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Bounds on exceptional Dehn filling

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Abstract

We show that for a hyperbolic knot complement, all but at most 12 Dehn fillings are irreducible with infinite word-hyperbolic fundamental group.

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1 Introduction

Thurston demonstrated that if one has a hyperbolic knot complement, all but finitely many Dehn fillings give hyperbolic manifolds [14]. The example with the largest known number of non-hyperbolic Dehn fillings is the figure-eight knot complement, which has 10 fillings which are not hyperbolic. It is conjectured that this is the maximal number that can occur. Call a manifold *hyperbolike* if it is irreducible with infinite word-hyperbolic fundamental group (this is stronger than Gordon's definition [9]). For example, manifolds with a Riemannian metric of negative sectional curvature are hyperbolike. Call a Dehn filling *exceptional* if it is not hyperbolike. We will consider the more amenable problem of determining the number of exceptional Dehn fillings on a knot complement. The geometrization conjecture would imply that hyperbolike manifolds are hyperbolic. Bleiler and Hodgson [4] showed that there are at most 24 exceptional Dehn fillings, using Gromov and Thurston's 2π–theorem and estimates on cusp size due to Colin Adams[1]. We will make an improvement on the 2π–theorem, and use improved lower bounds on cusp size due to Cao and Meyerhoff [6], to get an upper bound of 12 exceptional Dehn fillings. The inspiration for this work came from discussions with Zheng-Xu He. He has obtained bounds relating asymptotic crossing number to cusp geometry [11]. He remarked to me that his estimates could probably be improved, and this paper gives my attempt at such an improvement. Mark Lackenby [12] has obtained the same improvement of the 2π–theorem. My thesis gave implications about Dehn fillings being atoroidal, not word-hyperbolic [2]. Marc has an improved version of Gabai's ubiquity theorem which filled in a gap in an early draft of my thesis, and this is the argument which appears in this paper.

2 Definitions

The notation introduced here will be used throughout the paper. We will use $\mathrm{int}X$ to mean the interior of the space X, and $\mathcal{N}(X)$ will denote an open regular neighborhood of a subset $X \subset M$. M is a hyperbolic 3–manifold with a distinguished torus cusp. M has a compactification to a compact 3 manifold $\overline{M}$ with torus boundary, by adding the ends of geodesic rays which remain in the cusp. Let S be a surface of finite type (S may have both boundary and punctures), let $f\colon S \to M$ be a map such that every puncture maps properly into a cusp. This map might not necessarily be an embedding or an immersion. Using the terminology of [13], $f\colon S \to M$ is *incompressible* if every simple loop c in S for which $f(c)$ is homotopically trivial in M bounds a disk in S. The simple loop conjecture would imply that such a map f is π_1–injective, but this is not known for general 3–manifolds [7]. Let $U = [0, \infty) \times \mathbf{R}$. A boundary compression of f is a proper map $b\colon U \to M$ such that there is a map $b'\colon \partial U \to S$ with $f \circ b' = b_{|\partial U}$, $b'(\partial U)$ is a proper simple line in S which does not bound a properly embedded half-plane in S. f is *∂–incompressible* if it has no boundary compression. $f\colon S \to M$ is *essential* if it is incompressible and ∂–incompressible. $f\colon S \to M$ is *pleated* if the boundary components of S map to geodesics in M, and $\mathrm{int}S$ (the interior of S) is piecewise made of triangles which map under f to ideal hyperbolic geodesic triangles in M, so that the 1–skeleton $\cup\, \partial S$ forms a lamination in S. A pleated surface has an induced hyperbolic metric, where the lamination is geodesic.

For a cusped hyperbolic 3–manifold M, we may take an embedded neighborhood C of the cusp which is a quotient of an open horoball by the torus group, which we will call a *horocusp*. The closure of C might not be embedded, so by ∂C we will mean the torus obtained as the path closure of the Riemannian manifold C (not regarded as a subset of M). ∂C inherits a euclidean metric from M. If p a loop in C, let $l_C(p)$ denote the length of a euclidean geodesic loop homotopic to p in ∂C. A *slope* in ∂C is an equivalence class of embedded loops in ∂C. If α is a slope in ∂C, then $M(\alpha)$ denotes the Dehn filling along that slope, which is a manifold obtained by gluing a solid torus to $M \backslash C$ so that the loop represented by α bounds a disk in the solid torus. This is uniquely determined by the slope in ∂C.

