ISSN 1364-0380

Geometry & Topology
Volume 1 (1997) 41–50
Published: 23 October 1997

Spin^c–structures and homotopy equivalences

Robert E Gompf

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1082 USA

Email: `gompf@math.utexas.edu`

Abstract

We show that a homotopy equivalence between manifolds induces a correspondence between their spin^c–structures, even in the presence of 2–torsion. This is proved by generalizing spin^c–structures to Poincaré complexes. A procedure is given for explicitly computing the correspondence under reasonable hypotheses.

AMS Classification numbers Primary: 57N13, 57R15

Secondary: 57P10, 57R19

Keywords: 4–manifold, Seiberg–Witten invariant, Poincaré complex

Proposed: Ronald Stern
Seconded: Robion Kirby, Dieter Kotschick

Received: 16 May 1997
Accepted: 17 October 1997

1 Introduction

The theory of spin^c–structures has attained new importance through its recent application to the topology of smooth 4–manifolds. Among smooth, closed, oriented 4–manifolds (with $b_1 + b_+$ odd) a typical homeomorphism type contains many diffeomorphism types. The only invariants known to distinguish such diffeomorphism types are those arising from gauge theory, as pioneered by Donaldson (eg [1]). The most efficient approach currently known is to assign a *Seiberg–Witten* invariant (eg [6]) to any such 4–manifold X with a fixed spin^c–structure. To extract the most information from these invariants, one must understand how spin^c–structures transform under homeomorphisms. This is straightforward if $H^2(X;\mathbb{Z})$ has no 2–torsion (for example, if X is simply connected), for then the Chern class will distinguish any two spin^c–structures on X. The general case is less obvious, however. In high dimensions, a homeomorphism between smooth manifolds need not be covered by an isomorphism of their tangent bundles. While such isomorphisms always exist in dimension 4, they are not canonical, and automorphisms of the tangent bundle covering id_X may permute the spin^c–structures on X. (For example, such an automorphism over $\mathbb{R}P^3$ or $\mathbb{R}P^3 \times S^1$ can be constructed from the diffeomorphism $\mathbb{R}P^3 \to SO(3)$.) In this note, we show how to canonically assign to any orientation-preserving proper homotopy equivalence $X_1 \to X_2$ between manifolds a correspondence between spin^c–structures on X_1 and those on X_2.

Our approach is to generalize the theory of spin and spin^c–structures from $SO(n)$ to more general structure groups H. Most of the homotopy of $SO(n)$ does not enter into the theory. In fact, it suffices for H to be path connected with a nontrivial double cover so that we can generalize the definition $\text{spin}^c(n) = (\text{spin}(n)\times\text{spin}(2))/\mathbb{Z}_2$. The resulting theory generalizes the classical theory in the obvious way, for example, with spin^c–structures on a bundle ξ over X classified by $H^2(X;\mathbb{Z})$ whenever $W_3(\xi) = 0$ (Proposition 1). Ultimately, the map $BSO \to BSG$ of classifying spaces allows us to generalize spin^c–structures from smooth manifolds to Poincaré complexes, and the latter theory has the required functoriality with respect to homotopy equivalences by naturality of the Spivak normal fibration (Theorem 5). Under reasonable hypotheses, one can explicitly compute the correspondence of spin^c–structures induced by a homotopy equivalence; a procedure is given following Theorem 5. The concluding remarks include other characterizations of classical spin^c–structures.

Acknowledgements The author wishes to thank Larry Taylor for helpful discussions, and the Mathematical Sciences Research Institute for their hospitality. The author is partially supported by NSF grant DMS-9625654. Research at MSRI is supported in part by NSF grant DMS-9022140.

