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Ward's Solitons

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Abstract

Using the 'Riemann Problem with zeros' method, Ward has constructed exact solutions to a $(2+1)$–dimensional integrable Chiral Model, which exhibit solitons with nontrivial scattering. We give a correspondence between what we conjecture to be all pure soliton solutions and certain holomorphic vector bundles on a compact surface.

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1 Introduction

Nonlinear equations admitting soliton solutions in 3–dimensional space-time have been studied recently both numerically and analytically. See [4] and [6] for a discussion of solitons in planar models.

In this paper, we study an integrable model introduced by Ward which is remarkable in that it possesses interacting soliton solutions of finite energy [4, 6, 3]. This $\mathrm{SU}(N)$ chiral model with torsion term may be obtained by dimensional reduction and gauge fixing from the $(2+2)$ Yang–Mills equations [6] or more directly from the $(2+1)$ Bogomolny equations. Static solutions of the model correspond to harmonic maps of $\mathbb{R}^2 \to \mathrm{U}(N)$ which extend analytically to $\mathbb{S}^2$ iff they have finite energy.

The basic equations of Ward are

$$\frac{\partial}{\partial t}(J^{-1}\frac{\partial}{\partial t}J) - \frac{\partial}{\partial x}(J^{-1}\frac{\partial}{\partial x}J) - \frac{\partial}{\partial y}(J^{-1}\frac{\partial}{\partial y}J) + [J^{-1}\frac{\partial}{\partial y}J, J^{-1}\frac{\partial}{\partial t}J] = 0 \tag{1.1}$$

where $J\colon \mathbb{R}^3 \to \mathrm{SU}(N)$. To this equation Ward added the boundary condition:

$$J(r,\theta,t) = \mathbb{I} + \frac{1}{r}J_1(\theta) + \mathcal{O}\left(\frac{1}{r^2}\right) \qquad \text{as} \qquad r \to \infty; \tag{1.2}$$

we will assume $J_1(\theta)$ is continuous. Ward showed that analytic solutions to (1.1) correspond to doubly-framed holomorphic bundles on the open surface $T\mathbb{P}^1$. We will show that a neccessary and sufficient condition for the bundle to extend to the compactification $\widetilde{T\mathbb{P}}^1$, the second Hirzebruch surface is that J be analytic and that the operator

$$\frac{d}{du} + \tfrac{1}{2}(1+\cos\theta)\iota^*\left(J^{-1}\frac{\partial}{\partial x}J\right) + \tfrac{1}{2}\sin\theta\,\iota^*\left(J^{-1}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)J\right) \tag{1.3}$$

have null monodromy around $u \in \mathbb{R}\cup\{\infty\}$, where

$$\iota(u) \stackrel{\text{def}}{=} (\cos\theta\, u + x_0, \sin\theta\, u + y_0, 0), \tag{1.4}$$

for all $x_0, y_0 \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$, i.e. for all lines in $\mathbb{R}^2$. There is some evidence that our techniques can be applied to the case of nonanalytic solutions, but we will not do so here. We also leave open the question as to whether these are all the pure soliton solutions.

Before going on, consider the null monodromy of (1.3) in the U(1) case, i.e. for the usual d'Alembert equation. Let $j = \log J$ be some logarithm of a solution. The monodromy of (1.3) becomes

$$\int_{-\infty}^{\infty} [(1+\cos\theta)\iota^* j_x + \sin\theta(\iota^* j_y + \iota^* j_t)]\, du = 0$$

where $j_x = \frac{\partial j}{\partial x}$, etc. The fundamental theorem of calculus and the boundary condition (1.2) imply

$$\int_{-\infty}^{\infty} \cos\theta\, \iota^* j_x + \sin\theta\, \iota^* j_y \; du = 0.$$

Combining the two integrals with $\theta = \theta_0$ and $\theta = \theta_0 + \pi$, we obtain

$$0 = \int_{-\infty}^{\infty} \sin\theta_0\, \iota^* j_t \; du = \sin\theta_0 \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \iota^* j \; du$$

and

$$0 = \int_{-\infty}^{\infty} \iota^* j_x \; du.$$

The first statement is that the Radon transform of j on a space-plane is independent of time, and hence j is a harmonic function. Since j is also bounded (a result of (1.2)) it must be constant. This provides some support for the idea that (1.3) has null monodromy for pure soliton solutions only.

