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Embeddings from the point of view of immersion theory : Part II

Thomas G Goodwillie
Michael Weiss

Department of Mathematics, Brown University
Box 1917, Providence RI 02912–0001, USA
and
Department of Mathematics, University of Aberdeen
Aberdeen AB24 3UE, UK

Email: `tomg@math.brown.edu` and `m.weiss@maths.abdn.ac.uk`

Abstract

Let M and N be smooth manifolds. For an open $V \subset M$ let $\mathrm{emb}(V,N)$ be the space of embeddings from V to N. By the results of Goodwillie [4], [5], [6] and Goodwillie–Klein [7], the cofunctor $V \mapsto \mathrm{emb}(V,N)$ is *analytic* if $\dim(N) - \dim(M) \geq 3$. We deduce that its Taylor series converges to it. For details about the Taylor series, see Part I (reference [15]).

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0 Introduction

This is a continuation of [15]. The ideas in this second part are mostly due to Goodwillie; notes and illustrations mostly by Weiss. For introductions and notation generally speaking, see [14] and [15]. We fix smooth manifolds M^m and N^n, without boundary unless otherwise stated, and we assume $m \leq n$. As in [15] we write $\mathcal{O}$ for the poset of open subsets of M. Then $V \mapsto \mathrm{emb}(V, N)$ is a cofunctor from $\mathcal{O}$ to Spaces (= fibrant simplicial sets), and from [14] and [15] we have a fairly thorough understanding of its Taylor approximations, the cofunctors $V \mapsto T_k \,\mathrm{emb}(V, N)$. Here we show that the Taylor approximations are good approximations. The main result is Corollary 2.5. Sections 4 and 5 contain illustrations.

Following are some conventions.

Let R be a set with r elements. A functor $\mathcal{X}$ from the poset $\mathcal{P}_R$ of subsets of R to Spaces is an r–dimensional *cube* of Spaces [3, section 1]. The cube is k–*Cartesian* if the canonical map

$$\mathcal{X}(\emptyset) \longrightarrow \operatorname*{holim}_{\{S \mid \emptyset \neq S \subset R\}} \mathcal{X}(S)$$

is k–connected. A *Cartesian* cube is one which is k–Cartesian for all k. Since $\mathcal{P}_R$ as a category is isomorphic to its own opposite, we can use similar terminology for cofunctors from $\mathcal{P}_R$ to Spaces. Such a cofunctor will also be called a cube. It is k–*Cartesian* if the canonical map

$$\mathcal{X}(R) \longrightarrow \operatorname*{holim}_{\{S \mid S \subset R, S \neq R\}} \mathcal{X}(S)$$

is k–connected, and *Cartesian* if it is k–Cartesian for all k.

Let $\mathbb{R}^{n+2}_{\llcorner} := [0,\infty) \times [0,\infty) \times \mathbb{R}^n$. An $(n+2)$–dimensional *smooth manifold triad* is a paracompact Hausdorff space Q together with a maximal atlas consisting of open subsets $V_i \subset Q$ and open embeddings $\phi_i\colon V_i \to \mathbb{R}^{n+2}_{\llcorner}$ satisfying the following conditions:

- The union of the V_i is Q.
- The changes of charts $\phi_i(\phi_j)^{-1}$ are smooth where defined, and take points in $\mathbb{R}^{n+2}_{\llcorner}$ with vanishing first coordinate (vanishing second coordinate) to points with vanishing first coordinate (vanishing second coordinate).

Let $\partial_0 Q$ be the set of those $x \in Q$ which under some chart ϕ_i map to a point in $\mathbb{R}^{n+2}_{\llcorner}$ with vanishing first coordinate. Also, let $\partial_1 Q$ be the set of those $x \in Q$ which under some chart ϕ_i map to a point in $\mathbb{R}^{n+2}_{\llcorner}$ with vanishing second coordinate. Then Q is an $(n+2)$–dimensional smooth manifold with *corners*; its boundary is $\partial_0 Q \cup \partial_1 Q$ and its corner set is $\partial_0 Q \cap \partial_1 Q$.

The *handle index* of the manifold triad Q is the smallest integer a such that Q can be built from a collar on $\partial_0 Q$ by attaching handles of index $\leq a$. (It may happen that Q *is* a collar on $\partial_0 Q$. When that is the case we say that the handle index is $-\infty$.)

Example Suppose that P is smooth, with boundary, and let $f\colon P \to \mathbb{R}$ be smooth. If 0 is a regular value both for f and for $f|\partial P$, then the inverse image of $[0,\infty)$ under f is a manifold triad Q, with $\partial_0 Q = \partial P \cap Q$. (If $\partial P = \emptyset$, then of course $\partial_0 Q = \emptyset$.) Any $Q \subset P$ which can be obtained in this way will be called a *codimension zero subobject* of P.

1 Excision Estimates

Let Y be a smooth n–manifold with boundary. Let $Q_0, Q_1, \ldots, Q_r$ be smooth compact n–manifold triads with handle index q_i where $n - q_i \geq 3$. Suppose that smooth embeddings $e_i\colon \partial_0 Q_i \to \partial Y$ have been selected for $0 \leq i \leq r$, and that their images are pairwise disjoint. For $S \subset [r] = \{0, \ldots, r\}$ let Q_S be the disjoint union of the Q_i with $i \in S$. By $\mathrm{emb}(Q_S, Y)$ we mean the Space of smooth embeddings $f\colon Q_S \to Y$ which satisfy $f \mid \partial_0 Q_i = e_i$ and $f^{-1}(\partial Y) = \partial_0 Q_S$.