2 Generalized spin^c–structures

A naive approach to generalizing the theory of spin and spin^c–structures would be to define spin(H) to be a preassigned double cover of a path connected topological group H, and let spin$^c(H)$ denote the group spin$(H \times SO(2))$ diagonally double covering $H \times SO(2)$. One could then generalize the theory in the obvious way, using principal spin(H) and spin$^c(H)$–bundles, the natural epimorphisms from spin$^c(H)$ to H and $SO(2)$, and the involution of spin$^c(H)$ induced by conjugation on $SO(2) = U(1)$. However, to avoid the difficulties of adapting principal bundle theory to spherical fibrations, we translate the argument into the language of classifying spaces, replacing epimorphisms of groups with kernel $\mathbb{Z}_2$ or $SO(2)$ by fibrations of the corresponding classifying spaces with fiber $B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$ or $BSO(2) = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$, respectively. We remove the groups from the theory while keeping the suggestive notation, obtaining a theory of spin and spin^c–structures on bundles or fibrations classified by a universal bundle (fibration) $\xi_H \to BH$, where BH is homotopy equivalent to a simply connected CW–complex, and a nonzero class $w \in H^2(BH; \mathbb{Z}_2)$ is specified (corresponding to a choice of double cover of H). We can recover the classical theory by setting $BH = BSO(n)$ $(n \geq 2)$, with w the unique nonzero class $w^* \in H^2(BSO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Recall [8] that any map $f\colon X \to Y$ can be transformed into a fibration by replacing X by the space P of paths from X to Y in the mapping cylinder of f. The initial point fibration $p_0\colon P \to X$ has contractible fiber, and the endpoint fibration $p_1\colon P \to Y$ is homotopic to $f \circ p_0$. The fiber F of p_1 is homotopy equivalent to a CW–complex if X and Y are [4], and $p_0|F$ is a fibration with fiber the loop space ΩY.

Now let (BH, w) be as above. Then w defines epimorphisms $H_2(BH; \mathbb{Z}_2) \to \mathbb{Z}_2$ and hence $\varphi_w\colon \pi_2(BH) \to \mathbb{Z}_2$. We apply the previous paragraph to the map $BH \to K(\mathbb{Z}_2, 2)$ induced by φ_w, and let $B\mathrm{spin}(H, w)$ denote the fiber F. The fibration $B\mathrm{spin}(H, w) \to BH$ induces isomorphisms of $\pi_i(B\mathrm{spin}(H, w))$ with $\ker \varphi_w$ for $i = 2$ and $\pi_i(BH)$ otherwise, and its fiber is $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$. Now we define $B\mathrm{spin}^c(H, w)$ to be $B\mathrm{spin}(H^\times, w + w^*)$, where $BH^\times = BH \times BSO(2)$. We immediately obtain fibrations p_H and $p_{SO(2)}$ of $B\mathrm{spin}^c(H, w)$ over BH and $BSO(2)$, whose fibers are $B\mathrm{spin}(SO(2), w^*) = K(\mathbb{Z}, 2)$ and $B\mathrm{spin}(H, w)$, respectively, and each fibration restricted to the opposite fiber is the map arising from the definition of $B\mathrm{spin}(\cdot)$. (Compare with the projections of spin$^c(H, w)$ to H and $SO(2)$ on the level of groups.) By obstruction theory, complex conjugation on the second factor $BSO(2) = \mathbb{C}P^\infty$ of $BH^\times$ lifts uniquely from $BH^\times$ to a map on $B\mathrm{spin}^c(H, w)$ whose square is fiber homotopic to the identity, and the map is homotopic to conjugation on each $\mathbb{C}P^\infty$–fiber of p_H.

To define spin^c–structures over H, recall that an H–bundle (or fibration) $\xi \to X$ over a CW–complex is classified by a bundle map

$$\begin{array}{ccc} \xi & \xrightarrow{\tilde f} & \xi_H \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BH. \end{array}$$

For two choices of classifying map $\tilde f$, there is a canonical homotopy (up to homotopy rel 0,1) between the corresponding maps f, characterized by lifting to a homotopy of the maps $\tilde f$ through bundle maps. This allows us to define spin^c–structures in a manner independent of the choice of $\tilde f$.

Definition A *spin structure* on an H–bundle (fibration) ξ (relative to w) is a function assigning to each classifying bundle map $\tilde f\colon \xi \to \xi_H$ a homotopy class of lifts $\hat f\colon X \to B\mathrm{spin}(H,w)$ of $f\colon X \to BH$, such that for two choices of $\tilde f$ the canonical homotopy between the maps f lifts to a homotopy of the corresponding maps $\hat f$. A *spin^c–structure* is defined similarly with spin replaced by spin^c.

