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An Invariant of Smooth 4–Manifolds

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Abstract

We define a diffeomorphism invariant of smooth 4–manifolds which we can estimate for many smoothings of $\mathbb{R}^4$ and other smooth 4–manifolds. Using this invariant we can show that uncountably many smoothings of $\mathbb{R}^4$ support no Stein structure. Gompf [11] constructed uncountably many smoothings of $\mathbb{R}^4$ which do support Stein structures. Other applications of this invariant are given.

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1 An invariant of smooth 4–manifolds

We define an invariant of a smooth 4–manifold M, denoted $\gamma(M)$, by measuring the complexity of the smooth $\mathbb{R}^4$'s which embed in it. In our applications, we will have a smooth 4–manifold M and an *exhaustion* of M by smooth submanifolds: ie a collection of smooth submanifolds $W_0 \subset \cdots \subset W_{i-1} \subset \overset{\circ}{W}_i \subset W_i \subset \cdots \subset M$ with $M = \cup W_i$. Here $\overset{\circ}{W}_i$ denotes the interior of W_i. We will need to estimate $\gamma(M)$ from the $\gamma(W_i)$, specifically we want $\gamma(M) \leq \max_i\{\gamma(W_i)\}$. The definition has several steps.

First, given a smoothing of $\mathbb{R}^4$, E, define b_E, a non-negative integer or ∞, as follows. Let $\mathcal{S}p(E)$ denote the collection of smooth, compact, Spin manifolds N without boundary with hyperbolic intersection form such that E embeds smoothly in N. Define $b_E = \infty$ if $\mathcal{S}p(E) = \emptyset$. Otherwise define b_E so that

$$2b_E = \min_{N \in \mathcal{S}p(E)} \{\beta_2(N)\}$$

where β_2 denotes the second Betti number.

While b_E measures the complexity of a smooth $\mathbb{R}^4$, our ignorance of the properties of smoothings of $\mathbb{R}^4$ prevents us from using this invariant directly. For example, we can not rule out the possibility that there is a smoothing of $\mathbb{R}^4$, E_ω with the property that all its compact subsets embed in S^4 but $b_{E_\omega} > 0$. If we were to define $\gamma(M)$ as below using b_E directly, our inequality in the first paragraph would be violated. Hence we proceed as follows. For any smooth 4–manifold M, let $\mathcal{E}(M)$ denote the set of topological embeddings, $e\colon D^4 \to M$ satisfying two additional conditions. First we require $e(\partial D^4)$ to be bicollared and second we require the existence of a point $p \in \partial D^4$ so that e restricted to a neighborhood of p is smooth. These two conditions are introduced to make the proofs which follow work more smoothly. The smooth structure on M induces via e a smooth structure on the interior of D^4, denoted $\mathbb{R}^4_e$. Define $b_e = b_{\mathbb{R}^4_e}$. Define

$$\gamma(E) = \max_{e \in \mathcal{E}(E)} \{\, b_e \,\}$$

If M is Spin, define $\gamma(M)$ to be the maximum of $\gamma(E)$ where $E \subset M$ is an open subset homeomorphic to $\mathbb{R}^4$. Note $\gamma(M)$ also is the maximum of b_e for $e \in \mathcal{E}(M)$. If M is orientable but not Spin and has no compact dual to w_2, set $\gamma(M) = -\infty$. If there are compact duals to w_2, then define $\gamma(M)$ to be the maximum of $\gamma(M - F) - \dim_{\mathbf{F}_2} H_1(F^2; \mathbf{F}_2)$ where F runs over all smooth, compact surfaces in M which are dual to w_2. If M is not orientable, let $\gamma(M) = \gamma(\widetilde{M})$ where $\widetilde{M}$ denotes the orientable double cover of M. The non-oriented case will not be mentioned further.

Clearly $\gamma(M)$ is a diffeomorphism invariant. If M_2 is Spin and if $M_1 \subset M_2$, or even if every compact smooth submanifold of M_1 smoothly embeds in M_2, then clearly $\gamma(M_1) \leq \gamma(M_2)$. If M_2 is not Spin then such a result is false. Example 5.10 says $\gamma(CP^2) = 0$ and there are smoothings of Euclidean space, E with $E \subset CP^2$ and $\gamma(E)$ arbitrarily large.

Note M has no compact duals to w_2, if and only if $\gamma(M) = -\infty$. If M has a compact dual to w_2, say F, then embedding a standard smooth disk in $M - F$ shows that $\gamma(M) \geq -\dim_{\mathbf{F}_2} H_1(F; \mathbf{F}_2)$. In particular, if M is Spin, then $\gamma(M) \geq 0$. The proof of the inequality $\gamma(M) \leq \max_i\{\gamma(W_i)\}$ will be left to the reader. If M is Spin this inequality is an equality.

The precise set of values assumed by γ is not known. One extreme, $-\infty$, is assumed, but no example with $-\infty < \gamma(M) < 0$ is known. Turning to the non-negative part, ∞ is assumed and it follows from Furuta's work that any non-negative integer is assumed. Adding Taubes's work to Furuta's, there are uncountably many distinct smoothings of $\mathbb{R}^4$ with $\gamma(E) = n$ for each integer $0 \leq n < \infty$. (The case $n = 0$ uses work of DeMichelis and Freedman [3].) The referee has noticed that ∞ is assumed uncountably often.

We now turn to some applications of the invariant. In Section 4 we will estimate $\gamma(M)$ under a condition that is implied by the existence of a handlebody structure with no 3–handles. It is a theorem that Stein 4–manifolds have such a handlebody decomposition. Some version of this theorem goes back to Lefchetz with further work by Serre and Andreotti–Frankel. There is an excellent exposition of the Andreotti–Frankel theorem in Milnor [17, Section 7 pages 39–40]. It follows from Theorem 4.3 that if M is a Stein 4–manifold, $\gamma(M) \leq \beta_2(M)$. In particular, $\gamma(M) = 0$ if M is a Stein manifold homeomorphic to $\mathbb{R}^4$, Example 4.4. Since there are uncountably many smoothings of $\mathbb{R}^4$ with $\gamma > 0$, Corollary 5.5, there are uncountably many smoothings of $\mathbb{R}^4$ which support no Stein structure. In contrast, Gompf [11] has constructed uncountably many smoothings of $\mathbb{R}^4$ which do support Stein structures. They all embed smoothly in the standard $\mathbb{R}^4$ and hence have $\gamma = 0$ as required. These remarks represent some progress on Problem 4.78 [15].

The invariant can be used to show some manifolds can not be a non-trivial cover of any manifold. For example, if E is a smoothing of $\mathbb{R}^4$ and is a non-trivial cover of some other smooth manifold then, for $r \geq 2$, $\gamma(\natural^r E) = \gamma(E)$, where $\natural^r E$ denotes the end-connected sum, Definition 2.1, of r copies of E. By Example 5.6, for each integer $i > 0$ there are smoothings E_i of $\mathbb{R}^4$ with $\gamma(\natural^r E_i) > 2r\gamma(E_i)/3$, so these manifolds are not covers of any smooth 4–manifold. Gompf [15, Problem 4.79A] asks for smoothings of $\mathbb{R}^4$ which cover compact smooth manifolds. These E_i are ruled out, perhaps the first such examples known not to cover.