1.1 Theorem ([4], [5], [6], [7]) *The $(r+1)$–cube taking a subset S of $[r]$ to $\mathrm{emb}(Q_S, Y)$ is $(3 - n + \Sigma_{i=0}^{r}(n - q_i - 2))$–Cartesian, provided $r \geq 1$.*

Comment Assuming that $\mathrm{emb}(Q_{[r]}, Y)$ is nonempty, fix a base point f in $\mathrm{emb}(Q_{[r]}, Y)$. Let the image of f in $\mathrm{emb}(Q_S, Y)$ serve as base point for $\mathrm{emb}(Q_S, Y)$. Goodwillie shows in [6] that the cube $S \mapsto \Omega\,\mathrm{emb}(Q_S, Y)$ is $(2 - n + \Sigma_{i=0}^{r}(n - q_i - 2))$–Cartesian, provided $r \geq 1$. The delooped statement, Theorem 1.1 just above, has been proved by Goodwillie and Klein and will appear in [7].

We will need a slight generalization of 1.1 where the Q_i are allowed to have dimension $m \leq n$. For this and other purposes we need a lemma.

1.2 Lemma *Let $u\colon \mathcal{X} \to \mathcal{Y}$ be a map of $(r+1)$–cubes. That is, $\mathcal{X}$ and $\mathcal{Y}$ are functors from the poset of subsets of $[r]$ to Spaces, and u is a natural transformation. Suppose that $\mathcal{Y}$ is k–Cartesian and, for every $y \in \mathcal{Y}(\emptyset)$, the $(r+1)$–cube defined by*

$$S \mapsto \mathrm{hofiber}[\mathcal{X}(S) \xrightarrow{u} \mathcal{Y}(S)]$$

is k–Cartesian. (The homotopy fiber is taken over the image of y in $\mathcal{Y}(S)$.) Then $\mathcal{X}$ is k–Cartesian.

Proof of 1.2 Combine [3, 1.18] with [3, 1.6]. □

Now let Y and Q_i be as in 1.1, assuming however $\dim(Q_i) = m \le n$. As before, embeddings $e_i\colon \partial_0 Q_i \to \partial Y$ are specified and $n - q_i \ge 3$, where q_i is the handle index of Q_i. We want to show that the conclusion of 1.1 still holds. Without loss of generality, Y is a smooth submanifold of some $\mathbb{R}^t$. Then we can define maps

$$\mathrm{emb}(Q_S, Y) \longrightarrow \mathrm{map}(Q_S, G_{n-m}) \quad ; \quad f \mapsto \nu_f$$

where G_{n-m} is the direct limit of the Grassmannians of $(n-m)$–dimensional linear subspaces of $\mathbb{R}^u$, for $u \ge 0$, and where ν_f takes $x \in Q_S$ to the intersection of the tangent space of Y at $f(x)$ with the normal space of $f(Q_S)$ at $f(x)$. In other words, ν_f classifies the normal bundle of the embedding f. We have therefore a map of $(r+1)$–cubes:

$$(\bullet) \qquad \{S \mapsto \mathrm{emb}(Q_S, Y)\} \longrightarrow \{S \mapsto \mathrm{map}(Q_S, G_{n-m})\}.$$

Since the codomain cube in $(\bullet)$ is Cartesian, the contravariant analog of 1.2 tells us that the domain cube is k–Cartesian provided that, for every ξ in $\mathrm{map}(Q_{[r]}, G_{n-m})$, the cube of homotopy fibers

$$(\bullet\bullet) \qquad S \mapsto \mathrm{hofiber}_\xi\,[\,\mathrm{emb}(Q_S, Y)\} \longrightarrow \mathrm{map}(Q_S, G_{n-m})\,]$$

is k–Cartesian. Now it is easy to construct a homotopy equivalence from the homotopy fiber of $\mathrm{emb}(Q_S, Y) \to \mathrm{map}(Q_S, G_{n-m})$ to $\mathrm{emb}(Q'_S, Y)$ where Q'_S is the total space of the disk bundle on Q_S determined by ξ. This construction is natural in S, so $(\bullet\bullet)$ is k–Cartesian if the cube $S \mapsto \mathrm{emb}(Q'_S, Y)$ is k–Cartesian, which by 1.1 is the case if $k = (3 - n + \Sigma_{i=0}^r (n - q_i - 2))$. Therefore:

1.3 Observation *Theorem 1.1 generalizes to the situation where the Q_i have dimension $m \le n$.*

The corollary below is a technical statement about the cofunctor on $\mathcal{O}$ given by $V \mapsto \mathrm{emb}(V, N)$. Suppose that P is a smooth compact codimension zero subobject of M, and that $Q_0, \ldots, Q_r$ are pairwise disjoint compact codimension zero subobjects of $M \smallsetminus \mathrm{int}(P)$. For $S \subset \{0, \ldots, r\} = [r]$ let V_S be the interior of $P \cup Q_S$ where Q_S is the union of the Q_i for $i \in S$. We write V instead of $V_\emptyset$. Suppose that Q_i has handle index $q_i \le n - 3$.

1.4 Corollary *Assume $r \ge 1$. The $(r+1)$–cube taking a subset S of $[r]$ to $\mathrm{emb}(V_S, N)$ is $(3 - n + \Sigma_{i=0}^r (n - q_i - 2))$–Cartesian.*

