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$\mathbb{R}$–covered foliations of hyperbolic 3–manifolds

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Abstract

We produce examples of taut foliations of hyperbolic 3–manifolds which are $\mathbb{R}$–covered but not uniform — ie the leaf space of the universal cover is $\mathbb{R}$, but pairs of leaves are not contained in bounded neighborhoods of each other. This answers in the negative a conjecture of Thurston in [7]. We further show that these foliations can be chosen to be C^0 close to foliations by closed surfaces. Our construction underscores the importance of the existence of transverse regulating vector fields and cone fields for $\mathbb{R}$–covered foliations. Finally, we discuss the effect of perturbing arbitrary $\mathbb{R}$–covered foliations.

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1 $\mathbb{R}$–covered foliations that are not uniform

1.1 $\mathbb{R}$–covered foliations

Definition 1.1 We say a foliation $\mathcal{F}$ of a compact 3–manifold M is $\mathbb{R}$–*covered* if the pulled back foliation $\tilde{\mathcal{F}}$ of the universal cover $\tilde{M}$ of M is topologically the standard foliation of $\mathbb{R}^3$ by horizontal $\mathbb{R}^2$'s.

The first step in our construction is to produce a manifold M with an $\mathbb{R}$–covered foliation $\mathcal{F}$ which is not uniform. The condition that a foliation be $\mathbb{R}$–covered is a somewhat elusive one, and in particular it does not seem to be sufficient to find a cover $\hat{M}$ of M so that the leaf space of $\hat{\mathcal{F}}$ is $\mathbb{R}$. This is related to the question of when an infinite braid in $\mathbb{R}^3$ is trivial.

Lemma 1.1 *A taut foliation $\mathcal{F}$ of M is $\mathbb{R}$–covered iff $\mathcal{F}$ has no spherical or projective plane leaves, and for every arc α between two points $p, q \in M$ there is an arc $\hat{\alpha}$ homotopic to α rel. endpoints which is either contained in a leaf of $\mathcal{F}$ or which is everywhere transverse to $\mathcal{F}$.*

Proof If $\mathcal{F}$ is $\mathbb{R}$–covered, lift α to some $\tilde{\alpha}$ in $\tilde{M}$ and make it transverse there. If $\mathcal{F}$ is not $\mathbb{R}$–covered, either M is covered by $S^2 \times S^1$ or the leaf space L of $\tilde{\mathcal{F}}$ is a non-Hausdorff simply connected 1–manifold. This follows from a theorem of Palmeira, which says that the foliation of $\mathbb{R}^3$ by the universal cover of a taut foliation is topologically equivalent to a product of a foliation on $\mathbb{R}^2$ and $\mathbb{R}$, and therefore such a foliation of $\mathbb{R}^3$ by planes whose leaf space is $\mathbb{R}$ is a product foliation (see [5]). In particular, there are distinct leaves $\lambda, \mu \in L$ which cannot be joined by an immersed path in L. That is, the topology of L provides an obstruction to finding such a $\hat{\alpha}$ as above. □

Note that one does not have an analogue of Palmeira's theorem for arbitrary open 3–manifolds — that a foliation with leaf space $\mathbb{R}$ should be a product — and in fact this is not true. For example, remove from $\mathbb{R}^3$ foliated by horizontal $\mathbb{R}^2$'s a properly embedded bi-infinite transverse curve which does not intersect every leaf. This is where the difficulty resides in showing that a foliation is $\mathbb{R}$–covered by investigating an intermediate cover.

Many (most?) taut foliations of 3–manifolds are not $\mathbb{R}$–covered. In particular, by a theorem of Fenley, an $\mathbb{R}$–covered foliation of a hyperbolic 3–manifold has the property that in the universal cover, every leaf limits to all of S^2_∞. However, any compact leaf of a taut foliation which is not a fiber of a fibration over S^1 lifts to a quasi-isometrically embedded plane in the universal cover, and its limit set is a quasicircle (see [3] for a fuller discussion).

1.2 Uniform foliations

Definition 1.2 A taut foliation $\mathcal{F}$ of a compact 3–manifold M is *uniform* if in the pullback foliation $\tilde{\mathcal{F}}$ of the universal cover $\tilde{M}$, every two leaves λ, μ are a bounded distance apart. That is, there is some ϵ depending on λ, μ so that λ is contained in the ϵ–neighborhood of μ, and vice versa. A foliation $\mathcal{F}$ is obtained from a *slithering over* S^1 if there is a fibration $\phi\colon\ \tilde{M} \to S^1$ such that $\pi_1(M)$ acts as bundle maps of this fibration, and such that the foliation of $\tilde{M}$ by components of the fibers of ϕ agrees with $\tilde{\mathcal{F}}$. We will also refer to such a foliation, perhaps ungrammatically, as a slithering.

For additional details and definitions, see [7]. It is shown in [7] that a uniform foliation with every leaf dense is obtained from a slithering.

It is almost tautological from the definition of a slithering over S^1 that the action of $\pi_1(M)$ on the leaf space L of $\tilde{\mathcal{F}}$ is conjugate to a representation in $\widetilde{Homeo}(S^1)$, the universal central extension of $Homeo(S^1)$.

