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# Regular geodesic languages and the falsification by fellow traveler property

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**Abstract** We furnish an example of a finite generating set for a group that does not enjoy the falsification by fellow traveler property, while the full language of geodesics is regular.

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**Keywords** Regular language, falsification by fellow traveler property

## 1 Introduction

In this short note we answer the following question of Neumann and Shapiro from [5]:

**Question** Can one find a monoid generating set $A$ of a group $G$ so that the language of geodesics is regular but $A$ does not have the falsification by fellow traveler property?

The converse to this statement is Proposition 4.1 of their paper, which states that if $A$ has the falsification by fellow traveler property then the full language of geodesics on $A$ is regular, and this fact is the reason for the property's existence. Several authors have used the falsification by fellow traveler property as a route to finding other (geometric) properties of groups; Rebbechi uses a version of the property to prove that relatively hyperbolic groups are biautomatic [6], and the author exploits the property to prove that a certain class of groups is almost convex [4]. The author discusses various attributes and extensions of the property in [1, 2, 3].

In this article we answer the question via an example first given by Cannon to demonstrate that a group may have a regular language of geodesics with respect to one generating set but not another. Neumann and Shapiro include it in [5] to prove that the falsification by fellow traveler property is generating set dependent.

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## 2 Definitions

**Definition 2.1** (Finite state automaton; regular language) Let $A$ be a finite set of letters, and let $A^*$ be the set of all finite strings, including the empty string, that can be formed from the letters of $A$. A *finite state automaton* is a quintuple $(S, A, \tau, Y, s_0)$, where $S$ is a finite set of *states*, $\tau$ is a map $\tau : S \times A \to S$, $Y \subseteq S$ are the *accept states*, and $s_0 \in S$ is the *start state.* A finite string $w \in A^*$ is *accepted* by the finite state automaton if starting in the state $s_0$ and changing states according to the letters of $w$ and the map $\tau$, the final state is in $Y$. The set of all finite strings that are accepted by a finite state automaton is called the *language* of the automaton. A language $L \subseteq A^*$ is *regular* if it is the language of a finite state automaton.

Suppose $G$ is a group with finite generating set $A$. A word in $A^*$ represents a path in the Cayley graph based at any vertex. Define $d(a,b)$ to be the distance between two points $a$ and $b$ in the Cayley graph with respect to the path metric. Paths can be parameterized by non-negative $t \in \mathbb{R}$ by defining $w(t)$ as the point at distance $t$ along the path $w$ if $t$ is between 0 and the length of $w$, and the endpoint of $w$ otherwise.

**Definition 2.2** (The (asynchronous) fellow traveler property) Paths $u$ and $v$ are said to *$k$-fellow travel* if $d(u(t), v(t)) \leq k$ for all $t \geq 0$. They *asynchronously $k$-fellow travel* if there is a non-decreasing proper continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $d(u(t), v(\phi(t))) \leq k$. A language $L \subseteq A^*$ enjoys the *(asynchronous) fellow traveler property* if there is a constant $k$ such that for each $u, v \in L$ that start at the identity and end at distance 0 or 1 apart in the Cayley graph, $u$ and $v$ (asynchronously) $k$-fellow travel.

**Definition 2.3** (The (asynchronous) falsification by fellow traveler property) A finite generating set $A$ for a group $G$ has the *(asynchronous) falsification by fellow traveler property* if there is a constant $k$ such that every non-geodesic word in the Cayley graph of $G$ with respect to $A$ is (asynchronously) $k$-fellow traveled by a shorter word.

The property arises naturally in the context of geodesic regular languages, and the proof of Proposition 4.1 in [5] uses the property to build an appropriate finite state automaton. The author proves in [1] that the synchronous and asynchronous versions of the falsification by fellow traveler property are equivalent.