Proof Let $\bar{V}_S$ be the closure of V_S in M. Let $\mathrm{emb}(\bar{V}_S, N)$ be the inverse limit of the Spaces (simplicial sets) $\mathrm{emb}(U, N)$ where U ranges over the neighborhoods of V_S in M. The restriction from $\mathrm{emb}(\bar{V}_S, N)$ to $\mathrm{emb}(V_S, N)$ is a homotopy equivalence. Using this fact and 1.2, we see that it is enough to show that for every embedding $f\colon \bar{V} \to N$, the $(r+1)$–cube

$$S \mapsto \mathrm{hofiber}\,[\,\mathrm{emb}(\bar{V}_S, N) \xrightarrow{\mathrm{res}} \mathrm{emb}(\bar{V}, N)\,]$$

is $(3 - n + \Sigma_{i=0}^{r}(n - q_i - 2))$–Cartesian. The homotopy fibers are to be taken over the point f in $\mathrm{emb}(\bar{V}, N)$, which we fix for the rest of this proof. By the isotopy extension theorem, the restriction maps from $\mathrm{emb}(\bar{V}_S, N)$ to $\mathrm{emb}(\bar{V}, N)$ are Kan fibrations. Therefore it is enough to show that

$$S \mapsto \mathrm{fiber}\,[\,\mathrm{emb}(\bar{V}_S, N) \xrightarrow{\mathrm{res}} \mathrm{emb}(\bar{V}, N)\,]$$

is a $(3-n+\Sigma_{i=0}^{r}(n-q_i-2))$–Cartesian $(r+1)$–cube. Let $D(\bar{V})$ be the total space of a normal disk bundle for $\bar{V}$ in N, with corners rounded off, so that $D(\bar{V})$ is a smooth codimension zero subobject of N. Let Y be the closure of $N \smallsetminus D(\bar{V})$ in N. Let $\mathcal{X}(S) \subset \mathrm{emb}(Q_S, Y)$ be the Space of embeddings $g\colon Q_S \to Y$ for which the map $f \cup g$ from $\bar{V}_S = \bar{V} \cup Q_S$ to N is smooth. (Here all embeddings from Q_S to Y are prescribed on $\partial_0 Q_S$, as in 1.1 and 1.3.) The inclusions of $\mathcal{X}(S)$ in $\mathrm{emb}(Q_S, Y)$ and the fiber of the restriction map $\mathrm{emb}(\bar{V}_S, N) \to \mathrm{emb}(\bar{V}, N)$ are homotopy equivalences. Therefore it is enough to know that $S \mapsto \mathrm{emb}(Q_S, Y)$ is $(3 - n + \Sigma_{i=0}^{r}(n - q_i - 2))$–Cartesian, which we know from 1.3. □

2 Convergence

We begin with an abstraction. Suppose that G is a good cofunctor from $\mathcal{O}$ to Spaces [14, 2.2]. Fix an integer $\rho > 0$. Let P be a smooth compact codimension zero subobject of M, and let $Q_0, \ldots, Q_r$ be pairwise disjoint compact codimension zero subobjects of $M \smallsetminus \mathrm{int}(P)$. Suppose that Q_i has handle index $q_i < \rho$. Let $V_S = \mathrm{int}(P \cup Q_S)$ as in 1.4. Assume also $r \geq 1$.

2.1 Definition The cofunctor G is *ρ–analytic with excess c* if, in these circumstances, the $(r+1)$–cube $S \mapsto G(V_S)$ is $(c + \Sigma_{i=0}^{r}(\rho - q_i))$–Cartesian.

2.2 Example According to 1.4, the cofunctor $V \mapsto \mathrm{emb}(V, N)$ is $(n-2)$–analytic, with excess $3 - n$.

2.3 Theorem *Suppose that G is ρ–analytic with excess c, and that $W \in \mathcal{O}$ has a proper Morse function whose critical points are all of index $\leq q$, where $q < \rho$. Then $\eta_{k-1}\colon G(W) \to T_{k-1}G(W)$ is $(c + k(\rho - q))$–connected, for any $k > 1$.*

Proof With a homotopy inverse limit argument we can reduce to the case where W is tame (ie, is the interior of a compact codimension zero subobject of M) and $\bar{W}$ has a smooth handle decomposition with handles of index $\leq q$ only.

Case 1 : $q = 0$ Then W is the union of disjoint open m–balls W_i with $1 \leq i \leq \ell$ for some ℓ. For $S \subset \{1, \ldots, \ell\}$ we write $W_S = \cup_{i \in S} W_i$. If $\ell < k$ then η_{k-1} from $G(W)$ to $T_{k-1}G(W)$ is a homotopy equivalence by definition of $T_{k-1}G$. Assume therefore $\ell \geq k$. The diagram

$$T_\ell G(W) \xrightarrow{r_\ell} T_{\ell-1}G(W) \xrightarrow{r_{\ell-1}} \cdots \xrightarrow{r_k} T_{k-1}G(W)$$

can be identified up to homotopy equivalences with

$$J_\ell \to J_{\ell-1} \to \cdots \to J_k \to J_{k-1}$$

where J_t is the homotopy inverse limit of the $G(W_S)$ with $S \subset \{1, \ldots, \ell\}$ and $|S| \leq t$. (See also the last lines of the proof of [15, 9.1].) The fibration $J_t \to J_{t-1}$ is obtained by pullback from another fibration, namely the product over all $S \subset \{1, \ldots, \ell\}$ with $|S| = t$ of the fibration

$$p_S \colon \operatorname*{holim}_{\{R|R \subset S\}} G(W_R) \longrightarrow \operatorname*{holim}_{\{R|R \subset S, R \neq S\}} G(W_R).$$

All this is true without any special assumptions on G except goodness. But now we use the analyticity hypothesis and find that p_S is $(c + t\rho)$–connected where $t = |S|$. Therefore the composition

$$r_k r_{k+1} \ldots r_\ell \colon T_\ell G(W) \to T_{k-1}G(W)$$

is $(c + k\rho)$–connected. We can identify it with $\eta_{k-1} \colon G(W) \to T_{k-1}G(W)$ since η_ℓ from $G(W)$ to $T_\ell G(W)$ is a homotopy equivalence.