3–manifold topologists will be familiar with the short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{PSL(2,\mathbb{R})} \to PSL(2,\mathbb{R}) \to 0$$

This sits inside the short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{Homeo}(S^1) \to Homeo(S^1) \to 0$$

Informally, $\widetilde{Homeo}(S^1)$ is the group of homeomorphisms of $\mathbb{R}$ which are periodic with period 1.

Let Z be the generator of the center of $\widetilde{Homeo}(S^1)$. Then Z acts on L by translations, and by choosing an invariant metric on L for this action, the action of every element of $\pi_1(M)$ on L is periodic with some period equal to the translation length of Z. Z is known as *the slithering map.*

The condition that a foliation be uniform is reflected in the action of $\pi_1(M)$ on L in the following way: since leaves do not converge or diverge too much at infinity, holonomy cannot expand or contract the leaf space too much. Every compact interval in L can be included in a larger compact interval which is "incompressible": no element of $\pi_1(M)$ takes it to a proper subset or superset of itself. If every leaf is dense, L can be tiled with a countable collection of these incompressible intervals laid end to end, so that the slithering map Z acts on this tiling as a permutation.

Note that a foliation $\mathcal{F}$ may come from a slithering in many different ways. For instance, if $\mathcal{F}$ admits a nonsingular transverse measure then the leaf space of $\tilde{\mathcal{F}}$ inherits an invariant measure making it isometric to $\mathbb{R}$. Then for any real $t > 0$ there is a slithering

$$\phi_t\colon \tilde{M} \to S^1$$

defined by the composition

$$\tilde{M} \to L \to S^1$$

where the last map is reduction mod t. Since the action of $\pi_1(M)$ on L preserves the property of points being integral multiples of t apart, this map is a slithering.

However *generic* foliations come from a slithering in essentially at most one way. The slithering map Z commutes with the action of every element of $\pi_1(M)$. So, for instance, if an element α acts on L with isolated fixed points, the map Z must permute this fixed point set. If, further, the action of $\pi_1(M)$ on L is minimal (ie every leaf of $\mathcal{F}$ is dense), the map Z is determined uniquely up to taking iterates. That is, there is a *minimal* slithering $\phi\colon \tilde{M} \to S^1$ with the property that for every other slithering $\phi'\colon \tilde{M} \to S^1$ determining the same foliation, there is a finite cover $\psi\colon S^1 \to S^1$ so that $\phi = \psi \circ \phi'$.

This theory is developed in [7].

Following [7] we define some auxiliary structure that will be used to show that certain foliations are uniform or $\mathbb{R}$–covered.

Definition 1.3 Let X transverse to $\mathcal{F}$ be a vector field. Then X is *regulating* if the lifts of the integral curves of X to $\tilde{M}$ intersect every leaf of $\tilde{\mathcal{F}}$.

These lifts determine a one dimensional foliation of $\tilde{M}$. A leaf in this foliation and a leaf in $\tilde{\mathcal{F}}$ intersect in exactly one point, and consequently one can identify the leaf space of the one dimensional foliation with any fixed leaf of $\tilde{\mathcal{F}}$—that is, with $\mathbb{R}^2$. Such one dimensional foliations are called *product covered* in [2]. The main point for our purposes of this structure is the following theorem:

Theorem 1.2 *Suppose $\mathcal{F}$ is uniform (respectively $\mathbb{R}$–covered) and X is a transverse regulating vector field with a closed trajectory α. Then the restriction of $\mathcal{F}$ to $M - \alpha$ is also uniform (respectively $\mathbb{R}$–covered).*

Proof The universal cover $\tilde{M}$ of M is foliated as a product by $\tilde{\mathcal{F}}$ and the integral curves of X give this the structure of a product $\mathbb{R}^2 \times \mathbb{R}$ in such a way that $\pi_1(M)$ acts by elements of $Homeo(\mathbb{R}^2) \times Homeo(\mathbb{R})$. Let N be obtained from $\tilde{M}$ by removing the lifts of α. Then N is the cover of $M-\alpha$ corresponding to the subgroup of $\pi_1(M-\alpha)$ normally generated by the meridian of α. One sees from the structure of X that N is foliated as a product by infinitely punctured disks and therefore that $\tilde{N}$ is $\mathbb{R}^3$ foliated by horizontal $\mathbb{R}^2$'s. If $\mathcal{F}$ was uniform, any two leaves in N would be a finite distance apart. But leaves in $\tilde{N}$ correspond bijectively with leaves in N under the covering projection and therefore the same is true in $\tilde{N}$; that is, the restriction of $\mathcal{F}$ to $M-\alpha$ is uniform. □

Notice from the construction that if $\mathcal{F}$ came from a slithering, then the restriction of $\mathcal{F}$ to $M-\alpha$ comes from a slithering which agrees with the restriction of the slithering map on $\tilde{M}$ to the complement of the lifts of α.