We denote the sets of spin and spin^c–structures on an H–bundle ξ by $\mathcal{S}(\xi,w)$ and $\mathcal{S}^c(\xi,w)$, respectively. Note that in either case, any lift of a single f with a specified $\tilde f$ uniquely determines such a structure, but changing $\tilde f$ with f fixed may result in an automorphism of $\mathcal{S}(\xi,w)$ or $\mathcal{S}^c(\xi,w)$.

To define characteristic classes, let $Y \subset X$ be a possibly empty subcomplex, and let τ be a trivialization of $\xi|Y$. Then we can assume that the classifying map $f\colon X \to BH$ of ξ is constant on Y, and that τ determines the restriction $\tilde f|Y\colon \xi|Y \to \xi_H$. Set $w_2(\xi,\tau) = f^*(w) \in H^2(X,Y;\mathbb{Z}_2)$ and $W_3(\xi,\tau) = \beta w_2(\xi,\tau) \in H^3(X,Y;\mathbb{Z})$, where β is the Bockstein homomorphism. Any spin^c–structure $s \in \mathcal{S}^c(\xi,w)$ determines a homotopy class of lifts $\hat f\colon X \to B\mathrm{spin}^c(H,w)$ of f, and we define a trivialization $\hat\tau$ of $s|Y$ over τ to be a choice of $\hat f$ (within the given homotopy class) that is constant on Y, up to homotopies through such maps. (Equivalently, $\hat\tau$ is a spin^c–structure on X/Y that pulls back to s on X.) We define Chern classes by setting $c_1(s,\hat\tau) = \hat f^* p^*_{SO(2)}(c) \in H^2(X,Y;\mathbb{Z})$, where $c \in H^2(BSO(2),\mathbb{Z}) \cong \mathbb{Z}$ is the generator $c_1(\xi_{SO(2)})$. If Y is empty, we use the notation $w_2(\xi)$, $W_3(\xi)$, $c_1(s)$.

Proposition 1 *The set $\mathcal{S}(\xi,w)$ of spin structures on an H–bundle (or fibration) $\xi \to X$ is nonempty if and only if $w_2(\xi) = 0$. If so, then $H^1(X;\mathbb{Z}_2)$ acts freely and transitively on $\mathcal{S}(\xi,w)$. The set $\mathcal{S}^c(\xi,w)$ is nonempty if and only if $W_3(\xi) = 0$, and if so, then $H^2(X;\mathbb{Z})$ acts freely and transitively on it. For*

$s \in \mathcal{S}^c(\xi, w)$ and $a \in H^2(X;\mathbb{Z})$, we have $c_1(s+a) = c_1(s) + 2a$. Conjugation induces an involution on $\mathcal{S}^c(\xi, w)$ that reverses signs of Chern classes and the $H^2(X;\mathbb{Z})$–action. For $Y \subset X$ and $\hat{\tau}$ as above, $c_1(s, \hat{\tau})$ reduces modulo 2 to $w_2(\xi, \tau)$.

Thus, choosing a base point in $\mathcal{S}(\xi, w)$ or $\mathcal{S}^c(\xi, w)$ (if nonempty) identifies it with $H^1(X;\mathbb{Z}_2)$ or $H^2(X;\mathbb{Z})$.

