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Erratum 1
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Correction to "The Symmetry of Intersection Numbers in Group Theory"

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Theorem 3.1 is false as stated. The error in the argument occurs in the proof of Lemma 3.6. See below for a counterexample.

Lemma 3.6 asserts that, under suitable hypotheses, X crosses Y if and only if there is an element λ in Λ such that either $\lambda Y \subsetneq Y$ or $\lambda Y^* \subsetneq Y^*$. One of these implications is correct. If such a λ exists, then it is true that X must cross Y. However, I failed to give any argument for this, and I provide one below. The other implication is false. The mistake is contained in the second sentence of the second paragraph on page 28. A simple fix is to amend the statements of Theorem 3.1 and Lemma 3.6 to take this into account. Thus we need the additional hypothesis for Lemma 3.6 that if X crosses Y, then δX must contain points which are arbitrarily far from δY on each side of δY. We also need the additional hypothesis for Theorem 3.1 that if X crosses gY then δX must contain points which are arbitrarily far from δgY on each side of δgY. This technical assumption is often but not always satisfied.

Here is the half of the proof of Lemma 3.6 which was omitted. This asserts that if there is an element λ in Λ such that either $\lambda Y \subsetneq Y$ or $\lambda Y^* \subsetneq Y^*$, then X must cross Y. We will assume that $\lambda Y \subsetneq Y$, as the argument in the other case is essentially identical. As Y is associated to a splitting of G, it is easy to see that the distance of $\lambda^n \delta Y$ from δY must tend to infinity as $n \to \infty$. (For example, if $G = A *_C B$, and Y is the set of words in G which begin in $A - C$, then λ must begin in $A - C$ and end in $B - C$.) Now consider an element $g \in G$, and let d denote the distance of g from δY. Then d is also the distance of $\lambda^n g$ from $\lambda^n \delta Y$. Hence, for any element g of G, all translates $\lambda^n g$ lie in Y, for suitably large n. If we apply these statements to an edge of δX, and recall that δX is preserved by λ, we see that δX must contain points which are arbitrarily far from δY and lie in Y. By applying the same discussion to λ^{-1}, we see that δX must also contain points which are arbitrarily far from

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δY and lie in Y^*. Hence δX must contain points which are arbitrarily far from δY on each side of δY as required.

Now we come to the promised counterexample. Let G denote the fundamental group of the closed orientable surface M of genus two. Let D denote a simple closed curve on M which separates M into two once-punctured tori S and T and let D' denote a non-separating simple closed curve in the interior of S. Let W denote the surface obtained from S by removing a regular neighbourhood of D'. Let C denote a non-separating simple closed curve on M whose intersection number with D is two, and which is disjoint from D'. We will describe two splittings of G. The first will be the HNN splitting over an infinite cyclic group determined by C. The second will be the amalgamated free product splitting of G over $\pi_1(W)$ with vertex groups $\pi_1(S)$ and $\pi_1(W \cup T)$. These two splittings satisfy the hypotheses of Theorem 3.1. If one considers $\pi_1(C)$ as a subgroup of the splitting over $\pi_1(W)$, the minimal graph of groups obtained has no edges, because $\pi_1(C)$ is contained in $\pi_1(W \cup T)$ which is a vertex group. If one considers $\pi_1(W)$ as a subgroup of the HNN splitting determined by C, the minimal graph of groups obtained has at least one edge because $\pi_1(W)$ does not lie in a conjugate of any vertex group. (The graph in question has exactly one edge, but this fact is not needed here.) This shows that Theorem 3.1 must fail for this example, because the numbers of edges in these two graphs are not equal. It also true that Lemma 3.6 fails for this example. Let X and Y be the usual subsets of G associated to the two splittings. I claim that X crosses Y but δX does not contain points which are arbitrarily far from δY on each side of δY. To see this, consider the picture in the cover M_C of M whose fundamental group equals $\pi_1(C)$. This cover is an open annulus which contains a lift of C which we will continue to denote by C. As in section 2, we pick a generating set for G which can be represented by a bouquet of circles embedded in M, so that the pre-image in the universal cover $\widetilde{M}$ of M of this bouquet is a copy of the Cayley graph Γ of G, and we identify the vertices of this graph with G. Now let E denote the set of all vertices of $\pi_1(C)\backslash\Gamma$ in M_C which lie on one side of C. Then E represents an almost invariant subset of $\pi_1(C)\backslash G$ and the pre-image of E in Γ can be taken to be X. Now consider the picture in the cover M_W of M whose fundamental group equals $\pi_1(W)$. This cover consists of a lift of W, which we will continue to denote by W and open collars attached to the boundary components of W. Let F denote the set of all vertices of $\pi_1(W)\backslash\Gamma$ which lie in the union of W together with the collar attached to the component D of ∂W. Then F represents an almost invariant subset of $\pi_1(W)\backslash G$ and the pre-image of F in G can be taken to be Y. The pre-image in $\widetilde{M}$ of C is a line whose image in M_W is a properly embedded line meeting W in a compact arc which projects homeomorphically to $C \cap W$. Now inspection shows that each of the four sets $X^{(*)} \cap Y^{(*)}$ has infinite image in M_W so that X crosses Y but δX does not contain points which are arbitrarily far from δY on each side of δY.

The new version of Theorem 3.1 described here is, of course, rather unsatisfactory as the extra hypothesis is technical and it is not clear when it holds. However, there is a little more which can be said without any extra work. For it follows from the preceding discussion that the number of edges in each of the minimal graphs of groups described above is always less than or equal to the intersection number of the two splittings being considered.