1.3 Building uniform foliations from representations

Let F_g^n denote the surface of genus g with n punctures. Then F_1^1 is the punctured torus, and $\pi_1(F_1^1) = \mathbb{Z} * \mathbb{Z}$. Let α_l, β_l be standard generators for $\pi_1(F_1^1)$. Then we can choose a representation ρ of $\pi_1(F_1^1) \to \widetilde{Homeo}(S^1)$ by sending α_l to translation through length t and β_l to some monotone element perhaps with a periodic collection of fixed points, each distance 1 apart. Let M_l be the trivial circle bundle $M_l = F_1^1 \times S^1$ over the punctured torus, and pick a flat $Homeo(S^1)$ connection on this bundle whose holonomy realizes the representation ρ. Note that after fixing a trivialization of the product, the representation is well-defined in $\widetilde{Homeo}(S^1)$ and not just $Homeo(S^1)$. The distribution determined by this connection is integrable, by flatness, and integrates to give a foliation $\mathcal{F}_l$.

Geometrically, there is a foliation of $\tilde{F}_1^1 \times S^1$ by leaves $\tilde{F}_1^1 \times point$. $\pi_1(F_1^1)$ acts on this space by

$$(x,\theta) \to (\alpha(x), \rho(\alpha)(\theta))$$

which preserves the foliation. This foliation therefore descends to a foliation on

$$M_l = \tilde{F}_1^1 \times S^1/\pi_1(M)$$

transverse to the S^1 fibers.

This foliation of M_l comes from an obvious slithering $\phi\colon \tilde{F}_1^1 \times \mathbb{R} \to S^1$ which is just projection onto the second factor followed by the covering map $\mathbb{R} \to S^1$ corresponding to the circles in M_l. The action of $\pi_1(M_l)$ on the leaf space is exactly given by the representation ρ in $\widetilde{Homeo}(S^1)$ thought of as sitting in $Homeo(\mathbb{R})$.

Then the curve $\alpha_l \times 0$ sits in M_l transverse to $\mathcal{F}_l$, and there is also a foliation on $M_l - \alpha_l$ which we also denote by $\mathcal{F}_l$.

It is easy to see that for irrational choice of t the foliation $\mathcal{F}_l$ has every leaf dense.

Furthermore we have the following lemma:

Lemma 1.3 *The foliation $\mathcal{F}_l$ of $M_l - \alpha_l$ comes from a slithering. Furthermore, this slithering can be taken to be the restriction of $\phi\colon \tilde{M}_l \to S^1$ to the complement of the lifts of α_l in $\tilde{M}_l$. Moreover, by choosing $\rho(\beta_l)$ suitably generic, this slithering is minimal as defined above.*

Proof By theorem 1.2 it suffices to show there is a regulating vector field of M_l which agrees with α_l' when restricted to α. Since M_l is topologically just $F_1^1 \times S^1$ it has a projection map to F_1^1. Let H be the torus foliated by circles that is the preimage of the curve α on F_1^1 under this projection. Then $M_l - N(H)$ has an obvious codimension 2 foliation by S^1 fibers. We extend this foliation over $N(H)$, which can be parameterized as $S^1 \times S^1 \times [-1,1]$, by foliating each $S^1 \times S^1 \times *$ with parallel lines which rotate continuously from vertical (parallel to the $* \times S^1$ direction in M_l) on the boundary to horizontal (parallel to α) on H, always staying transverse to $\mathcal{F}_l$. It is obvious that this is a foliation by regulating curves, and we denote its associated unit tangent vector field by X_l.

If we choose $\rho(\beta_l)$ to be generic and close to the identity with isolated fixed points, the slithering defined in the statement of the theorem is minimal. $\square$

On another punctured torus with basis for π_1 given by α_r, β_r we pick another representation σ in $\widetilde{Homeo}(S^1)$ so that $\sigma(\alpha_r)$ is translation through s, where again s is irrational and incommensurable with t, and $\sigma(\beta_r)$ is some random element which commutes with Z but not with α_r. Then we can form $M_r = F_1^1 \times S^1$ foliated as above, and remove α_r from M_r to produce another foliated manifold with a slithering.

Let M be obtained by gluing $M_l - \alpha_l$ and $M_r - \alpha_r$ along the torus boundaries of neighborhoods of α_l and α_r respectively. Denote this torus in the sequel by $S \subset M$. Each piece $M_l - \alpha_l$, $M_r - \alpha_r$ admits a regulating vector field X_l, X_r as constructed in lemma 1.3. We perform this gluing in such a way that the foliations of the boundary tori by meridional circles agree. If we like, we can perform the gluing so that the restriction of the slithering maps for the left and the right foliations commute, when restricted to their action on the leaf space of the universal cover of S.

Then

$$\pi_1(M) = \pi_1(M_l - \alpha_l) *_{\mathbb{Z}\oplus\mathbb{Z}} \pi_1(M_r - \alpha_r)$$

acts on $\mathbb{R}$ by the amalgamated action of each piece on the leaf space of its respective universal cover, each canonically identified with the leaf space of the foliation of $\tilde{S}$.