Proof The first two sentences are immediate from obstruction theory, since the fiber of $B\mathrm{spin}(H, w) \to BH$ is $K(\mathbb{Z}_2, 1)$. In fact, $w_2(\xi, \tau)$ is the obstruction to lifting f to a map $\hat{f}: X \to B\mathrm{spin}(H, w)$ with $\hat{f}|Y$ constant, as can be seen by first considering the case where Y contains the 1–skeleton of X. Similarly, $H^2(X;\mathbb{Z})$ acts as required on $\mathcal{S}^c(\xi, w)$ (when nonempty) via difference classes, since the fiber of p_H is $K(\mathbb{Z}, 2)$. Now recall that $B\mathrm{spin}^c(H, w) = B\mathrm{spin}(H^\times, w + w^*)$ with $BH^\times = BH \times BSO(2)$. Thus, a lift of f to $\hat{f}: X \to B\mathrm{spin}^c(H, w)$ with $\hat{f}|Y$ constant is the same as a choice of complex line bundle $L \to X$ with a trivialization τ_L over Y, together with a spin structure on the bundle $\xi \times L \to X$ (classified by $BH \times BSO(2)$) whose defining lift $\hat{f}$ is constant on Y. The resulting spin^c–structure s with trivialization $\hat{\tau}$ over τ will satisfy $c_1(s, \hat{\tau}) = c_1(L, \tau_L)$, since $p_{SO(2)} \circ \hat{f}$ is the classifying map of L. Such a structure exists if and only if $0 = w_2(\xi \times L, \tau \times \tau_L) = w_2(\xi, \tau) + w_2(L, \tau_L)$, or equivalently $w_2(\xi, \tau) = w_2(L, \tau_L) = c_1(L, \tau_L)|_2$. Thus, $\mathcal{S}^c(\xi, w)$ is nonempty if and only if $w_2(\xi)$ has a lift to $H^2(X;\mathbb{Z})$, ie $W_3(\xi) = 0$, and any $c_1(s, \hat{\tau})$ reduces mod 2 to $w_2(\xi, \tau)$. Given $s, s' \in \mathcal{S}^c(\xi, w)$, the difference class $d(s, s')$ takes coefficients in $\pi_2(K(\mathbb{Z}, 2))$, where $K(\mathbb{Z}, 2)$ is the fiber of p_H. Since $(p_{SO(2)})_*: \pi_2(K(\mathbb{Z}, 2)) \to \pi_2(BSO(2))$ is multiplication by 2, we have $2d(s, s') = c_1(s') - c_1(s)$. Equivalently, $c_1(s + a) = c_1(s) + 2a$ for $a = d(s, s')$. The assertion about conjugation is clear from the way it lifts to $B\mathrm{spin}^c(H, w)$. □

Now suppose we are given pairs (BH, w) and (BH', w') as before, and a map $h: BH \to BH'$ covered by a bundle map $\tilde{h}: \xi_H \to \xi_{H'}$, with $h^* w' = w$. Then any H–bundle $\xi \to X$ determines an H'–bundle $\xi' \to X$ with the same w_2 and W_3, and h determines maps $B\mathrm{spin}(H, w) \to B\mathrm{spin}(H', w')$ and $B\mathrm{spin}^c(H, w) \to B\mathrm{spin}^c(H', w')$. We obtain canonical equivariant identifications $\mathcal{S}(\xi, w) \cong \mathcal{S}(\xi', w')$ and $\mathcal{S}^c(\xi, w) \cong \mathcal{S}^c(\xi', w')$, and the latter preserves Chern classes and conjugation. On the other hand, given an H–bundle map $\tilde{g}: \xi_1 \to \xi_2$ covering $g: X_1 \to X_2$, we have induced maps $g^*: \mathcal{S}(\xi_2, w) \to \mathcal{S}(\xi_1, w)$ and $g^*: \mathcal{S}^c(\xi_2, w) \to \mathcal{S}^c(\xi_1, w)$ that are equivariantly equivalent to $g^*: H^1(X_2;\mathbb{Z}_2) \to H^1(X_1;\mathbb{Z}_2)$ and $g^*: H^2(X_2;\mathbb{Z}) \to H^2(X_1;\mathbb{Z})$ when the domains are nonempty, and characteristic classes and conjugation are preserved in the obvious way. If g is a homotopy equivalence, then the maps g^* are isomorphisms.

Examples 2 (a) If $h\colon BSO(m) \to BSO(n)$, $2 \le m \le n$, is induced by the usual inclusion of groups, we recover the stabilization-invariance of classical spin and spin^c–structures. We are free to pass to the limiting group SO, eliminating the dependence on n.

(b) An oriented topological n–manifold X has the homotopy type of a CW–complex, and it has a tangent bundle classified by a map into the universal bundle over $BSTOP(n)$ (eg [3]). There is a canonical map $h\colon BSO(n) \to BSTOP(n)$ that corresponds to interpreting $\xi_{SO(n)}$ as a topological bundle and is a π_2–isomorphism of simply connected spaces. We immediately obtain a theory of spin and spin^c–structures on oriented topological manifolds by using their tangent bundles (stabilized if $n < 2$). As before, the theory is stabilization-invariant, and we can pass to the limiting case of $BSTOP$. On smooth manifolds, the new theory canonically reduces via h to the classical theory. However, any orientation-preserving homeomorphism $g\colon X_1 \to X_2$ induces an isomorphism of topological tangent bundles, hence, isomorphisms $g^*\colon \mathcal{S}(X_2) \cong \mathcal{S}(X_1)$ and $g^*\colon \mathcal{S}^c(X_2) \cong \mathcal{S}^c(X_1)$ as above.