Case 2 : $q > 0$ We induct on q. For every q–handle Q_u in the handle decomposition of $\bar{W}$ choose a smooth chart $\phi_u \colon Q_u \cong \mathbb{D}^q \times \mathbb{D}^{m-q}$ and distinct points $x_{u,i}$ in the interior of $\mathbb{D}^q$ for $1 \leq i \leq k$. Let $A_{u,i} \subset W$ be the inverse image of $\{x_{u,i}\} \times \mathrm{int}(\mathbb{D}^{m-q})$ under ϕ_u. Each $A_{u,i}$ is a closed smooth codimension q submanifold of W, meeting the core of handle Q_u transversely in one point. For $S \subset \{1, \ldots, k\}$ let A_S be the union of all $A_{u,i}$ with $i \in S$ and u arbitrary. Our analyticity assumption on G implies that the cube defined by $S \mapsto G(W \smallsetminus A_S)$ is $(c + k(\rho - q))$–Cartesian. The cube $S \mapsto T_{k-1}G(W \smallsetminus A_S)$ is Cartesian ($=\infty$–Cartesian). For each nonempty S, the manifold $W \smallsetminus A_S$ has an isotopy equivalent tame codimension zero submanifold with a handle decomposition where all handles have indices $< q$. Therefore by inductive assumption, the map $\eta_{k-1} \colon G(W \smallsetminus A_S) \to T_{k-1}G(W \smallsetminus A_S)$ is $(c + k(\rho - q + 1))$–connected, provided $S \neq \emptyset$. It follows with [3, 1.22] that η_{k-1} induces a map

$$\operatorname{holim} G(W \smallsetminus A_S) \longrightarrow \operatorname{holim} T_{k-1}G(W \smallsetminus A_S)$$

which is $(c+k(\rho-q+1)-k+1)$–connected (both homotopy inverse limits are over *nonempty* $S\subset\{1,\ldots,k\}$). Combining this with our "Cartesian–ness" estimates for the cubes $S\mapsto G(W\smallsetminus A_S)$ and $S\mapsto T_{k-1}G(W\smallsetminus A_S)$, where S again denotes an arbitrary subset of $\{1,\ldots,k\}$, we can conclude that $G(W)\to T_{k-1}G(W)$ is indeed $(c+k(\rho-q))$–connected. □

2.4 Corollary *Suppose that G is ρ–analytic. If $\rho>m$, then the canonical map $G(W)\longrightarrow \mathrm{holim}_k\, T_kG(W)$ is a homotopy equivalence for every W in $\mathcal{O}$. In general, the map $G(W)\to \mathrm{holim}_k\, T_kG(W)$ is a homotopy equivalence if W has a proper Morse function whose critical points are all of index $<\rho$.*

2.5 Corollary *Let $G(W)=\mathrm{emb}(W,N^n)$ for open $W\subset M^m$. If m is less than $n-2$, then $G(W)\simeq \mathrm{holim}_k\, T_kG(W)$ for all W. If $m=n-2$, then $G(W)\simeq \mathrm{holim}_k\, T_kG(W)$ provided W has no compact component. In general: suppose that W has a proper Morse function whose critical points are all of index $\le q$, where $q<n-2$. Then $\eta_k\colon G(W)\to T_kG(W)$ is $(k(n-2-q)-q+1)$–connected for $k\ge 1$. Consequently $G(W)\simeq \mathrm{holim}_k\, T_kG(W)$.*

Proof This follows from 2.4, 2.3 and 2.2 with $\rho=n-2$ and $c=3-n$. □

2.6 Corollary *Let $f\colon G_1\to G_2$ be a natural transformation between good cofunctors on $\mathcal{O}$. Suppose that G_1 and G_2 are both ρ–analytic and f from $G_1(W)$ to $G_2(W)$ is a homotopy equivalence whenever W is a tubular neighborhood of a finite set ($W\in\mathcal{O}j$ for some j). Then $f\colon G_1(W)\to G_2(W)$ is a homotopy equivalence for any W which has a proper Morse function with critical points of index $<\rho$ only.*

Proof The hypothesis on f implies that $T_kf\colon T_kG_1\to T_kG_2$ is an equivalence. □

3 Taylor Approximations of Analytic Cofunctors

As in the preceding section, G is a good cofunctor from $\mathcal{O}$ to Spaces.

3.1 Proposition *Suppose that G is homogeneous of degree k where $k\ge 0$. If $G(V)$ is $(c-1+k\rho)$–connected for every V in $\mathcal{O}k$, and $\rho\ge m$, then G is ρ–analytic with excess c.*

Remark If V in $\mathcal{O}k$ has $< k$ components, then $G(V)$ is contractible and therefore automatically $(c-1+k\rho)$–connected. The values $G(V)$ for V in $\mathcal{O}k$ with exactly k components can be regarded as the fibers of the *classifying (quasi)–fibration* for G. See [15, section 8] for the classification of homogeneous cofunctors on $\mathcal{O}$.

Proof The case $k=0$ is trivial, so we assume $k \geq 1$. Choose r and V_S for $S \subset [r]$ as in 2.1. Again write $V = V_{\emptyset}$. We must show that the cube $S \mapsto G(V_S)$ is a $(c+\Sigma_{i=0}^{r}(\rho - q_i))$–Cartesian $(r+1)$–cube. The case $r \geq k$ is again trivial, so we assume that $r < k$ and proceed by *downward* induction on r. Also the cases where $q_i = -\infty$ for some i are trivial, so we assume $q_i \geq 0$ for all i.

Case 1 : $q_i = 0, \forall i$ We do an induction on the number of handles in a handle decomposition of $\bar{V}$. If $\bar{V}$ is empty and $r < k-1$ then $G(V_S)$ is contractible for all S, so there is nothing to prove. If $\bar{V}$ is empty and $r = k-1$, then $G(V_S)$ is contractible except possibly when $S = [r]$, in which case it is $(c-1+k\rho)$–connected; then the cube $S \mapsto G(V_S)$ is $(c+k\rho)$–Cartesian, which means, $(c+\Sigma(\rho - q_i))$–Cartesian.

