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# Exponential separation in 4–manifolds

Vyacheslav S Krushkal

*Department of Mathematics, Yale University*
*New Haven, CT 06520-8283, USA*

Email: `krushkal@math.yale.edu`

**Abstract**

We use a new geometric construction, grope splitting, to give a sharp bound for separation of surfaces in 4–manifolds. We also describe applications of this technique in link-homotopy theory, and to the problem of locating $\pi_1$–null surfaces in 4–manifolds. In our applications to link-homotopy, grope splitting serves as a geometric substitute for the Milnor group.

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Open problems in the classification theory of topological four–manifolds, for "large" fundamental groups, have been reformulated in terms of immersions of surfaces in 4–manifolds, cf [1], [3]. Two related properties of immersed surfaces are important in this discussion: disjointness, and vanishing of the double point loops in the fundamental group of the ambient 4–manifold. More precisely, these questions concern more general 2–complexes, *capped gropes*, that naturally arise in this context. Capped gropes are assembled of several surface stages, capped with disks with self-intersections. Each double point determines an element of the fundamental group of the ambient 4–manifold $M$, and a central question is whether one can find a $\pi_1$–*null* capped grope (so that each double point loop is contractible in $M$.) A closely related question asks whether a collection of surfaces, intersecting the caps of a given grope, may be pushed off it by a homotopy, without creating intersections between different surfaces. It follows from the work of Freedman–Teichner [4] that such problems may be solved if the number of group elements (respectively, the number of surfaces) is bounded by the exponential function $2^h - 1$ in the grope height $h$.

In the present paper we describe a new construction, *grope splitting*, which may be thought of as a tool for organizing intersections between surfaces and capped gropes. This construction is used to give a new proof of the results on separation of surfaces, and locating $\pi_1$–null capped gropes, mentioned above. The argument in [4] relies on algebraic theory of link homotopy, and is an indirect existence proof. Our proof is more transparent geometrically, and it gives an explicit construction of the resulting surfaces. We also point out that the bound for separation of surfaces is sharp.

This exponential result is one of the main ingredients of the theorem [4] that the classification techniques (4–dimensional surgery and 5–dimensional $s$–cobordism conjectures) hold in topological category for fundamental groups of subexponential growth. A new geometric proof of this theorem is presented in [7]. (Also see the Appendix in that paper for a revised version of [4].) The conjectures for arbitrary fundamental groups remain open; the new viewpoint presented here should be helpful in clarifying the problem.

In our applications to link-homotopy, the operation of grope splitting replaces the Milnor group, used in the original proofs. Here grope splitting is used to show that certain links are (colored) link-homotopic. (Of course, Milnor group gives in general a more precise algebraic information about links, but such generality is not needed for the questions considered here.) In particular, we present a new, geometric proof of the Grope Lemma, see Theorem 2 below.

We follow the terminology and notations of [4]. In particular, $g$ denotes a grope (the underlying 2–complex), while the capital letter $G$ indicates the use of its untwisted 4–dimensional thickening. The body of a capped grope $g^c$ is denoted by $g$. We refer the reader to [3] for definitions and a discussion of the properties of gropes. The operations that are used extensively in this paper (justifying the informal name for theorem 3 below) are *contraction*, sometimes also referred to as *symmetric surgery*, and *pushoff*, which are described in detail in [3, section 2.3]. We remark that these operations are suited perfectly for the purpose of separating surfaces in 4–manifolds at the expense of introducing self-intersections.

**Theorem 1** (Exponential separation) *Let $(g^c,\gamma)$ be a capped grope of height $h$, properly immersed in a 4–manifold $M$, and let $\Sigma_1,\dots,\Sigma_{2^h-1}$ be properly immersed surfaces in $M$ which are pairwise disjoint, and are also disjoint from the body of $g^c$. Then, given a regular neighborhood $N$ of $g^c$ in $M$, the collection of surfaces $\{\Sigma_i\}$ is homotopic to $\{\Sigma_i'\}$ with homotopy supported in $N$, and $N$ contains an immersed disk $\Delta$ on $\gamma$ such that all surfaces $\Delta,\Sigma_1',\dots,\Sigma_{2^h-1}'$ are pairwise disjoint. Moreover, the surfaces $\{\Sigma_i\}$ stay pairwise disjoint during the homotopy.*

*The bound $2^h-1$ on the number of surfaces $\Sigma$, for which this conclusion holds in general, is sharp.*

The term *proper immersion* of a capped grope usually incorporates the condition that the body $g$ is embedded, and that the cap interiors are disjoint from $g$, so only cap–cap intersections are allowed. This assumption is not needed in theorem 1. The necessary condition is that the surfaces $\Sigma$ may intersect only the caps of $g^c$, but not the body $g$.