To generalize to homotopy equivalences, we need one further construction. Suppose we are given a bundle map

$$\begin{array}{ccc} \xi_H \times \xi_{H'} & \xrightarrow{\tilde{k}} & \xi_{H''} \\ \downarrow & & \downarrow \\ BH \times BH' & \xrightarrow{k} & BH'' \end{array}$$

with $k^*(w'') = w + w'$. Then a pair of bundles $\xi, \xi' \to X$ classified by BH, BH' determine an H''–bundle $\xi'' \to X$, and w_2 and W_3 add.

Proposition 3 *A trivialization of ξ'' induces equivariant isomorphisms $k_*\colon \mathcal{S}(\xi, w) \to \mathcal{S}(\xi', w')$ and $k_*\colon \mathcal{S}^c(\xi, w) \to \mathcal{S}^c(\xi', w')$, and the latter preserves conjugation and Chern classes.*

Proof By obstruction theory, the map k uniquely determines a map $\hat{k}$ making the diagram

$$\begin{array}{ccc} B\mathrm{spin}(H, w) \times B\mathrm{spin}(H', w') & \xrightarrow{\hat{k}} & B\mathrm{spin}(H'', w'') \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_2} \\ BH \times BH' & \xrightarrow{k} & BH'' \end{array}$$

commute, and a similar diagram is induced for spin^c via the map $k \times k_0$, where $k_0\colon BSO(2) \times BSO(2) \to BSO(2)$ induces addition on π_2. The diagrams determine a map $k_\#\colon \mathcal{S}(\xi,w)\times\mathcal{S}(\xi',w')\to\mathcal{S}(\xi'',w'')$ and similarly for $\mathcal{S}^c$. In the latter case, $k_\#$ commutes with conjugation and adds Chern classes. In either case, $\hat{k}$ restricts to addition on the homotopy groups of the fibers of p_1 and p_2, so difference classes add under $k_\#$, and for suitably chosen base points $k_\#$ is given by addition on $H^1(X;\mathbb{Z}_2)$ or $H^2(X;\mathbb{Z})$ whenever its domain is nonempty. Now a trivialization of ξ'' determines a trivial spin^c–structure $s''\in\mathcal{S}^c(\xi'',w'')$. Since $W_3(\xi)+W_3(\xi')=W_3(\xi'')=0$, it follows that $\mathcal{S}^c(\xi,w)$ is nonempty if and only if $\mathcal{S}^c(\xi',w')$ is. For each $s\in\mathcal{S}^c(\xi,w)$ there is a unique "inverse" $s'\in\mathcal{S}^c(\xi',w')$ with $k_\#(s,s')=s''$. Let $k_*(s)$ equal the conjugate of s'. Then $k_*\colon\mathcal{S}^c(\xi,w)\to\mathcal{S}^c(\xi',w')$ is an equivariant isomorphism, and it preserves conjugation and Chern classes since s'' is conjugation-invariant with $c_1(s'')=0$. A similar procedure (with $k_*(s)=s'$) works for spin structures. □

Example 4 Any oriented, smooth n–manifold X admits a unique isotopy class of proper embeddings in $\mathbb{R}^N$ for N sufficiently large. This determines a normal bundle νX that is unique up to stabilization. Since the tangent bundle τX satisfies $\tau X\oplus\nu X=\tau\mathbb{R}^N|X$ and the latter bundle is canonically trivial, the obvious map $BSO(n)\times BSO(N-n)\to BSO(N)$ determines canonical equivariant identifications $\mathcal{S}(\nu X,w^*)\cong\mathcal{S}(\tau X,w^*)$ and $\mathcal{S}^c(\nu X,w^*)\cong\mathcal{S}^c(\tau X,w^*)$, the latter preserving Chern classes and conjugation.

