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Examples of Riemannian manifolds with positive curvature almost everywhere

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Abstract

We show that the unit tangent bundle of S^4 and a real cohomology CP^3 admit Riemannian metrics with positive sectional curvature almost everywhere. These are the only examples so far with positive curvature almost everywhere that are not also known to admit positive curvature.

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0 Introduction

A manifold is said to have quasi-positive curvature if the curvature is nonnegative everywhere and positive at a point. In analogy with Aubin's theorem for manifolds with quasi-positive Ricci curvature one can use the Ricci flow to show that any manifold with quasi-positive scalar curvature or curvature operator can be deformed to have positive curvature.

By contrast no such result is known for sectional curvature. In fact, we do not know whether manifolds with quasi-positive sectional curvature can be deformed to ones with positive curvature almost everywhere, nor is it known whether manifolds with positive sectional curvature almost everywhere can be deformed to have positive curvature. What is more, there are only two examples ([12] and [11]) of manifolds with quasi-positive curvature that are not also known to admit positive sectional curvature. The example in [11] is a fake quaternionic flag manifold of dimension 12. It is not known if it has positive curvature almost everywhere. The example in [12] is on one of Milnor's exotic 7-spheres. It was asserted without proof, in [12], that the Gromoll–Meyer metric has positive sectional curvature almost everywhere, but this assertion was disproven by Mandell, a student of Gromoll ([16], cf also [25]). Thus there is no known example of a manifold with positive sectional curvature at almost every point that is not also known to admit positive curvature. We will rectify this situation here by proving the following theorem.

Theorem A *The unit tangent bundle of S^4 admits a metric with positive sectional curvature at almost every point with the following properties.*

- **(i)** *The connected component of the identity of the isometry group is isomorphic to $SO(4)$ and contains a free S^3–subaction.*
- **(ii)** *The set of points where there are 0 sectional curvatures contains totally geodesic flat 2–tori and is the union of two copies of $S^3 \times S^3$ that intersect along a $S^2 \times S^3$.*

Remark In the course of our proof we will also obtain a precise description of the set of 0–curvature planes in the Grassmannian. This set is not extremely complicated, but the authors have not thought of a description that is succinct enough to include in the introduction.

Remark In the sequel to this paper, [25], the second author shows that the metric on the Gromoll–Meyer sphere can be perturbed to one that has positive sectional curvature almost everywhere. In contrast to [25] the metric we construct here is a perturbation of a metric that has zero curvatures at every point.

By taking a circle subgroup of the free S^3–action in Theorem A(i) we get the following.

Corollary B *There is a manifold M^6 with the homology of CP^3 but not the cohomology ring of CP^3, that admits a metric with positive sectional curvature almost everywhere.*

There are also flat totally geodesic 2–tori in M^6 so as a corollary of Lemma 4.1 in [22] (cf also Proposition 3 in [5]) we have the following.

Corollary C *There are no perturbations of our metrics on the unit tangent bundle and M^6 whose sectional curvature is positive to first order.*

That is, there is no smooth family of metrics $\{g_t\}_{t\in\mathbb{R}}$ with g_0 the metric in Theorem A or Corollary B so that

$$\frac{d}{dt} sec_{g_t}(P)|_{t=0} > 0$$

for all planes P that satisfy $sec_{g_0}(P) = 0$.

Remark Although the unit tangent bundle of S^4 is a homogeneous space (see below), the metric of Theorem A is obviously inhomogeneous. What is more if the unit tangent bundle of S^4 admits a metric with positive sectional curvature, then it must be for some inhomogenous metric see [6]. The space in Corollary B is a biquotient of $Sp(2)$. It follows from [19, Theorem 6] that M^6 does not have the homotopy type of a homogeneous space.

Before outlining the construction of our metric we recall that the S^3–bundles over S^4 are classified by $\mathbb{Z}\oplus\mathbb{Z}$ as follows ([14], [21]). The bundle that corresponds to $(n,m)\in\mathbb{Z}\oplus\mathbb{Z}$ is obtained by gluing two copies of $\mathbb{R}^4\times S^3$ together via the diffeomorphism $g_{n,m}\colon (\mathbb{R}^4\backslash\{0\})\times S^3 \longrightarrow (\mathbb{R}^4\backslash\{0\})\times S^3$ given by

$$g_{m,n}(u,v) \longrightarrow \left(\frac{u}{|u|^2}, \frac{u^m v u^n}{|u|^{n+m}}\right), \tag{0.1}$$

where we have identified $\mathbb{R}^4$ with $\mathbb{H}$ and S^3 with $\{v\in\mathbb{H} \mid |v|=1\}$. We will call the bundle obtained from $g_{m,n}$ "the bundle of type (m,n)", and we will denote it by $E_{m,n}$.

Translating Theorem 9.5 on page 99 of [15] into our classification scheme (0.1) shows that the unit tangent bundle is of type $(1,1)$. We will show it is also the quotient of the S^3–action on $Sp(2)$ given by

$$A_{2,0}\left(p, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} pa & pb \\ pc & pd \end{pmatrix}.$$

(It was shown in [20] that this quotient is also the total space of the bundle of type $(2,0)$, so we will call it $E_{2,0}$.)

The quotient of the biinvariant metric via $A_{2,0}$ is a normal homogeneous space with nonnegative, but not positive sectional curvature. To get the metric of Theorem A we use the method described in [8] to perturb the biinvariant of $Sp(2)$ using the commuting S^3–actions

$$A^u(\, p_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} p_1 a & p_1 b \\ c & d \end{pmatrix}$$
$$A^d(\, p_2, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ p_2 c & p_2 d \end{pmatrix}$$
$$A^l(\, q_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a\bar{q}_1 & b \\ c\bar{q}_1 & d \end{pmatrix}$$
$$A^r(\, p_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b\bar{q}_2 \\ c & d\bar{q}_2 \end{pmatrix}.$$

We call the new metric on $Sp(2)$, $g_{\nu_1,\nu_2,l_1^u,l_1^d}$, and will observe in Proposition 1.14 that $A_{2,0}$ is by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. Our metric on the unit tangent bundle is the one induced by the Riemannian submersion $(Sp(2), g_{\nu_1,\nu_2,l_1^u,l_1^d}) \xrightarrow{q_{2,0}} Sp(2)/A_{2,0} = E_{2,0}$.

In section 1 we review some generalities of Cheeger's method. In section 2 we study the symmetries of $E_{2,0}$. In section 3 we analyze the infinitesimal geometry of the Riemannian submersion $Sp(2) \xrightarrow{p_{2,1}} S^7$, given by projection onto the first column. This will allow us to compute the curvature tensor of the metric, g_{ν_1,ν_2}, obtained by perturbing the biinvariant metric via A^l and A^r. In section 4 we compute the A–tensor of the Hopf fibration $S^7 \longrightarrow S^4$, because it is the key to the geometry of g_{ν_1,ν_2}. In section 5 we specify the zero curvatures of g_{ν_1,ν_2}, and in section 6 we describe the horizontal space of $q_{2,0}\colon Sp(2) \longrightarrow E_{2,0}$ with respect to g_{ν_1,ν_2} and hence (via results from section 1) with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. In section 7 we specify the zero curvatures of $E_{2,0}$, first with respect to g_{ν_1,ν_2} and then with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$, proving Theorem A. In section 8 we establish the various topological assertions that we made above, that $E_{2,0}$ is the total space of the unit tangent bundle and that while the cohomology modules of $E_{2,0}/S^1$ are the same as CP^3's the ring structure is different. Using these computations we will conclude that $E_{2,0}$ and $E_{2,0}/S^1$ do not have the homotopy type of any known example of a manifold of positive curvature.

We assume that the reader has a working knowledge of O'Neill's "fundamental equations of a submersion" [18] and the second author's description of the tangent bundle of $Sp(2)$, [24], we will adopt results and notation from both papers, in most cases without further notice. It will also be important for the reader to keep the definitions of the five S^3 actions A^u, A^d, A^l, A^r and $A_{2,0}$ straight. To assist we point out that the letters u, d, l and r stand for "up", "down", "left" and "right" and are meant to indicate the row or column of $Sp(2)$ that is acted upon.

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Dedicated to Detlef Gromoll on his sixtieth birthday

1 Cheeger's Method

In [8] a general method for perturbing the metric on a manifold, M, of nonnegative sectional curvature is proposed. Various special cases of this method were first studied in [4] and [7].

If G is a compact group of isometries of M, then we let G act on $G \times M$ by

$$g(p, m) = (pg^{-1}, gm). \tag{1.1}$$

If we endow G with a biinvariant metric and $G \times M$ with the product metric, then the quotient of (1.1) is a new metric of nonnegative sectional curvature on M. It was observed in [8], that we may expect the new metric to have fewer 0 curvatures and symmetries than the original metric.

In this section we will describe the effect of certain Cheeger perturbations on the curvature tensor of M. Most of the results are special cases of results of [8], we have included them because they can be described fairly succinctly and are central to all of our subsequent computations.

The quotient map for the action (1.1) is

$$q_{G\times M}\colon (p, m) \mapsto pm. \tag{1.2}$$

The vertical space for $q_{G\times M}$ at (p, m) is

$$V_{q_{G\times M}} = \{(-k, k) \mid k \in \mathfrak{g}\}$$

where the $-k$ in the first factor stands for the value at p of the killing field on G given by the circle action

$$(\exp(tk), p) \mapsto p \exp(-kt) \tag{1.3}$$

and the k in the second factor is the value of the killing field

$$\frac{d}{dt} \exp(tk)m \tag{1.4}$$

on M at m.

Until further notice all Cheeger perturbations under consideration will have the property:

$$\text{For all } k_1, k_2 \in \mathfrak{g} \text{ if } \langle (k_1, 0), (k_2, 0) \rangle = 0, \text{ then } \langle (0, k_1), (0, k_2) \rangle = 0, \tag{1.5}$$

where $\mathfrak{g}$ is the Lie algebra of G. Notice that A^u, A^d, A^r and A^l have this property.

In this case the horizontal space for $q_{G\times M}$ is the direct sum

$$H_{q_{G\times M}} = \{(\lambda_2^2 k, \lambda_1^2 k) \ |k \in \mathfrak{g}\} \oplus (\ \{0\} \times H_{O_G} \ ), \tag{1.6}$$

where $\lambda_1 = |(-k, 0)|$ and $\lambda_2 = |(0, k)|$ and H_{O_G} is the space that is normal to the orbit of G. The image of $(\lambda_2^2 k, \lambda_1^2 k)$ under $dq_{G\times M}$ is

$$dq_{G\times M}(\lambda_2^2 k, \lambda_1^2 k) = \frac{d}{dt} p \exp(\lambda_2^2 kt) \exp(\lambda_1^2 kt) m|_{t=0} = dL_{p,*}(\ (\lambda_1^2 + \lambda_2^2) k \ ). \tag{1.7}$$

It follows that the effect of Cheeger's perturbation is to keep H_{O_G} perpendicular to the orbits of G, to keep the metric restricted to H_{O_G} unchanged and to multiply the length of the vector $dL_{p,*}(\ k \ )$ by the factor

$$\frac{\sqrt{(\lambda_2^2 \lambda_1)^2 + (\lambda_1^2 \lambda_2)^2}}{(\lambda_1^2 + \lambda_2^2)\lambda_2} = \sqrt{\frac{\lambda_1^4 + \lambda_2^2 \lambda_1^2}{(\lambda_1^2 + \lambda_2^2)^2}} = \sqrt{\frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2}} \to 1 \text{ as } \lambda_1 \to \infty. \tag{1.8}$$

If b is a fixed biinvariant metric on G and l_1 is a positive real number, then we let g_{l_1} denote the metric we obtain on M via the Riemannian submersion $q_{G\times M}$: $G \times M \longrightarrow M$ when the metric on the G–factor in $G \times M$ is $l_1^2 b$. When $G = S^3$, b will always be the unit metric. As pointed out in (1.8), g_{l_1} converges to the original metric, g, as $l_1 \to \infty$. To emphasize this point we will also denote g by g_∞.

Let TO_G denote the tangent distribution in M to the orbits of G. Then any plane P tangent to M can be written as

$$P = span\{z + k^a, \zeta + k^b\}, \tag{1.9}$$

where $z, \zeta \in H_{O_G}$ and $k^a, k^b \in TO_G$. We let $\hat{P}$ denote the plane in $T(G \times M)$ that is horizontal with respect to $q_{G\times M}$ and satisfies $dp_2(\hat{P}) = P$ where $p_2 \colon G \times M \longrightarrow M$ denotes the projection onto the second factor. If $\xi \in TM$, then $\hat{\xi} \in T(G \times M)$ has the analogous relationship to ξ as $\hat{P}$ has to P. As pointed out in [8] we have the following result.