Briefly, the idea of the proof is the following. Consider the special case, when each body surface of $g^c$ has genus 1 (thus $g^c$ has $2^h$ caps), and when each cap intersects a single surface. Then there must be two caps intersecting the same surface, say $\Sigma_i$. We keep just these two caps for $g^c$, and using surgery and contraction/pushoff, $g^c$ is made disjoint from $\Sigma_i$ as well. The general situation is reduced to this special case via the operation of *grope splitting*, explained in lemma 4. As another application of grope splitting, we present a new proof of the *Grope Lemma*:

**Theorem 2** *Two $n$–component links in $S^3$ are link homotopic if and only if they cobound disjointly immersed annulus-like gropes of class $n$ in $S^3\times I$.*

This result was originally stated in [2], in the case when one of the links is trivial. In the generality as stated here, Grope lemma is proved in [8], using Milnor group. We refer the reader to [2], [8] for the background, and for applications of this result to the surgery conjecture. Note that the grope *class* corresponds to the index in the lower central series of a group, while the grope *height* in theorems 1, 3 corresponds to the index in the derived series. Thus a grope of height $h$ has class $2^h$. To prove the Grope lemma, cap each grope by any transverse map of the disks into $S^3 \times I$. Note that a grope of class $n$ has $n$ caps, while there are only $n-1$ gropes bounded by other link components. Now the argument using grope splitting, identical to the proof of Theorem 1, gives disjoint maps of annuli (singular link concordance.) $\square$

The proof of theorem 1 also implies the result on $\pi_1$–null immersions:

**Theorem 3** (Exponential contraction/pushoff [4, Theorem 3.5])
*Let $\phi\colon \pi_1 G^c \longrightarrow \pi$ be a group homomorphism with $(G^c, \gamma)$ a Capped Grope of height $h$. If $\phi$ maps the double point loops of $G^c$ to a set of cardinality at most $2^h-1$ in $\pi$ then $G^c$ contains a disk on $\gamma$ which is $\pi_1$–null under $\phi$.*

Note that while finding a $\pi_1$–null disk in theorem 3 is similar to finding a disk disjoint from other surfaces in theorem 1, the converse – showing that $2^h-1$ is the sharp bound in theorem 3 – remains a central unsolved problem. (If the bound $2^h-1$ in theorem 3 could be replaced by $2^h$, then one would find an embedded disk in the "model" capped gropes, cf [4].)

In theorem 3, we allow cap–cap and cap–body intersections of $g^c$, but it is important that the body $g$ on its own has no self-intersections. (More precisely, we allow only $\pi_1$–null self-intersections of the body.) Also note that the proof goes through for any, not necessarily untwisted, thickening $G^c$ of $g^c$.

We now make a brief digression to discuss the proof of Exponential contraction/pushoff in [4]. For a given Capped Grope $G^c$ of height $h$, the proof constructs, for each cap $C$, $2^h-1$ dual spheres in $G^c$ with certain crucial disjointness properties. If these dual spheres are used to resolve cap intersections, then the grope is not left intact, but it has to be completely contracted, using all its caps. More precisely, the dual spheres are built in the "complete" contraction. (Perhaps this point is not stated clearly in the exposition of [4].) It is the construction of these dual spheres that requires developing the theory of colored link homotopy, to show that a certain colored link $L$ is colored homotopically trivial. In the present paper we do not follow that path, but prove theorem 3 directly. We note that our proof can also be used to show that that particular link $L$ is trivial. (It also implies that each dual sphere is *embedded*, so the intersections only occur between *different* spheres of the same color.) The link $L$ is a certain colored ramified iterated Bing double of the Hopf link, which arises as a Kirby handle diagram of the Grope $G$. We say a few more words about this link at the end of the proof of theorem 1. The theory of colored Milnor groups introduced in [4] can be used under more general circumstances, for example for the purpose of distinguishing non-homotopic links, but this generality is not used in the proof of Exponential contraction/pushoff.