Theorem 5 *Let $(X,\partial X)$ be an oriented, possibly noncompact Poincaré pair. There is a canonical procedure for defining sets $\mathcal{S}(X)$ and $\mathcal{S}^c(X)$ of spin and spin^c–structures on X having the structure described in Proposition 1 (with respect to the usual classes $w_2(X)$ and $W_3(X)$). For $(X,\partial X)$ a smooth manifold, the theory is canonically equivariantly equivalent to the standard one (preserving Chern classes and conjugation). For pairs $(X_i,\partial X_i)$ as above, any orientation-preserving, pairwise, proper homotopy equivalence $g\colon(X_1,\partial X_1)\to(X_2,\partial X_2)$ induces equivariant isomorphisms $g^*\colon\mathcal{S}(X_2)\cong\mathcal{S}(X_1)$ and $g^*\colon\mathcal{S}^c(X_2)\cong\mathcal{S}^c(X_1)$, the latter preserving Chern classes and conjugation, and the construction is functorial for such maps g.*

Proof The pair $(X,\partial X)$ has a canonical *Spivak normal fibration* [7] defined by embedding $(X,\partial X)$ pairwise and properly in half-space $\mathbb{R}^N\times([0,\infty),\{0\})$ (uniquely for N sufficiently large), and making a fibration out of the collapsing map of the boundary of a regular neighborhood. The resulting oriented spherical fibration over X is classified by a fiber-preserving map into the universal spherical fibration, whose base space stabilizes to BSG. As in Example 2(b), there is a canonical map $h\colon BSO\to BSG$ induced by the spherical fibrations $\xi_{SO(n)}-(0\text{–section})$, and h is a π_2–isomorphism of simply connected spaces. We immediately obtain $\mathcal{S}(X)$, $\mathcal{S}^c(X)$ and characteristic classes satisfying Proposition 1, using the Spivak fibration and BSG. (The resulting classes $w_2(X)$ and $W_3(X)$ are well known.) For $(X, \partial X)$ a smooth manifold, the theory is canonically equivalent (via h) to that of the stable normal bundle, which is the usual theory over the tangent bundle by Example 4. A homotopy equivalence g as above induces a fiber-preserving map of the corresponding Spivak fibrations, and hence, the required maps g^*. □

The map $g^*\colon \mathcal{S}^c(X_2) \cong \mathcal{S}^c(X_1)$ induced by a homotopy equivalence can frequently be computed explicitly. We consider the case where X_2 contains a 1–dimensional subcomplex with a regular neighborhood N_2 that is a manifold, such that $H^2(X_2, N_2; \mathbb{Z})$ has no 2–torsion. We also assume that $g\colon X_1 \to X_2$ restricts to a homeomorphism from $N_1 = g^{-1}(N_2)$ to N_2. These conditions are always satisfied if g is a homeomorphism between smooth manifolds, for example by taking N_2 to be a neighborhood of the 1–skeleton of X_2. Now the map $g^*\colon H^*(X_2, N_2) \cong H^*(X_1, N_1)$ is an isomorphism. A (stable) trivialization τ_2 of the tangent bundle of N_2 (or equivalently, of the stable normal bundle) pulls back via $g|N_1$ to a trivialization τ_1 over N_1, and $g^* w_2(X_2, \tau_2) = w_2(X_1, \tau_1)$. Given spin^c–structures $s_i \in \mathcal{S}^c(X_i)$, pick any trivializations $\hat{\tau}_i$ of $s_i|N_i$ over τ_i. Then by Proposition 1, $g^* c_1(s_2, \hat{\tau}_2) - c_1(s_1, \hat{\tau}_1)$ reduces to zero mod 2. Since $H^2(X_1, N_1; \mathbb{Z})$ has no 2–torsion, there is a unique class $\delta(s_1, s_2) \in H^2(X_1, N_1; \mathbb{Z})$ with $2\delta(s_1, s_2) = g^* c_1(s_2, \hat{\tau}_2) - c_1(s_1, \hat{\tau}_1)$. If we change $\hat{\tau}_i$ with τ_i fixed, then $\delta(s_1, s_2)$ changes by the coboundary of a cochain in N_1, so it represents a class $d(s_1, s_2) \in H^2(X_1; \mathbb{Z})$ that depends only on s_1 and s_2 (τ_i fixed). But $\delta(s_1, s_2)$ vanishes for $s_1 = g^* s_2$ and $\hat{\tau}_1$ given by pulling back $\hat{\tau}_2$, and a change of s_i changes $2\delta(s_1, s_2)$ by twice the corresponding relative difference class (by the addition formula of Proposition 1 applied to X_i/N_i). Thus, $d(s_1, s_2)$ is precisely the difference class $d(s_1, g^* s_2)$, in a form accessible to computation.