Topologically, M is a graph manifold obtained from four copies of $F_0^3 \times S^1$. To see this, observe that $M_l - N(H)$ (with notation as above) is exactly $(F_1^1 - N(\alpha)) \times S^1$ which is $F_0^3 \times S^1$. Also, observe that $N(H) = S^1 \times S^1 \times [-1,1]$ in many different ways, including a way in which α is $S^1 \times * \times 0$. It follows that $N(H) - \alpha$ is topologically also $F_0^3 \times S^1$. However, these foliations by circles cannot be made to agree on the boundary tori of different pieces, and M is not a Seifert fibered space.

Let $\mathcal{F}$ denote the induced foliation of M. Is $\mathcal{F}$ $\mathbb{R}$–covered? To establish that it is indeed $\mathbb{R}$–covered, it will suffice to show that each piece is uniform and admits a regulating transverse vector field X_l, X_r which agree on the gluing torus to make a global regulating vector field X.

Lemma 1.4 *$\mathcal{F}$ is an $\mathbb{R}$–covered foliation of M.*

Proof Let $\hat{M}$ foliated by $\hat{\mathcal{F}}$ be the cover of M obtained by taking copies of the universal covers of M_l and M_r, drilling out countably many copies of the lifts of α_l and α_r, then gluing along the boundary components. Then the regulating vector fields X_l, X_r lift to $\hat{M}$ to give a global trivialization of this manifold as a product of an infinite genus Riemann surface with $\mathbb{R}$. This implies that the universal cover of $\hat{M}$ is the standard $\mathbb{R}^3$ foliated by $\mathbb{R}^2$'s, and we see therefore that $\mathcal{F}$ is $\mathbb{R}$–covered. □

Since every leaf of $\mathcal{F}$ is dense in M, if $\mathcal{F}$ were uniform it would come from a slithering by [7]. However, we have seen that the action of $\pi_1(M)$ on the leaf space of the universal cover is the amalgamation of the actions of $\pi_1(M_l - \alpha_l)$ and $\pi_1(M_r - \alpha_r)$ along their gluing $\mathbb{Z} \oplus \mathbb{Z}$. It follows that there is no single translation $\mu \in Homeo(\mathbb{R})$ (ie an element without fixed points) which commutes with both β_l and β_r for sufficiently generic choice of $\sigma(\beta_r)$ and $\rho(\beta_l)$, since the periods of the left and right slithering maps are incommensurable.

More explicitly, let us fix a lift $\tilde{S}$ of the torus S to $\tilde{M}$ which divides a piece of $\tilde{M}$ which is a lift of $M_l - \alpha_l$ from a piece which is a lift of $M_r - \alpha_r$. Let us identify the leaf space L with the leaf space of $\tilde{S}$. Let $\alpha \in \pi_1(M)$ corresponding to the longitude of S preserve $\tilde{S}$ and act on L as a translation. Fix a metric on L such that α acts as translation through a unit length. Let Z_l and Z_r be the translations in $Homeo(L)$ corresponding to the slithering map of $M_l - \alpha_l$ and $M_r - \alpha_r$ thought of as acting on the leaf space of $\tilde{S}$. Then Z_l acts as translation through length $\frac{1}{s}$ and Z_r acts as translation through length $\frac{1}{t}$. By minimality, the only translations in $Homeo(L)$ that commute with $\rho(\beta_l)$ are multiples of Z_l, whereas the only translations that commute with $\sigma(\beta_r)$ are multiples of Z_r. It follows that no translation commutes with both elements, and $\mathcal{F}$ does not come from a slithering.

We have therefore proved the following theorem:

Theorem 1.5 *$\mathcal{F}$ as above is $\mathbb{R}$–covered but not uniform.*

2 Lorentz cone fields

The following definition is from [7]:

Definition 2.1 On any manifold M, a *(Lorentz) cone field C* transverse to a codimension one foliation $\mathcal{F}$ is the field of timelike vectors (ie with positive norm) for a (continuously varying) form on TM of signature $(n-1,1)$ such that $T\mathcal{F}_p$ are spacelike. A cone field is *regulating* if every complete curve X with $X' \in C$ is regulating for an $\mathbb{R}$–covered $\mathcal{F}$.

Regulating cone fields are discussed in [7], and shown to exist for all uniform foliations. We show that the example constructed in the previous section admits a regulating cone field.

Each piece $M_l - \alpha_l$, $M_r - \alpha_r$ admits a regulating cone field C_l, C_r which is degenerate along the boundary torus, coming from the restriction of the regulating cone fields on M_l, M_r which are tangent to α_l and α_r. Let C denote the cone field on M which agrees with C_l and C_r away from a collar of the separating torus, and which near this separating torus is sufficiently thin so that every curve which crosses this collar must wind a distance at least T, as measured in periods of the longitude, transverse to the foliation.