Another use of the invariant is to construct countably many distinct smoothings on various non-compact 4–manifolds. The genesis of the idea goes back to Gompf [10] and has recently been employed again by Bižaca and Etnyre [1]. In the present incarnation of the idea, one constructs a smoothing, M, with $\gamma(M)=\infty$ and an exhaustion of M by manifolds $M(\rho)$, $0\leq\rho<\infty$, homeomorphic to M. One then proves $\gamma(M(\rho_1))\leq\gamma(M(\rho_2))<\infty$ for all $0\leq\rho_1<\rho_2<\infty$ and $\lim_{\rho\to\infty}\gamma(M(\rho))=\gamma(M)$. It follows that $\gamma(M(\rho))$ takes on infinitely many values. Some hypotheses are necessary: see Theorem 6.4 for details. In particular, Example 3.6, there exists a Stein 4–manifold M with $H_1(M;\mathbb{Z})=\mathbb{Q}$ and $H_2(M;\mathbb{Z})=0$ and this manifold has at least countably many smoothings. This may be the first example of a non-compact, connected manifold with many smoothings and no topological collar structure on any end. For a non-compact, orientable 4–manifold M with no 3 or 4–handles and with finitely generated but infinite $H^2(M;\mathbb{Z})$, Gompf has a second construction of countably many distinct smoothings [12, after Theorem 3.1]. The precise relationship between Gompf's smoothings and ours is not clear.

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2 Some properties of the invariant

The main result in this section is that $\gamma(M)=\gamma(M-K)$ for K a 1–complex, Proposition 2.5 below. Along the way, we show that $e(D^4)$ can be engulfed in a manifold with only zero, one and two handles for any $e\in\mathcal{E}(M)$. These results follow from some nice properties of embeddings in $\mathcal{E}(M)$.

Our first result introduces terminology used later. If E is a smoothing of $\mathbb{R}^4$ and if $K\subset E$ is compact, then there exist $e\in\mathcal{E}(E)$ with $K\subset e(\mathring{D}^4)$. This follows from Freedman's work [7] which says that given any smoothing of $\mathbb{R}^4$, say E, there is a homeomorphism $h\colon\mathbb{R}^4\to E$ which is smooth almost everywhere. In particular, h of any standard ball in $\mathbb{R}^4$ is an element of $\mathcal{E}(E)$. We say that some property of smoothings of $\mathbb{R}^4$ holds *for all sufficiently large balls* in E provided there is some compact set K such that for every $e\in\mathcal{E}(E)$ with $K\subset e(\mathring{D}^4)$, $\mathbb{R}^4_e$ is a smoothing of $\mathbb{R}^4$ with this same property.

To describe our next result, we recall a standard definition.

Definition 2.1 Given two non-compact smooth 4–manifolds M_1 and M_2, define the *end-connected sum* of M_1 and M_2 as follows [10]. First take smooth embeddings of $[0,\infty)$ into each M_i. Thicken each ray up to a tubular neighborhood. These tubular neighborhoods are diffeomorphic to $(\mathbb{R}^4_+, \mathbb{R}^3)$, where $\mathbb{R}^4_+$ denotes the standard half space, $\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_4 \geq 0\}$. The end-connected sum, denoted $M_1 \natural M_2$, is obtained by removing the interiors of these tubular neighborhoods and gluing the two $\mathbb{R}^3$'s together by an orientation reversing diffeomorphism. It is well defined up to diffeomorphism once rays are chosen in M_1 and M_2, but we suppress this dependence in our notation. A ray determines an end in M_i and and it follows from a result of Wall [19] that properly homotopic rays are isotopic. Hence the end-connected sum depends only on the proper homotopy class of the chosen rays.

Elements in $\mathcal{E}(M)$ behave well with respect to end-connected sum. We will use the bicollaring of the boundary of elements in $\mathcal{E}(M)$ several times, but here is the only place that the smooth point on the boundary is really useful. One could have defined the invariant without requiring the smooth point on the boundary and then use Freedman as above to modify the proof below. Of course it also follows from Freedman's work that this other invariant is the same as the one defined.

Proposition 2.2 *Let M be connected and let $e_i \in \mathcal{E}(M)$, $i = 1, \ldots, T$ for some finite integer $T > 1$. If $e_i(D^4) \cap e_j(D^4) = \emptyset$ for $i \neq j$ then there is an element $e \in \mathcal{E}(M)$ such that $\mathring{D}^4_e$ is the end-connected sum of all the $\mathring{D}^4_{e_i}$. If there is a surface $F \subset M$ such that $e_i(D^4) \cap F = \emptyset$, $i = 1, \ldots, T$, then e can be chosen so that $e(D^4) \cap F = \emptyset$.*

Proof One uses the smooth points in ∂D^4 to connect the various $e_i(D^4)$ by thickening up arcs. See Gompf [10] or [14, page 96] for more details. □

Next we show that elements in $\mathcal{E}(M)$ can be engulfed in handlebodies with no 3 or 4–handles.

Proposition 2.3 *Let M be a smooth 4–manifold and let $e \in \mathcal{E}(M)$. Then there exists a smooth compact submanifold, $V \subset M$ such that $e(D^4) \subset V$ and V has a handlebody decomposition with no 3 or 4–handles.*

Proof Let $W \subset M$ be a smooth, codimension 0 submanifold with $e(D^4)$ in its interior. We may assume W is connected and hence has a handlebody decomposition with only one 0–handle and no 4–handles. Consider the handlebody decomposition beginning with ∂W. Since $\pi_1\big(W - e(D^4)\big) = \pi_1(W)$, there are smoothly embedded compact 1–manifolds in $W - e(D^4)$ which are homotopic to the cores of the 1–handles. Wall [19] shows how to do an isotopy to bring the cores of the 1–handles into $W - e(D^4)$. Let W_2 denote the 0, 1 and 2–handles building from the other direction. Then after the isotopy, $e(D^4)$ is contained in the interior of W_2. □

General position arguments now yield

Proposition 2.4 *Let $K \subset M^4$ be a PL proper embedding of a locally-finite complex of dimension ≤ 1. Then $\gamma(M) = \gamma(M - K)$.*

Proof If $F \subset M$ is a 2–manifold, then we can do an isotopy to get K and F separated. Moreover, $F \subset M - K$ is dual to w_2 if and only if F is dual in M to w_2. Hence we need only consider $F \subset M - K$ dual to w_2 in M.

Now $\gamma(M) = -\infty$ if and only if $\gamma(M - K) = -\infty$. Hereafter, assume neither is. Fix $e \in \mathcal{E}(M - F)$. Use Proposition 2.3 to get a compact codimension 0 submanifold, V, with $e(D^4) \subset \overset{\circ}{V}$ and $V \subset M - F$ such that V has no 3 or 4–handles. Now do an isotopy to move V off of K. This shows that there is an $e' \in \mathcal{E}(M - K - F)$ with $b_e = b_{e'}$. □

3 Few essential 3–handles

We say that a smooth manifold M has *few essential k–handles* provided M has an exhaustion, $W_0 \subset W_1 \subset \cdots$ where each W_i is a compact, codimension 0, smooth submanifold such that $H_k(W_{i+1}, W_i; \mathbb{Z}) = 0$ for each $i \geq 0$.