Remarks (a) Spin^c–structures have several other convenient characterizations. As we observed in proving Proposition 1, a spin^c–structure on $\xi \to X$ is the same as a line bundle L and spin structure on $\xi \times L \to X$. For a different approach, recall that Milnor [5] observed that a spin structure on an oriented vector bundle over a CW–complex is equivalent (after stabilizing if necessary) to a trivialization over the 1–skeleton that can be extended over the 2–skeleton, just as an orientation is a trivialization over the 0–skeleton that extends over the 1–skeleton. Similarly, a spin^c–structure over an oriented vector bundle is equivalent (after stabilizing if the fiber dimension is odd or ≤ 2) to a complex structure over the 2–skeleton that can be extended over the 3–skeleton. To see this, observe that the map of classifying spaces induced by inclusion $i\colon U(n) \to SO(2n)$

lifts canonically to a map $j\colon BU(n) \to B\mathrm{spin}^c(SO(2n), w^*)$ by first lifting the map $id \times B\det : BU(n) \to BU(n) \times BSO(2)$ to $B\mathrm{spin}^c(U(n), i^*w^*)$. (In fact, the corresponding diagram exists on the group level.) Thus, any complex structure determines a spin^c–structure (and the correspondence preserves c_1 and conjugation). For $n \geq 2$, this correspondence is bijective for 2–complexes and surjective for 3-complexes, since the map j has a 2–connected fiber. The observation now follows from the fact that restriction induces a bijection from spin^c–structures to those over the 2–skeleton extending over the 3–skeleton. The same remark applies to bundles classified by $BSTOP$ or BSG if we define a complex structure to be a lift of the classifying map to BU.

(b) The Wu relations are known to hold for Poincaré complexes. In particular, for a compact, oriented 4–dimensional Poincaré complex X (without boundary) we have $w_2(X) \cup x = x \cup x$ for all $x \in H^2(X; \mathbb{Z}_2)$. The usual argument [2] then shows that $W_3(X) = 0$, so all such complexes admit spin^c–structures.