Proposition 1.10 *Let λ_1^a and λ_2^a denote the lengths of the killing fields on G and M corresponding to k^a via the procedures described in (1.3) and (1.4). Let λ_1^b and λ_2^b have the analogous meaning with respect to k^b.*

- **(i)** *If the curvature of P is positive with respect to g_∞, then the curvature of*

$$dq_{G\times M}(\hat{P}) = span\{\, z + \frac{\lambda_1^{a2} + \lambda_2^{a2}}{\lambda_1^{a2}} k^a,\ \zeta + \frac{\lambda_1^{b2} + \lambda_2^{b2}}{\lambda_1^{b2}} k^b \,\} \tag{1.11}$$

  is positive with respect g_{l_1}.
- **(ii)** *The curvature of $dq_{G\times M}(\hat{P})$ is positive with respect to g_{l_1} if the A–tensor, $A^{q_{G\times M}}$, of $q_{G\times M}$ is nonzero on $\hat{P}$.*
- **(iii)** *If $G = S^3$, then the curvature of $dq_{G\times M}(\hat{P})$ is positive if the projection of P onto TO_G is nondegenerate.*
- **(iv)** *If the curvature of $\hat{P}$ is 0 and $A^{q_{G\times M}}$ vanishes on $\hat{P}$, then the curvature of $dq_{G\times M}(\hat{P})$ is 0.*

Proof (ii) and (iv) are corollaries of O'Neill's horizontal curvature equation.

To prove (i) notice that the curvature of $dq_{G\times M}(\hat{P})$ is positive if the curvature of its horizontal lift, $\hat{P}$, is positive. The curvature of $\hat{P}$ is positive if its image, P, under dp_2 has positive curvature, proving (i).

Let $p_1 \colon G \times M \longrightarrow G$ be the projection onto the first factor. The curvature of $\hat{P}$ is also positive if its image under dp_1 is positively curved. If $G = S^3$, then this is the case, provided the image of $\hat{p}$ is nondegenerate, proving (iii). □

Using (1.8) we get the following.

Proposition 1.12 *Let $A_H \colon H \times M \longrightarrow M$ be an action that is by isometries with respect to both g_∞ and g_{l_1}. Let H_{A_H} denote the distribution of vectors that are perpendicular to the orbits of A_H.*

P is in H_{A_H} with respect to g_∞ if and only if $dq_{G\times M}(\hat{P})$ is in H_{A_H} with respect to g_{l_1}.

Proof Just combine our description of g_{l_1} with the observation that if we square the expression in (1.8) we get the reciprocal of the quantity in (1.11). $\square$

Ultimately we will be studying Cheeger Perturbations via commuting group actions, A_{G_1}, A_{G_2}, that individually have property (1.5). Generalizing our formulas to this situation is a simple matter once we observe the following result.

Proposition 1.13 *Let $G_1 \times G_2$ act isometrically on (M, g). Fix biinvariant metrics b_1 and b_2, on G_1 and G_2. Let g_1 and g_2 be the metrics obtained by doing Cheeger perturbations of (M, g) with G_1 and G_2 respectively.*

- **(i)** *G_2 acts by isometries on (M, g_1) and G_1 acts by isometries on (M, g_2).*
- **(ii)** *Let $g_{1,2}$ denote the metric obtained by doing the Cheeger perturbation with G_2 on (M, g_1) and let $g_{2,1}$ denote the metric obtained by doing the Cheeger perturbation with G_1 on (M, g_2). Then*

$$g_{1,2} = g_{2,1}.$$

In fact $g_{1,2}$ coincides with the metric obtained by doing a single Cheeger perturbation of (M, g) with $G_1 \times G_2$.

- **(iii)** *(k_u^1, k_u^2, u) is horizontal for $G_1 \times G_2 \times M \overset{q_{G_1 \times G_2 \times M}}{\longrightarrow} M$ with respect to $b_1 \times b_2 \times g$ if and only if (k_u^1, u) is horizontal for $G_1 \times M \overset{q_{G_1 \times M}}{\longrightarrow} M$ with respect to $b_1 \times g$ and (k_u^2, u) is horizontal for $G_2 \times M \overset{q_{G_2 \times M}}{\longrightarrow} M$ with respect to $b_2 \times g$.*

Proof The proof of (i) is a routine exercise in the definitions which we leave to the reader.

Part (i) gives us a commutative diagram

$$\begin{array}{ccc} G_1 \times G_2 \times M & \xrightarrow{id \times q_{G_1 \times M}} & G_2 \times M \\ {\scriptstyle id \times q_{G_2 \times M}} \downarrow & & \downarrow {\scriptstyle q_{G_2 \times M}} \\ G_1 \times M & \xrightarrow{q_{G_1 \times M}} & M \end{array}$$

of Riemannian submersions from which (ii) readily follows.

It follows from the diagram that if (κ_1, κ_2, u) is horizontal for $q_{G_1 \times G_2 \times M}$ with respect to $b_1 \times b_2 \times g$, then (κ_1, u) is horizontal for $q_{G_1 \times M}$ with respect to $b_1 \times g$ and (κ_2, u) is horizontal for $q_{G_2 \times M}$ with respect to $b_2 \times g$. This proves the "only if" part of (iii).

The "if" part of (iii) follows from the "only if" part and the observation that $H_{q_{G_1 \times G_2 \times M}} \cap T(G_1 \times G_2) = 0$ via a dimension counting argument. $\square$

Rather than changing the metric of $E_{2,0}$ directly with a Cheeger perturbation, we will change the metric on $Sp(2)$ and then mod out by $A_{2,0}$. The constraint to this approach is that the Cheeger perturbations that we use can not destroy the fact that $A_{2,0}$ is by isometries.

Fortunately it was observed in [8] that if the metric on the G factor in $G \times M$ is biinvariant, then G acts by isometries with respect to g_{l_1}. Therefore we have the following result.

Proposition 1.14 *Let $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ denote a metric obtained from the biinvariant metric on $Sp(2)$ via Cheeger's method using the $S^3 \times S^3 \times S^3 \times S^3$–action, $A^u \times A^d \times A^l \times A^r$.*

Then $A^u \times A^d \times A^l \times A^r$ is by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. In particular, $A_{2,0}$ is by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$.

We include a proof of Proposition 1.14 even though it follows from an assertion on page 624 of [8]. We do this to establish notation that will be used in the sequel, and because the assertion in [8] was not proven.

Proof Throughout the paper we will call the tangent spaces to the orbits of A^l and A^r, V_1 and V_2. The orthogonal complement of $V_1 \oplus V_2$ with respect to the biinvariant metric will be called H. According to Proposition 2.1 in [24], $Sp(2)$ is diffeomorphic to the pull back of the Hopf fibration $S^7 \xrightarrow{h} S^4$ via $S^7 \xrightarrow{a\circ h} S^4$, where $a\colon S^4 \longrightarrow S^4$ is the antipodal map and $S^7 \xrightarrow{h} S^4$ is the Hopf fibration that is given by right multiplication by S^3. Moreover, the metric induced on the pull back by the product of two unit S^7's is biinvariant. Through out the paper our computations will be based on perturbations of the biinvariant metric, $b_{\frac{1}{\sqrt{2}}}$, induced by $S^7(\frac{1}{\sqrt{2}}) \times S^7(\frac{1}{\sqrt{2}})$, where $S^7(\frac{1}{\sqrt{2}})$ is the sphere of radius $\frac{1}{\sqrt{2}}$.

Observe that if k is a killing field on S^3 whose length is 1 with respect to the unit metric, then the corresponding killing field on $Sp(2)$ with respect to either A^l or A^r has length $\frac{1}{\sqrt{2}}$ with respect to $b_{\frac{1}{\sqrt{2}}}$. It follows from this that the quantity (1.8) is constant when we do a Cheeger perturbation on $b_{\frac{1}{\sqrt{2}}}$ via either A^l or A^r. Thus the effect of these Cheeger perturbations is to scale V_1 and V_2, and to preserve the splitting $V_1 \oplus V_2 \oplus H$ and $b_{\frac{1}{\sqrt{2}}}|_H$. The amount of the scaling is < 1 and converges to 1 as the scale, l_1, on the S^3–factor in $S^3 \times Sp(2)$ converges to ∞ and converges to 0 as $l_1 \to 0$. We will call the resulting scales on V_1 and V_2, ν_1 and ν_2, and call the resulting metric g_{ν_1,ν_2}. With this convention the biinvariant metric $b_{\frac{1}{\sqrt{2}}}$ is $g_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}$.

It follows that g_{ν_1,ν_2} is the restriction to $Sp(2)$ of the product metric $S^7_{\nu_1} \times S^7_{\nu_2}$ where S^7_ν denotes the metric obtained from $S^7(\frac{1}{\sqrt{2}})$ by scaling the fibers of h by $\nu\sqrt{2}$. Since A^l and A^r are by symmetries of h in each column, they are by isometries on $S^7_{\nu_1} \times S^7_{\nu_2}$ and hence also on $(Sp(2), g_{\nu_1,\nu_2})$.

Let $g_{l_1^u,l_1^d}$ denote the metric obtained from $b_{\frac{1}{\sqrt{2}}}$ via the Cheeger perturbation with $A^u \times A^d$ when the metric on the $S^3 \times S^3$–factor in $S^3 \times S^3 \times Sp(2)$ is $S^3(l_1^u) \times S^3(l_1^d)$. An argument similar to the one above, using rows instead of columns, shows that $A^u \times A^d$ is by isometries on $(Sp(2), g_{l_1^u,l_1^d})$.

Since $A^u \times A^d$ commutes with $A^l \times A^r$, it follows that $A^u \times A^d$ acts by isometries on $(Sp(2), g_{\nu_1,\nu_2})$. Doing a Cheeger perturbation with $A^u \times A^d$ on $(Sp(2), g_{\nu_1,\nu_2})$ produces a metric $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ which can also be thought of as obtained from $Sp(2)$ via a single Cheeger perturbation with $A^l \times A^r \times A^u \times A^d$. Since $A^l \times A^r$ acts by isometries on $(Sp(2), g_{\nu_1,\nu_2})$ and commutes with $A^u \times A^d$, $A^l \times A^r$ acts by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$.

But $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ can also be obtained by first perturbing with $A^u \times A^d$ and then perturbing with $A^l \times A^r$. So repeating the argument of the proceeding paragraph shows that $A^u \times A^d$ acts by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. $\square$

It follows that $q_{2,0}\colon Sp(2) \longrightarrow E_{2,0} = Sp(2)/A_{2,0}$ is a Riemannian submersion with respect to both g_{ν_1,ν_2} and $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. We will abuse notation and call the induced metrics on $E_{2,0}$, g_{ν_1,ν_2} and $g_{\nu_1,\nu_2,l_1^u,l_1^d}$.

2 Symmetries and their effects on $E_{2,0}$

Since $A^l \times A^r$ commutes with $A_{2,0}$ and is by isometries on $(Sp(2), g_{\nu_1,\nu_2,l_1^u,l_1^d})$, it is by isometries on $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_1^d})$. However on the level of $E_{2,0}$ it has a kernel that at least contains $\mathbb{Z}_2$. To see this just observe that the action of $(-1,-1)$ on $Sp(2)$ via $A^l \times A^r$ is the same as the action of -1 via $A_{2,0}$. It turns out that the kernel is exactly $\mathbb{Z}_2$, and that $A^l \times A^r$ induces the $SO(4)$–action whose existence was asserted in Theorem A(i).

Proposition 2.1 **(i)** *A^l and A^r act freely on $E_{2,0}$.*

(ii) *$E_{2,0}/A^l$ is diffeomorphic to S^4 and the quotient map $p_{2,0}\colon E_{2,0} \longrightarrow E_{2,0}/A^l$ is the bundle of type $(2,0)$.*

(iii) *A^r acts by symmetries of $p_{2,0}$. The induced map on S^4 is the join of the standard $\mathbb{Z}_2$–ineffective S^3–action on S^2 with the trivial action on S^1.*

(iv) *The kernel of the action $A^l \times A^r$ on $E_{2,0}$ is generated by $(-1,-1)$, so $A^l \times A^r$ induces an effective $SO(4)$–action on $E_{2,0}$.*

Proof It is easy to see that $A_{2,0} \times A^l$ and $A_{2,0} \times A^r$ are free on $Sp(2)$ and hence that A^l and A^r are free on $E_{2,0}$, proving (i).

It was observed in [12] that $Sp(2)/(A_{2,0} \times A^l) = E_{2,0}/A^l$ is diffeomorphic to S^4 and in [20] that $p_{2,0}$ is the bundle of type $(2,0)$. Combining these facts proves (ii).

Part (iii) is a special case of Proposition 5.5 in [24].