**Proof of Theorem 1** Consider first the special case when all body surfaces of $g$ have genus one, and each cap of $g^c$ intersects just one of the $\Sigma_i$'s. This case captures the essence of the bound in theorems 1 and 3. Since there are $2^h$ caps and $2^h-1$ surfaces $\{\Sigma_i\}$, at least two of the caps $C_1$, $C_2$ intersect the same surface $\Sigma_{i_0}$ (and they are disjoint from all other surfaces.) Consider these two caps, and for the rest of the proof disregard all other caps of $g^c$. Suppose that $(C_1,\ C_2)$ is a dual pair of caps, so they are attached to the symplectic pair of circles in an $h$-th stage surface of $g$. In this case contract $C_1$ and $C_2$ and push $\Sigma_{i_0}$ off the contraction to get $\Sigma'_{i_0}$. Consider the disk $\Delta$ on $\gamma$ which "uses" only the contraction of $C_1$ and $C_2$, and not the other caps. This disk is gotten by successive surgeries along the branch of $g$ which leads from $\gamma$ to the tips $C_1$ and $C_2$; all other caps and surfaces in $g^c$ are disregarded. The disk $\Delta$ and the new surfaces $\{\Sigma'_i\}$ satisfy the conclusion of the theorem. Note that $g^c$ is not framed, so "parallel copies" (perturbations) of the surface stages of $g^c$, which are used in the surgeries and contractions, may intersect. This is not important in our argument, since the goal is to find only an immersed disk.

If the caps $C_1$ and $C_2$ are not dual, still disregard all other caps, surger two top stage surfaces, which are capped by $C_1$ and $C_2$ respectively, along these caps and continue surgering until the two new caps become dual. This reduces the situation to the previous case.

Now consider the general case, with surfaces of an arbitrary genus, and when each cap may intersect several different surfaces $\Sigma_i$. We will need for the proof the following operation of *grope splitting*, so we make a digression to explain it in detail.

**Lemma 4** (Grope splitting) *Let $(g^c,\gamma)$ be a capped grope in $M^4$, and let $\Sigma_1,\ldots,\ \Sigma_n$ be surfaces in $M$, disjoint from the body of $g^c$, but perhaps intersecting its caps. Then, given a regular neighborhood $N$ of $g^c$ in $M$, there is a capped grope $(g^c_{\text{split}},\gamma)\subset N$, such that each cap of $g^c_{\text{split}}$ intersects at most one of the surfaces $\Sigma$, and each body surface, above the first stage, of $g^c_{\text{split}}$ has genus $1$.*