Remarks 3.1 By excision, $H_k(W_{i+1}, W_i; \mathbb{Z}) = H_k(\overline{W_{i+1} - W_i}, \partial W_i; \mathbb{Z})$ so if each pair $(\overline{W_{i+1} - W_i}, \partial W_i)$ has a handlebody decomposition with no k–handles then M has few essential k–handles. We say *few* essential k–handles, because $H_k(W_0; \mathbb{Z})$ may be non-zero.

In the next section, we will use a "few essential 3–handles" hypothesis to estimate the invariant γ. The key remark needed is the next result.

Proposition 3.2 *Suppose M^4 is smooth, orientable, connected, non-compact and has few essential 3–handles. Then $\beta_2(W_i) \leq \beta_2(M)$ for all $i \geq 0$.*

Proof By the Universal Coefficients Theorem, $H_3(W_i, W_{i-1}; \mathbb{Q}) = 0$. (Note by Poincaré duality $H_3(W_i, W_{i-1}; \mathbb{Q}) = 0$ implies $H_3(W_i, W_{i-1}; \mathbb{Z}) = 0$ as well, so we get no better result by only requiring $H_3(W_i, W_{i-1}; \mathbb{Q}) = 0$ in our definition of few essential 3–handles.) It follows from induction and the long exact sequence of the triple $(W_{i+j}, W_{i+j-1}, W_i)$ that $H_3(W_{i+j}, W_i; \mathbb{Q}) = 0$ for all $i \geq 0$ and $j > 0$. Letting j go to ∞, we see $H_3(M, W_i; \mathbb{Q}) = 0$. Hence $H_2(W_i; \mathbb{Q}) \to H_2(M; \mathbb{Q})$ is injective for all $i \geq 0$. □

The rest of this section is devoted to examples. The first two use Remarks 3.1.

Example 3.3 For any connected, non-compact 3–manifold M^3, $M^3 \times \mathbb{R}$ has few essential 3–handles. Indeed, find a handlebody decomposition of M with no 3–handles and this gives an evident handlebody decomposition of $M \times \mathbb{R}$ with no 3–handles.

Example 3.4 If M^4 is a smooth manifold with a Stein structure, then M has few essential 3–handles. As remarked in Section 1, a theorem of Andreotti and Frankel [17, Section 7 pages 39–40] shows that there is a proper Morse function, $f\colon M \to [0, \infty)$ with no critical points of index 3 or 4.

Gompf [11] proves a partial converse: if there is a proper Morse function $f\colon M \to [0, \infty)$ with no critical points of index 3 or 4 then there is a topological embedding of M inside itself so that the induced smoothing on M supports a Stein structure. Gompf's result leads to a full characterization of the homotopy type of Stein manifolds.

Theorem 3.5 *Any Stein 4–manifold has the homotopy type of a countable, locally-finite CW complex of dimension ≤ 2. Conversely, any countable, locally-finite CW complex of dimension ≤ 2 is the homotopy type of a Stein 4–manifold.*

Proof Any manifold has the homotopy type of the complex built from a Morse function, so the Andreotti and Frankel result shows Stein manifolds have the homotopy type of countable, locally-finite CW complexes of dimension ≤ 2.

Conversely, given any countable, locally-finite CW complex of dimension ≤ 2, there is a smooth orientable 4–manifold with a Morse function with critical points of index ≤ 2 of the same homotopy type. By Gompf, some smoothing of this manifold supports a Stein structure. □

We can now use standard homotopy-theoretic constructions to produce examples of manifolds with few essential 3–handles.

Example 3.6 Let G be any countable group. Then G has a countable presentation: there are generators x_g, one for each element $g \in G$ and relations $r_{g,h} = x_g x_h (x_{gh})^{-1}$, one for each element $(g, h) \in G \times G$. Put the relations in bijection with the positive integers. Choose new generators $X_{g,i}$, one for each $g \in G$ and each integer $i > 0$. Define new relations, $X_{g,i} X_{g,i+1}^{-1}$ and $R_i = X_{g,i} X_{h,i} (X_{gh,i})^{-1}$ where the i–th relation in the first presentation is

$r_{g,h}$. Use this second presentation to construct a connected, countable, locally-finite CW complex of dimension 2 with $\pi_1 \cong G$. From Theorem 3.5, there is a Stein manifold with $\pi_1 \cong G$. One can select a set of relations which precisely generate the image of the relations in the free abelian quotient of the generators. This gives a new group $\hat{G}$ and an epimorphism $\hat{G} \to G$ which induces an isomorphism $H_1(\hat{G};\mathbb{Z}) \to H_1(G;\mathbb{Z})$. The resulting 2–complex has $H_2 = 0$. Hence there is a Stein manifold M with $H_1(M;\mathbb{Z})$ any countable abelian group and $H_2(M;\mathbb{Z}) = 0$.

Here are two ways to construct additional examples.

Example 3.7 If M_1 is diffeomorphic at infinity to M_2 and if M_2 has few essential 3–handles, then M_1 has few essential 3–handles. If M_1 and M_2 have few essential 3–handles, then $M_1 \natural M_2$ has few essential 3–handles.

4 Some estimates of $\gamma(M)$

Definition 4.1 An oriented 4–manifold has an intersection form on $H_2(M;\mathbb{Z})$. We say M is *odd* if there is an element whose intersection with itself is odd: otherwise we say M is *even*. Spin implies even, but there are manifolds like the Enriques surface which are not Spin but are even.

Theorem 4.2 *Let M be the interior of an orientable, smooth, compact manifold with boundary (which may be empty). Then*

$$\gamma(M) \leq \beta_2(M) - \begin{cases} 0 & M \text{ even} \\ 1 & M \text{ odd}. \end{cases}$$

Proof It suffices to deal with each component of M separately so assume M is connected. If $\partial M = \emptyset$ replace M by $M - D^4$ where D^4 denotes a smooth standard disk. By Proposition 2.4, $\gamma(M) = \gamma(M - D^4)$ so hereafter assume $\partial M \neq \emptyset$. Since M is compact, there are compact duals to w_2. Let F be a fixed dual to w_2, and then fix an $e \in \mathcal{E}(M - F)$. Note it suffices to prove $b_e - d_1(F) \leq \beta_2(M)$, M even, (or $\beta_2(M) - 1$, M odd) where $d_1(F) = \dim_{\mathbf{F}_2} H_1(F;\mathbf{F}_2)$. Let $\overline{M}$ denote the compact manifold with boundary whose interior is M. One can add 1–handles to F inside $M - e(D^4)$ to ensure F is connected without changing $d_1(F)$ so assume F is connected. Let $U = \overline{M} - F$ and let N be the double of U. Since U is Spin, N is a closed, compact Spin manifold with signature 0. The composition $r\colon N \subset U \times [0,1] \xrightarrow{proj.} U$ splits the inclusion, so

$$H_*(N;\mathbb{Q}) = H_*(U;\mathbb{Q}) \oplus H_*(U, \partial U;\mathbb{Q})$$

since $(U,\partial U)\to(N,U)$ is an excision map. Hence $b_e\le\ell$ where

$$\begin{aligned}2\ell+2=&\chi(N)+2\beta_1(N)=2\cdot\big(\chi(M)-\chi(F)+\beta_1(N)\big)\\=&2\cdot\big(\chi(M)+d_1(F)+\beta_1(N)-2\big)\end{aligned}$$

so $b_e-d_1(F)\le\chi(M)+\beta_1(N)-3$.