It follows from (iii) that if (q,p) is in the kernel of $A^l \times A^r$, then $p = -1$. On the other hand, A^l is the principal S^3–action for $E_{2,0} \xrightarrow{p_{2,0}} S^4$. Combining these facts we see that the kernel has order ≤ 2, and we observed above that $(-1,-1)$ is in the kernel. $\square$

The $O(2)$ action on $Sp(2)$, $A_{O(2)}\colon O(2) \times Sp(2) \longrightarrow S(2)$, that is given by

$$A_{O(2)}\colon (A,U) \mapsto AU$$

commutes with $A_{2,0}$ and so is by symmetries of $q_{2,0}\colon Sp(2) \longrightarrow E_{2,0}$. Since it also commutes with $A^l \times A^r$ it acts by isometries on $(E_{2,0}, g_{\nu_1,\nu_2})$. It is also acts by symmetries of $p_{2,0}$ according to Proposition 5.5 in [24]. We shall see, however, that it is not by isometries on $(Sp(2), g_{\nu_1,\nu_2,l_1^u,l_1^d})$. (Note that it commutes with neither A^u nor A^d. This is one of the central ideas behind the curvature computations in the proof of Theorem A.)

We may nevertheless use $A_{O(2)}$ to find the 0–curvatures of $(E_{2,0}, g_{\nu_1,\nu_2})$. (We will then use Proposition 1.10 to see that most of them are not 0 with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$.)

A special case of Proposition 5.7 in [24] is the following.

Proposition 2.2 *Every point in $E_{2,0}$ has a point in its orbit under $A_{SO(2)} \times A^l \times A^r$ that can be represented in $Sp(2)$ by a point of the form*

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix}, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix} \right), \tag{2.3}$$

with $t \in [0, \frac{\pi}{4}], re(\alpha) = 0$ and $|\alpha| = 1$.

We will call points of the form (2.3) representative points.

Notational Convention We have seen that A^r, A^l and $A_{SO(2)}$ all induce actions on $E_{2,0}$, and that A^r and $A_{SO(2)}$ even induce actions on S^4. To simplify the exposition we will make no notational distinction between these actions and their induced actions. Thus A^r stands for an action on $Sp(2)$, $E_{2,0}$, or S^4. The space that is acted on will be clear from the context.

3 The curvature tensor of $(Sp(2),\ g_{\nu_1,\nu_2})$

We will study the curvature of $(Sp(2),\ g_{\nu_1,\nu_2})$ by analyzing the geometry of the riemannian submersion $p_{2,1}\colon Sp(2) \longrightarrow S^7$ given by projection onto the first column.

The moral of our story is that the geometry of $p_{2,1}$ resembles the geometry of $h\colon S^7 \longrightarrow S^4$ very closely. $p_{2,1}$ is a principal bundle with a "connection metric". That is the metric is of the form

$$\langle X,\ Y\rangle_{t,\omega} = \langle dp_E(X),\ dp_E(Y)\rangle_B + t^2\langle \omega(X),\ \omega(Y)\rangle_G,$$

where $G \hookrightarrow E \xrightarrow{p_E} B$ is a principal bundle, $\langle\ ,\ \rangle_G$ is a biinvariant metric on G, $\langle\ ,\ \rangle_B$ is an arbitrary metric on B, and $\omega\colon VE \longrightarrow TE$ is a connection map. In the case of $p_{2,1}$, $G = S^3$ and the action is given by A^r.

It will be important for us to understand how the infinitesimal geometry of $p_{2,1}$ changes as ν_1 and ν_2 change. Combining 2.1–2.3 of [17] with a rescaling argument we get the following.

Proposition 3.1 *Let $G \hookrightarrow E \xrightarrow{p_E} B$ be a principal bundle with a connection metric $\langle\ ,\ \rangle_{t,\omega}$. Let ∇^t, A^t and R^t denote the covariant derivative, A–tensor and curvature tensor of $\langle\ ,\ \rangle_{t,\omega}$. If the paremeter t is omitted from the superscript of one of the objects ∇^t, A^t, R^t, or $\langle\ ,\ \rangle_{t,\omega}$, then implicity that parameter has value 1.*

If e_1, e_2 and Z are horizontal fields and σ, U and τ are vertical fields, then:

(i) *The fibers of p_E are totally geodesic.*

(ii) $A^t_{e_1}e_2 = A_{e_1}e_2$, $A^t_{e_1}\sigma = t^2 A_{e_1}\sigma$, $\nabla^t_{e_1}e_2 = \nabla_{e_1}e_2$, $(\nabla^t_{e_1}\sigma)^v = (\nabla_{e_1}\sigma)^v$, $(\nabla^t_{\sigma}\tau)^v = (\nabla_{\sigma}\tau)^v$,

(iii) $\langle R^t(e_1,e_2)e_2,e_1\rangle_t = \langle R^B(dp_E(e_1),dp_E(e_2))dp_E(e_2),dp_E(e_1)\rangle$
$- 3t^2\|A_{e_1}e_2\|^2$,

(iv) $\langle R^t(\sigma,\tau)\tau,\sigma\rangle_t = t^2\langle R(\sigma,\tau)\tau,\sigma\rangle$,

(v) $\langle R^t(e_1,\sigma)\sigma,e_1\rangle_t = t^4\|A_{e_1}\sigma\|^2$,

(vi) $\langle R^t(\sigma,\tau)U,e_1\rangle_t = 0$, *and*

(vii) $\langle R^t(e_1,e_2)Z,\sigma\rangle_t = t^2\langle R(e_1,e_2)Z,\sigma\rangle$.

The next step is to determine the A and T tensors of $p_{2,1}$.

The vertical space of $p_{2,1}$ is

$$V_{p_{2,1}} = \{\ (0,\sigma) \in TSp(2) \mid \sigma \in V_h\} \equiv V_2,$$

where V_h is the vertical space for the Hopf fibration $h\colon S^7 \longrightarrow S^4$ that is given by left quaternionic multiplication. We will also denote vectors in $V_{p_{2,1}}$ by $(0,\tau)$, and we will often abuse notation and write just τ for $(0,\tau)$.

The T–tensor Given two vector fields, $(0,\sigma_1)$, $(0,\sigma_2)$, with values in $V_{p_{2,1}}$ we compute

$$T^{p_{2,1}}_{(0,\sigma_1)}(0,\sigma_2) = (0,\nabla_{\sigma_1}\sigma_2)^h = (0,0)$$

since V_h is totally geodesic. Therefore

$$T \equiv 0. \tag{3.2}$$

The A–tensor $H_{p_{2,1}}$ splits as the direct sum $H_{p_{2,1}} = V_1 \oplus H$ where V_1 and H are as defined on page 339. So any vector field with values in $H_{p_{2,1}}$ can be written uniquely as $z+w$ where z takes values in H and w takes values in V_1. Given two such vector fields z_1+w_1 and z_2+w_2 we compute

$$\begin{aligned} A^{p_{2,1}}_{z_1+w_1} z_2 + w_2 = (\nabla_{z_1+w_1} z_2 + w_2)^v = (\nabla_{z_1} z_2)^v = \\ (\ 0,\ (\ \nabla^{S^7_{\nu_2}}_{dp_2^2(z_1)} dp_2^2(z_2)\ )^v) = (\ 0,\ \frac{1}{2}[dp_2^2(z_1), dp_2^2(z_2)]^v\ ) = \\ (\ 0,\ \nabla^1_{dp_2^2(z_1)} dp_2^2(z_2)\ ), \end{aligned} \tag{3.3}$$

where $p_2^2\colon Sp(2) \longrightarrow S^7$ denotes the projection of $Sp(2)$ onto its last factor. The second equality is due to the fact that a vector in V_2 is 0 in the first entry and a vector in V_1 is 0 in the last entry.

The upshot of (3.2) and (3.3) is that the T and A tensors of $p_{2,1}$ are essentially the "T and A tensors of h in the last factor". The only difference is that $A^{p_{2,1}}$ has a 3–dimensional kernel, V_1, and A^h has kernel $= 0$.

This principal also holds for the vertizontal A–tensor. If $\sigma = (0,\sigma)$ is a vector field with values in V_2 and $z+w$ is a vector field with values in $H_{p_{2,1}} = H \oplus V_1$, then

$$A^{p_{2,1}}_{z+w}\sigma = (\ 0,\ \nabla^{S^7_{\nu_2}}_{dp_2^2(z)}\sigma)^h = 2\nu_2^2(\ 0, \nabla^{S^7(1)}_{dp_2^2(z)}\sigma\ )^h. \tag{3.4}$$

The only components of the curvature tensor of $Sp(2)$ which are not mentioned in (3.1) are those of the form $\langle R(e_1,\sigma)e_2,\tau\rangle$ and $\langle R(\sigma,\tau)e_1,e_2\rangle$. According to formulas $\{2\}$ and $\{2'\}$ in [18] these are

$$\langle R(e_1,\sigma)e_2,\tau\rangle = -\langle(\nabla_\sigma A)_{e_1}e_2,\tau\rangle - \langle A_{e_1}\sigma, A_{e_2}\tau\rangle$$

and

$$\begin{aligned} &\langle R(\sigma,\tau)e_1,e_2\rangle = \\ -\langle(\nabla_\sigma A)_{e_1}e_2,\tau\rangle - \langle A_{e_1}\sigma, A_{e_2}\tau\rangle &+ \langle(\nabla_\tau A)_{e_1}e_2,\sigma\rangle + \langle A_{e_1}\tau, A_{e_2}\sigma\rangle. \end{aligned} \tag{3.5}$$

(We use the opposite sign convention for the curvature tensor than O'Neill.)

If we choose e_1 and e_2 to be basic horizontal fields the first equation simplifies to

$$\begin{aligned} &\langle R(e_1,\sigma)e_2,\tau\rangle = \\ -\langle\nabla_\sigma(A_{e_1}e_2),\tau\rangle &+ \langle A_{\nabla_\sigma e_1}e_2,\tau\rangle + \langle A_{e_1}\nabla^t_\sigma e_2,\tau\rangle - \langle A_{e_1}\sigma, A_{e_2}\tau\rangle = \\ -\langle\nabla_\sigma(A_{e_1}e_2),\tau\rangle &- \langle A_{e_2}(\nabla_{e_1}\sigma)^h,\tau\rangle + \langle A_{e_1}(\nabla_{e_2}\sigma)^h,\tau\rangle - \langle A_{e_1}\sigma, A_{e_2}\tau\rangle = \\ -\langle\nabla_\sigma(A_{e_1}e_2),\tau\rangle &+ \langle A_{e_1}\sigma, A_{e_2}\tau\rangle - \langle A_{e_2}\sigma, A_{e_1}\tau\rangle - \langle A_{e_1}\sigma, A_{e_2}\tau\rangle = \\ &-\langle\nabla_\sigma(A_{e_1}e_2),\tau\rangle - \langle A_{e_2}\sigma, A_{e_1}\tau\rangle, \end{aligned} \tag{3.6}$$

where the superscript ^{h} denotes the component in H. It appears in the formula because the orthogonal complement of H in the horizontal space for $p_{2,1}$, V_1, is the kernel of A.

Similarly (3.5) simplifies to

$$\begin{aligned} &\langle R(\sigma,\tau)e_1,e_2\rangle = \\ -\langle\nabla_\sigma(A_{e_1}e_2),\tau\rangle - \langle A_{e_2}\sigma, A_{e_1}\tau\rangle &+ \langle\nabla_\tau(A_{e_1}e_2),\sigma\rangle + \langle A_{e_2}\tau, A_{e_1}\sigma\rangle. \end{aligned} \tag{3.7}$$

According to O'Neill, the curvatures with exactly three horizontal terms are

$$\begin{aligned} \langle R(e_1,e_2)e_3,\sigma\rangle &= -\langle(\nabla_{e_3}A)_{e_1}e_2,\sigma\rangle = \\ -\langle\nabla_{e_3}(A_{e_1}e_2),\sigma\rangle &+ \langle A_{(\nabla_{e_3}e_1)^h}e_2,\sigma\rangle + \langle A_{e_1}(\nabla_{e_3}e_2)^h,\sigma\rangle. \end{aligned} \tag{3.8}$$

Through out the rest of this section we will let the superscript ^{v} denote the component in V_1.

By (3.3) and the fact that the analogous equation holds for the Hopf fibration we have

$$\begin{aligned} &\langle R(e_1^h,e_2^h)e_3^h,\sigma\rangle = \\ -\langle\nabla_{e_3^h}(A_{e_1^h}e_2^h),\sigma\rangle &+ \langle A_{(\nabla_{e_3^h}e_1^h)^h}e_2^h,\sigma\rangle + \langle A_{e_1^h}(\nabla_{e_3^h}e_2^h)^h,\sigma\rangle = 0. \end{aligned} \tag{3.9}$$

Of course

$$\langle R(e_1^v, e_2^v)e_3, \sigma\rangle = 0 \tag{3.10}$$

since V_1 is the kernel of A. Using this again we get

$$\langle R(e_1^h, e_2^v)e_3^v, \sigma\rangle = \langle A_{e_1^h}(\ \nabla_{e_3^v} e_2^v\ )^h, \sigma\rangle = 0, \tag{3.11}$$

where the last equality is because $(\ \nabla_{e_3^v} e_2^v\ )^h = 0$, since the orbits of A^l are totally geodesic.