Theorem 2.1 *C as defined above is a regulating cone field for $\mathcal{F}$.*

Proof We will do our calculations in $\hat{M}$, using the fact that $\hat{M}$ admits a regulating vector field coming from X_l, X_r which preserves each uniform piece. Observe that $\hat{M}$ has a decomposition into a countable collection of covering spaces of $M_l - \alpha_l$, $M_r - \alpha_r$ which we denote L_i, R_i for some particular choice of indices i. Also label the separating cylinders, all of them lifts of the gluing torus in M, as A_i for some index i. Let d_i denote the metric on L, the leaf space of $\hat{\mathcal{F}}$, given by the transverse measure on $\mathcal{F}|_S$ scaled so that the curves $\alpha_l = \alpha_r$ have period 1 and so that the left and right slithering maps act by translations of this measure, as measured in the cylinder A_i. Then we claim the following lemma:

Lemma 2.2 *If A_i and A_j are separated by n cylinders A_k then for any two leaves $\lambda, \mu \in L$,*

$$|d_i(\lambda,\mu) - d_j(\lambda,\mu)| \leq (n+1)\max(\frac{1}{s},\frac{1}{t})$$

Proof The proof follows immediately by induction once we show the result for $n=0$. If A_i and A_j bound a single L_i or R_i, then the fact that the pieces L_i and R_i slither over S^1 implies that $d_i(\lambda,\mu)$ and $d_j(\lambda,\mu)$ differ by at most one period of the slithering, which in terms of the measure on the boundary torus, is either $\frac{1}{s}$ or $\frac{1}{t}$ depending on whether we are in an L_i or an R_i. □

Let $r = \max(\frac{1}{s},\frac{1}{t})$.

Now we will show that any curve γ supported by C makes definite progress relative to any given d_i, and therefore relative to all of them. In particular, it is regulating. After re-ordering indices, any such infinite curve in $\hat{M}$, starting on some leaf $\lambda \in L_1$, can be broken up into segments $\gamma_1, \gamma_2, \dots$ where each γ_i is contained in L_i or R_i (according to sign). It is clear that if there are only finitely many γ_i (ie γ crosses only finitely many separating annuli) then eventually γ can be seen to be regulating, since it is supported by some lift of C_r or C_l. So we suppose there are infinitely many γ_i. Let Y_i be the union of the first i segments of γ. Let Z_i be the *shadow* of Y_i on λ; that is, the curve on λ obtained by projecting $\hat{M}$ to λ along the integral curves of X. Let Y_i' be the integral curve of X interpolating between the endpoint of Z_i and the endpoint of Y_i. Then $d_i(Y_j) = d_i(Y_j')$ for all i,j since the curves Y_j and Y_j' have endpoints on the same pair of leaves.

We can estimate $d_1(Y_1') \geq T$ by hypothesis on C. It follows that $d_2(Y_1') \geq T - r$. But then $d_2(Y_2') \geq 2T - r$ and so $d_3(Y_2') \geq 2T - 2r$. Continuing in this way and by induction, we get $d_n(Y_n') \geq nT - nr$. But then

$$d_1(Y_n) = d_1(Y_n') \geq nT - 2nr$$

and one can see that by choosing $T \gg 2r$ the curve γ makes arbitrary progress relative to some *fixed* d_i, and therefore is regulating. $\square$

An instructive analogy is given by the comparison between imperial and metric weights and measures: suppose I have a small object which I can measure to the nearest inch or to the nearest centimeter. Then I get estimates which vary greatly compared to the length of the object. If the object is much bigger, the estimates are comparatively better. My regulating curve above makes definite progress, even though its progress is translated into inches, then centimeters, then inches, then centimeters ... rounding down every time.

2.1 A hyperbolic example

Theorem 2.3 *Suppose $M,\mathcal{F}$ is any compact oriented 3–manifold with a co-orientable $\mathbb{R}$–covered foliation, and suppose that $\mathcal{F}$ admits a transverse regulating Lorentz cone field C. Let γ be any simple closed curve supported by C. If $M_n(\gamma)$ is obtained by taking an n–fold branched cover over γ, and $\mathcal{F}_n(\gamma)$ denotes the pullback foliation, then $\mathcal{F}_n(\gamma)$ is $\mathbb{R}$–covered. Moreover, $\mathcal{F}_n(\gamma)$ is uniform iff $\mathcal{F}$ is.*

Proof The point of having a regulating cone field C is that for any γ supported as above, there is a regulating vector field X of M so that $X|_\gamma = \gamma'$. This follows immediately from obstruction theory, once one notices that sections of C are contractible; eg use a partition of unity.

Now in $\tilde{M}$, the universal cover of M, γ lifts to a collection of bi-infinite regulating curves, and $\tilde{M}_n(\gamma)$ is the universal orbifold cover of $\tilde{M}$ where we declare that there are order n cone angles along the lifts of γ. Let $\tilde{\mathcal{F}}_n(\gamma)$ be the pullback foliation in that universal orbifold cover and let λ, μ be two leaves there. They can be joined by some arc α in the complement of the lifts of the cone locus, which projects to an arc $\pi(\alpha)$ in $\tilde{M}$. By homotoping $\pi(\alpha)$ rel. endpoints along integral curves of X, we can make it transverse to $\tilde{\mathcal{F}}$ *without crossing any lift of* γ. Then this perturbed $\pi(\alpha)$ lifts to a perturbed α in $\tilde{M}_n(\gamma)$ transverse to $\tilde{\mathcal{F}}_n(\gamma)$, thereby demonstrating that $\mathcal{F}_n(\gamma)$ is $\mathbb{R}$–covered, as required.