We now proceed to compute

$$H_1(N;\mathbb{Q})=H_1(U;\mathbb{Q})\oplus H_1(U,\partial U;\mathbb{Q})=H_1(U;\mathbb{Q})\oplus H^3(U;\mathbb{Q}).$$

Denote the disk–sphere bundle pair for the normal bundle of $F\subset M$ by $\big(D(F),S(F)\big)$. The next two sequences are exact:

$$H^3(\overline{M})\to H^3(U)\to H^4(\overline{M},U)\cong H^4\big(D(F),S(F)\big)\cong H_0(F)$$

$$(**)\qquad H^2(F)\cong H_2\big(D(F),S(F)\big)\cong H_2(\overline{M},U)\to H_1(U)\to H_1(\overline{M})$$

Since $H_0(F)=\mathbb{Q}$ and $H^2(F)=\mathbb{Q}$ or 0, $\beta_1(U)\le\beta_1(M)+1$ and $\beta_3(U)\le\beta_3(M)+1$. It follows that $b_e-d_1(F)\le\chi(M)+\beta_1(M)+\beta_3(M)-1=\beta_2(M)$.

If the surface dual to w_2 is not orientable, then, in $(**)$ above, $H^2(F)=0$, so $\beta_1(U)\le\beta_1(M)$ and $b_e-d_1(F)\le\beta_2(M)-1$. If F is orientable, then $H_2(F)=H_2(\overline{M},U)=\mathbb{Q}$. Since the form is odd, there is an element $x\in H_2(\overline{M})$ of odd self-intersection. It follows that the image of x in $H_2(\overline{M},U)$ is non-zero, so again $\beta_1(U)\le\beta_1(M)$ and $b_e-d_1(F)\le\beta_2(M)-1$. □

Theorem 4.3 *If M is orientable with few essential 3–handles, then*

$$\gamma(M)\le\beta_2(M)-\begin{cases}0 & M\ \textit{even}\\ 1 & M\ \textit{odd}.\end{cases}$$

Proof If there are no compact duals to w_2 the required inequality is clear, so assume we have compact duals. Fix a compact dual F and an $e\in\mathcal{E}(M-F)$ and show $b_e-d_1(F)\le\beta_2(M)$, M even ($\beta_2(M)-1$ M odd). If $\beta_2(M)=\infty$, the required inequality is immediate, so assume it is finite. Use Proposition 3.2 to find a compact smooth manifold with boundary, say $W^4\subset M$, with $F\amalg e(D^4)\subset\overset{\circ}{W}$ and $\beta_2(W)\le\beta_2(M)$. Then F is dual to $w_2(W)$ so $b_e-d_1(F)\le\gamma(W)$ and Theorem 4.2 implies $\gamma(W)\le\beta_2(W)$ and since $\beta_2(W)\le\beta_2(M)$, $\gamma(W)\le\beta_2(M)$. If the form on M is odd, then we can choose W so large that the form on W is also odd and then $\gamma(W)\le\beta_2(M)-1$. □

Example 4.4 If M is a smoothing of $\mathbb{R}^4$ with few essential 3–handles, then $\gamma(M)=0$. Hence, $\gamma(M)=0$ if M supports a Stein structure or if M is diffeomorphic to $N^3\times\mathbb{R}$. In particular, the standard smoothing of $\mathbb{R}^4$ has $\gamma=0$ and hence any smoothing of $\mathbb{R}^4$ which embeds in it also has $\gamma=0$.

Remark 4.5 In the next section, we show many smoothings of $\mathbb{R}^4$ have $\gamma > 0$. It follows that these smoothings must have infinitely many 3–handles in *any* handlebody decomposition. Even more must be true. For any exhaustion of such a smoothing of $\mathbb{R}^4$, $H_3(W_{i+1}, W_i; \mathbb{Z})$ must be non-zero for infinitely many i.

5 Examples of γ for smoothings of $\mathbb{R}^4$

First observe that for a smoothing of $\mathbb{R}^4$, γ depends only on the behavior at infinity.

Theorem 5.1 *Let E_1 and E_2 be smoothings of $\mathbb{R}^4$ and suppose E_1 embeds at ∞ in E_2 (ie there is an open subset of E_2 which is diffeomorphic at ∞ to E_1). Then $\gamma(E_1) \leq \gamma(E_2)$.*

Proof If $E_3 \subset E_2$ is the submanifold which is diffeomorphic at ∞ to E_1, it follows that E_3 is homeomorphic to $\mathbb{R}^4$. It suffices to prove $\gamma(E_1) = \gamma(E_3)$.

Let $F\colon E_1 - V_1 \to E_3 - V_2$ be a representative of the diffeomorphism at ∞. Let $e_i \in \mathcal{E}(E_1)$ be a sequence so that $e_i(D^4)$ form an exhaustion and further assume $V_1 \subset e_0(\mathring{D}^4)$. Let $W_i \subset E_3$ be $F\bigl(e_i(D^4) - V_1\bigr) \cup V_2$. The W_i are an exhaustion of E_3 and $W_i = \hat{e}_i(D^4)$ for elements $\hat{e}_i \in \mathcal{E}(E_3)$. Check $(\mathbb{R}^4_{e_i} - V_1) \cup V_2 = \mathbb{R}^4_{\hat{e}_i}$. Next check $b_{e_i} = b_{\hat{e}_i}$: if $\mathbb{R}^4_{e_i}$ embeds in N, then $\mathbb{R}^4_{\hat{e}_i}$ embeds in $N' = (N - V_1) \cup V_2$ and N' is homeomorphic to N. $\square$

It is easy to estimate γ for an end-connected sum.

