I)$. Then β also commutes with the rotations in Γ (this means that for every $x \in M$, every rotation $\rho \in \Gamma$ of $\mathbb{R}^3$ and the corresponding rotation ρ' of $\mathbb{R}P^2$, $\beta(\rho(x)) = \rho'(\beta(x))$). Denote $\gamma = \beta \circ \alpha : K \longrightarrow \mathbb{R}P^2$. It is easy to see that γ commutes with the rotations. Consider ∂D as a subspace $\partial D = \mathbb{R}P^1 \subset \mathbb{R}P^2$ of a projective plane, and consider $\partial L = S^1 \times \partial D$ as the subset of $T = S^1 \times \mathbb{R}P^2$ induced by the inclusion $\partial D \subset \mathbb{R}P^2$. Fix a in S^1. By Proposition 2.1 extend γ restricted to $\{a\} \times \partial D$ to a map $\mu : \{a\} \times RP^2 \longrightarrow RP^2$ and by the rotations of S^1 and $\mathbb{R}P^2$ extend the map μ to the map $\kappa : T = S^1 \times \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$. Note that κ agrees with γ on $\partial L = S^1 \times \partial D$ and therefore the maps γ and κ define the map $\nu : M_1 \longrightarrow \mathbb{R}P^2$ from the first modification of $\mathbb{R}P^3$. This map ν induces the required retraction $r : M_1 \longrightarrow \mathbb{R}P^2$. □

The second modification M_2 of $\mathbb{R}P^3$

Let B^3 be the unit ball in $\mathbb{R}^3$ and let $L \subset B^3$ be the subset of B^3 consisting of the points lying in the cylinder $x^2 + y^2 \leq 1/4$. Notice that $R = q_3(L)$ is a solid torus in $\mathbb{R}P^3$ as the map $B^2 \longrightarrow B^2$ sending z to $-z$ is isotopic to the identity. Set $D = R \cap \mathbb{R}P^2$. Represent R as $S^1 \times D$ such that $\{a\} \times D$ is identified with D. The *second modification* M_2 of $\mathbb{R}P^3$ is obtained by removing the interior of R and attaching $S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$ via the inclusion $S^1 \times \partial D \longrightarrow S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$, where ∂D is identified with $\mathbb{R}P^1$.

Proposition 2.6 *Let M_2 be the second modification of $\mathbb{R}P^3$. The inclusion $i : \mathbb{R}P^2 \cap M_2 \hookrightarrow \mathbb{R}P^2$ extends to a map $g : M_2 \longrightarrow \mathbb{R}P^2$.*

Proof We use the notation that we introduced above defining the notion of the second modification of $\mathbb{R}P^3$. Denote $H = \mathbb{R}P^3 \setminus (R \setminus \partial R)$. Since the center of B^3 does not belong to $q_3^{-1}(H)$, the radial projection sends $q_3^{-1}(H)$ to ∂B^3 and hence the radial projection induces the corresponding map $\alpha : H \longrightarrow \mathbb{R}P^2$ which extends the map i. Recall that R is represented as $S^1 \times D$. Then $\partial R = S^1 \times \partial D$. Fix $a \in S^1$ and $b \in \partial D$. Notice that $\alpha|\{a\} \times \partial D$ is null-homotopic and $\alpha|S^1 \times \{b\}$ is not null-homotopic. By 2.4 one can extend $\alpha|\partial R$ over $S^1 \times \mathbb{R}P^2$. Any such extension is null-homotopic when restricted to $\{a\} \times \mathbb{R}P^1$, so it can be extended over $\{a\} \times \mathbb{R}P^3$ (see 2.2). Pasting all those extensions gives the desired map $g : M_2 \longrightarrow \mathbb{R}P^2$. □

3 Proof of Theorem 1.5

Lemma 3.1 *Suppose X is a compactum of dimension at most three and mod 2 dimension $\dim_{\mathbb{Z}_2} X$ of X equals 1. A map $f : A \longrightarrow S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$*

extends over X if and only if $\pi \circ f$ extends over X, where $\pi : S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3 \longrightarrow S^1$ is the projection onto the first coordinate.

Proof Only one direction is of interest. Pick an extension $g : X \to S^1$ of $\pi \circ f$. Let $\pi_2 : S^1 \times \mathbb{R}P^3 \longrightarrow \mathbb{R}P^3$ be the projection. Since $\mathrm{edim} X \le \mathbb{R}P^\infty$ implies $\mathrm{edim} X \le \mathbb{R}P^3$ (see 1.1), the composition $\pi_2 \circ f$ extends over X to $h : X \longrightarrow \mathbb{R}P^3$. The diagonal $G : X \longrightarrow S^1 \times \mathbb{R}P^3$ of g and h can be pushed rel. A to the 3-skeleton of $S^1 \times \mathbb{R}P^3$ which is exactly $S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$ under standard CW structures of S^1 and $\mathbb{R}P^3$. The resulting map $X \longrightarrow S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$ is an extension of f. □

Corollary 3.2 *Suppose X is a compactum of dimension at most three and A is a closed subset of X. If mod 2 dimension $\dim_{\mathbb{Z}_2} X$ of X equals 1, then any map $f : A \longrightarrow \mathbb{R}P^1$ followed by the inclusion $\mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$ extends over X.*

Proof Let $i : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^3$ be the inclusion. Extend $i \circ f : A \longrightarrow \mathbb{R}P^3$ to $G : X \longrightarrow \mathbb{R}P^3$. Let R be the solid torus as in the definition of the second modification of $\mathbb{R}P^3$. Put $Y = G^{-1}(R)$ and $B = G^{-1}(\partial R)$. The map $g : B \to \partial R = S^1 \times \mathbb{R}P^1$ induced by G extends to $H : Y \longrightarrow S^1 \times \mathbb{R}P^2 \cup \{a\} \times \mathbb{R}P^3$ in view of 3.1. Pasting $G|(X \setminus G^{-1}(int(R))$ and H results in an extension $F : X \to M_2$ of f. By 2.6 the inclusion $\mathbb{R}P^2 \cap M_2 \hookrightarrow \mathbb{R}P^2$ extends to a map $g : M_2 \longrightarrow \mathbb{R}P^2$. Notice that $g \circ F$ is an extension of $i \circ f$. □

Corollary 3.3 *Suppose X is a compactum of dimension at most three. If mod 2 dimension $\dim_{\mathbb{Z}_2} X$ of X equals 1, then any map $f : A \longrightarrow \mathbb{R}P^2$ followed by the inclusion $\mathbb{R}P^2 \longrightarrow M_1$ from the projective plane to the first modification of $\mathbb{R}P^3$ extends over X.*

Proof Let $i : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^3$ be the inclusion. Extend $i \circ f : A \longrightarrow \mathbb{R}P^3$ to $G : X \longrightarrow \mathbb{R}P^3$. Let L be the solid torus as in the definition of the first modification of $\mathbb{R}P^3$. Put $Y = G^{-1}(L)$ and $B = G^{-1}(\partial L)$. The map $g : B \to \partial L = S^1 \times \mathbb{R}P^1$ induced by G extends to $H : Y \longrightarrow S^1 \times \mathbb{R}P^2$ in view of 3.2. Pasting $G|(X \setminus G^{-1}(int(L)))$ and H results in an extension $F : X \to M_1$ of f. □

Since $\mathbb{R}P^2$ is a retract of M_1 (see 2.5), Corollary 3.3 does indeed imply Theorem 1.5.

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