Also

$$\begin{aligned} &\langle R(e_1^h, e_2^v)e_3^h, \sigma\rangle = \\ -\langle \nabla_{e_3^h}(A_{e_1^h} e_2^v), \sigma\rangle + \langle A_{(\nabla_{e_3^h} e_1^h)^h} e_2^v, \sigma\rangle &+ \langle A_{e_1^h}(\nabla_{e_3^h} e_2^v)^h, \sigma\rangle = \\ \langle A_{e_1^h}(\nabla_{e_3^h} e_2^v)^h, \sigma\rangle &= -\langle (\nabla_{e_3^h} e_2^v)^h, A_{e_1^h}\sigma\rangle. \end{aligned} \tag{3.12}$$

Keeping in mind that H is the horizontal space for $h \circ p_{2,1}$ and that V_1 is in the vertical space for $h \circ p_{2,1}$ we can rewrite the right hand side of (3.12) and get

$$\langle R(e_1^h, e_2^v)e_3^h, \sigma\rangle = -\langle A_{e_3^h}^{h\circ p_{2,1}} e_2^v, A_{e_1^h}\sigma\rangle. \tag{3.13}$$

Using (3.13) and the antisymmetry of the curvature tensor we get

$$\langle R(e_1^v, e_2^h)e_3^h, \sigma\rangle = \langle A_{e_3^h}^{h\circ p^{2,1}} e_1^v, A_{e_2^h}\sigma\rangle. \tag{3.14}$$

Combining the first Bianchi identity with (3.13) and (3.14) we find that

$$\begin{aligned} \langle R(e_1^h, e_2^h)e_3^v, \sigma\rangle &= -\langle R(e_3^v, e_1^h)e_2^h, \sigma\rangle - \langle R(e_2^h, e_3^v)e_1^h, \sigma\rangle = \\ &-\langle A_{e_2^h}^{h\circ p_{2,1}} e_3^v, A_{e_1^h}\sigma\rangle + \langle A_{e_1^h}^{h\circ p_{2,1}} e_3^v, A_{e_2^h}\sigma\rangle. \end{aligned} \tag{3.15}$$

It turns out that the sum of the two terms on the right is always 0 and hence that

$$\langle R(e_1^h, e_2^h)e_3^v, \sigma\rangle = 0. \tag{3.16}$$

This can be seen by direct computation combining Proposition 4.1 (below) with section 6 of [24], but there are many details to check. A less direct, but much quicker proof begins by using the second equation in 3.1(ii) to observe that the right hand side of (3.15) is $4\nu_1^2\nu_2^2$ times the corresponding curvature quantity for the biinvariant metric. Therefore it suffices to prove (3.16) for the biinvariant metric. To do this observe that distributions V_1 and V_2 have the form

$$V_{1,Q} = \left\{ L_{Q,*}\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; \beta \in \mathbb{H} \text{ and } re(\beta) = 0 \right\}$$

$$V_{2,Q} = \left\{ L_{Q,*}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \;\middle|\; \beta \in \mathbb{H} \text{ and } re(\beta) = 0 \right\},$$

where $L_{Q,*}$ denotes the differential of left translation by Q.

Thus $[E_3^v, \Sigma] = 0$, where E_3^v and Σ denote the extensions of e_3^v and σ to left invariant fields. Therefore

$$\langle R(e_1^h, e_2^h)e_3^v, \sigma\rangle = -\frac{1}{4}\langle [E_1^h, E_2^h],\ [E_3^v, \Sigma]\rangle = 0,$$

where E_1^h and E_2^h denote the extensions of e_1^h and e_2^h to left invariant fields, proving (3.16).

Combining (3.10), (3.11), and (3.16) gives us

$$\langle R(e_1, e_2)e_3^v, \sigma\rangle = 0. \tag{3.17}$$

A review of the results of this section shows that the only difference between the infinitesimal geometries of h and $p_{2,1}$ is the fact that for $p_{2,1}$ the terms (3.13) and (3.14) are not always 0 but for h the curvatures terms with exactly three horizontal vectors are always 0. This is the root cause of many of the 0 curvatures in $(Sp(2), g_{\nu_1,\nu_2})$.

4 The A–tensor of h

In section 3 we showed that the A–tensor of $p_{2,1}$ is essentially the A–tensor of the Hopf fibration, h, in the last factor. A^h certainly ought to be very well known, but the authors are not aware of any computation of it in the literature. So for completeness we prove the following.

Proposition 4.1 *Let N be a point in S^7.*

(i) *The vertical space of h at N, $V_{h,N}$, is*

$$\{N\beta \mid \beta \in \mathbb{H}, re(\beta) = 0\}.$$

(ii) *If z is in the horizontal space, $H_{h,N}$, for h at N, then*

$$A_z^h N\beta = z\beta.$$

Proof A proof of this for the case of the complex Hopf fibration can be found in [18]. The proof for the quaternionic case is nearly identical.

(i) is an immediate consequence of the definition h.

To prove (ii) we let a superscript ^{h} denote the horizontal component with respect to h and we compute

$$(\nabla_z^{S^7} N\beta)^h = (\nabla_z^{\mathbb{R}^8} N\beta)^h = ((\nabla_z^{\mathbb{R}^8} N)\beta)^h = (z\beta)^h = z\beta,$$

where for the last equality we have used the fact that $H_{h,N}$ is invariant under right quaternionic multiplication. □

5 The Curvature of Generic Planes in Sp(2)

Because the metric g_{ν_1,ν_2} is a simpler metric than $g_{\nu_1,\nu_2 l_1^u, l_1^d}$, we study its 0 curvatures in this section. Where all statements about curvatures of $Sp(2)$ are understood to be with respect to g_{ν_1,ν_2}, for some $\nu_1, \nu_2 < \frac{1}{\sqrt{2}}$.

A plane P tangent to $Sp(2)$ has the form

$$P = span\{z + w, \zeta + v\}, \tag{5.1}$$

where $z, \zeta \in H$ and $w, v \in V_1 \oplus V_2$.

Theorem 5.2

(i) *The curvature of P is positive if the projection of P on to one of H, V_1, or V_2 is two dimensional.*

(ii) *If all three projections in (i) are degenerate, then the curvature of P is 0, if its projection onto H is 0.*

(iii) *If all three projections in (i) are degenerate and the projection of P onto H is one dimensional, then we may assume that $z \neq 0$ and $\zeta = 0$. The curvature of P is positive unless*

$$A_z^{h\circ p_{2,1}} v^1 + A_z^{p_{1,2}} v^2 = 0. \tag{5.3}$$

(iv) *If all of the hypotheses of (iii) hold and (5.3) holds, then the curvature of P is 0.*

Remark Since v^2 is vertical for $h \circ p_{1,2}$ and $A_z^{p_{1,2}} v^2 = A_z^{h\circ p_{1,2}} v^2$, (5.3) can be thought of as $A_z^{h\circ p_{1,2}}(v^1 + v^2) = 0$. Since the fibers of $h \circ p_{1,2}$ are totally geodesic this version of (iii) is a corollary of 3.1(v). We give an alternative proof below.

Proof of (iii) and (iv) If the projection of P onto H, V_1 and V_2 is degenerate, then by replacing the second vector with the appropriate linear combination of the vectors in (5.1) we may assume that P has the form

$$P = span\{z + w^1 + w^2,\ v^1 + v^2\}, \tag{5.4}$$

where $w^1, v^1 \in V_1$ and $w^2, v^2 \in V_2$ are multiples of each other.

By replacing the first vector in (5.4) by the appropriate vector in P we can further assume that $w^2 = 0$, so P has the form

$$P = span\{z + w^1, v^1 + v^2\},$$

where w^1 and v^1 are multiples of each other.

It follows that

$$\begin{aligned} &\langle\ R(z + w^1,\ v^1 + v^2)\ v^1 + v^2,\ z + w^1\ \rangle = \\ &\langle\ R(z,\ v^1)\ v^1,\ z\ \rangle + 2\langle\ R(z,\ v^1)\ v^2,\ z\ \rangle + \langle\ R(z,\ v^2)\ v^2,\ z\ \rangle = \\ &\|\ A_z^{h\circ p_{2,1}} v^1\ \|^2 + 2\langle\ A_z^{h\circ p_{2,1}} v^1,\ A_z^{p_{1,2}} v^2 \rangle + \|\ A_z^{p_{1,2}} v^2\ \|^2 = \\ &\|\ A_z^{h\circ p_{2,1}} v^1 + A_z^{p_{1,2}} v^2\ \|^2, \end{aligned} \tag{5.5}$$

where we have used the results of section 3 to conclude that many terms are 0.

(iii) and (iv) follow from (5.5). □

Notation Since the A–tensors in (5.5) are essentially the A–tensors of h in the first and second factor we will shorten the notation and set

$$A^1 = A^{h\circ p_{2,1}} \text{ and}$$
$$A^2 = A^{p_{1,2}},$$

and we will denote the inner product in (5.5) by

$$\langle\ A_z^{h\circ p_{2,1}} v^1,\ A_z^{p_{1,2}} v^2 \rangle = \langle\ A_z^1 v^1,\ A_z^2 v^2 \rangle_a.$$

The idea behind this notation is that we are taking a certain type of inner product between the A–tensors of h in the first and second factors. The subscript $_a$ is meant to remind us of the role that the antipodal map of S^4 plays in determining this inner product.

Proof of (i) If the projection of P onto either V_1 or V_2 is nondegenerate, then according to 1.10(iii) the curvature of P is positive.

So we may assume that the projection of P onto H is nondegenerate and the projections onto both V_1 and V_2 are degenerate. Since the projection of P onto

V_2 is degenerate we may replace the first vector with another vector in P that satisfies

$$w^2 = 0. \tag{5.6}$$

Since the projection onto V_1 is also degenerate we may replace our vectors by appropriate linear combinations to get that either

$$v^1 = 0 \text{ or } w^1 = 0. \tag{5.7}$$

If $v^1 = 0$, P is spanned by

$$\{z + w^1, \zeta + v^2\}.$$

It follows that

$$\begin{aligned}\langle R(z + w^1, \zeta + v^2)\zeta + v^2, z + w^1\rangle_{g_{\nu_1,\nu_2}} \geq \\ |z|^2|\zeta^\perp|^2 + |A^1_\zeta w^1|^2 + |A^2_z v^2|^2 + \\ \langle R(z,\zeta)v^2, w^1\rangle + \langle R(z,v^2)\zeta, w^1\rangle + \langle R(w^1,v^2)\zeta, z\rangle + \langle R(w^1,\zeta)v^2, z\rangle + \\ \langle R(z,v^2)v^2, w^1\rangle + \langle R(w^1,v^2)v^2, z\rangle,\end{aligned}$$

where $\zeta^\perp$ denotes the component of ζ that is perpendicular to z, and we have relied heavily on results of the previous two sections to conclude that many of the components of this curvature are 0.

Using (3.17) and the symmetries of the curvature tensor we see that the fourth and the sixth terms on the right are 0. The last two terms are also 0. One way to see this is to combine (3.6) with the fact that w^1 is in the kernel of $A^{p_{2,1}}$.

Use (3.13) to evaluate the other two terms to get

$$\begin{aligned}\langle R(z + w^1, \zeta + v^2)\zeta + v^2, z + w^1\rangle_{g_{\nu_1,\nu_2}} \geq \\ |z|^2|\zeta^\perp|^2 + |A^1_\zeta w^1|^2 + |A^2_z v^2|^2 - 2\langle A^1_z w^1, A_\zeta v^2\rangle_a.\end{aligned} \tag{5.8}$$

For $\xi \in TSp(2)$ let $|\xi|_{1,1} = |dp_{2,1}\xi|_1$ and $|\xi|_{1,2} = |dp^2_2\xi|_1$ where $|\cdot|_1$ denotes the unit norm on S^7. Combining 3.1(v), (3.4) and (4.1) and meticulously chasing through the definitions we get that

$$\begin{aligned}|A^1_\zeta w^1|^2 = \nu_1^4\ |\zeta|^2_{1,1}\ |w^1|^2_{1,1}, \\ |A^2_z v^2|^2 = \nu_2^4\ |z|^2_{1,2}\ |v^2|^2_{1,2}, \text{ and} \\ |\ 2\langle A^1_z w^1, A_\zeta v^2\rangle_a\ | \leq 2\nu_1^2\ \nu_2^2\ |\zeta|_{1,1}\ |w^1|_{1,1}\ |z|_{1,2}\ |v^2|_{1,2}.\end{aligned} \tag{5.9}$$

From (5.9) we see that sum of the last three terms in (5.8) is always nonnegative so

$$\langle R(z + w^1, \zeta + v^2)\zeta + v^2, z + w^1\rangle_{g_{\nu_1,\nu_2}} \geq |z|^2|\zeta^\perp|^2 > 0, \tag{5.10}$$

where the last inequality is due to our hypothesis that the projection of P onto H is nondegenerate.