Lemma 5.2 *For E_1 and E_2 any two smoothings of $\mathbb{R}^4$,*

$$\max\bigl\{\gamma(E_1), \gamma(E_2)\bigr\} \leq \gamma(E_1 \natural E_2) \leq \gamma(E_1) + \gamma(E_2) \ .$$

Proof The lower bound follows since $E_i \subset E_1 \natural E_2$. To see the upper bound, consider the ordinary connected sum, $E_1 \# E_2$. One can embed a smooth $\mathbb{R}^1$ meeting the S^3 in the connected sum transversely in one point so that $E_1 \natural E_2$ and $E_1 \# E_2 - \mathbb{R}^1$ are diffeomorphic. By Proposition 2.4, $\gamma(E_1 \natural E_2) = \gamma(E_1 \# E_2)$. Now, given any embedding $e \in \mathcal{E}(E_1 \# E_2)$, there are embeddings $e_i \in \mathcal{E}(E_i)$, each of which contains the disk used to form the connected sum and so that $e_1(D^4) \# e_2(D^4)$ contains $e(D^4)$ in its interior. If each $e_i(D^4)$ embeds in N_i, $e_1(D^4) \# e_2(D^4)$ embeds in $N_1 \# N_2$. $\square$

Donaldson [5] began work on the 11/8–th's conjecture (Problem 4.92 [15]), one version of which says that any smooth Spin 4–manifold, N, with form $2s\mathbf{E}_8 \perp t\mathbf{H}$ must have $t \geq 3s$. Here $\mathbf{E}_8$ is the even, unimodular form of rank 8 and index -8 and $\mathbf{H}$ is the rank 2 hyperbolic form. While this conjecture is still unsolved in general Donaldson [5] proved $t > 0$ and Furuta [9] has shown that $t > 2s$. Recall that both these results require no condition on π_1, which accounts for our lack of π_1 conditions on the manifolds in $\mathcal{S}p(E)$ from Section 1.

We say that a smoothing E of $\mathbb{R}^4$ is *semi-definite* provided there is a positive integer k and a compact, closed, topological Spin 4–manifold M with form $2s\mathbf{E}_8$, $s > 0$, so that some smoothing of $M - pt$ is diffeomorphic at ∞ to $\natural^k E$. If the k and s are important we will say E is (k,s)–semi-definite. If E_1 embeds at ∞ in E_2 and if E_1 is (k,s)–semi-definite, then E_2 is (k,s)–semi-definite. If E is (k,s)–semi-definite, then all sufficiently large balls in E are (k,s)–semi-definite. We say E is (k,s)–simple-semi-definite if the M can be chosen to be simply-connected.

Theorem 5.3 *If E is (k,s)–semi-definite, then $\gamma(E) > 2s/k$ and*

$$\lim_{r\to\infty} \gamma(\natural^r E) = \infty.$$

Proof If $\gamma(E) = \infty$ the result is clear, so assume $\gamma(E) < \infty$. Select a large topological ball in $\natural^k E$ so that the smoothing on its interior, E_1, is also diffeomorphic at ∞ to a smoothing of $M - pt$, say V. By selecting the ball large enough, $\gamma(\natural^k E) = \gamma(E_1)$. By definition, E_1 embeds in a smooth, closed, compact, Spin manifold, N^4, whose intersection form is $\gamma(E_1)\mathbf{H}$. Choose a compact set $K \subset V$ and a ball $\Delta \subset E_1$ so that $V - K$ and $E_1 - \Delta$ are diffeomorphic and use the diffeomorphism to glue $N - \Delta$ and V together along $V - K$. The resulting manifold is smooth and has form $2s\mathbf{E}_8 \perp \gamma(E_1)\mathbf{H}$. The case $\gamma(E_1) = 0$ is forbidden by Donaldson [5] and by Lemma 5.2, $k\gamma(E) \geq \gamma(E_1) > 0$.

By Furuta [9] it further follows that $\gamma(\natural^k E) > 2s$. Since $\natural^{\ell k} E$ is diffeomorphic at ∞ to a smoothing of $M_\ell - pt$ where M_ℓ is the connected sum of ℓ copies of M, Furuta's result implies $\gamma(\natural^{k\ell} E) > 2s\ell$. Since $\gamma(\natural^r E)$ is a non-decreasing function of r, the limit exists and is ∞. □

The next result follows from work of Taubes [18].

Theorem 5.4 *If E is a simple-semi-definite smoothing of $\mathbb{R}^4$ then any sufficiently large ball in E is not diffeomorphic to any larger ball containing it in its interior.*

Proof If not, then for any compact set $K \subset E$ there exists a pair of balls $e_i \in \mathcal{E}(E)$ with $K \subset e_1(\overset{\circ}{D^4}) \subset e_1(D^4) \subset e_2(\overset{\circ}{D^4})$ and with $\mathbb{R}^4_{e_1}$ diffeomorphic to $\mathbb{R}^4_{e_2}$. We may further require that $\mathbb{R}^4_{e_1}$ is (k,s)–semi-definite, where E is (k,s)–semi-definite. We can take the end-connected sum in such a way that $\natural^k \mathbb{R}^4_{e_1} \subset \natural^k \mathbb{R}^4_{e_2}$ Hence there is a smoothing of $M - pt$ which is diffeomorphic at ∞ to $\natural^k \mathbb{R}^4_{e_1}$, where the form on M is $2s\mathbf{E}_8$. But this is forbidden by [18]. □

Corollary 5.5 *If E is a simple-semi-definite smoothing of $\mathbb{R}^4$ with $\gamma(E) < \infty$ then there are uncountably many simple-semi-definite smoothings of $\mathbb{R}^4$ with γ equal to $\gamma(E)$.*

Proof All sufficiently large balls inside of E are simple-semi-definite and have γ equal to $\gamma(E)$. □

Example 5.6 *There exists an $e \in \mathcal{E}(S^2 \times S^2)$ so that $\mathbb{R}^4_e$ is a $(3,1)$–simple-semi-definite smoothing of $\mathbb{R}^4$ with $\gamma = 1$. Hence $2r/3 < \gamma(\natural^r \mathbb{R}^4_e) \le r$. For each integer $n \ge 1$ there exists an r_n such that $\gamma(\natural^{r_n} \mathbb{R}^4_e) = n$.*

Proof Consider the standard D^4 with a Hopf link in its boundary. Attach two Casson handles with 0 framing to this Hopf link and call the interior a *Casson wedge*. Let M^4 denote the Kummer surface or any other simply-connected smooth Spin manifold with form $2\mathbf{E}_8 \perp 3\mathbf{H}$.

Casson's results [2] allow us to construct a particular Casson wedge, C, so that $C \amalg C \amalg C \subset M$. We can also find $C \subset S^2 \times S^2$. Freedman [7] constructs a topological embedding of $S^2 \vee S^2 \subset C$ and shows that $E = S^2 \times S^2 - S^2 \vee S^2$ is homeomorphic to $\mathbb{R}^4$.

By construction $\gamma(E) \le 1$ and $\natural^3 E$ is diffeomorphic at ∞ to a smoothing of $M' - pt$, where M' is the simply-connected topological manifold with intersection form $E_8 \perp E_8$ constructed by Freedman. By Theorem 5.3, $\gamma(E) > 0$. Now choose $e \in \mathcal{E}(S^2 \times S^2)$ so that $e(D^4) \subset E$ is so large that $\gamma(\mathbb{R}^4_e) = \gamma(E) = 1$ and $\mathbb{R}^4_e$ is $(3,1)$–semi-definite.