If $\mathcal{F}$ was not uniform, there would be a pair of leaves in $\tilde{\mathcal{F}}$ which diverge at infinity. They lift to leaves with the same property in $\tilde{\mathcal{F}}_n(\gamma)$. Alternatively, the uniformity or lack thereof can be seen in the action of $\pi_1(M_n(\gamma))$ on $\mathbb{R}$. $\square$

We return to the example M that we constructed in the previous section. In M, it is clear that we can choose a curve γ supported by C whose complement is hyperbolic. For, we can certainly do this in each side of M, and then by crossing back and forth across the separating torus, we can arrange for the complement of γ to be atoroidal. In particular, by choosing $\rho(\beta_l)$ and $\sigma(\beta_r)$ to be sufficiently close to translations, the regulating cone fields C_l and C_r can be as "squat" as we like, and we have a great deal of freedom in our choice of the restriction of γ to the complement of a collar neighborhood of S.

For example, if ρ, σ are chosen so that every element acts on the leaf space as a translation through a rational distance, $\mathcal{F}$ would be a surface bundle over a circle; in a product bundle, a curve whose projection to S^1 is a homeomorphism and whose projection to the base surface fills up the surface (ie complementary regions for the geodesic representative are disks) has atoroidal complement. Similar curves are easily found in any surface bundle, and one can arrange for them to wind several times in the circle direction when they pass through some reducing torus. The point is that any sufficiently complicated curve will suffice. Then for nearby choices of ρ, σ, such a curve will still be regulating and contained in the regulating Lorentz cone field, as we show in the following lemma:

Lemma 2.4 *Suppose $\mathcal{F}_l'$ is a transversely measured foliation of $M_l - \alpha_l$ as above coming from some representation $\rho(\beta_l) =$ translation, and let C' be a transverse Lorentz cone field for $\mathcal{F}_l'$ appropriately degenerate near α_l. Then for slitherings $\mathcal{F}_l$ coming from sufficiently close choices of $\rho(\beta_l)$, the cone field C' is regulating for $\mathcal{F}_l$.*

Proof We need to check for sufficiently mild perturbations $\mathcal{F}_l$ of $\mathcal{F}_l'$ that any curve supported by C' passes through at least one period of the slithering of $\mathcal{F}_l$, since then it must pass through arbitrarily many such periods and therefore be regulating.

Observe first that any transverse Lorentz cone field is regulating for a transversely measured foliation, since one can uniformly compare distance along a curve supported by the cone field and distance with respect to the transverse measure.

For $\mathcal{F}_l$ sufficiently close to $\mathcal{F}_l'$, C' is a transverse Lorentz cone field for $\mathcal{F}_l$. The codimension 2 foliation X_l described in lemma 1.3 is regulating for every choice of $\rho(\beta_l)$, and we assume that this foliation lifts to the vertical foliation of $\mathbb{R}^3$ by point $\times\, \mathbb{R}$. We choose co-ordinates on $\mathbb{R}^3$ so that the regulating curves are of the form $x=$ const., $y=$ const. and the leaves of $\tilde{\mathcal{F}}_l'$ are of the form $z=$ const.

The slithering on $M_l-\alpha_l$ comes from the slithering on M_l associated with the circle bundle as above. We assume that the circles of the Seifert fibration away from $N(H)$ lift to vertical arcs of length 1.

With this choice of co-ordinates, for any foliations $\mathcal{F}_l$ constructed from a representation as above and for every x,y,z, the points (x,y,z) and $(x,y,z+1)$ in the universal cover are on leaves which are one period of the slithering apart.

For a point p in $\tilde{M}_l$ given in co-ordinates by (x,y,z), let λ_p be the leaf of $\tilde{\mathcal{F}}_l$ through p, and μ_p the leaf of $\tilde{\mathcal{F}}_l$ through $(x,y,z+1)$. Then the leaves λ_p and μ_p differ by a translation parallel to the z-axis of length 1. Note that this translation need in no way correspond to the action of an element of $\pi_1(M_l)$ on $\tilde{M}_l$. The light cone of C' through p intersects the horizontal plane passing through $(x,y,z+1)$ in a compact region. For sufficiently small perturbations of $\rho(\beta_l)$, the leaf μ_p will be a small perturbation of the horizontal plane through p, and the intersection of the light cone of C' through p with μ_p will also be a compact region.

Now, $M_l-\alpha_l$ is not compact, but we can consider its double N, foliated by the double of $\mathcal{F}_l'$, and equipped with a transverse Lorentz cone field obtained by doubling C' which is degenerate along the boundary components of $M_l-\alpha_l$. The compactness of N implies that for a sufficiently small perturbation $\mathcal{F}_l'$ of $\mathcal{F}_l$, the intersections as above will be compact for all p. This implies that a curve supported by C' will need to go only a bounded distance before traveling a full period of the slithering. In particular, any bi-infinite curve supported by C' will travel through infinitely many periods of the slithering in either direction and will therefore be regulating, which is what we wanted to show. □

Remark 2.1 What is really essential to notice in the above set up is that the leaves passing through (x,y,z) and $(x,y,z+1)$ in the universal cover were one period of the slithering away from each other for *all* $\mathcal{F}_l$. In some sense, all the slitherings are determined by the structure of the Seifert fibration where they originated. For a generic perturbation of a uniform foliation, one has no control over how the slithering map may vary, or even whether the perturbed foliations are uniform at all.