For nonempty (but still tame) $\bar{V}$, choose a handle decomposition. If $\bar{B}$ is the cocore of a handle of index p in $\bar{V}$, and $B := \bar{B} \cap V \cong \mathbb{R}^{m-p}$, then the $(r+2)$–cube

$$\{G(V_S) \mid S \subset [r]\} \longrightarrow \{G(V_S \smallsetminus B) \mid S \subset [r]\}$$

is $(c+(\rho-p)+(r+1)\rho)$-Cartesian by the downward induction on r. The $(r+1)$–subcube $\{G(V_S \smallsetminus B) \mid S \subset [r]\}$ is $(c+(r+1)\rho)$–Cartesian by the upward induction on the number of handles of $\bar{V}$. Noting that $\rho \geq m \geq p$ and using [3, 1.6] we conclude that the cube $\{G(V_S) \mid S \subset [r]\}$ is also $(c+(r+1)\rho)$–Cartesian.

Case 2 : $q_0 > 0$ Let $\bar{A}$ and $\bar{B}$ be two parallel but disjoint cocores for the handle $Q_0 \subset \bar{V}_{[r]}$ and let $\bar{C}$ be a strip between $\bar{A}$ and $\bar{B}$. Let $A = \bar{A} \cap V$, $B = \bar{B} \cap V$, $C = \bar{C} \cap V$ so that the triad $(C; A, B)$ is homeomorphic to the triad $([0,1] \times \mathbb{R}^{m-q_0}; \{0\} \times \mathbb{R}^{m-q_0}, \{1\} \times \mathbb{R}^{m-q_0})$. Consider the diagram of r–cubes

$$\begin{tikzcd}
\{G(V_S \smallsetminus C)\} & \{G(V_S \smallsetminus (A \cup B))\} \arrow[l, "f"'] & \{G(V_S \smallsetminus B)\} \arrow[l, "g"'] \\
& \{G(V_S \smallsetminus A)\} \arrow[u] & \{G(V_S)\} \arrow[l, "h"'] \arrow[u]
\end{tikzcd}$$

where now S runs through subsets of $[r]$ containing the element 0. (Each arrow in the diagram is a natural transformation of r–cubes, induced by appropriate inclusions.) We have to show that the arrow h (which is a map of r–cubes) is a $(c+\Sigma(\rho-q_i))$–Cartesian $(r+1)$–cube. It is enough [3, 1.6] to show that g is a $(c+\Sigma(\rho-q_i))$–Cartesian $(r+1)$–cube and the square with horizontal arrows g and h is a $(c+\Sigma(\rho-q_i))$–Cartesian $(r+2)$–cube. But the composition fg is a Cartesian cube, because for each S containing 0 the inclusion $V_S \smallsetminus C \to V_S \smallsetminus B$ is an isotopy equivalence. And f is a $(c+1+\Sigma(\rho-q_i))$–Cartesian $(r+1)$–cube by induction on q_0 (see comments below if this raises doubts). Using [3, 1.8] now, we see that g is indeed a $(c+\Sigma(\rho-q_i))$–Cartesian $(r+1)$–cube. What about the square? By the downward induction on r it is a $(c+(\rho-q_0)+\Sigma(\rho-q_i))$–Cartesian $(r+2)$–cube. This is good enough since $\rho \geq m \geq q_0$. □

Comment Let $\bar{U}$ be the complement of a nice tubular neighborhood of $\bar{A}\cup\bar{B}$ in $\bar{V}_{[r]}$, and $U = \mathrm{int}(\bar{U})$ so that U is tame. Then $\bar{U}$ can be obtained from $\bar{V}$ by attaching $r+1$ handles whose indices are $q_0-1, q_1, \dots, q_r$. In fact the intersection of $\bar{U}$ with $\bar{C}$ is the cocore of the first handle in the list, with index q_0-1.

The $(r+1)$–cube determined as in 2.1 by the handle decomposition of $\bar{U}$ relative to $\bar{V}$ can be identified with the cube f above. More precisely, the two cubes are "identified" by a natural transformation from one to the other which is a termwise homotopy equivalence.

3.2 Corollary *If G is ρ–analytic with excess c, then so is T_kG.*

Proof We proceed by induction on k. Fix V and V_S for $S \subset [r]$ as in 2.1, with $r \geq 0$. We must prove that the $(r+1)$–cube $S \mapsto T_kG(V_S)$ is $(c+\Sigma_{i=0}^r(\rho-q_i))$–Cartesian. For $k=0$ this is correct.

Suppose $k \geq 1$ and let $z \in T_{k-1}G(V_{[r]})$. For open $U \subset V_{[r]}$ let $L_k^zG(U)$ be the homotopy fiber of the forgetful map $T_kG(U) \to T_{k-1}G(U)$ over the point obtained from z by restriction. The cofunctor $U \mapsto L_k^zG(U)$ on open subsets $U \subset V_{[r]}$ is homogeneous of degree k. If $U \subset V_{[r]}$ is a disjoint union of open m–balls, then $L_k^zG(U)$ is homotopy equivalent to the total homotopy fiber of the cube $Y \mapsto G(U_Y)$ where Y runs over subsets of $\pi_0(U)$; see [15, 9.1]. By the assumption on G, this implies that $L_k^zG(U)$ is $(c+k\rho-1)$–connected. We are now in a position to use 3.1, and conclude that L_k^zG is ρ–analytic with excess c, just like G. Now the induction step is easy: we can make a fibration sequence up to homotopy of $(r+1)$–cubes

$$\{L_k^zG(V_S)\} \longrightarrow \{T_kG(V_S)\} \longrightarrow \{T_{k-1}G(V_S)\}\,.$$

By induction, the right–hand cube is $(c+\Sigma(\rho-q_i))$–Cartesian; the left-hand one also is, for arbitrary z, because we just proved it; and therefore the one in the middle is $(c+\Sigma(\rho-q_i))$–Cartesian, by lemma 1.3. □

3.3 Corollary *Let G be ρ–analytic with excess c. Suppose that $V \in \mathcal{O}$ has a proper Morse function with critical points of index $\leq q$ only, where $q < \rho$. Then r_k from $T_kG(V)$ to $T_{k-1}G(V)$ is $(c + k(\rho - q))$–connected, for $k \geq 0$.*

Remark Note that 3.3 is suggested but not implied by 2.3, since $\eta_{k-1} = r_k\eta_k$. Also 3.3 does not imply 2.3 since it only depends on the behavior of G on $\bigcup \mathcal{O}k$.