It remains to consider the case $w^1 = 0$ in (5.7). If so, our plane is spanned by

$$\{z, \zeta + v^1 + v^2\}.$$

Arguing as in (5.8) and (5.10) we can show that

$$\begin{gathered}\langle R(z, \zeta + v^1 + v^2)\zeta + v^1 + v^2, z\rangle \geq \\ |z|^2|\zeta^{\perp}|^2 + |A_z^1 v^1|^2 + |A_z^2 v^2|^2 + 2\langle A_z^1 v^1, A_z^2 v^2\rangle_a \geq \\ |z|^2|\zeta^{\perp}|^2 > 0.\end{gathered}$$

□

Proof of (ii) If $P = span\{v^1, w^2\}$ where $v^1 \in V_1$ and $w^2 \in V_2$, then $curv(P) = 0$ because the T-tensor of $p_{2,1}$ is 0 and V_1 is the kernel of the A-tensor of $p_{2,1}$. □

6 Horizontal space of $q_{2,0}$

From this point forward we will use the notation from section 6 in [24] for specific tangent vectors to $Sp(2)$. Recall that $t \in [0, \frac{\pi}{4}]$ is one of the coordinates of our representative points (see Proposition 2.3).

Proposition 6.1 *For $t \in [0, \frac{\pi}{4})$ the horizontal space of $q_{2,0}$ with respect to g_{ν_1,ν_2} is spanned by*

$$\begin{gathered}\{\ (x,x),\ \ (-y,y),\ \ (\eta_1, \eta_1 + \tan(2t)\frac{\vartheta_1}{\nu_2^2}),\ \ (\eta_2, \eta_2 + \tan(2t)\frac{\vartheta_2}{\nu_2^2}), \\ (-\frac{\mathfrak{v}}{\nu_1^2}, \frac{\mathfrak{v}}{\nu_2^2}),\ \ (-\frac{\vartheta_1}{\nu_1^2}, \frac{\vartheta_1}{\nu_2^2}),\ \ (-\frac{\vartheta_2}{\nu_1^2}, \frac{\vartheta_2}{\nu_2^2})\ \}.\end{gathered}$$

Notation We will call the seven vectors in (6.1) $x^{2,0}$, $y^{2,0}$, $\eta_1^{2,0}$, $\eta_2^{2,0}$, $\mathfrak{v}^{2,0}$, $\vartheta_1^{2,0}$, and $\vartheta_2^{2,0}$ respectively. We will call the set consisting of the first four $basis(H_{2,0})$, and the set consisting of the last three $basis(V_{2,0})$.

Sketch of Proof The proof is just straight forward computations of inner products. Many of the required computations were done in Proposition 6.5 of [24]. □

A corollary of 5.2(i) is the following.

Corollary 6.2 *Let P be a plane in $E_{2,0}$ whose t–coordinate is in $[0, \frac{\pi}{4})$.*

If the horizontal lift of P with respect to g_{ν_1,ν_2} has a nondegenerate projection onto either $H_{2,0}$ or $V_{2,0}$, then P is positively curved with respect to g_{ν_1,ν_2}.

Proof We will combine Theorem 5.2(i) with the following observations.

- **(a)** Any plane in $H_{2,0}$ has a nondegenerate projection (with respect to the splitting $H \oplus V_1 \oplus V_2$) onto H.
- **(a′)** $V_{2,0} \subset V_1 \oplus V_2$.

It follows from (a) and (a′) that if the projection of P onto $H_{2,0}$ is nondegenerate, then so is the projection onto H (with respect to the splitting $H \oplus V_1 \oplus V_2$).

- **(b)** Any plane in $V_{2,0}$ has a nondegenerate projection (with respect to the splitting $H \oplus V_1 \oplus V_2$) onto V_1.
- **(b′)** $H_{2,0} \subset H \oplus V_2$.

It follows from (b) and (b′) that if the projection of P onto $V_{2,0}$ is nondegenerate, then so is the projection of P onto V_1 (with respect to the splitting $H \oplus V_1 \oplus V_2$). □

Proposition 6.3 *For $t = \frac{\pi}{4}$ the horizontal space of $q_{2,0}$ with respect to g_{ν_1,ν_2} is spanned by*

$$(x,x),\ (-y,y),\ (\,0,\ \frac{\vartheta_1}{\nu_2^2}),\ (\,0,\ \frac{\vartheta_2}{\nu_2^2}),$$
$$(-\frac{\mathfrak{v}}{\nu_1^2},\ \frac{\mathfrak{v}}{\nu_2^2}\,),\ (-\frac{\vartheta_1}{\nu_1^2},\ \frac{\vartheta_1}{\nu_2^2}\,),\ (-\frac{\vartheta_2}{\nu_1^2},\ \frac{\vartheta_2}{\nu_2^2}\,).$$

Sketch of proof Again the proof is just straight forward computations of inner products. In fact the computations that proved (6.1) will suffice for all of the vectors except $(\,0,\ \frac{\vartheta_1}{\nu_2^2})$ and $(\,0,\ \frac{\vartheta_2}{\nu_2^2})$. These are just multiples of the limits of $\frac{\eta_1^{2,0}}{|\eta_1^{2,0}|}$ and $\frac{\eta_2^{2,0}}{|\eta_2^{2,0}|}$ as $t \to \frac{\pi}{4}$. □

Notational Convention Let $\pi\colon M \longrightarrow B$ be a riemannian submersion. To simplify the exposition we will make no notational distinction between a vector that is tangent to B and its horizontal lift to M.

Remark Since the seven vectors listed in (6.3) are limits of (function multiples of) the seven vectors listed in (6.1) we will call them $x^{2,0}$, $y^{2,0}$, $\eta_1^{2,0}$, $\eta_2^{2,0}$,

$\mathfrak{v}^{2,0}$, $\vartheta_1^{2,0}$, and $\vartheta_2^{2,0}$. Notice that via $q_{2,0} \circ p_{2,0}$, $x^{2,0}$ and $y^{2,0}$ project to the unit normals of S^2_{im}, where S^2_{im} is the 2–sphere in S^4 that is fixed by the action induced on S^4 by A^r via $p_{2,0}$: $E_{2,0} \longrightarrow S^4$ (cf the notational remark in section 6 of [24]). Because of this their definitions do not seem to be very canonical when $t = \frac{\pi}{4}$. The approach we have taken is to define them at representative points with $t = \frac{\pi}{4}$ as extensions of their definitions at representative points when t is generic. We will extend these definitions (as needed) to all points with $t = \frac{\pi}{4}$ by letting the isometry group act.

Also notice that the vectors $(\frac{\vartheta_1}{\nu_1^2}, 0), (\frac{\vartheta_2}{\nu_1^2}, 0)$ are in $H_{q_{2,0}}$ when $t = \frac{\pi}{4}$. Notice furthermore that at $t = \frac{\pi}{4}$

$$curv_{\nu_1,\nu_2}(\, (\, \frac{\vartheta_i}{\nu_1^2}, \, 0 \,), \eta_j^{2,0} \,) = 0, \tag{6.4}$$

for any $(\, \frac{\vartheta_i}{\nu_1^2}, \, 0 \,) \in span\{(\, \frac{\vartheta_1}{\nu_1^2}, \, 0 \,), (\, \frac{\vartheta_2}{\nu_1^2}, \, 0 \,)\}$ and any $\eta_j^{2,0} \in span\{\eta_1^{2,0}, \eta_2^{2,0}\}$. Here and in the sequel $curv(v, w)$ stands for $\langle R(v, w)w, v\rangle$.

7 Zero Curvatures of $(E_{2,0}, g_{\nu_1,\nu_2})$ and $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_1^d})$

It follows from (6.2) that any 0–curvature planes for $(E_{2,0},\ g_{\nu_1,\nu_2})$ have the form

$$P = span\{\zeta, w\}, \tag{7.1}$$

where $\zeta \in H_{2,0}$, $w \in V_{2,0}$ and $t \in [0, \frac{\pi}{4})$. About such planes we will prove the following.

Proposition 7.2

(i) *For $t \in [0, \frac{\pi}{4})$ the 0–curvature planes of $(E_{2,0}, g_{\nu_1,\nu_2})$ satisfy (7.1) and also,*

$$\begin{aligned} \zeta &\in span\{x^{2,0}, \eta^{2,0}\} \\ w &\in span\{\vartheta^{2,0}\}, \end{aligned} \tag{7.3}$$

where $\eta^{2,0}$ is a convex combination of $\{\eta_1^{2,0}, \eta_2^{2,0}\}$ and $\vartheta^{2,0}$ is the same convex combination of $\{\vartheta_1^{2,0}, \vartheta_2^{2,0}\}$.

(ii) *When $t = \frac{\pi}{4}$ the planes of the form (6.4) have 0 curvature in $Sp(2)$ with respect to g_{ν_1,ν_2}, and, in addition, given any convex combination z of $\{x^{2,0}, y^{2,0}\}$ and any convex combination w of $\{ \, (\, -\frac{\vartheta_1}{\nu_1^2}, \, 0 \,), \, (\, -\frac{\vartheta_2}{\nu_1^2}, \, 0 \,) \, \}$,* *there is a unique convex combination* $v_{w,z}$ *of* $\{\ (\ 0,\ \frac{\vartheta_1}{\nu_1^2}\ ),\ (\ 0,\ \frac{\vartheta_2}{\nu_1^2}\ )\ \}$, *so that*

$$curv_{(Sp(2),g_{\nu_1,\nu_2})}(z+\lambda w, w+v_{w,z})=0 \tag{7.4}$$

for all $\lambda \in \mathbb{R}$.
There are no other zero curvatures when $t=\frac{\pi}{4}$.

Proof To prove (i) split ζ into $\zeta = z+\eta$ where $z \in span\{x^{2,0}, y^{2,0}\}$ and $\eta \in span\{\eta_1^{2,0}, \eta_2^{2,0}\}$. To be more concrete suppose that η is a multiple of $\eta_1^{2,0}$.

Suppose w has the form $w = (-\frac{N_1\beta}{\nu_1^2}, \frac{N_2\beta}{\nu_2^2})$ for some purely imaginary unit quaternion β. Notice that

$$\langle A^1_{y^{2,0}}(-\frac{N_1\beta}{\nu_1^2}),\ A^2_{y^{2,0}}\frac{N_2\beta}{\nu_2^2}\rangle = 1, \tag{7.5}$$

and

$$\langle A^1_{y^{2,0}}(-\frac{N_1\beta}{\nu_1^2}),\ A^2_{\xi}\frac{N_2\beta}{\nu_2^2}\rangle = \langle A^1_{\xi}(-\frac{N_1\beta}{\nu_1^2}),\ A^2_{y^{2,0}}\frac{N_2\beta}{\nu_2^2}\rangle = 0, \tag{7.6}$$

if $\xi \in span\{x^{2,0}, \eta_1^{2,0}, \eta_2^{2,0}\}$. Combining 5.2(iii),(iv) with (7.5) and (7.6) we get that the curvature of P is positive except possibly if $\zeta \in span\{x^{2,0}, \eta\}$.

On the other hand, the projection of P onto V_2 is nondegenerate unless $\eta = 0$ or

$$p_{V_2}(\eta) \text{ and } p_{V_2}(w) \text{ are multiples of each other.} \tag{7.7}$$

So P is positively curved unless $\eta = 0$ or w is a multiple of the same convex combination of $\{\vartheta_1^{2,0}, \vartheta_2^{2,0}\}$ as η is of $\{\eta_1^{2,0}, \eta_2^{2,0}\}$.

To complete the proof of (i) it remains to consider the case when $\zeta = x^{2,0}$ in which it is easy to see, using 5.2(iii),(iv) that P is positively curved unless $w \in span\{\vartheta_1^{2,0}, \vartheta_2^{2,0}\}$.