We explain (in §4) how the boundary conditions can be interpreted in terms of the extension of the holomorphic bundle to the fibrewise compactification ($\widetilde{T\mathbb{P}^1}$) when J satisfies (1.2) and (1.3) has null monodromy, and to infinite points for fibres not above the equator in $\mathbb{P}^1$ (i.e. $\{\lambda \in \mathbb{C} \cup \{\infty\} : |\lambda| \neq 1\}$), when J satisfies (1.2) alone.

When (1.3) does have null monodromy, Serre's GAGA principle tells us that the associated bundles are algebraic. This explains the algebraic nature of the solutions constructed so far, and was a strong motivation for proving the main theorem.

Main Theorem *There are bijections between the sets of*

1) *analytic solutions* J *of* (1.1) *satisfying* (1.2) *for which* (1.3) *has null monodromy; and*

2) *holomorphic rank N bundles $\mathcal{V} \to T\mathbb{P}^1$ which are real in the sense that they admit a lift*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\tilde{\sigma}} & \mathcal{V} \\ \downarrow & & \downarrow \\ T\mathbb{P}^1 & \xrightarrow{\sigma} & T\mathbb{P}^1 \end{array} \quad \begin{array}{c} \textit{of the} \\ \textit{antiholomorphic} \\ \textit{involution} \end{array} \quad \begin{array}{l} \sigma^*\lambda = 1/\bar{\lambda} \\ \sigma^*\eta = -\bar{\lambda}^{-2}\bar{\eta} \end{array} \tag{1.5}$$

(where λ and η are standard base and fibre coordinates of $T\mathbb{C} \subset T\mathbb{P}^1$) and which extend to bundles on the singular quadric cone $T\mathbb{P}^1 \cup \{\infty\}$, such that restricted to real sections (sections invariant under the real structure) $\mathcal{V}$ is trivial, and restricted to the compactified tangent planes $T_\lambda\mathbb{P}^1 \cup \{\infty\}$ for $|\lambda| = 1$, $\mathcal{V}$ is trivial, with a fixed, real framing.

Remark 1.6 The null monodromy of (1.3) makes sense for initial conditions on a space-plane $\{t = t_0\}$. It follows from the proof that the initial value problem with null-monodromy initial conditions has an analytic solution extending forward and backward to all time, i.e. it cannot blow up in finite time.

Construction of solutions

There are currently three methods of solving this system. The first method of Ward was to give a twistor correspondence between solutions of (1.1) and holomorphic bundles on $T\mathbb{P}^1$, the holomorphic tangent space to the complex projective line. This led to the construction of noninteracting soliton solutions. Thereafter, numerical simulations of these solutions by Sutcliffe led to his discovery of interacting soliton solutions. Exact solutions with two interacting solitons were then constructed by Ward using a Zakharov–Shabat procedure. Using this procedure, more general solutions were constructed by Ioannidou concurrently with the present work. In a future paper, we will present a closed-form expression for all solutions satisfying (1.1), (1.2) with null (1.3) monodromy, including all known exact soliton solutions. This will build on the monad-theoretic work in [1].

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2 Zero Curvature and the Bogomolny equations

Ward's equations are not a reduction in the sense of dimensional reduction. We obtain them from the Bogomolny equations by fixing a gauge.

On $\mathbb{R}^{2+1}$, the Bogomolny equations for a connection $\nabla = d + A$ and a Higgs field (section of the adjoint bundle) Φ are

$$-\nabla_t \Phi = [\nabla_x, \nabla_y] \tag{2.1a}$$
$$\nabla_x \Phi = [\nabla_y, \nabla_t] \tag{2.1b}$$
$$\nabla_y \Phi = [\nabla_t, \nabla_x]. \tag{2.1c}$$

They are completely integrable, and can be written in the form

$$[\nabla_{\bar z} + \frac{i\lambda}{2}\nabla_t - \frac{\lambda}{2}\Phi, \nabla_z - \frac{i}{2\lambda}\nabla_t - \frac{1}{2\lambda}\Phi] = 0 \quad \text{for all } \lambda \in \mathbb{C}^*. \tag{2.2}$$

When $|\lambda| = 1$ this is the curvature for an underlying connection on a family of planes. Integrating it, we obtain a circle of special gauges in which

$$\Phi = \Re\lambda\, A_x + \Im\lambda\, A_y$$
$$A_t = \Im\lambda\, A_x - \Re\lambda\, A_y.$$

Ward's equations are equations for the gauge transformation from the $\lambda = -1$ gauge to the $\lambda = 1$ gauge. We will call the $\lambda = -1$ gauge the *standard gauge.*