Since the set $\{\gamma(\natural^r \mathbb{R}^4_e)\}$ is unbounded and since $0 \le \gamma(\natural^{r+1} \mathbb{R}^4_e) - \gamma(\natural^r \mathbb{R}^4_e)) \le 1$ by Lemma 5.2, every integer $n \ge 1$ is assumed by some $\gamma(\natural^r \mathbb{R}^4_e)$. □

Remark 5.7 If E is any $(3,1)$–semi-definite smoothing of $\mathbb{R}^4$ with $\gamma(E) = 1$, it follows as above that for $n \ge 1$ there is an r_n such that $n = \gamma(\natural^{r_n} E)$. Moreover, $r_n = n$ for $1 \le n \le 3$. If the 11/8-th's Conjecture holds, then $r_n = n$ for all $n \ge 1$.

Let U denote the interior of the universal half-space constructed in [8] or any other smoothing of $\mathbb{R}^4$ into which all others embed. Then clearly we have:

Corollary 5.8 $\gamma(U) = \infty$.

The above results give some information for compact manifolds.

Example 5.9 If M is the connected sum of s copies of $S^2 \times S^2$, then $2s/3 < \gamma(M) \le s$ and $\gamma(M) = s$ if $s \le 3$ (Example 5.6).

Example 5.10 The case $M = CP^2$ is more interesting. It is clear $\gamma(CP^2) = 0$: the upper bound comes from Theorem 4.2 while the lower bound comes from a smooth disk missing the CP^1 dual to w_2. More interestingly, let M_s be the connected sum of $2s$ copies of Freedman's $\mathbf{E}_8$ manifold [7]. Then there is a homeomorphism $h\colon CP^2 \# M_s \to CP^2 \# 16s\overline{CP}^2$. Apply h to the $S^2 = CP^1 \subset CP^2$ to see a locally flat topological embedding $\iota\colon S^2 \to CP^2 \# 16s\overline{CP}^2$ whose complement is M_s. A neighborhood of $\iota(S^2)$ is homeomorphic to $CP^2 - pt$. With a bit more care, we can choose ι and a topological embedding $\iota_0\colon S^2 \to CP^2$ so that ι and ι_0 have diffeomorphic neighborhoods. Hence $E_s = CP^2 - \iota_0(S^2)$ is homeomorphic to $\mathbb{R}^4$ and diffeomorphic at ∞ to a smoothing of $M_s - pt$. See Kirby [14, page 95] for more details. By construction, E_s is $(1,s)$–simple-semi-definite, so Theorem 5.3 shows $\gamma(E_s) \ge 2s$. Hence there are $e \in \mathcal{E}(CP^2)$ with b_e arbitrarily large. The above calculation of γ shows that a dual to w_2 in the complement of the image of e must have large H_1.

As pointed out by the referee, more mileage is available from this example. With a bit of care, it can be arranged so that $E_s \subset E_{s+1} \subset CP^2$. If we let $E_\infty \subset CP^2$ denote the union, then $\gamma(E_\infty) = \infty$ and again with care, E_∞ can be extended to have a smooth patch of boundary $\mathbb{R}^3$. Fix an orientation reversing homeomorphism between $\mathbb{R}^4$ and some simple-semi-definite smoothing of $\mathbb{R}^4$ and let $E(\rho)$ denote the smoothing inherited by the ball of radius ρ. Following Ding's use of Taubes's periodic ends theorem, [4], note that for sufficiently large ρ, $E_\infty \natural E(\rho_1)$ is diffeomorphic to $E_\infty \natural E(\rho_2)$ if and only if $\rho_1 = \rho_2$. Since $\gamma(E_\infty \natural E) = \infty$, there are uncountably many distinct smoothings of $\mathbb{R}^4$ with $\gamma = \infty$. Together with 5.5 and 5.6 this shows that for each $0 \le n \le \infty$ there are uncountably many distinct smoothings of $\mathbb{R}^4$ with $\gamma = n$.

6 Smoothings of other 4–manifolds

In this section we will start with a non-compact smooth manifold M, construct a family of smoothings $M(\rho)$ and prove that $\gamma\big(M(\rho)\big)$ takes on countably many distinct values. We will start with a smoothing of M satisfying some hypotheses which will be clarified later. Then we form $M \natural U$, where U is a smooth $\mathbb{R}^4$ into which all others embed. Without much trouble, we see

$\gamma(M\natural U) = \infty$. For positive real numbers ρ we construct a family of open submanifolds, $M(\rho) \subset M\natural U = M(\infty)$ which exhaust $M(\infty)$. Again without much trouble, we show that $\gamma\big(M(\rho_1)\big) \leq \gamma\big(M(\rho_2)\big)$ for $\rho_1 < \rho_2$ and that $\gamma\big(M(\infty)\big) = \max_\rho\{\,\gamma\big(M(\rho)\big)\,\}$. Finally (the hardest part) we show $\gamma\big(M(\rho)\big) < \infty$ for $\rho < \infty$. At this point, it is easy to show that there is a countable collection of $\rho_1 < \cdots < \rho_i < \cdots < \infty$ so that $\gamma\big(M(\rho_1) < \cdots < \gamma\big(M(\rho_i)\big) < \cdots < \infty$ so we have our countable collection of distinct smoothings.

To show $\gamma\big(M(\rho)\big) < \infty$ will require constructing a 4–manifold with few essential 3–handles. To be able to do this, we need some structure to start with on M. We actually need our structure on $M\natural U$ and this manifold never has few essential 3–handles so we need some structure close to few essential 3–handles but weaker. Roughly, we require an exhaustion of M by topological manifolds W_i with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$ so that the embedding $\partial W_i \to M$ is smooth except for a finite number of 3–disks. The precise condition follows next.

We say that an exhaustion of M by topological manifolds W_i is *almost-smooth* provided there exists a space $\Delta = \amalg_{k=0}^{T} D^3$ and a proper embedding $\tau\colon \Delta \times [-1,\infty) \to M$ which is smooth near $(\partial\Delta) \times [-1,\infty)$ and a function $\mu\colon M \to [-4,\infty)$ satisfying the following. Let $Y = M - \tau\big(\mathring{\Delta} \times (-1,\infty)\big) - \tau(\Delta \times -1)$ and note Y is a smooth manifold with boundary $(\partial\Delta) \times (-1,\infty)$. We require

(1) $W_i = \mu^{-1}\big([-4,i]\big)$,

(2) $\mu \circ \tau\colon \Delta \times [-1,\infty) \to [-4,\infty)$ is projection followed by inclusion,

(3) μ restricted to Y is smooth and the integers in $(-1,\infty)$ are regular values.

Note that $\mathring{Y} = M - \tau\big(\Delta \times [-1,\infty)\big)$ is homeomorphic to M. Indeed $M(\rho)$ will be $M - \tau\big(\Delta \times [\rho,\infty)\big)$ for some almost-smooth exhaustion. The replacement for "few essential 3–handles" will be an almost-smooth exhaustion with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$.

Our first requirement on our structure follows.

Remark 6.1 If M has an almost-smooth exhaustion with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$, then so does $M\natural E$ for any E homeomorphic to $\mathbb{R}^4$. Just add one more 3–disk to Δ and construct τ and μ so that the end-connect sum takes place inside the added cylinder.