# 3 The Example

Let $G$ be the split extension of $\mathbb{Z}^2$, generated by $\{a, b\}$, by $\mathbb{Z}_2$, generated by $\{t\}$, such that $t$ conjugates $a$ to $b$ and $b$ to $a$, with presentation

$$\langle a, b, t \mid t^2 = 1, ab = ba, tat = b \rangle.$$

Performing one Tietze transformation (removing $b = tat$) we obtain

$$\langle a, t \mid t^2 = 1, atat = tata \rangle.$$

Let $A = \{a^{\pm 1}, t^{\pm 1}\}$ be the inverse-closed generating set corresponding to this presentation. The Cayley graph for $G$ with respect to $A$ is shown in Figure 1. We can consider the vertices of the graph as either being in the top or the

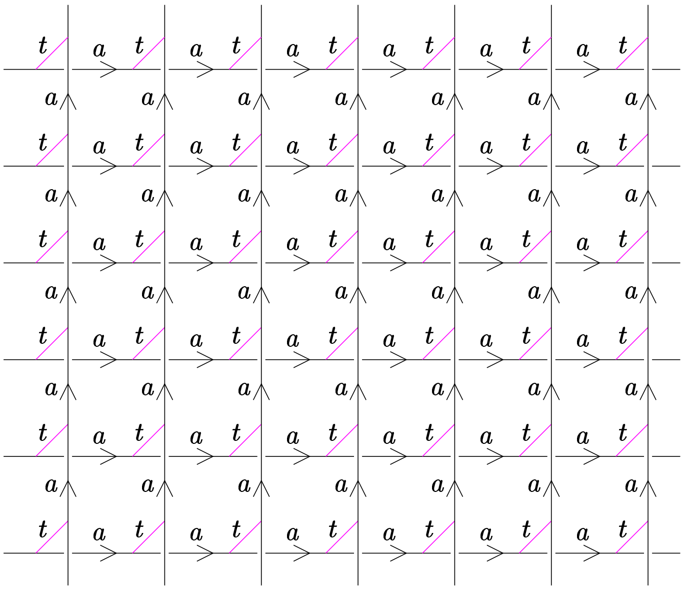

Figure 1: The Cayley graph for $G = \langle a, t \mid t^2 = 1, atat = tata \rangle$

bottom layer, where edges labeled $t$ link top and bottom layers. We declare the identity vertex to lie in the bottom. Each vertex can also be given a coordinate $(x, y)$ where $x$ is the distance in the East-West direction from the identity, and

$y$ is the distance in the North-South direction. In this way each vertex (group element) is uniquely specified by the triple $(x, y, \mathrm{bottom})$ or $(x, y, \mathrm{top})$.

For example, the identity has the coordinate $(0, 0, \mathrm{bottom})$, the word $a^3ta^4$ has the coordinate $(3, 4, \mathrm{top})$, the word $a^3ta^4t$ has the coordinate $(3, 4, \mathrm{bottom})$, and the word $ta^3ta^4$ has the coordinate $(4, 3, \mathrm{bottom})$.

**Lemma 3.1** *Each word from $1$ to a vertex in the top layer has an odd number of $t$ letters, and each word from $1$ to a vertex in the bottom layer has an even number of $t$ letters.*

**Proof** Suppose a vertex has the coordinate $(x, y, \mathrm{top})$. Then $a^xta^y$ is a word to this vertex. Suppose $w$ is any other word to this vertex. Then $wa^{-y}ta^{-x} =_G 1$ so under the map which sends $t$ to $t$ and $a$ to $1$ this word must be sent to an even power of $t$, so $w$ has an odd number of $t$ letters. Similarly if a vertex lies in the bottom layer there is a word $a^xta^yt$ to it from $1$. If $w$ is any other word to this vertex then $wta^{-y}ta^{-x} =_G 1$ gets sent to an even power of $t$ so $w$ has an even number of $t$ letters. □

**Lemma 3.2** *The word $ta^nta^mt$ for any $m, n \in \mathbb{Z}$ is not geodesic.*

**Proof** The word $ta^nta^mt$ can be written as $a^mta^n$ which is shorter. □

**Lemma 3.3** *Each vertex in the top layer has a unique geodesic to it from the identity of the form $a^xta^y$, where $(x, y, \mathrm{top})$ is the coordinate of the vertex.*