(c) As in the classical case, we have a canonical map $i : B\mathrm{spin}(H, w) \to B\mathrm{spin}^c(H, w)$ as the fiber of $p_{SO(2)}$ (induced by inclusion of groups), inducing a map $\alpha\colon \mathcal{S}(\xi, w) \to \mathcal{S}^c(\xi, w)$ that is equivariantly equivalent (when the domain is nonempty) to the Bockstein homomorphism $\beta\colon H^1(X; \mathbb{Z}_2) \to H^2(X; \mathbb{Z})$. The image $\mathrm{Im}\ \alpha$ is the set of spin^c–structures with $c_1 = 0$, or equivalently, the set of conjugation-invariant structures. To verify that α has the stated equivariance and image, note that we can either consider i to be an inclusion into the fixed set of conjugation or replace it by a fibration p. Over each point in BH, i and p will restrict to the canonical inclusion and fibration $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$, respectively, both of which represent the unique nontrivial homotopy class of maps in $[\mathbb{R}P^\infty, \mathbb{C}P^\infty]$. For a fixed classifying map $\tilde{f}\colon \xi \to \xi_H$, spin structures $s_1, s_2 \in \mathcal{S}(\xi, w)$ determine lifts $\hat{f}_1, \hat{f}_2\colon X \to B\mathrm{spin}(H, w)$. We can assume that these agree over the 0–skeleton and that $p \circ \hat{f}_1$, $p \circ \hat{f}_2$ agree over the 1–skeleton, giving us obstruction cochains $d(s_1, s_2) \in C^1(X; \mathbb{Z}_2)$ and $d(\alpha s_1, \alpha s_2) \in C^2(X; \mathbb{Z})$. Now $d(\alpha s_1, \alpha s_2)$ evaluated on a 2–cell c is the element of $\pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ given by $p \circ \hat{f}_2(c) - p \circ \hat{f}_1(c)$. Since the boundary operator $\pi_2(\mathbb{C}P^\infty) \to \pi_1(\mathbb{R}P^1)$ of p is multiplication by 2, the same coefficient is obtained as $\frac{1}{2}\langle d(s_1, s_2), \partial c\rangle = \langle \beta d(s_1, s_2), c\rangle$. Thus, we obtain the required equivariance $d(\alpha s_1, \alpha s_2) = \beta d(s_1, s_2)$. To compute $\mathrm{Im}\ \alpha$, first note that any $s \in \mathrm{Im}\ \alpha$ is conjugation-invariant (since i is) with $c_1 = 0$. If $\mathcal{S}(\xi, w)$ is nonempty, fix $s \in \mathrm{Im}\ \alpha$ and let s' be any spin^c–structure that either is conjugation-invariant or satisfies $c_1(s') = 0$. By Proposition 1, $2d(s, s') = 0$, so $d(s, s') \in \mathrm{Im}\ \beta$ and $s' \in \mathrm{Im}\ \alpha$. It now suffices to show that when $\mathcal{S}(\xi, w)$ is empty, no spin^c–structure has $c_1 = 0$ or is conjugation-invariant. The first assertion is obvious since $c_1|_2 = w_2 \neq 0$. For the remaining assertion, choose $s \in \mathcal{S}^c(\xi, w)$ with conjugate $\bar{s}$. Since $\pi_1(\mathbb{C}P^\infty, \mathbb{R}P^\infty) = 0$, we can assume that the lift $\hat{f}\colon X \to B\mathrm{spin}^c(H,w)$ determined by s maps the 1–skeleton X_1 into $i(B\mathrm{spin}(H,w))$, which is fixed by conjugation. Thus, $\hat{f}$ and its conjugate determine a difference cochain $d(s,\bar{s}) \in C^2(X,\mathbb{Z})$. Since $\pi_2(\mathbb{C}P^\infty) \to \pi_2(\mathbb{C}P^\infty,\mathbb{R}P^\infty)$ is multiplication by 2 on $\mathbb{Z}$, we can change $d(s,\bar{s})$ by any coboundary by changing $\hat{f}|X_1\colon X_1 \to i(B\mathrm{spin}(H,w))$. Thus, if $s=\bar{s}$ we can assume that $d(s,\bar{s})=0$, so over each 2–cell, $\hat{f}$ is conjugation-invariant up to homotopy rel ∂. But conjugation fixes only 0 in $\pi_2(\mathbb{C}P^\infty,\mathbb{R}P^\infty)$, so $\hat{f}$ can then be homotoped into $i(B\mathrm{spin}(H,w))$, ie $s \in \mathrm{Im}\,\alpha$.

References

[1] **S Donaldson**, **P Kronheimer**, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK (1990)

[2] **F Hirzebruch**, **H Hopf**, *Felder von Flächenelementen in 4–dimensionalen Mannigfaltigkeiten*, Math. Ann. 136 (1958) 156–172

[3] **R Kirby**, **L Siebenmann**, *Foundational essays on topological manifolds, smoothings and triangulations*, Ann. Math. Studies 88, Princeton Univ. Press, Princeton, NJ (1977)

[4] **J Milnor**, *On spaces having the homotopy type of a CW–complex*, Trans. Amer. Math. Soc. 90 (1959) 272–280

[5] **J Milnor**, *Spin structures on manifolds*, L'Ensignement Math. 8 (1962) 198–203

[6] **J Morgan**, *The Seiberg–Witten equations and applications to the topology of smooth 4–manifolds*, Princeton Univ. Press, Princeton, NJ (1996)

[7] **M Spivak**, *Spaces satisfying Poincaré duality*, Topology 6 (1967) 77–102

[8] **G Whitehead**, *Elements of homotopy theory*, Graduate Texts in Math. 61, Springer-Verlag, Berlin (1978)