We saw in (6.3) that at points where $t = \frac{\pi}{4}$,

$$\begin{aligned} H_{q_{2,0}} = span\{&(x,x),\ (-y,y),\ (\ 0,\ \frac{\vartheta_1}{\nu_2^2}),\ (\ 0,\ \frac{\vartheta_2}{\nu_2^2}), \\ &(-\frac{\mathfrak{v}}{\nu_1^2},\ \frac{\mathfrak{v}}{\nu_2^2}\ ),\ (-\frac{\vartheta_1}{\nu_1^2}, 0\ ),\ (-\frac{\vartheta_2}{\nu_1^2},\ 0\ )\}. \end{aligned} \tag{7.8}$$

Using (7.8) it is easy to see that the only planes in $H_{q_{2,0}} \cap (V_1 \oplus V_2)$ whose projections onto both V_1 and V_2 are degenerate are those of the form (6.4).

It follows that the only other possible 0's are of the type described in 5.2(iii),(iv). Combining this with straightforward computations of A–tensors completes the proof of (ii), and even supplies explicit solutions for $v_{w,z}$ in terms of w and z. For example when $z=x^{2,0}$ and $w=(\ -\frac{\vartheta_1}{\nu_1^2},\ 0\ )$, $v_{w,z}=(\ 0,\ \frac{\vartheta_1}{\nu_2^2}\ )$, and when $z=y^{2,0}$ and $w=(\ -\frac{\vartheta_1}{\nu_1^2},\ 0\ )$, $v_{w,z}=(\ 0,\ -\frac{\vartheta_1}{\nu_2^2}\ )$. □

Remark We have shown that the planes in (7.2) have 0 curvature in $(Sp(2), g_{\nu_1,\nu_2})$. They also have zero curvature in $(E_{2,0}, g_{\nu_1,\nu_2})$. To verify this we have to evaluate the A–tensor of $q_{2,0}$ on these planes. It turns out that it is 0. We will not verify this since most of these planes become positively curved with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$.

Now we study the zero curvatures of $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_1^d})$. To do this we adopt the following convention.

Notational Convention Let $H_{q_{2,0},\ g_{\nu_1,\nu_2}}$ denote the horizontal space of $q_{2,0}$ with respect to g_{ν_1,ν_2} and $H_{q_{2,0},\ g_{\nu_1,\nu_2,l_1^u,l_1^d}}$ the horizontal space of $q_{2,0}$ with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. Let $q^{diag}_{A^u\times A^d}\colon S^3\times S^3\times Sp(2)\longrightarrow Sp(2)$ be the quotient map given in (1.1) for $A^u\times A^d$. According to (1.12) P is in $H_{q_{2,0},\ g_{\nu_1,\nu_2}}$ if and only if $dq^{diag}_{A^u\times A^d}(\hat{P})$ is in $H_{q_{2,0},\ g_{\nu_1,\nu_2,l_1^u,l_1^d}}$. To keep the notation simpler we will think of this correspondence as a parameterization of $H_{q_{2,0},\ g_{\nu_1,\nu_2,l_1^u,l_1^d}}$ by $H_{q_{2,0},\ g_{\nu_1,\nu_2}}$ and we will denote vectors and planes in $H_{q_{2,0},\ g_{\nu_1,\nu_2 l_1^u,l_1^d}}$ by the corresponding vectors and planes in $H_{q_{2,0},\ g_{\nu_1,\nu_2}}$. We will do this without any further mention or change in notation. For example $x^{2,0}$ now stands for $dq^{diag}_{A^u\times A^d}(\hat{x}^{2,0})$.

With this convention it follows from 1.10(i) that the zero curvatures of $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ are a subset of the zero curvatures of g_{ν_1,ν_2}. We will show that most of the planes in (7.2) have nondegenerate projections onto the orbits of A^u and hence are positively curved with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ according to 1.10(iii). The root cause of this is that $A_{SO(2)}$ is not by isometries with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$. This seems extremely plausible because $A_{SO(2)}$ commutes with neither A^u nor A^d. A rigorous proof that it does not act by isometries follows from our curvature computations.

In any case we no longer know that $A_{SO(2)}$ is by isometries, so (2.3) are no longer our representative points.

The representative points are now

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\left(\begin{pmatrix} \cos t \\ \alpha\sin t \end{pmatrix},\ \begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix}\right)=$$

$$\left(\begin{pmatrix} \cos\theta\cos t + \alpha\sin\theta\sin t \\ -\sin\theta\cos t + \alpha\cos\theta\sin t \end{pmatrix}, \begin{pmatrix} \alpha\cos\theta\sin t + \sin\theta\cos t \\ -\alpha\sin\theta\sin t + \cos\theta\cos t \end{pmatrix}\right) \equiv (N_1(\theta,t),\ N_2(\theta,t)), \tag{7.9}$$

for all $t \in [0, \frac{\pi}{4}]$ and for all $\theta \in [0,\pi)$. The interval for θ is $[0,\pi)$ because $A_{SO(2)}$ is $\mathbb{Z}_2$–ineffective.

Using this we will show the following.

Proposition 7.10 *For any $t \in [0, \frac{\pi}{4})$ the planes of the form*

$$P = span\{\eta^{2,0}, \vartheta^{2,0}\} \tag{7.11}$$

have positive curvature with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ unless $\theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{\pi}{2}$.

Here $\eta^{2,0}$ and $\vartheta^{2,0}$ stand for (the same) convex combinations of $\{\eta_1^{2,0}, \eta_2^{2,0}\}$ and $\{\vartheta_1^{2,0}, \vartheta_2^{2,0}\}$ respectively.

Proof The main idea is that most planes are eliminated because their projection onto the orbits of A^u are nondegenerate. Combining 1.10(iii) with (1.13) we see that it is actually enough to check that a projection is degenerate with respect to $b_{\frac{1}{\sqrt{2}}}$.

The tangent space to the orbit of A^u is spanned by the $b_{\frac{1}{\sqrt{2}}}$–orthogonal basis

$$\{ U_\alpha, U_{\gamma_1}, U_{\gamma_2} \} \equiv$$
$$\left\{ \left(\begin{pmatrix} \alpha(\cos\theta\cos t + \alpha\sin\theta\sin t) \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha(\alpha\cos\theta\sin t + \sin\theta\cos t) \\ 0 \end{pmatrix}\right), \right.$$
$$\left(\begin{pmatrix} \gamma_1(\cos\theta\cos t + \alpha\sin\theta\sin t) \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1(\alpha\cos\theta\sin t + \sin\theta\cos t) \\ 0 \end{pmatrix}\right),$$
$$\left.\left(\begin{pmatrix} \gamma_2(\cos\theta\cos t + \alpha\sin\theta\sin t) \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_2(\alpha\cos\theta\sin t + \sin\theta\cos t) \\ 0 \end{pmatrix}\right) \right\}, \tag{7.12}$$

where γ_1 and γ_2 are purely imaginary, unit quaternions that satisfy $\gamma_1\gamma_2 = \alpha$.

Before we can find the projections we also need a formula for (η_1, η_1) along the orbits of $A_{SO(2)}$. At points of the form (7.9) (η_1, η_1) is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\left(\begin{pmatrix} -\gamma_1\sin t \\ \gamma_2\cos t \end{pmatrix}, \begin{pmatrix} \gamma_2\cos t \\ -\gamma_1\sin t \end{pmatrix}\right) =$$
$$\left(\begin{pmatrix} -\gamma_1\cos\theta\sin t + \gamma_2\sin\theta\cos t \\ \gamma_1\sin\theta\sin t + \gamma_2\cos\theta\cos t \end{pmatrix}, \begin{pmatrix} \gamma_2\cos\theta\cos t - \gamma_1\sin\theta\sin t \\ -\gamma_2\sin\theta\cos t - \gamma_1\cos\theta\sin t \end{pmatrix}\right).$$

To find the projections we compute

$$\begin{aligned}
&\langle (\eta_1, \eta_1), U_{\gamma_1}\rangle = \\
&\left\langle \left(\begin{pmatrix} -\gamma_1 \cos\theta \sin t + \gamma_2 \sin\theta \cos t \\ \gamma_1 \sin\theta \sin t + \gamma_2 \cos\theta \cos t \end{pmatrix}, \begin{pmatrix} \gamma_2 \cos\theta \cos t - \gamma_1 \sin\theta \sin t \\ -\gamma_2 \sin\theta \cos t - \gamma_1 \cos\theta \sin t \end{pmatrix} \right), \right. \\
&\left. \left(\begin{pmatrix} \gamma_1 \cos\theta \cos t - \gamma_2 \sin\theta \sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma_2 \cos\theta \sin t + \gamma_1 \sin\theta \cos t \\ 0 \end{pmatrix} \right) \right\rangle = \\
&\frac{1}{2}(-\cos^2\theta \sin t \cos t - \sin^2\theta \cos t \sin t - \cos^2\theta \sin t \cos t - \sin^2\theta \cos t \sin t) = \\
&- \sin t \cos t = -\frac{1}{2} \sin 2t.
\end{aligned} \tag{7.13}$$

According to (6.1) we should compute the inner products of $(-\frac{\vartheta_1}{.5}, \frac{\vartheta_1}{.5})$ and $(-\frac{\vartheta_2}{.5}, \frac{\vartheta_2}{.5})$ with the vectors in (7.12) with respect to $b_{\frac{1}{\sqrt{2}}}$. These are the same as the inner products of $(-\vartheta_1, \vartheta_1)$ and $(-\vartheta_2, \vartheta_2)$ with the vectors in (7.12) with respect to b_1, where $b_1 = 2b_{\frac{1}{\sqrt{2}}}$. We will compute the latter inner products since the notation is simpler. We will only do this explicitly for $(-\vartheta_1, \vartheta_1)$, since the computations for $(-\vartheta_2, \vartheta_2)$ are the same modulo obvious changes in notation.

$$\begin{aligned}
&\langle (0, \vartheta_1), U_{\gamma_1}\rangle = \\
&\left\langle \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_2 \cos\theta \sin t + \gamma_1 \sin\theta \cos t \\ Blah \end{pmatrix} \right), \right. \\
&\left. \left(\begin{pmatrix} \gamma_1 \cos\theta \cos t - \gamma_2 \sin\theta \sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma_2 \cos\theta \sin t + \gamma_1 \sin\theta \cos t \\ 0 \end{pmatrix} \right) \right\rangle = \\
&- \cos^2\theta \sin^2 t + \sin^2\theta \cos^2 t,
\end{aligned} \tag{7.14}$$

where "*Blah*" stands for a nonzero, but irrelevant term.

Combining the previous 2 equations we get

$$\langle \eta_1^{2,0}, U_{\gamma_1}\rangle = -\frac{1}{2} \sin 2t + \tan(2t)(-\cos^2\theta \sin^2 t + \sin^2\theta \cos^2 t) \tag{7.15}$$

$$\begin{aligned}
&\langle U_{\gamma_1}, (-\vartheta_1, 0)\rangle = \\
&\left\langle \left(\begin{pmatrix} \gamma_1 \cos\theta \cos t - \gamma_2 \sin\theta \sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma_2 \cos\theta \sin t + \gamma_1 \sin\theta \cos t \\ 0 \end{pmatrix} \right), \right. \\
&\left. \left(\begin{pmatrix} -\gamma_1 \cos\theta \cos t - \gamma_2 \sin\theta \sin t \\ Blah \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \right\rangle = \\
&- \cos^2\theta \cos^2 t + \sin^2\theta \sin^2 t.
\end{aligned} \tag{7.16}$$

Combining (7.14) and (7.16) we get

$$\langle (-\vartheta_1, \vartheta_1), U_{\gamma_1}\rangle =$$

$$\begin{aligned}
&-\cos^2\theta\sin^2 t+\sin^2\theta\cos^2 t-\cos^2\theta\cos^2 t+\sin^2\theta\sin^2 t=\\
&\quad-\cos^2\theta(\ \sin^2 t+\cos^2 t\ )+\sin^2\theta(\ \cos^2 t+\sin^2 t\ )=\\
&\quad-\cos 2\theta.
\end{aligned}\tag{7.17}$$

$$\begin{aligned}
&\langle U_{\gamma_2},\ (\ \eta_1,\eta_1\ )\rangle=\\
&\Bigg\langle\ \left(\left(\begin{array}{c}\gamma_2\cos\theta\cos t+\gamma_1\sin\theta\sin t\\ 0\end{array}\right),\left(\begin{array}{c}\gamma_1\cos\theta\sin t+\gamma_2\sin\theta\cos t\\ 0\end{array}\right)\right),\\
&\left(\left(\begin{array}{c}-\gamma_1\cos\theta\sin t+\gamma_2\sin\theta\cos t\\ \gamma_1\sin\theta\sin t+\gamma_2\cos\theta\cos t\end{array}\right),\left(\begin{array}{c}\gamma_2\cos\theta\cos t-\gamma_1\sin\theta\sin t\\ -\gamma_2\sin\theta\cos t-\gamma_1\cos\theta\sin t\end{array}\right)\right)\ \Bigg\rangle=\\
&\frac{1}{2}(\ \cos^2 t\cos\theta\sin\theta-\sin^2 t\cos\theta\sin\theta-\sin^2 t\cos\theta\sin\theta+\cos^2 t\cos\theta\sin\theta\ )=\\
&\frac{1}{4}\sin 2\theta(\ 2\cos^2 t-2\sin^2 t\ )=\\
&\frac{1}{2}\sin 2\theta\cos 2t.
\end{aligned}\tag{7.18}$$

$$\begin{aligned}
&\langle U_{\gamma_2},(\ 0,\ \vartheta_1\ )\rangle=\\
&\Bigg\langle\ \left(\left(\begin{array}{c}\gamma_2\cos\theta\cos t+\gamma_1\sin\theta\sin t\\ 0\end{array}\right),\left(\begin{array}{c}\gamma_1\cos\theta\sin t+\gamma_2\sin\theta\cos t\\ 0\end{array}\right)\right),\\
&\left(\left(\begin{array}{c}0\\ 0\end{array}\right),\left(\begin{array}{c}\gamma_2\cos\theta\sin t+\gamma_1\sin\theta\cos t\\ blah\end{array}\right)\right)\ \Bigg\rangle=\\
&2\sin\theta\cos\theta\sin t\cos t=\frac{1}{2}\sin 2\theta\sin 2t.
\end{aligned}\tag{7.19}$$