If J is the gauge transformation, (2.1b) is Ward's equation (1.1), and in the standard gauge, $\nabla = d + A$ and Φ are

$$\begin{aligned} -A_x = \Phi &= \frac{1}{2} J^{-1}\partial_x J \\ A_y = A_t &= \frac{1}{2} J^{-1}(\partial_y + \partial_t) J. \end{aligned} \tag{2.3}$$

Conversely, given J, we can form (∇, Φ) in this way. Moreover, if J satisfies (1.2) and has null (1.3) monodromy, the resulting map $J(z, t = 0, \lambda)\colon \mathbb{R}^2 \times \mathbb{S}^1 \to \mathrm{SU}(N)$ extends to a based map $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathrm{SU}(N)$. This associates a topological charge in $\pi^3(\mathrm{SU}(N)) = \mathbb{Z}$ to any such solution J.

Conjecture 2.4 *This topological degree can be defined for all finite-energy solutions, and is equal to the energy minus the effect of Lorentz boosting, internal spinning and radiation.*

3 Twistor constructions of Ward and Hitchin

Hitchin showed that the set of oriented geodesics on an odd-dimensional real manifold has a complex structure ([2]). In particular, the set of lines in $\mathbb{R}^3$ is isomorphic as a complex manifold to the holomorphic tangent bundle of the complex projective line. Using this equivalence he shows that solutions to the Bogomolny equations correspond to holomorphic bundles on $T\mathbb{P}^1$.

Very briefly, given a solution (∇, Φ) to the Bogomolny equations, one associates to a line the vector space of covariant constant frames of the modified connection $\nabla - i\Phi$ on the line. This is a complex bundle. The operator $\nabla_{\bar{\eta}}$ where η represents a holomorphic fibre coordinate on $T\mathbb{P}^1$ commutes with $\nabla - i\Phi$, and hence descends to a $\bar{\partial}$–operator on the bundle.

The key point is the commuting of the two operators and after a recombination, this can be written as a zero curvature condition. See [2] for a full account.

4 The holomorphic bundle

Given a solution J, let $(\nabla = d + A, \Phi)$ be the solution to the Bogomolny equations, in the standard gauge, as in (2.3). The extension to the compactification requires one argument near the equator ($|\lambda| = 1$) (which requires null (1.3) monodromy) and another on the open hemispheres.

4.1 Away from the equator

Consider the z–plane, $\{t = 0\}$, and the 'projection':

$$T\mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^2 \times \mathbb{S}^2 \tag{4.2}$$

onto this plane.

The zero curvature connection has a characteristic direction in this plane, and the appropriate linear combination of the operators in (2.2) gives the $\bar{\partial}$–operator for a rank N bundle $\mathcal{V}_\lambda \to T_\lambda\mathbb{P}^1$:

$$\overline{\nabla}_\eta \stackrel{\text{def}}{=} (1+\lambda^2)\partial_x + i(1-\lambda^2)\partial_y + (1+\lambda)^2 A_x + i(1-\lambda^2)(A_y + A_t). \tag{4.3}$$

The kernel of this operator is the set of holomorphic sections of a bundle with respect to the complex variable

$$\eta = \frac{z - \lambda^2\bar{z}}{1 - \lambda\bar{\lambda}}. \tag{4.4}$$

Together with $\partial_{\bar\lambda}$, this defines an operator

$$\overline{\nabla}\colon\ \mathrm{gl}(\mathbb{C}^N) \to \mathrm{gl}\Big(\mathbb{C}^N \otimes T^{(0,1)}\{|\lambda|<1, \eta\in\mathbb{C}\}\Big).$$

Since $\overline{\nabla}_\eta$ depends holomorphically on λ, $\overline{\nabla}^2 = 0$. Under the assumption that $J \in C^1(\mathbb{R}^3)$ plus boundary conditions (1.2), $\overline{\nabla}$ will be continuous on $\{|\lambda|<1, \eta\in\mathbb{C}\}$ which we identify with $\{|\lambda|<1, z\in\mathbb{R}^2\}$. Near $z=\infty$

$$-\overline{\eta}^2\overline{\nabla}_\eta = \partial_{1/\overline{\eta}} + C_1(\lambda,\theta)r^2A_x + C_2(\lambda,\theta)r^2(A_y+A_t)$$

where $z = re^{i\theta}$, and the functions C_1 and C_2 are bounded in θ for each fixed λ, i.e. they are polynomials in $\sin\theta$ and $\cos\theta$. The boundary conditions (1.2) for J imply

$$\begin{aligned} A_x &= J^{-1}(\cos\theta\,\partial_r + \frac{\sin\theta}{2ir}\partial_\theta)J \\ &= 1/r^2 A'_x(1/r,\theta,t) \end{aligned} \tag{4.5}$$

where A'_x is continuous near $z=\infty$, and similarly for A_y and A_t. Hence $\overline{\nabla}_\eta$ is continuous with a bounded singularity at $z=\infty$.