Before introducing the main theorem, we discuss some examples.

Example 6.2 If $(N, \partial N)$ is a topological manifold, then $\overset{\circ}{N}$ has a smoothing with an almost-smooth exhaustion with $H_*(W_{i+1}, W_i; \mathbb{Z}) = 0$ for all $*$. Indeed, one can smooth $\overset{\circ}{N}$ so that there is a topological collar which is smooth off a cylinder $D^3 \times [-1, \infty)$. Here are the details. Smooth $N - pt$ and choose a smooth collar $c\colon \partial N \times [-2, \infty] \to N$ so that $c(\partial N \times \infty) = \partial N$. Let $\mu\colon N \to [-4, \infty]$ be chosen to be smooth on $\mu^{-1}\big((-3, \infty]\big)$ and projection on the collar. Pick a smooth $\iota\colon \Delta = D^3 \subset \partial N$ and let $\tau\colon \Delta \times [-1, \infty) \xrightarrow{\iota \times id} \partial N \times [-1, \infty) \xrightarrow{c} \overset{\circ}{N}$. Note that $\mu^{-1}\big([-4, -2]\big)$ is a compact submanifold of $\overset{\circ}{N}$ with a smooth boundary in the smooth structure on $\overset{\circ}{N} - pt$; note further that pt is in its interior. Let $X = \mu^{-1}\big([-4, -2]\big) \cup c\big(\iota(D^3) \times [-2, \infty)\big)$ and note $\overset{\circ}{N} - \overset{\circ}{X} \subset \overset{\circ}{N} - pt$ is a smooth submanifold. Let M be the smoothing on $\overset{\circ}{N}$ given by smoothing X rel boundary and extending over $\overset{\circ}{N} - \overset{\circ}{X}$. As things now stand, μ may not be smooth on the space Y in the definition, but it can clearly be altered to be smooth while leaving it fixed on $\mu^{-1}\big[-1, \infty)\big)$. Clearly (2) and (3) are satisfied and $\overline{W_{i+1} - W_i} = \partial N \times [i, i+1]$ so $H_*(W_{i+1}, W_i; \mathbb{Z}) = 0$.

Here is a way to produce new examples.

Example 6.3 By Kirby and Siebenmann [16] the stable isotopy classes of smoothings of M are in one-to-one correspondence with elements $\alpha \in H^3(M; \mathbf{F}_2)$. To construct such a smoothing, select a proper embedding $\mathbb{R}^1 \to M$ which is dual to α. Extend to an embedding $\hat{\alpha}\colon D^3 \times \mathbb{R}^1 \to M$. Freedman [6] has constructed a smooth proper homotopy $D^3 \times \mathbb{R}^1$, $(D^3 \times \mathbb{R}^1)_\Sigma$, and [7] a homeomorphism $h\colon (D^3 \times \mathbb{R}^1)_\Sigma \to D^3 \times \mathbb{R}^1$ which is the identity on the boundary and represents the non-zero element in $H^3(D^3 \times \mathbb{R}^1; \mathbf{F}_2)$. Remove the image of $\hat{\alpha}$ and replace it by $(D^3 \times \mathbb{R}^1)_\Sigma$ to get $M_{\hat{\alpha}}$ and let $h_\alpha\colon M_{\hat{\alpha}} \to M$ be the evident homeomorphism. Now let M have an almost-smooth exhaustion, τ and μ. The two ends of $\mathbb{R}^1$ and $\hat{\alpha}$ determine two ends of M (which may be the same). We can further guarantee that the image of $\hat{\alpha}$ misses the image of τ. Let Δ' be Δ with two more 3–disks added. One can choose $\tau'\colon \Delta' \times [-1, \infty) \to M_{\hat{\alpha}}$ and μ' so that τ' and μ' give an almost-smooth exhaustion for of $M_{\hat{\alpha}}$ with the same topological manifolds W_i.

Theorem 6.4 *Let M be a non-compact, orientable 4–manifold with an almost-smooth exhaustion with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$.*
If M satisfies $\dim_{\mathbf{F}_2} H_2(M; \mathbf{F}_2) + \beta_2(M) < \infty$, then there are at least countably many distinct smoothings of M in each stable isotopy class.

Proof By 6.3 there is a smoothing of M in a given stable isotopy class which possesses an almost-smooth exhaustion still satisfying $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$. Use M to denote this smoothing. If U is any $\mathbb{R}^4$ into which all others smoothly embed, form $M\natural U$. This is a smoothing of M in the same stable isotopy class and has an almost-smooth exhaustion $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$ by 6.1. Fix τ and μ to give an almost-smooth exhaustion of $M\natural U$ with $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$.

For each ρ, $0 \le \rho < \infty$, let

$$M(\rho) = M\natural U - \tau\big(\Delta \times [\rho, \infty)\big)$$

and observe each $M(\rho)$ is homeomorphic to M and since $M(\rho)$ is an open subset of $M\natural U$ it is a smooth manifold. Let $M(\infty) = M\natural U$. Note that if $\rho_1 < \rho_2$, $M(\rho_1) \subset M(\rho_2)$.

Since $\dim_{\mathbf{F}_2} H_2(M; \mathbf{F}_2) < \infty$, all the $M(\rho)$ have compact duals to w_2 and if $F \subset M(\rho_1)$ is a compact dual to w_2, then it is a compact dual to w_2 in $M(\rho_2)$ whenever $\rho_1 < \rho_2$. Hence $\gamma\big(M(\rho)\big)$ is a non-decreasing function of ρ. Fix a compact dual to w_2 in $M(0)$, say F_0. Given any integer S, there exist balls $e \in \mathcal{E}(U)$ with $b_e > S$ and there exists a T such that the image of e lies in $M(\rho)$ for all $\rho > T$. For $\rho > T$, $\gamma\big(M(\rho)\big) > S - \dim_{\mathbf{F}_2} H_1(F_0; \mathbf{F}_2)$ so $\lim_{\rho\to\infty} \gamma\big(M(\rho)\big) = \infty$.

It remains to prove $\gamma\big(M(\rho)\big) < \infty$ if $\rho < \infty$. To do this, we construct a manifold $X(\rho)$ so that $X(\rho)$ has few essential 3–handles. We show that $\gamma\big(M(\rho)\big) \le \gamma\big(X(\rho)\big)$ and that $\beta_2\big(X(\rho)\big) < \infty$. Theorem 4.3 implies that $\gamma\big(X(\rho)\big) < \infty$.

To construct $X(\rho)$ first note that we can deform μ to a function $\mu_X\colon M\natural U \to [-4, \infty)$ which is smooth everywhere, equal to μ on Y and has the non-negative integers as regular values.

Let k_ρ denote the first integer so that $\mu^{-1}\big([-4, \rho]\big) \subset \mu_X^{-1}\big([-4, k_\rho]\big)$. Let $Z(\rho) = \tau(\Delta \times [-1, \infty) \cap \mu_X^{-1}\big([k_\rho, \infty)\big)$.