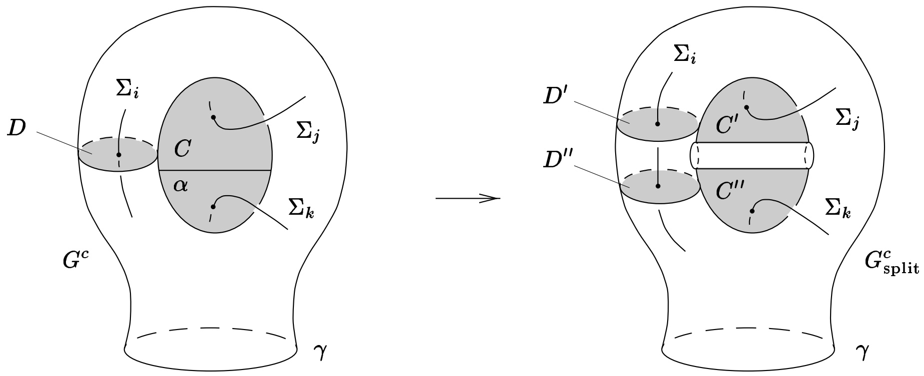


Figure 1: Grope splitting

**Proof** First assume that $N$ is the untwisted thickening of $g^c$, $N = G^c$, and moreover let $g^c$ be a model capped grope (without double points). Let $C$, $D$ be a dual pair of its caps, and let $\alpha$ be an arc in $C$ with endpoints on the boundary of $C$. (In our applications, $\alpha$ will be chosen to separate intersection points of $C$ with different surfaces $\Sigma_j$, $\Sigma_k$, as shown in figure 1.) Recall that the untwisted thickening $N$ of $g^c$ is defined as the thickening in $\mathbb{R}^3$, times the interval $I$. We consider the 3–dimensional thickening, and surger the top-stage surface of $g$, which is capped by $C$ and $D$, along the arc $\alpha$. The cap $C$ is divided by $\alpha$ into two disks $C'$, $C''$ which serve as the caps for the new grope; their dual caps $D'$, $D''$ are formed by parallel copies of $D$. This operation increases the genus of this top-stage surface by 1; note that if some surface $\Sigma_i$ intersected the cap $D$ of $G^c$, it will intersect both caps $D'$, $D''$ of $G^c_{\text{split}}$.

We described this operation for a model capped grope; a splitting of a capped grope with double points is defined as the image of this operation in $N$/plumbings (where the arcs $\alpha$ are chosen to avoid the double points.) Also note that the same construction works for any (not necessarily untwisted) thickening $N$: all that one needs is a line subbundle of the normal bundle of the disk $C$ in $N$, restricted to $\alpha$. The fact that the new caps $D'$ and $D''$ may intersect is not important here.

Continue the proof of lemma 4 by dividing each cap $C$ by arcs $\{\alpha\}$, so that each component of $C \smallsetminus \cup\alpha$ intersects at most one surface in the collection $\{\Sigma_i\}$, and splitting $g^c$ along all these arcs. (At most $n$ arcs are needed for each cap.) The result is illustrated in figure 2. We apply the same operation to the surfaces in the $(h-1)$-st stage of the grope, separating each top stage surface by arcs into

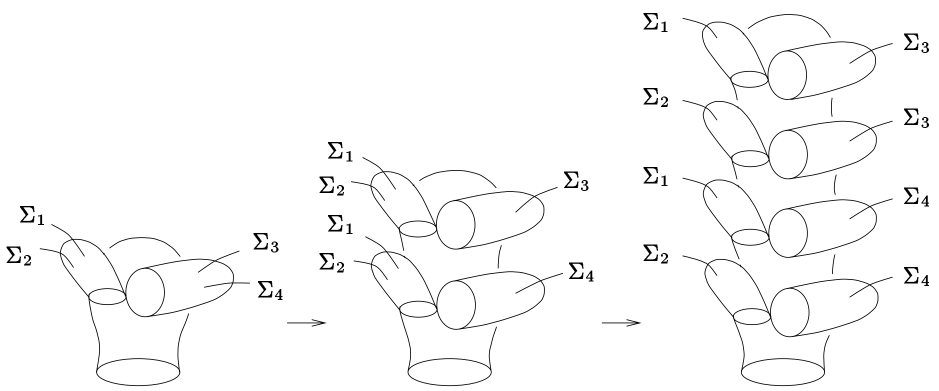


Figure 2: Example of a splitting of a top stage surface of $g^c$

genus 1 pieces. This procedure is performed inductively, descending to the first stage of $g^c$. For example, if originally each body surface of $g$ had genus one, and each cap intersected all $n$ surfaces $\{\Sigma_i\}$, then after this complete splitting procedure the first stage surface will have genus $n^{2^h}$. $\square$

We continue the proof of theorem 1 in the general case, applying this complete grope splitting procedure to $g^c$. Separate the first stage surface by arcs into genus one pieces and treat each one of them separately, as in the special (genus one) case, considered above. If one of the caps of $g^c$ is disjoint from all surfaces $\Sigma_i$, the result for that genus one piece follows trivially. The disk $\Delta$ bounding $\gamma$ is obtained as the union of disks produced by the genus one pieces.