This implies that the coefficient is $L^p_{\mathrm{loc}}(\mathbb{S}^2)$ for $0<p\le\infty$ which is sufficient to show that iterating convolution with the Cauchy kernel produces local holomorphic gauges. Since the data vary holomorphically in λ, the gauges can be used to define a holomorphic structure on $\mathcal{V}\to(\widetilde{T\mathbb{P}^1}\cap\{|\lambda|<1\})$.

Remark 4.6 The extension to the compactified nonequatorial fibres does not require the null (1.3) monodromy, and thus gives a necessary but not necessarily sufficient condition for a bundle to represent a solution satisfying the weak boundary condition.

4.7 Null monodromy and the equator

In the last section, we found a '$\bar\partial$–operator' hidden in the zero curvature condition (2.2). Away from the poles, we can make a different recombination of the operators, which on the equator can be written in the manifestly real form

$$\cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y} + \tfrac{1}{2}(1+\cos\theta)J^{-1}\frac{\partial}{\partial x}J + \tfrac{1}{2}\sin\theta\ J^{-1}\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)J. \tag{4.8}$$

Under the assumption that J is analytic, this represents an $\mathbb{S}^1 \times \mathbb{R}$–family of first order ODEs on the line which vary analytically with the parameter $\theta \in \mathbb{R}$. The boundary condition (1.2) implies that the functions $r^2 J^{-1}\frac{\partial}{\partial x}J$ and $r^2 J^{-1}\frac{\partial}{\partial y}J$ are bounded on $\mathbb{R}^2$, which means that $J^{-1}dJ$ has at worst a bounded discontinuity on $\mathbb{S}^2$, the conformal compactification of a space plane. Since the L^1 norm is the natural norm in this context, we can convert all the integrals on infinite lines to integrals over compact circles through $\infty \in \mathbb{S}^2$. It follows that the coefficients vary continuously in L^1 with the choice of line, and it makes sense, given θ, to solve the whole family of ODEs on parallel lines giving a function $\mathbb{S}^2 \to \mathrm{U}(N)$, which is continuous at ∞ iff (1.3) has null monodromy.

The result is an analytic map from $\{\theta \in \mathbb{S}^1\}$ to $C^0(\mathbb{S}^2, \mathrm{U}(N))$. By analytic, we mean that it can be expanded in local power series in θ with coefficients in $C^0(\mathbb{S}^2, \mathrm{U}(N))$, which converge in some neighbourhood with respect to the L^∞ norm (measured pointwise by geodesic distance from the unit in $\mathrm{U}(N)$). This follows from the fact that the operator (4.8) is analytic in θ and hence has a power series which (in particular) converges in the L^1 norm, and the integration map which solves the initial value problem is an absolutely continuous map, i.e. the L^∞ norm of the solution is bounded by the L^1 norm of the integrand.

The resulting analytic map

$$\mathbb{S}^1 \to C^0(\mathbb{S}^2, \mathrm{U}(N)),$$

can be continued to an analytic map

$$\{1 - \epsilon < |\lambda| < 1 + \epsilon\} \to C^0(\mathbb{S}^2, \mathrm{GL}(N)),$$

on some annulus containing the equator. Since (4.8) is the 'real form' of the 'holomorphic' equation (4.3), this solution defines a global trivialisation of the bundle $\mathcal{V}$ on a deleted neighbourhood of the equator, and we can use it to define the holomorphic structure of the bundle over the equator. Grauert's Theorem implies that the bundle is trivial on generic fibres.

To see this rigorously, observe that (4.3) and (4.8) can both be completed to the system (2.2) by adding a second operator which has nonzero $\frac{\partial}{\partial t}$ component. The solution to (4.8) has a unique extention to a neighbourhood of $\{t = t_0\}$ and the extension is in the kernel of this second operator. The resulting solution is a solution to (2.2) and hence a solution to (4.3). The important point is that null (1.3) monodromy insures that the solution is defined on the compactification of $\{t = t_0\}$ to a sphere, otherwise the resulting holomorphic trivialisation would have been for a neighbourhood in $T\mathbb{P}^1$ and not in $\widetilde{T\mathbb{P}}^1$.