Combining (7.18) and (7.19) we get

$$\begin{aligned}
&\langle U_{\gamma_2},\ \eta_1^{2,0}\rangle=\\
&\frac{1}{2}\sin 2\theta\cos 2t+\frac{1}{2}\tan 2t\sin 2\theta\sin 2t=\\
&\frac{\sin 2\theta}{2}(\ \cos 2t+\tan 2t\sin 2t\ ).
\end{aligned}\tag{7.20}$$

$$\begin{aligned}
&\langle U_{\gamma_2},(-\vartheta_1,0)\rangle=\\
&\Bigg\langle\ \left(\left(\begin{array}{c}\gamma_2\cos\theta\cos t+\gamma_1\sin\theta\sin t\\ 0\end{array}\right),\left(\begin{array}{c}\gamma_1\cos\theta\sin t+\gamma_2\sin\theta\cos t\\ 0\end{array}\right)\right),\\
&\left(\left(\begin{array}{c}-\gamma_1\cos\theta\cos t-\gamma_2\sin\theta\sin t\\ Blah\end{array}\right),\left(\begin{array}{c}0\\ 0\end{array}\right)\right)\Bigg\rangle=\\
&-2\cos\theta\sin\theta\cos t\sin t=-\frac{1}{2}\sin 2\theta\sin 2t.
\end{aligned}\tag{7.21}$$

Combining (7.19) and (7.21) we see that

$$\langle(-\vartheta_1,\ \vartheta_1\ ),\ U_{\gamma_2}\rangle=0.\tag{7.22}$$

Combining (7.17), (7.20), and (7.22) we see that the projection of P onto TO_{A^u} is nondegenerate unless $\cos 2\theta = 0$ or $\sin 2\theta = 0$, proving the proposition. $\square$

Remark One might hope to show that the planes from (7.10) with $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$, and $\frac{3\pi}{4}$ are positively curved by studying the projections onto TO_{A^d}. These are also nondegenerate for most values of θ. Unfortunately they are degenerate precisely when $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$, and $\frac{3\pi}{4}$.

Proposition 7.23 *Let $\eta^{2,0}$ and $\vartheta^{2,0}$ be as in (7.10). For $t \in [0, \frac{\pi}{4})$ if*

$$P = span\{\zeta, \vartheta^{2,0}\} \tag{7.24}$$

where $\zeta \in span\{x^{2,0}, \eta^{2,0}\}$, then the curvature of P is positive with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ unless $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$, or $\frac{3\pi}{4}$.

Proof Along the orbit of $SO(2)$ the formula for $x^{2,0}$ is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix}, \begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \right) =$$
$$\left(\begin{pmatrix} -\cos\theta\sin t + \alpha\sin\theta\cos t \\ \sin\theta\sin t + \alpha\cos\theta\cos t \end{pmatrix}, \begin{pmatrix} \alpha\cos\theta\cos t - \sin\theta\sin t \\ -\alpha\sin\theta\cos t - \cos\theta\sin t \end{pmatrix} \right).$$

So

$$\langle U_\alpha, x^{2,0} \rangle =$$
$$\Big\langle \left(\begin{pmatrix} \alpha\cos\theta\cos t - \sin\theta\sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -\cos\theta\sin t + \alpha\sin\theta\cos t \\ 0 \end{pmatrix} \right),$$
$$\left(\begin{pmatrix} -\cos\theta\sin t + \alpha\sin\theta\cos t \\ \sin\theta\sin t + \alpha\cos\theta\cos t \end{pmatrix}, \begin{pmatrix} \alpha\cos\theta\cos t - \sin\theta\sin t \\ -\alpha\sin\theta\cos t - \cos\theta\sin t \end{pmatrix} \right) \Big\rangle =$$
$$\frac{1}{2}(\cos^2 t\sin\theta\cos\theta + \sin^2 t\cos\theta\sin\theta + \sin^2 t\cos\theta\sin\theta + \cos^2 t\sin\theta\cos\theta) =$$
$$\frac{1}{2}\sin 2\theta, \tag{7.25}$$

Similar computations show that

$$\langle U_{\gamma_i}, x^{2,0} \rangle = \langle U_\alpha, \eta_1^{2,0} \rangle = \langle U_\alpha, \vartheta_1^{2,0} \rangle = 0. \tag{7.26}$$

Combining (7.17), (7.25), and (7.26) we see that the projection of P onto TO_{A^u} is nondegenerate unless $\sin 2\theta = 0$ or $\cos 2\theta = 0$, proving the proposition. $\square$

Remark One might hope that there really is positive curvature on these planes when $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$ because perhaps the projection of these planes onto TO_{A^d} is nondegenerate. While this is true for most values of θ, it is false precisely when $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$.

Finally we study the zero curvatures of $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_2^d})$ at points when $t = \frac{\pi}{4}$. At such points the action induced by $A_{SO(2)}$ on S^4 via $p_{2,0}$: $E_{2,0} \longrightarrow S^4$ fixes S^2_{im}, so the orbits of $A_{SO(2)}$ are vertical with respect to $p_{2,0}$ when $t = \frac{\pi}{4}$. Combining this with the fact that A^l acts transitively on the fibers of $p_{2,0}$, we see that to study the curvature of these points we may assume that $\theta = 0$, in other words we are back to representative points of the form (2.3) with $t = \frac{\pi}{4}$. At such points we get the following as a corollary of (7.14), (7.16), (7.19) and (7.21).

Corollary 7.27 *When $t = \frac{\pi}{4}$ and $\theta = 0$ the planes of the form (6.4) have positive curvature except if they have the form*

$$P = span\{(\ \vartheta, 0\ ),\ (0, \vartheta)\} \tag{7.28}$$

where ϑ is any convex combination of $\{\vartheta_1, \vartheta_2\}$.

To determine which of the planes in (7.4) still have 0–curvature we prove the following.

Proposition 7.29 *At points with $t = \frac{\pi}{4}$ and $\theta = 0$, the only planes of the form (7.4) that have 0–curvature with respect to $g_{\nu_1,\nu_2,l_1^u,l_1^d}$ are those of the form*

$$P = span\{x^{2,0} + \lambda(-\vartheta, 0), (-\vartheta, \vartheta)\} \tag{7.30}$$

and

$$P = span\{y^{2,0} + \lambda(-\vartheta, 0), (\vartheta, \vartheta)\} \tag{7.31}$$

where ϑ stands for any convex combination of $\{\vartheta_1, \vartheta_2\}$ and λ is any real number.

Proof The point is that if z is any other convex combination of $\{x^{2,0}, y^{2,0}\}$, then the projection of the plane in (7.4) onto TO_{A^u} is nondegenerate. To see this we will need to compute the projection of $y^{2,0}$ onto TO_{A^u}. Although we will only need to know its values for $(t,\theta)=(\frac{\pi}{4},0)$, we will do the computation for arbitrary (t,θ), since it may be of interest to some readers.

Along the orbit of $SO(2)$ the formula for $y^{2,0}$ is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\left(\begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix},\begin{pmatrix} -\cos t \\ -\alpha\sin t \end{pmatrix}\right)=$$
$$\left(\begin{pmatrix} \alpha\cos\theta\sin t+\sin\theta\cos t \\ -\alpha\sin\theta\sin t+\cos\theta\cos t \end{pmatrix},\begin{pmatrix} -\cos\theta\cos t-\alpha\sin\theta\sin t \\ \sin\theta\cos t-\alpha\cos\theta\sin t \end{pmatrix}\right).$$

So

$$\langle U_\alpha, y^{2,0}\rangle =$$
$$\left\langle \left(\begin{pmatrix} \alpha\cos\theta\cos t-\sin\theta\sin t \\ 0 \end{pmatrix},\begin{pmatrix} -\cos\theta\sin t+\alpha\sin\theta\cos t \\ 0 \end{pmatrix}\right),\right.$$
$$\left.\left(\begin{pmatrix} \alpha\cos\theta\sin t+\sin\theta\cos t \\ -\alpha\sin\theta\sin t+\cos\theta\cos t \end{pmatrix},\begin{pmatrix} -\cos\theta\cos t-\alpha\sin\theta\sin t \\ \sin\theta\cos t-\alpha\cos\theta\sin t \end{pmatrix}\right)\right\rangle =$$
$$\frac{1}{2}(\cos^2\theta\sin t\cos t-\sin^2\theta\cos t\sin t+\cos^2\theta\sin t\cos t-\sin^2\theta\cos t\sin t) =$$
$$\frac{1}{2}\sin t\cos t(2\cos^2\theta-2\sin^2\theta)=$$
$$\frac{1}{2}\sin 2t\cos 2\theta. \tag{7.32}$$

Similar computations show that

$$\langle U_{\gamma_i}, y^{2,0}\rangle = \langle U_\alpha,(\vartheta,0)\rangle = \langle U_\alpha,(0,\vartheta)\rangle = 0. \tag{7.33}$$

Combining (7.32) and (7.33) with (7.14), (7.16), (7.19), (7.21), (7.25) and (7.26) we see that if P is any plane of the form of (7.4) and not of the form of either (7.30) or (7.31), then the projection of P onto TO_{A^u} is nondegenerate and hence P is positively curved. □

Remark As was the case with our previous results one could study projections onto TO_{A^d} to prove (7.27) and (7.29). Unfortunately these projections are degenerate on the same planes as the projections onto TO_{A^u} and hence do not give us additional positive curvature.

The curvature computations for the proof of Theorem A are completed with the following.

Proposition 7.34

(i) *The planes in (7.11), (7.24), (7.28), (7.30) and (7.31) all have* 0*–curvature in* $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_1^d})$.

(ii) *The plane spanned by $x^{2,0}$ and $\vartheta_1^{2,0}$ in $E_{2,0}$ is tangent to a totally geodesic flat 2–torus.*

(iii) *Let Z be the pointwise zero locus in $(E_{2,0}, g_{\nu_1,\nu_2,l_1^u,l_1^d})$, then Z is the union of two copies of $S^3 \times S^3$ that intersect along a copy of $S^2 \times S^3$.*

Sketch of proof To prove (i) we must check that the A–tensors of $q^{diag}_{A^u \times A^d}$ and $q_{2,0}$ vanish on these planes. These computations are rather long, but are quite straight forward so we leave them to the reader.

Recall [24] that $S^1_{\mathbb{R}}$ denotes the circle in S^4 that is fixed by the action induced from A^r via $p_{2,0}$.

It is easy to see that $x^{2,0}$ and $\vartheta_1^{2,0}$ span a 2–torus, T, in $Sp(2)$ whose tangent plane is horizontal with respect to $q_{2,0}$ at every point. Since T is horizontal, $q_{2,0}|_T$ is a covering map. The order of the covering is at least two because -1 leaves T invariant under $A_{2,0}$. We show next that the order is exactly two.

Let c_x and $c_{\vartheta_1^{2,0}}$ be the geodesic circles that are generated by $x^{2,0}$ and $\vartheta_1^{2,0}$. Then with respect to the join decomposition $S^4 = S^2_{im} * S^1_{\mathbb{R}}$, $p_{2,0}(c_x)$ is a radial circle and for $t > 0$, $p_{2,0}(c_{\vartheta_1^{2,0}})$ is an intrinsic geodesic in a copy of S^2. Using these observations we see that a point in $p_{2,0}(q_{2,0}(T))$ has only two preimages under $p_{2,0} \circ q_{2,0}$ from which it follows that $q_{2,0}|_T$ is a two fold covering.