Proof of 3.3 We identify $r_k\colon T_kG \to T_{k-1}G$ with $\eta_{k-1}\colon T_kG \mapsto T_{k-1}(T_kG)$ (using [15, 6.1]). Now 2.3 can be applied, with T_kG in place of G, because T_kG is ρ–analytic with excess c according to 3.2. □

4 Haefliger's Theory

Here we explain how the calculus of embeddings contains the Haefliger theory of smooth embeddings in the metastable range (which Haefliger unfortunately calls the *stable range*). First we recall Haefliger's theory. With M and N as before, we can make a commutative diagram

$$(*)\qquad \begin{array}{ccc} \mathrm{emb}(M,N) & \xrightarrow{\subset} & \mathrm{map}(M,N) \\ \big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f\times f} \\ \mathrm{ivmap}^{\mathbb{Z}/2}(M\times M, N\times N) & \xrightarrow{\subset} & \mathrm{map}^{\mathbb{Z}/2}(M\times M, N\times N) \end{array}$$

where all mapping Spaces in sight consist of smooth maps. Specifically, the expression $\mathrm{ivmap}^{\mathbb{Z}/2}(\dots)$ stands for a Space of *strictly isovariant* smooth maps, ie, equivariant smooth maps g with the properties

$$\begin{aligned} g^{-1}(\Delta_N) &= \Delta_M \\ (T_xg)^{-1}(T_{g(x)}\Delta_N) &= T_x\Delta_M \quad \text{for any } x \in \Delta_M\,. \end{aligned}$$

(The second of these properties can be reformulated as follows: the vector bundle morphism induced by g, from the normal bundle of Δ_M in $M\times M$ to the normal bundle of Δ_N in $N\times N$, is a fiberwise monomorphism.)

4.1 Theorem [8] *If $2n > 3(m+1)$ and $n \geq 3$, square $(*)$ is 1–Cartesian.*

Remark In Haefliger's set–up, the space corresponding to $\mathrm{ivmap}^{\mathbb{Z}/2}(\dots)$ consists of the (smooth or continuous) equivariant maps $g\colon M\times M \to N\times N$ for which $g^{-1}(\Delta_N) = \Delta_M$. The extra condition on infinitesimal behavior that we have added does affect the homotopy type. However, the comparison map is 1–connected (proved in [8] and especially [9]), so that 4.1 is correct with either definition.

4.2 Lemma *If $m \leq n$, the cofunctor E taking $V \in \mathcal{O}$ to the homotopy pullback of*

$$\mathrm{ivmap}^{\mathbb{Z}/2}(V\times V, N\times N) \xrightarrow{\subset} \mathrm{map}^{\mathbb{Z}/2}(V\times V, N\times N) \leftarrow \mathrm{map}(V,N)$$

is polynomial of degree ≤ 2.

Proof By [15, 2.5] it suffices to verify that each of the three terms from which the homotopy pullback is made is polynomial of degree ≤ 2, as a cofunctor in V. Clearly $V \mapsto \mathrm{map}(V,N)$ is polynomial of degree ≤ 1. The remaining two terms can be handled as in [15, 2.4]. We omit the details. □

4.3 Corollary *Let $F(V) = \mathrm{emb}(V,N)$. The canonical morphism $F \to E$ is a second order Taylor approximation (induces an equivalence $T_2F \to T_2E$).*

Proof By [15, 5.1] and 4.2 above it suffices to verify that $F(V) \to E(V)$ is a homotopy equivalence whenever V is diffeomorphic to a disjoint union of ≤ 2 copies of $\mathbb{R}^m$. But this is rather obvious. □

We see that Haefliger's theorem, 4.1, can also be deduced from 4.3 and 2.5. In fact, 2.5 tells us that $F(M) \to E(M) \simeq T_2F(M)$ is s–connected with

$$s = 2(n-2-m) - m + 1 = 2n - 3(m+1)\,.$$

Again, this is not a new result. It is a reformulation of the main result of Dax's thesis [2].

5 An Application

The convergence statements 2.5 and 4.4 can be generalized mildly so that certain cases "with boundary" are included. What we have in mind is example 10.1 of [15]. Suppose therefore that M^m and N^n are smooth manifolds with boundary, $m \leq n$, and that a smooth embedding $g\colon \partial M \to \partial N$ has been selected. Let $\mathcal{O}$ be the poset of open subsets of M containing ∂M. For $V \in \mathcal{O}$ let $\mathrm{emb}(V,N)$ be the Space of (neat and smooth) embeddings $V \to N$ which agree with g near $V \cap \partial M$. The cofunctor $V \mapsto \mathrm{emb}(V,N)$ on $\mathcal{O}$ is *good*. A calculus of good cofunctors from $\mathcal{O}$ to Spaces is outlined in section 10 of [15].