Since for a transversely measured foliation, *any* transverse Lorentz cone field is regulating, we can choose our curve γ to be any curve transverse to a measured foliation $\mathcal{F}_l'$ with hyperbolic complement which is sufficiently steep near the separating torus, and then choose the representation $\rho(\beta_l)$ to be sufficiently close to a translation that γ stays in a regulating cone field.

Since $M-\gamma$ is hyperbolic, for sufficiently large n an n–fold branched cover of M over γ is hyperbolic. One cannot be sure that an n–fold branched cover will always exist, but at least one has a hyperbolic orbifold structure on M with cone angle $2\pi/n$ along γ, and by Selberg's lemma (see [6]) one knows this is finitely (orbifold) covered by a genuine hyperbolic manifold which is a branched cover of M along γ.

By the discussion above, the induced foliation is $\mathbb{R}$–covered but not uniform. Moreover, by choosing $\rho(\beta_l)$ and $\sigma(\beta_r)$ sufficiently close to translations, the foliation $\mathcal{F}$ is as close to a transversely measured foliation as we like. Passing to a branched cover preserves this fact. Since transversely measured foliations of 3–manifolds are arbitrarily close (as 2–plane fields) to surface bundles over S^1, we have proved:

Theorem 2.5 *There exist foliations of hyperbolic 3–manifolds which are $\mathbb{R}$–covered but not uniform. They can be chosen arbitrarily close to surface bundles over circles.*

Remark 2.2 It is clear that the construction outlined above can be made in some generality. One can construct $\mathbb{R}$–covered but not uniform foliations by plumbing together uniform foliations along boundary tori in numerous ways. In a great number of cases, these will admit regulating transverse cone fields, and by branching as above one can produce many atoroidal examples. One can easily arrange for these examples to be closed; for instance, by doubling M before removing a curve with atoroidal complement in the examples constructed above.

Another construction, explained in detail in [7], involves choosing a train track with integer weights supported by a regulating cone field, then plumbing the leaves along the train track with a surface of genus given by the track weight. These plumbed surfaces can be "twisted" by a surface automorphism under the monodromy around loops of the train track. Thurston expects that these examples are sufficiently flexible to allow one to prescribe the homology of M.

3 General $\mathbb{R}$–covered foliations

3.1 Regulating vector fields

For a general $\mathbb{R}$–covered foliation, we do not know whether or not there exists a transverse Lorentz cone field.

In what follows, $\mathcal{F}$ will be an $\mathbb{R}$–covered taut foliation of a 3–manifold M with hyperbolic leaves.

We show in [1] that the circles at infinity of each leaf in the universal cover of M can be included in a topological *cylinder at infinity* C_∞ on which $\pi_1(M)$ acts by homeomorphisms.

The following theorem is proved in [1]:

Theorem 3.1 *With notation as above, there is a global trivialization of C_∞ as $S^1 \times \mathbb{R}$ so that the action of $\pi_1(M)$ preserves the horizontal and vertical foliations of C_∞ by $S^1 \times$ point and point $\times \mathbb{R}^1$.*

In [8] it is suggested that *all* taut foliations should have a pair of essential laminations Λ_+, Λ_- transverse to each other and to the foliation which intersect each leaf in a (1–dimensional) geodesic lamination. In the case of $\mathbb{R}$–covered foliations, these laminations should come from a pair of 1–dimensional laminations $\hat{\Lambda}_+, \hat{\Lambda}_-$ of the universal S^1 described in the theorem above which are invariant under the action of $\pi_1(M)$.

Let the universal S^1 bound a hyperbolic plane D, and let $\hat{\Lambda}_+, \hat{\Lambda}_-$ be the associated geodesic laminations in D. Then since each circle at infinity is canonically associated with this S^1, each point in $\hat{\Lambda}_+ \cap \hat{\Lambda}_-$ determines a unique point in each leaf of $\tilde{\mathcal{F}}$. Similarly, each segment of $\hat{\Lambda}_\pm$ between points of intersection determines a unique geodesic segment in each leaf, and each complementary region determines a unique geodesic polygon in each leaf. If one fixes some canonical geometric parameterization of the family of convex geodesic polygons of a fixed combinatorial type, this parameterization gives rise to a canonical identification of each leaf with each other leaf, preserving the stratification outlined above. The fibers of this identification give a one-dimensional foliation transverse to $\tilde{\mathcal{F}}$, and the tangent vectors to this foliation are a regulating vector field.

It is easy to see that non-quadrilateral complementary regions in D give rise to solid cylinders in $\tilde{M}$ which cover solid tori in M, since their cores are isolated.