The proof that the bound $2^h-1$, for which the conclusion of the theorem holds in general, is sharp, is an elementary and well-known calculation in Massey products, or Milnor's $\bar{\mu}$–invariants. Consider the model capped grope $g^c$ (without double points) of height $h$ with each body surface of genus one and with the caps $C_1, \dots, C_{2^h}$, and consider $2^h$ surfaces $\Sigma_1, \dots, \Sigma_{2^h}$ such that for each $i$, the cap $C_i$ intersects $\Sigma_i$. The untwisted thickening of the model grope $g^c$ is the four–ball $B^4$; the attaching circle $\gamma$ of $g^c$ and the intersection of the surfaces $\Sigma_i$ with $\partial B^4 = S^3$ form the Borromean rings, shown in Figure 3, in the case $h=1$, cf [5], [3, 12.2]. The picture for larger $h$ is obtained by iterative Bing doubling (cf section 7 in [6]) of the link components, other than $\gamma$. At each step of the iteration, the caps are replaced by genus one capped surfaces (copies of figure 3.) The components of the resulting link do not bound disjoint maps of disks in $B^4$ since any iterated Bing double of the Hopf link has non-trivial

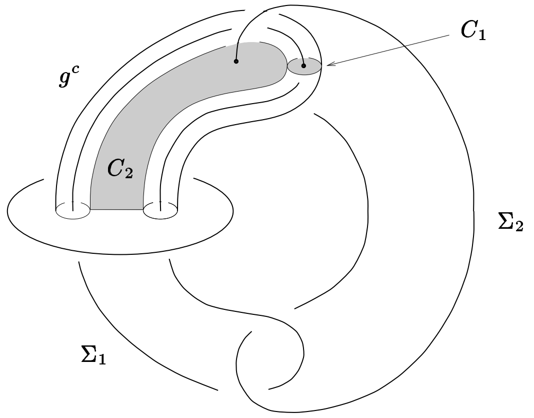


Figure 3: $\gamma$ bounds a capped surface

$\bar{\mu}$–invariants [9]. $\square$

**Proof of Theorem 3** This is similar to the proof of Theorem 1 above, only instead of separating intersections of the caps with different surfaces, the grope splitting procedure will now be used to separate intersection points among the caps of $g^c$, which correspond to different group elements in $\pi$. Recall from [3, 2.9] that the new group element created by the operation of pushoff from the elements $f$ and $g$ is $f \cdot g^{-1}$, thus only trivial double point loops are created during the final step.

When one applies the grope splitting in order to separate the selfintersections of $g^c$, rather than intersections of $g^c$ with other surfaces, one cannot achieve the situation shown in figure 2 where each cap has precisely one double point. Indeed, splitting a cap $C$ requires using two copies of the dual cap $D$, making it impossible to achieve progress in this respect. However, the double point loops, produced by the parallel copies $D'$, $D''$ of $D$, give the same group element, and the new intersections $D' \cap \Sigma_i$, $D'' \cap \Sigma_i$ do not need to be separated in $\Sigma_i$.

Another subtlety concerns ordering the sheets at each intersection point (if one ordering gives an element $g$ in $\pi$, switching them gives $g^{-1}$.) The statement of theorem 3 implicitly contains a choice of the first sheet at each intersection point. The proof of theorem 1, followed here without a change, would only give the bound $2^{h-1} - 1$. Thus a slight correction is necessary in the situation after the grope splitting is completed, when two caps $C_1$ and $C_2$ on the same branch $B$ have intersections with some other caps, representing the same non-trivial element $g$, but where $C_1$ is considered as the first sheet for its intersection point, $C_2$ is considered as the second sheet for its intersection, and $g \neq g^{-1}$. If both elements $g$, $g^{-1}$ are on the list of $2^h - 1$ elements, then this problem does not arise. Suppose $g$ is on the list, and $g^{-1}$ is not. Take the cap $C_1$ (labeled as the first sheet) and surger the grope along the branch leading to $C_1$ (all other caps, including $C_2$, of this branch are discarded.) Let $C$ be a cap, lying on some branch $B'$, intersecting $C_1$, giving the group element $g$. Note that $C$ is considered as the second sheet for this intersection, and it will be discarded when this operation is applied to the corresponding branch $B'$ of $g^c_{\text{split}}$. Hence this procedure is consistent, giving raise at the end to a $\pi_1$–null disk $\Delta$.

Note that theorem 3 holds for any (not necessarily untwisted) four–dimensional thickening of $g^c$. The "parallel copies" of the surfaces, that have to be taken for surgeries and contractions in the proof, are just perturbations of the originals. The resulting singularities are acceptable, since their double point loops are trivial in $\pi_1$. □

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