5.1 Fact *Let $F(V) = \mathrm{emb}(V,N)$ for $V \in \mathcal{O}$. If $m < n-2$, then $F(V) \simeq \mathrm{holim}_k\, T_kF(V)$ for all V. In general, $F(V) \simeq \mathrm{holim}_k\, T_kF(V)$ provided V has a handle decomposition relative to a collar on ∂V, with possibly infinitely many handles, all of index $\leq q$, where $q < n-2$. In this case $\eta_k\colon F(V) \to T_kF(V)$ is $(k(n-2-q)-q+1)$–connected.*

The proof is essentially identical with that of 2.5. We omit it and turn to some examples. Suppose that M is the unit interval I, and suppose for simplicity that it comes with a preferred smooth and neat embedding $I \to N$ (which we treat as an inclusion). Conditions on N^n will be added later; for now, the only condition is $n \geq 4$. Then $V \mapsto \mathrm{emb}(V, N)$ is a functor from $\mathcal{O}$ to based Spaces. *Notation:* $\iota X = X \smallsetminus \partial X$ for a manifold with boundary X.

Define $F(V)$ as in 5.1, for open $V \subset I$ containing the boundary. Our first and most important task is to understand the classifying fibration p_k for $L_k F$, assuming $k \geq 2$. Its base space is

$$\binom{\iota I}{k} \;\cong\; \mathbb{R}^k \,.$$

Its fiber over some S (subset of ιI with k elements) is the total homotopy fiber of the k–cube of pointed spaces

$$(*) \qquad R \mapsto \mathrm{emb}(R, \iota N) \qquad (R \subset S)\,.$$

Let $x \in S$ be the minimal element. In 5.2 and 5.3 below, $x \in S \subset I \subset N$ serves as base point of ιN.

5.2 Lemma *The total homotopy fiber of the k–cube $(*)$ is homotopy equivalent to the total homotopy fiber of the $(k-1)$–cube of pointed spaces*

$$R \mapsto \iota N \smallsetminus R \qquad (R \subset S, x \notin R)\,.$$

Proof The idea is to use a Fubini principle: The total homotopy fiber of the k–cube $(*)$ is homotopy equivalent to the total homotopy fiber of the $(k-1)$–cube

$$R \mapsto \mathrm{hofiber}[\mathrm{emb}(R \cup x, \iota N) \to \mathrm{emb}(R, \iota N)] \qquad (R \subset S, x \notin R)\,.$$

The restriction map $\mathrm{emb}(R \cup x, \iota N) \to \mathrm{emb}(R, \iota N)$ is a fibration, so that its homotopy fiber can be replaced by its fiber (over the base point), which is $\iota N \smallsetminus R$. Note that $R \subset S \subset \iota I \subset \iota N$. □

5.3 Corollary *The total homotopy fiber of the cube $(*)$ is homotopy equivalent to the total homotopy fiber of a cube of the form*

$$R \mapsto \iota N \vee (R_+ \wedge \mathbb{S}^{n-1}) \qquad (R \subset S \smallsetminus x)\,.$$

Remark The cube in 5.3 is contravariant like $(*)$, ie, the maps in it are collapsing maps, not inclusion maps. The base point of ιN is still x.

Proof Choose a smooth embedding $e\colon \iota N \to \iota N$, isotopic to the identity, such that $e(\iota N)$ and $S \smallsetminus x$ are disjoint and $e(x) = x$. For each $y \in S \smallsetminus x$ choose a smooth embedding $f_y\colon \mathbb{D}^n \to \iota N$ which maps the base point $(1, 0, \ldots, 0)$ to x and the center $(0, \ldots, 0)$ to y, and avoids all other points of S. Then for each $R \subset S \smallsetminus x$, the map

$$\iota N \vee (R_+ \wedge \mathbb{S}^{n-1}) \vee ((S \smallsetminus R \smallsetminus x)_+ \wedge \mathbb{D}^n) \longrightarrow \iota N \smallsetminus R$$

which is e on ιN and f_y on $\{y\}_+ \wedge \mathbb{S}^{n-1}$ and on $\{y\}_+ \wedge \mathbb{D}^n$ (where applicable) is a homotopy equivalence. It is also contravariantly natural in the variable R. Further, for each $R \subset S \smallsetminus x$, the collapse map

$$\iota N \vee (R_+ \wedge \mathbb{S}^{n-1}) \vee ((S \smallsetminus R \smallsetminus x)_+ \wedge \mathbb{D}^n) \longrightarrow \iota N \vee (R_+ \wedge \mathbb{S}^{n-1})$$

is a homotopy equivalence, again contravariantly natural in the variable R. We have now reduced 5.3 to 5.2. □

In certain cases the Hilton–Milnor theorem [11] can be used to simplify 5.3 further. See [16] for the meaning of *basic words*, which we use in 5.4 below. For a basic word w in the letters $z_1, \ldots, z_k$ we let $\alpha(w)$ be the number of letters distinct from z_1, and $\beta(w)$ the number of letters equal to z_1. For a pointed CW–space Y the smash product $Y \wedge Y \cdots \wedge Y$ with j factors will be abbreviated $Y^{(j)}$; the convention for $j = 0$ is $Y^{(0)} = \mathbb{S}^0$.

5.4 Corollary *Suppose that $N \simeq \Sigma Y$, where Y is a connected pointed CW–space. Then the loop space of the total homotopy fiber of $(*)$ is homotopy equivalent to a weak product (union of finite products)*

$$\prod_w{}' \Omega\Sigma^{1+\alpha(w)(n-2)} Y^{(\beta(w))}$$

taken over all basic words w in the letters $z_1, \ldots, z_k$ which involve all the letters except possibly z_1.

Before giving the proof, we must restate the Hilton–Milnor theorem. Let $X_1, \ldots, X_k$ be connected pointed CW–spaces. For a basic word w in the letters $z_1, \ldots, z_k$ denote by $w(X_1, \ldots, X_k)$ the space obtained by substituting X_i for z_i and $\wedge$ for multiplication; for example, if $k = 3$ and $w = (z_2 z_1) z_1$, then $w(X_1, X_2, X_3)$ is $(X_2 \wedge X_1) \wedge X_1$.