The fields $x^{2,0}$ and $\vartheta_1^{2,0}$ are both parallel along T. Straightforward computation shows that the results of transporting them from a point $z \in T$ to $-z$ agrees with their images under the differential of the map induced on T by -1 via $A_{2,0}$. Therefore $q_{2,0}|_T$ is a 2–fold orientation preserving cover, and $q_{2,0}(T)$ is a torus rather than a Klein bottle.

It follows from (i) that $q_{2,0}(T)$ is flat and computations similar to those in the proof of (i) show that $q_{2,0}(T)$ is totally geodesic.

Combining part (i) with (7.2), (7.10) and (7.23), we see that Z consists of the points with $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$, or $\frac{3\pi}{4}$ or $t = \frac{\pi}{4}$. Using the join structure $S^4 = S^1_{\mathbb{R}} * S^2_{im}$ we see that the union of the points with $\theta = 0$, $\theta = \frac{\pi}{2}$ and $t = \frac{\pi}{4}$ is a 3–sphere in S^4. (Keep in mind that $A_{SO(2)}$ is $\mathbb{Z}_2$–ineffective on S^4.) For similar reasons the union of the points with $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$ and $t = \frac{\pi}{4}$ is another 3–sphere. These two three spheres intersect along the common 2–sphere S^2_{im} and their inverse images via $p_{2,0}$ are of course diffeomorphic to $S^3 \times S^3$. Finally notice that every point in this inverse image has zero curvatures because $p_{2,0}$ is the quotient map of A^r and A^r acts by isometries. □

The example, M^6, of Corollary B is

$$M^6 = E_{2,0}/A^l_\beta$$

where β is any purely imaginary, unit quaternion, and A^l_β is the circle subaction of A^l that is generated by β. When $\alpha = \beta$ the torus $q_{2,0}(T) \subset E_{2,0}$ is normal to the orbit of A^l_β. Some further curvature computations show that its image in M^6 is totally geodesic and flat.

8 Topological Computations

Proposition 8.1 *$E_{2,0}$ is diffeomorphic to the total space of the unit tangent bundle of S^4. In fact, the Riemannian submersion $p_{1,1}\colon E_{2,0} \longrightarrow S^4$ that is given by*

$$p_{1,1}\colon orbit\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \tilde{h}(c,d)$$

is bundle isomorphic to the unit tangent bundle of S^4, where $\tilde{h}\colon S^7 \longrightarrow S^4$ is the Hopf fibration given by left multiplication.

Proof Translating Theorem 9.5 on page 99 of [15] into our classification scheme (0.1) shows that the unit tangent bundle is the bundle of type $(1,1)$. We will show via direct computations (similar to those in [12]) that $(E_{2,0}, p_{1,1})$ is also the bundle of type $(1,1)$.

As in [24] we define $\phi\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ by

$$\phi(u) = \frac{1}{\sqrt{1+|u|^2}},$$

and we define explicit bundle charts $h_1, h_2\colon \mathbb{R}^4 \times S^3 \longrightarrow E_{2,0}$ by

$$h_1(u,q) = orbit\begin{pmatrix} -q & qu \\ \bar{u} & 1 \end{pmatrix}\phi(u)$$

and

$$h_2(v,r) = orbit\begin{pmatrix} -r\bar{v} & r \\ 1 & v \end{pmatrix}\phi(v).$$

h_1 and h_2 are embeddings onto the open sets

$$U_1 = orbit\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid d \neq 0 \}$$

and

$$U_2 = orbit\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \neq 0 \}.$$

In fact their inverses are given by

$$h_1^{-1}(\, orbit \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\frac{\bar{c}d}{|d|^2}, \frac{-\bar{d}a}{|d||a|})$$

and

$$h_2^{-1}(\, orbit \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\frac{\bar{c}d}{|c|^2}, \frac{\bar{c}b}{|c||b|}).$$

Thus

$$h_2^{-1} \circ h_1(u,q) = h_2^{-1}(orbit \begin{pmatrix} -q & qu \\ \bar{u} & 1 \end{pmatrix} \phi(u)) = \\ (\frac{u}{|u|^2}, \frac{uqu}{|u|^2}).$$

So $(E_{2,0}, p_{1,1})$ is the bundle of type $(1,1)$ and hence is the unit tangent bundle of S^4. □

Next we compute the homology and first few homotopy groups of S^3–bundles over S^4 to distinguish $E_{2,0}$ from the known examples of 7–manifolds with positive sectional curvature.

Proposition 8.2 *Let $E_{m,-n}$ denote the S^3–bundle over S^4 of type $(m,-n)$.*

(i) *The integral homology groups of $E_{m,-n}$ are*

$$H_0(E_{m,-n}, \mathbb{Z}) \cong H_7(E_{m,-n}, \mathbb{Z}) \cong \mathbb{Z}$$
$$H_3(E_{m,-n}, \mathbb{Z}) \cong \frac{\mathbb{Z}}{(m-n)Z}, \text{ and}$$
$$H_q(E_{m,-n}, \mathbb{Z}) \cong \{0\} \text{ for all } q \neq 0,3,7,$$

if $m \neq n$.

(ii) $\pi_1(E_{m,-n}) \cong \pi_2(E_{m,-n}) \cong \{0\}$ *and* $\pi_3(E_{m,-n}) \cong \frac{\mathbb{Z}}{(m-n)\mathbb{Z}}$

(iii) $E_{2,0}$ *does not have the homotopy type of any known example of a manifold with positive curvature.*

Proof To compute the homology of $E_{m,-n}$ we decompose it as

$$E_{m,-n} = D^4 \times S^3 \cup_{g_{m,-n}} D^4 \times S^3 \tag{8.3}$$

where the gluing map $g_{m,-n}\colon S^3 \times S^3 \longrightarrow S^3 \times S^3$ is given by $g_{m,-n}(u,v) = (u, u^m v u^{-n})$. From (8.3) and the Seifert–Van Kampen theorem it follows that $E_{m,-n}$ is simply connected.

The Mayer–Vietoris sequence for the decomposition (8.3) is

$$\begin{aligned} \cdots \to H_q(S^3 \times S^3) &\overset{\Phi_q}{\to} H_q(D^4 \times S^3) \oplus H_q(D^4 \times S^3) \\ &\overset{\Psi_q}{\to} H_q(E_{m,-n}) \overset{\Gamma_q}{\to} H_{q-1}(S^3 \times S^3) \overset{\Phi_{q-1}}{\to} \cdots \end{aligned} \tag{8.4}$$

Since $H_2(D^4 \times S^3) \cong H_1(S^3 \times S^3) \cong \{0\}$, $H_2(E_{m,-n}) \cong \{0\}$.

Also since $H_2(S^3 \times S^3) \cong \{0\}$, $H_3(E_{m,-n}) \cong \frac{H_3(D^4\times S^3)\oplus H_3(D^4\times S^3)}{image(\Phi_3)}$. The next step is to compute $image(\Phi_3)$.

Suppose $\{\alpha \times 1, 1 \times \alpha\}$ is a set of generators for $H_3(S^3 \times S^3)$, and $\{(\beta,0),(0,\beta)\}$ generates $H_3(D^4 \times S^3) \oplus H_3(D^4 \times S^3)$. Then using the fact that the map $P_k\colon S^3 \longrightarrow S^3$ given by $P_k(q) = q^k$ has degree k we can see that

$$\begin{aligned} &\Phi_3(\alpha \times 1) = (m-n)(0,\beta) \quad \text{and} \\ &\Phi_3(1 \times \alpha) = (\beta, 0) + (0,\beta). \end{aligned}$$

(To evaluate the first map represent $\alpha \times 1$ by $S^3 \times \{1\}$.)

Put another way, the matrix of Φ_3 with respect to our sets of generators is

$$\begin{pmatrix} 0 & 1 \\ m-n & 1 \end{pmatrix}. \tag{8.5}$$

From this point a routine algebraic computation shows that

$$H_3(E_{m,-n}) \cong \frac{H_3(D^4 \times S^3) \oplus H_3(D^4 \times S^3)}{image(\Phi_3)} \cong \frac{\mathbb{Z}}{(m-n)\mathbb{Z}}.$$

It follows from (8.5) that Φ_3 is injective if $m \neq n$, and hence we get using (8.4) that

$$H_4(E_{m,-n}) \cong H_5(E_{m,-n}) \cong H_6(E_{m,-n}) \cong 0,$$

completing the proof of (i).

Part (ii) is a corollary of (i).

The known examples of simply connected 7–manifolds with positive curvature are given in [3], [9] and [2]. With the exception of S^7 and the example, V_1, of Berger none of them are 2–connected. Since $H_3(E_{2,0},\mathbb{Z}) \cong \mathbb{Z}_2$, $E_{2,0}$ is not a homotopy sphere. It is not homotopy equivalent V_1 since, according to Proposition 40.1 in [3], V_1 is not a $\mathbb{Z}_5$–cohomology sphere. □

Our last topological computation is the following (cf also [13] Corollary 3.9).

Proposition 8.6

- **(i)** *M^6 has the same integral cohomology modules as CP^3 but not the same integral cohomology algebra.*
- **(ii)** *M^6 does not have the homotopy type of any known example of a manifold of positive curvature.*

Proof From the long exact homotopy sequence of the fibration $S^1 \hookrightarrow E_{2,0} \longrightarrow M^6$ we see that $\pi_1(M^6) \cong 0$ and $\pi_2(M^6) \cong \mathbb{Z}$. Thus $H^1(M^6,\mathbb{Z}) \cong 0$ and $H^2(M^6,\mathbb{Z}) \cong \mathbb{Z}$. To compute $H^3(M^6,\mathbb{Z})$ and the cup products we appeal to the Gysin sequence

$$\cdots \to H^1(M^6,\mathbb{Z}) \to H^3(M^6,\mathbb{Z}) \to H^3(E_{2,0},\mathbb{Z}) \to H^2(M^6,\mathbb{Z}) \to \cdots$$

of the fiber bundle $S^1 \hookrightarrow E_{2,0} \longrightarrow M^6$ with integer coefficients. We already know that $H^1(M^6,\mathbb{Z}) = 0$. Moreover, it follows from the universal coefficient theorem that $H^3(E_{2,0},\mathbb{Z}) = 0$ since $H_2(E_{2,0},\mathbb{Z})$ has no torsion and $H_3(E_{2,0},\mathbb{Z})$ is a torsion group. Thus the Gysin sequence shows that $H^3(M^6,\mathbb{Z}) = 0$ and hence that M^6 has the same integral cohomology groups as $\mathbb{C}P^3$. To see that M^6 does not have the cohomology ring of $\mathbb{C}P^3$ we look at the Gysin sequence again:

$$\begin{aligned} 0 = H^3(E_{2,0},\mathbb{Z}) \overset{\pi_*}{\to} H^2(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^4(M^6,\mathbb{Z}) \overset{\pi^*}{\to} & \\ H^4(E_{2,0},\mathbb{Z}) \overset{\pi_*}{\to} H^3(M^6,\mathbb{Z}) = 0&, \\ 0 = H^5(E_{2,0},\mathbb{Z}) \overset{\pi_*}{\to} H^4(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^6(M^6,\mathbb{Z}) \overset{\pi^*}{\to} H^6(E_{2,0},\mathbb{Z}) = 0&. \end{aligned}$$

We therefore have

$$\begin{gathered} 0 \to H^2(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^4(M^6,\mathbb{Z}) \overset{\pi^*}{\to} \mathbb{Z}_2 \to 0, \\ 0 \to H^4(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^6(M^6,\mathbb{Z}) \to 0. \end{gathered}$$

This means that $H^2(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^4(M^6,\mathbb{Z})$ maps the generator x for $H^2(M^6,\mathbb{Z})$ to twice a generator y for $H^4(M^6,\mathbb{Z})$. If we let $e = kx$ we therefore have $x \cup kx = 2y$. Therefore, it suffices to show that $k = \pm 1$ in order to show that the cohomology ring of M^6 is not that of $\mathbb{C}P^3$. We have that $H^4(M^6,\mathbb{Z}) \overset{\cup e}{\to} H^6(M^6,\mathbb{Z})$ is an isomorphism. Thus $(x \cup kx) \cup kx = k^2 x \cup x \cup x$ must also be twice a generator for $H^6(M^6,\mathbb{Z})$. Therefore, if $k = \pm 2$ we would have that $\pm 2x \cup x \cup x$ generates $H^6(M^6,\mathbb{Z})$. This, however, is impossible.

Besides S^6 and $\mathbb{C}P^3$, there are two examples given in [23] and [10] of simply connected 6–manifolds with positive curvature. Neither of these has the cohomology modules of $\mathbb{C}P^3$ so M^6 does not have the homotopy type of any known example. □

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