4.9 Reality

Reality of the associated bundle is independent of the boundary conditions and gauge fixing, and is implied by the analogous property for arbitrary solutions of the Bogomolny equations. The simplest way to see it in this case is via the formula

$$\lambda^2 \overline{\tilde{f}^{-1}\sigma^* \overline{\nabla}_\eta \tilde{f}\tilde{f}^{-1}}^t = \overline{\nabla}_\eta \overline{(\sigma^* \tilde{f})}^t$$

for a local gauge, $\tilde{f}$, which shows that holomorphic gauges are transformed into antiholomorphic gauges of the dual bundle.

4.10 The section at infinity and the framing

Over a (possibly pinched) tubular neighbourhood of G_∞, the section at infinity, the iterative Cauchy-kernel argument defines a holomorphic framing. The radius of the tubular neighbourhood depends on an energy estimate and is nonzero away from the equator. Since the data are holomorphic in λ, the result is holomorphic in base and fibre directions and on G_∞ agrees with the trivialisation coming from integrating (1.3) from infinity. The resulting trivialization of $\mathcal{V}|_{G_\infty}$ defines the canonical framing. Grothendieck's theorem on formal functions implies that any bundle trivial on a rational curve of negative self-intersection is trivial on a neighbourhood of the curve. So the bundle is actually trivial on a neighbourhood of G_∞.

5 Inverse construction : compact twistor fibration

The inverse construction follows the inverse construction of ∇, Φ due to Hitchin. To accommodate the boundary condition, we need to extend the twistor fibration (and definition of J) to a compact twistor fibration.

The first step is to embed $T\mathbb{P}^1$ as the nonsingular part of the singular quadric $Q \stackrel{\text{def}}{=} \{\beta^2 = \alpha\gamma\} \subset \mathbb{P}^3$ by

$$(\lambda, \eta) \mapsto [1, -2i\lambda, -\lambda^2, -\eta] = [\alpha, \beta, \gamma, \delta]$$

(in terms of affine coordinates $\eta\frac{d}{d\lambda} \in T\mathbb{P}^1$ and homogeneous coordinates on $\mathbb{P}^3$). Since the bundle is trivial on a (complex) neighbourhood of the section at infinity, $\mathcal{V}$ pushes down via the collapsing map $\widetilde{T\mathbb{P}}^1 \to Q$ ($G_\infty \to$ singular point) to a bundle on Q.

The next step is to construct the compact double twistor fibration:

$$\begin{array}{ccc} & X \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{c} a\alpha + b\beta + c\gamma + d\delta = 0 \\ \beta^2 = \alpha\gamma \end{array} \right\} \subset \mathbb{P}^3 \times \mathbb{P}^3 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{R}^{2+1} \subset \mathbb{P}^3 & & Q \end{array}$$

Grauert's Theorem implies that pulling $\mathcal{V}$ back to X and pushing it forward to $\mathbb{P}^3$ gives a coherent sheaf which we assume is locally-free on a neighbourhood of $\mathbb{R}^{2+1} \subset \mathbb{C}^3 \subset \mathbb{P}^3$. (We will show in a future paper that this assumption is unneccessary, i.e. that real bundles which are trivial on equatorial fibres are necessarily trivial on real sections.) Call the new sheaf $\mathcal{W} \to \mathbb{P}^3$. Fixing a fibre $P_\lambda \subset \widetilde{T\mathbb{P}}^1$ such that $\mathcal{V}|_{P_\lambda}$ is trivial, the composition

$$\mathcal{W}_y = H^0(G_y, \mathcal{V}) \stackrel{\mathrm{eval}}{\cong} \mathcal{V}|_{G_y \cap P_\lambda} \stackrel{\mathrm{eval}}{\cong} H^0(P_\lambda, \mathcal{V}) \cong \mathbb{C}^N,$$

where $G_y \stackrel{\mathrm{def}}{=} \pi_{2*}\pi_1^{-1}(y)$, gives a natural frame of $\mathcal{W}|_Y$,

$$Y \stackrel{\mathrm{def}}{=} \{y \in \mathbb{P}^3 : (\pi_2^*\mathcal{V})|_{\pi_1^{-1}(y)} \text{ is trivial}\}.$$

In particular, the standard gauge comes from the fixed framing of $\mathcal{V}|_{P_{-1}}$, and J is the gauge transformation from the P_{-1} to the P_1 framing. It follows that J extends meromorphically to $\mathbb{P}^3$.