5.5 Hilton–Milnor Theorem $\Omega\Sigma(X_1 \vee \cdots \vee X_k) \simeq \prod'_w \Omega\Sigma(w(X_1, \ldots, X_k))$, *a weak product taken over the basic words w in the letters $z_1, \ldots, z_k$.*

Proof of 5.4 We identify S in 5.3 and 5.4 with $\{1,2,\ldots,k\}$, noting that S has a preferred ordering since it is a subset of I. Fix $X_1 = Y$ as in 5.4, and for $i > 1$ regard X_i as a variable with only two possible values, $\mathbb{S}^{n-2}$ and $*$. Letting the X_i vary with these constraints, we find that

$$\Omega\Sigma(X_1 \vee \cdots \vee X_k)$$

runs through the loop spaces of the vertices of the cube in 5.3. Each of these loop spaces can therefore be decomposed according to Hilton–Milnor. Moreover, the decomposition given by Hilton–Milnor is natural, a point which is stressed in [16], and we conclude that Ω of the cube in 5.3 splits into many separate factors indexed by the basic words w in the letters $z_1 \ldots, z_k$. Each factor is a $(k-1)$–cube in its own right, of the form

$$(**) \qquad R \mapsto \Omega\Sigma(w(Y, X_2(R), \ldots, X_k(R)) \qquad (R \subset \{2,\ldots,k\})$$

where $X_i(R) = \mathbb{S}^{n-2}$ if $i \in R$ and $X_i(R) = *$ otherwise. If w omits one of the letters $z_2, \ldots, z_k$, then the total homotopy fiber of $(**)$ is contractible (by the Fubini priciple, also used in the proof of 5.2). Otherwise, the cube $(**)$ has only one nontrivial vertex (the initial one) and its total homotopy fiber is therefore that vertex. □

5.6 Corollary *Suppose again that $N \simeq \Sigma Y$ as in 5.4. The terms of the Taylor tower of* $\mathrm{emb}(I,N)$ *are as follows, for $k \geq 2$:*

$$L_k\,\mathrm{emb}(I,N) \simeq \prod_w{}' \Omega^k \Sigma^{1+\alpha(w)(n-2)} Y^{(\beta(w))}$$

where the weak product is over all basic words w in the letters $z_1, \ldots, z_k$ which involve all the letters except possibly z_1.

Proof The k–th term of the Taylor tower can be described as the space of sections with compact support of a fibration p_k. See [15, section 10]. We found that the base space of p_k is homeomorphic to $\mathbb{R}^k$, so that the section space in question is the k–fold loop space of any of the fibers. The fibers are described in 5.3, and their loop spaces in 5.4. Add a prefix Ω^{k-1} to both sides of the formula in 5.4. □

5.7 Summary *Suppose that $N^n \simeq \Sigma Y$ for a connected pointed CW–space Y, and $n \geq 4$, and I is neatly embedded in N. Then* $\mathrm{emb}(I,N)$ *is homotopy equivalent to the homotopy inverse limit of a certain diagram*

$$\cdots \to T_k\,\mathrm{emb}(I,N) \to T_{k-1}\,\mathrm{emb}(I,N) \to \ldots T_1\,\mathrm{emb}(I,N)\,.$$

Here $T_1 \operatorname{emb}(I,N)$ is $\operatorname{imm}(I,N)$, the Space of smooth immersions which agree with the inclusion near ∂I. For $k \geq 2$, the homotopy fiber of the forgetful map $T_k \operatorname{emb}(I,N) \to T_{k-1} \operatorname{emb}(I,N)$ is homotopy equivalent to a weak product

$$\prod_w{}' \Omega^k \Sigma^{1+\alpha(w)(n-2)} Y^{(\beta(w))}$$

taken over all basic words w in the letters $z_1, \ldots, z_k$ which involve all the letters except possibly z_1.

5.8 Remarks In 5.7, let $N = \mathbb{R}^{n-1} \times I$. There is a distinguished neat embedding $I \to \mathbb{R}^{n-1} \times I$ which identifies I with $0 \times I$. There is also a fibration sequence up to homotopy

$$\operatorname{emb}(I, \mathbb{R}^{n-1} \times I) \longrightarrow \operatorname{emb}(\mathbb{S}^1, \mathbb{S}^n) \xrightarrow{f} \mathrm{O}(n+1)/\,\mathrm{O}(n-1)\,.$$

The arrow f takes a smooth embedding $e\colon \mathbb{S}^1 \to \mathbb{S}^n$ to the orthonormal 2–frame in $\mathbb{R}^{n+1}$ consisting of $e(*)$ and the unit tangent vector to $e(\mathbb{S}^1)$ at $e(*)$. Consequently 5.7 calculates $\operatorname{emb}(\mathbb{S}^1, \mathbb{S}^n)$ if $n \geq 4$, up to extension problems in the homotopy category. We do not know what the extensions are, but we do hope that homotopy theorists will be intrigued.

Unfortunately our convergence result does not cover the approximation map from $\operatorname{emb}(I, \mathbb{R}^2 \times I)$ to $\operatorname{holim}_k T_k \operatorname{emb}(I, \mathbb{R}^2 \times I)$. With or without convergence, we get a map of component sets

$$\pi_0 \operatorname{emb}(I, \mathbb{R}^2 \times I) \to \lim_k \pi_0 T_k \operatorname{emb}(I, \mathbb{R}^2 \times I)\,.$$

It is generally believed [1] that this is closely related to Vassiliev's theory of knot invariants [12], [13]. Conversely, Kontsevich [10] used ideas related to Vassiliev's theory to prove results similar to 5.7, more specifically, to set up a spectral sequence converging to the rational cohomology of $\operatorname{emb}(\mathbb{S}^1, \mathbb{R}^n)$ for $n \geq 4$. He was able to show that the spectral sequence collapses at the E_2 term. One hopes that this result can be reformulated and perhaps even re–proved in calculus language.

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