**Proof** By Lemma 3.1 any geodesic to the vertex with coordinate $(x, y, \mathrm{top})$ has an odd number of $t$ letters. If a word has three or more $t$ letters then it has a subword of the form $ta^nta^mt$ which is not geodesic by Lemma 3.2. So any geodesic to this vertex has exactly one $t$ letter, so is of the form $a^ita^j$. This path has coordinate $(i, j, \mathrm{top})$ so it must be that $i = x$ and $j = y$. □

**Proposition 3.4** *$A$ does not have the falsification by fellow traveler property.*

**Proof** Suppose by way of contradiction that $A$ has the falsification by fellow traveler property with positive constant $k$, and choose $n >> k$. Consider the word $w = ta^nta^nt$ which ends at the coordinate $(n, n, \mathrm{top})$. Any word that ends at this vertex must move East at least $n$ units and North at least $n$ units, and by Lemma 3.1 must have an odd number of $t$ letters. If it has just one $t$ letter then it must be the unique geodesic $a^nta^n$ which clearly does not $k$-fellow travel $w$. Otherwise it has three or more $t$ letters, so has length at least $2n + 3$ so is not shorter than $w$, so we are done. □

**Theorem 3.5** *There is a group and finite generating set such that the language of all geodesics is regular but fails to have the falsification by fellow traveler property.*

**Proof** By Proposition 3.4 the group $G$ with generating set $A$ fails the falsification by fellow traveler property.

Consider the language $L = \{a^x, a^x t a^y, a^{x_1} t a^y t a^{x_2} : x, x_1, x_2, y \in \mathbb{Z}, x_1 \cdot x_2 \geq 0\}$. $L$ is the language of the finite state automaton in Figure 2. All states are accept states.

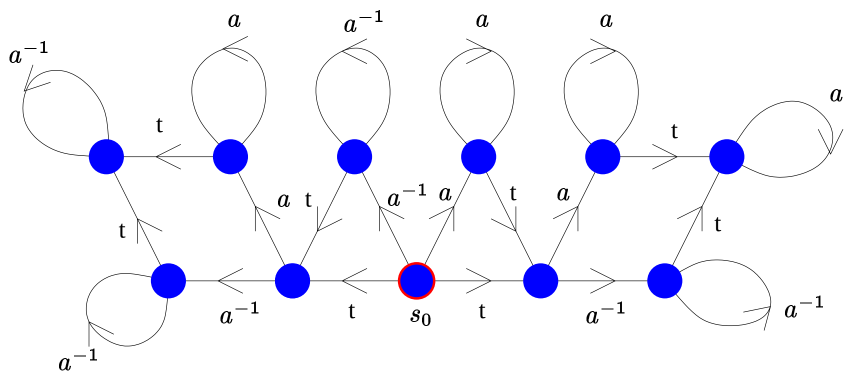


Figure 2: A finite state automaton accepting the language $L$

We now show that $L$ is the language of all geodesics on $A$ for $G$. By Lemma 3.3 every group element corresponding to a vertex in the top layer has a unique geodesic representative of the form $a^x t a^y$. Otherwise the group element corresponds to a vertex in the bottom, so has an even number of $t$ letters. If a word has more than two $t$ letters then it has a subword of the form $ta^n t a^m t$ which is not geodesic by Lemma 3.2, so a geodesic word for a bottom element has either zero $t$ letters, so is of the form $a^x$, or has two $t$ letters, so is of the form $a^{x_1} t a^y t a^{x_2}$. If $x_1$ and $x_2$ don't have the same sign then we can find a shorter word $a^{(x_1+x_2)} t a^y t$. Otherwise $a^{x_1} t a^y t a^{x_2}$ is a geodesic to a vertex with coordinate $(x_1 + x_2, y, \text{bottom})$. Notice that this gives a family of $x_1 + x_2 + 1$ geodesics to this vertex. □

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