First form $M\natural U - Z(\rho)$ and see that this is a smooth manifold with boundary $\mu_X^{-1}(k_\rho)$ union $\tau\big(\partial\Delta \times [k_\rho, \infty)\big)$. Let $X(\rho) = \big(M\natural U - Z(\rho)\big) \cup \mu_X^{-1}(k_\rho) \times [k_\rho, \infty)$.

Next check that $M(\rho) \subset X(\rho)$ and that any dual to w_2 in $M(\rho)$ will be a dual to w_2 in $X(\rho)$. This follows because topologically $X(\rho)$ can be obtained from $M(\rho)$ by adding a Spin manifold along an embedded $\mathbb{R}^3$. It now follows that $\gamma\big(M(\rho)\big) \le \gamma\big(X(\rho)\big)$.

Next check that $X(\rho)$ has few essential 3–handles.

For all $i \ge k_\rho$, $\mu_X^{-1}\big([-4, i]\big) = W'_i$ is a smooth submanifold and $\overline{W'_{i+1} - W'_i}$ is built topologically from $\overline{W_{i+1} - W_i}$ by removing a copy of $\Delta \times [i, i+1]$ and replacing it with a copy of $\mu_X^{-1}(k_\rho) \times [i, i+1]$. Since $H_3(W_{i+1}, W_i; \mathbb{Z}) = 0$, $H_3(W'_{i+1}, W'_i; \mathbb{Z}) = 0$.

Finally, use the Mayer–Vietoris sequence to see $\beta_2\big(X(\rho)\big)$ is finite.

Remark 6.5 Some sort of finiteness is needed to make the proof of Theorem 6.4 work. If M is the connected sum of infinitely many copies of $S^2 \times S^2$, all the $M(s)$ constructed above are diffeomorphic. To see this note that there is a smooth proper h–cobordism, H, between $\mathbb{R}^4$ and $\overset{\circ}{U}(s)$. Embed a ray cross $[0,1]$ properly into H and remove a tubular neighborhood. Do the same with $M \times [0,1]$ and glue the two manifolds together along the boundary $\mathbb{R}^3 \times [0,1]$. The result is a smooth proper h–cobordism between M and $M(s)$. This proper h–cobordism is trivial by the usual argument involving summing a proper h–cobordism with sufficiently many $S^2 \times S^2 \times [0,1]$. Note that we are not saying that M has only one smoothing but any different smoothing will have to have only finitely many smooth $S^2 \vee S^2$'s.

F Ding and Ž Bižaca and J Etnyre have obtained results on existence of smoothings which overlap the above results. If M is the interior of a topological manifold with a compact boundary component such that this component smoothly embeds in k copies of CP^2, then F. Ding [4] shows that there exists an $S > 0$ so that all the $M(s)$ are distinct for $s \geq S$. Bižaca and Etnyre [1] show there are at least countably many distinct smoothings of M among the $M(s)$ if M is a topological manifold with a compact boundary component. The Bižaca and Etnyre proof is clearly a precursor of the proof above.

An example not covered by either Ding or Bižaca and Etnyre can be constructed as follows. Use Example 3.6 to construct an M with few $\mathbf{F}_2$–essential 3–handles such that $H_1(M;\mathbb{Z}) = \mathbb{Q}$ and $H_2(M;\mathbb{Z}) = 0$. Then M has at least countably many distinct smoothings, possibly the first such example with one end which is not topologically collared.

7 Infinite Covers

In the theorems below, we will use the hypothesis that M is an infinite cover to bound $\lim\limits_{r\to\infty} \gamma(\natural^r \mathbb{R}_e)$ for $e \in \mathcal{E}(M)$. All the examples for which we know this limit satisfy $\lim\limits_{r\to\infty} \gamma(\natural^r \mathbb{R}_e) = 0$ or ∞.

Theorem 7.1 *Let M be a non-compact smooth Spin 4–manifold with an action by $\mathbb{Z}$ which is smooth and properly discontinuous. Then, for any $e \in \mathcal{E}(M)$ and any integer $0 \leq r < \infty$, $\gamma(\natural^r \mathbb{R}^4_e) \leq \gamma(M)$. If $\gamma(M) < \infty$ then for some $e \in \mathcal{E}(M)$, $\lim\limits_{r\to\infty} \gamma(\natural^r \mathbb{R}^4_e) = \gamma(M)$.*

Proof Properly discontinuous implies that the orbit space N is a smooth manifold and that the map $M \to N$ is a covering projection. Given any $e \in \mathcal{E}(M)$, we can find some element $g \in \mathbb{Z}$ such that all the translates of $e(D^4)$ by the powers of g are disjoint, so all the $g^i \circ e \in \mathcal{E}(M)$. For any integer $r > 0$, use Proposition 2.2 to find $e' \in \mathcal{E}(M)$ such that $\overset{\circ}{D}{}^4_{e'} = \natural^r \overset{\circ}{D}{}^4_e$. The result follows. □

Remark 7.2 Let M be a smooth manifold homeomorphic to $\mathbb{R}^4$. If M is a non-trivial cover of some other 4–manifold then M has a properly-discontinuous $\mathbb{Z}$ action. Gompf [15, Problem 4.79A] asks for such smoothings and remarks that most smoothings of $\mathbb{R}^4$ are not universal covers of compact manifolds because there are only countably many smooth compact manifolds but there are uncountably many smoothings of $\mathbb{R}^4$. Example 5.6 gives the first concrete examples of smoothings of $\mathbb{R}^4$ which can not be universal covers.

Theorem 7.3 *Let N be a compact smooth Spin 4–manifold with an infinite, residually-finite fundamental group. Let M denote the universal cover and let $\mathcal{C}$ denote the set of finite sheeted covers. Then*

$$\gamma(\natural^r \mathbb{R}^4_e) \le \gamma(M) = \max_{P \in \mathcal{C}} \{\gamma(P)\}$$

for all $e \in \mathcal{E}(M)$.

Proof Since there are elements of infinite order in $\pi_1(M)$, Theorem 7.1 implies $\gamma(\natural^r \mathbb{R}^4_e) \le \gamma(M)$. Let $e \in \mathcal{E}(M)$. It follows from the residual finiteness of $\pi_1(N)$ that there is a finite cover P so that the composite $D^4 \xrightarrow{e} M \to P$ represents an element in $\mathcal{E}(P)$. Hence $\gamma(M) \le \max_{P\in\mathcal{C}}\{\gamma(P)\}$. Given any $e \in \mathcal{E}(P)$, e lifts to an infinite number of disjoint copies in M, so $\max_{P\in\mathcal{C}}\{\gamma(P)\} \le \gamma(M)$. □

Example 7.4 If N is a smoothing of $S^3 \times S^1$ then $\gamma(M) = 0$. If N is a smoothing of the four torus then $\gamma(M) \le 3$. To see these results just observe that all the finite covers, P, of N are homeomorphic to N. These P are compact, Spin and have signature 0. Hence $\gamma(P) = 0$ if $N = S^3 \times S^1$ and $\gamma(P) \le 3$ for N the four torus. Now apply Theorem 7.3.

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