The cores of these solid tori are necessarily regulating, and therefore define elements of $\pi_1(M)$ which act on the leaf space by translation.

In fact, Thurston has communicated to this author a sketch of a proof that an $\mathbb{R}$–covered foliation admits *some* transverse lamination which intersects every leaf in geodesics (though not necessarily a pair of such). In [1] we can show by methods slightly different to those above that this assumption is enough to imply that there exists a regulating vector field transverse to $\mathcal{F}$ which can be chosen to have closed orbits. In short, we have the following theorem:

Theorem 3.2 *Let $\mathcal{F}$ be an $\mathbb{R}$–covered foliation. Then there exists a regulating vector field transverse to $\mathcal{F}$ which can be taken to have closed orbits. These orbits determine elements $\alpha \in \pi_1(M)$ which act on the leaf space of $\tilde{\mathcal{F}}$ without fixed points. Branched covers of M over these closed orbits give new $\mathbb{R}$–covered foliations.*

One hopes these results are all pieces of a unified picture tying the intrinsic geometry of $\mathbb{R}$–covered foliations to the extrinsic geometry of the foliated manifolds containing them. Not all the details of this picture are yet visible. Nevertheless, it seems worthwhile to make this picture as explicit as possible.

3.2 Instability of $\mathbb{R}$–covered foliations

Despite the positive results of the previous sections, it seems that the property of being $\mathbb{R}$–covered is quite delicate. The following example is suggestive.

Let M be a hyperbolic surface bundle over a circle with fiber F and pseudo-Anosov monodromy $\psi\colon F \to F$. Let $\mathcal{F}$ be the induced foliation by surfaces. Let γ be a simple closed curve on F so that $\gamma \cap \psi(\gamma) = \emptyset$. Note that it is easy to show that there exist such examples, by first choosing any γ, ψ, F and then using the fact that surface groups are LERF to lift to a finite regular cover where γ and its image are disjoint.

Let $\hat{M}$ be the $\mathbb{Z}$–cover of M defined by the circle direction. Topologically, $\hat{M}$ is $F \times \mathbb{R}$ foliated by closed surfaces $F \times$ point. Let the group $\mathbb{Z}$ generated by the deck translation, which we denote Ψ, act by $\Psi(x,t) = (\psi(x), t+1)$.

Let A be the annulus $\gamma \times [-\epsilon, 1+\epsilon]$. Then by construction, A and its translates are disjoint. Let $\mu\colon [-\epsilon, 1+\epsilon] \to [-\epsilon, 1+\epsilon]$ be a homeomorphism close to the identity which moves every point except the endpoints up some small amount. Then we can cut open $\hat{M}$ along A and its translates, and shear the foliation on one side by the translation $\mathrm{id} \times \mu$ to get a new foliation $\mathcal{G}$. This can certainly be done in such a way that $\mathcal{G}$ is arbitrarily close to $\mathcal{F}$.

Now whenever an integral curve of $\mathcal{G}$ passes through A or a translate in the "positive" direction, it will be sheared upward (relative to $\mathcal{F}$) by μ.

Let us suppose that by choosing a suitable path in $\mathcal{G}$, we can arrange that a curve starting at some $(x_0, 0)$ can get to $(x_1, 1)$ in length t, as measured in $\mathcal{G}$, by winding sufficiently many times through A. Now, the curve can continue to wind around $\Psi(A)$, and after moving a distance $2t$, it can reach $(x_2, 2)$, and so on. Remember that there is a transverse regulating vector field X to $\mathcal{F}$ given by tangents to the curves point $\times \mathbb{R}$.

Since ψ is pseudo-Anosov, when we compare arclength at (x,t) and $(x,0)$ by projection along X, we see that a vector of norm λ^t at (x,t) might project to a vector of norm 1 at $(x,0)$, where λ is the multiplier of ψ on the invariant transverse measure of the unstable lamination of F. Hence, as measured in $\mathcal{G}$, a curve γ beginning at $(x,0)$ could have length nt but its projection to $F, 0$ could have length as little as $\sum_{i=1}^{n} t/\lambda^i$. In particular, a curve in $\mathcal{G}$ could "escape to infinity" while its projection to $(F,0)$ could move only a finite distance. By picking two points $(x,0)$ and (x,n) sufficiently far apart, and moving them by curves in $\mathcal{G}$ joined by integral curves of X, it seems plausible that one could find a path in $\mathcal{G}$ where holonomy was not defined after some finite time, suggesting that $\mathcal{G}$ was not $\mathbb{R}$–covered.

Of course, there are problems with making this concrete: distances in $\mathcal{G}$ are only magnified in the direction of the stable lamination as we go upwards; perhaps to make the curves cross through A and its translates sufficiently often, we need to go in both stable and unstable directions. Moreover, even if one could show that holonomy failed to be defined for all time along integral curves of X, it does not rule out the possibility that $\mathcal{G}$ is still $\mathbb{R}$–covered and X is merely not regulating, although the author understands that very recently Sérgio Fenley has shown that a pseudo-Anosov flow transverse to an $\mathbb{R}$–covered foliation should always be regulating ([4]). A similar result is also proved in [1].

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