In terms of projective coordinates $[a,b,c,d]$ on $\mathbb{P}^3$, the 'finite' hyperplane sections $\{[a,b,c,1]\} = \mathbb{C}^3 \subset \mathbb{P}^3$ represent the sections $\{\eta = a - 2ib\lambda - c\lambda^2\}$ of $T\mathbb{P}^1$. The 'infinite' hyperplanes $\{[a,b,c,0]\}$ represent the completion of the linear system on $\widetilde{T\mathbb{P}}^1$ to include the family of divisors $G_{[a,b,c,0]} \stackrel{\mathrm{def}}{=} G_\infty + P_{\lambda_0} + P_{\lambda_{-1}}$ (where $a - 2ib\lambda_i - c\lambda_i^2 = 0$). We know that the set of such hyperplane sections over which $\mathcal{V}$ is trivial is open and includes the circle $\{G_\infty + 2P_\lambda : \lambda \in \mathbb{S}^1\}$. The intersection $G_{[a,b,c,0]} \cap P_\lambda$ is either $P_\lambda \cap G_\infty$ or P_λ. Since P_λ was taken so that $\mathcal{V}|_{P_\lambda}$ is trivial, the definition of the standard and P_λ frames extends to an open set of points of the plane at infinity in $\mathbb{P}^3$, and they agree on this set by definition. In particular, J_θ, the transformation from the P_{-1} frame to the $P_{e^{i\theta}}$ frame is the identity on the infinite points. Since $J_{e^{i\theta}}$ is in the kernel of (1.3) and is defined on compactified space planes, (1.3) has null monodromy.

Since J is analytic by construction, we can use power series: Let b/a, c/a, d/a be affine coordinates on $\mathbb{P}^3$ centred at a point at infinity. J is defined on an open set in this coordinate chart containing $(0,1,0)$. The plane at infinity is cut out by the equation $d/a = 0$. Since $J|_{\{d/a=0\}} = \mathbb{I}$, we can expand J in a power series

$$\begin{aligned} J &= \mathbb{I} + \sum_{\substack{i\geq 1\\ j,k\geq 0}} \left(\frac{d}{a}\right)^i \left(\frac{b}{a}\right)^j \left(\frac{c}{a}\right)^k J_{ijk} \\ &= \mathbb{I} + \left(\frac{d}{a}\right) \sum_{k\geq 0} \left(\frac{c}{a}\right)^k J_{10k} \\ &\quad + \left(\frac{d}{a}\right)^2 \left(\frac{b}{d}\right) \sum_{\substack{j\geq 1\\ k\geq 0}} \left(\frac{b}{a}\right)^{j-1} \left(\frac{c}{a}\right)^k J_{ijk} \\ &\quad + \left(\frac{d}{a}\right)^2 \sum_{\substack{i\geq 2\\ j,k\geq 0}} \left(\frac{d}{a}\right)^{i-2} \left(\frac{b}{a}\right)^j \left(\frac{c}{a}\right)^k J_{ijk} \\ &= \mathbb{I} + 1/rJ_1(\theta) + 1/r^2 J_2(\theta,t) + 1/r^2 J_3(\theta,1/r,t) \end{aligned} \tag{5.1}$$

where we have used $d/a = 1/z = 1/re^{-i\theta}$, $b/a = -2it/re^{i\theta}$, $c/a = e^{-2i\theta}$ in terms of cylindrical coordinates on $\mathbb{C}^3$, which shows that J satisfies the required boundary conditions (1.2).

This completes the proof that solutions of Ward's equations satisfying the boundary conditions (1.2) with null (1.3) monodromy are in one to one correspondence with framed holomorphic bundles over $\widetilde{T\mathbb{P}^1}$ which satisfy a reality and certain triviality conditions.

Remark 5.2 In a future paper, we will use monads to show that triviality on equatorial fibres plus reality implies triviality on real fibres.

Remark 5.3 It follows from (5.1) that the energy decays as $\frac{1}{r^4}$ as $r \to \infty$, as Ward observed for his solutions. This is a property of analytic functions on $\mathbb{S}^2 \times \mathbb{R}$ which are constant on $\{\infty\} \times \mathbb{R}$.

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