A theorem of Gromov [10] implies that for a closed manifold M, $\pi_1(M)$ is *word-hyperbolic* if for a metric on M, M has a linear isoperimetric inequality. That is for a metric on M, there is a constant V so that for any map of a disk $d\colon D^2 \to M$, $\mathrm{area}(D) \leq V\, \mathrm{length}(\partial D)$ in the induced metric on D. Gromov has shown that for such a manifold, $\pi_1 M$ has no $\mathbb{Z} + \mathbb{Z}$ subgroup and has a solvable word problem. A theorem of Bestvina and Mess implies that the universal cover $\tilde{M}$ has a compactification to a ball [3] (if the fundamental group is infinite and the manifold is irreducible). Thurston's geometrization conjecture would imply that M has a hyperbolic structure, that is a Riemannian metric of constant sectional curvature -1.

3 Essential Surfaces

In this section, we show how to obtain singular essential surfaces in a knot complement coming from the ambient manifold. The results are similar to those of Ulrich Oertel [13], but we do not worry about embeddedness of the boundary components. Marc Lackenby [12] and Zheng-Xu He [11] have also obtained similar results to the following lemmas. The idea is that if there is an essential sphere or if the core of a Dehn filling has finite order in the fundamental group, so that some multiple of the core bounds a disk, then the surface can be homotoped so that its intersection with the knot complement is essential.

Lemma 3.1 (Essential punctured spheres) *Let M^3 be a compact 3–manifold and take a knot $k \subset M$ with $N = M \setminus \mathcal{N}(k)$, such that ∂N is incompressible in N and N is irreducible. Let $f\colon S \to M$ be a singular map of a sphere or disk. If S is a sphere, then f is a homotopically non-trivial map into M. If S is a disk, then its boundary maps to a homotopically non-trivial curve in $\mathcal{N}(k)$. Then we can find a surface T and a mapping $g\colon T \to M$, with the same properties as above, such that g is transverse to ∂N and $g^{-1}(N)$ is essential in N.*

Proof First notice that we may take f transverse to k, so that $f^{-1}\mathcal{N}(k)$ is a collection of disks in S and an annular neighborhood of the boundary in the disk case (we will call these *dots*). Then we will induct on $|f^{-1}\mathcal{N}(k)| =$ the number of dots. Suppose there is an essential simple closed curve c in $f^{-1}(N) \equiv \hat{S}$, which bounds a disk D in N (ie, there is a homeomorphism $d'\colon c \to \partial D$ and a map $d\colon D \to N$ with $d_{|\partial D} \circ d' = f$). Then since S is either a disk or a sphere, c bounds a disk E in $\hat{S}$ which must meet $\mathcal{N}(k)$, since c is essential in $\hat{S}$. Surger $f\colon S \to M$ along $d\colon D \to M$. That is, create a new surface S' by splitting S along c, and glue in two copies of D to the two new boundary components (corresponding to two copies of c) by gluing using the homeomorphism $d'\colon c \to \partial D$, then form a mapping $f'\colon S' \to M$ by using f or d on the relevant pieces of S'. One component of S' may have image in M under f' a homotopically trivial sphere, so we get rid of it. In case S is a sphere, there are two choices for the disk E bounding c in S. At least one choice will surger to an essential sphere in M, so we keep this one. We then have a surface which has fewer dots. Replace S with this surface, which we will still call S, and f with the restriction of f' to this subsurface, which we will still call f.

Suppose there is an arc α which is embedded and essential in $\hat{S}$ which bounds a boundary compression for $\hat{S}$. That is, there is a map $d'\colon \partial U \to \alpha \subset \hat{S}$ and a map $d\colon U \to N$ such that $f \circ d' = d_{|\partial U}$. There are two types of boundary compressions:

(1) α connects different dots in S, so we push S along the boundary compression. That is, we split S along the arc $\overline{\alpha}$ and make a new surface

S' by gluing two copies of the disk $\overline{U}$ to the new boundary components using the homeomorphism $d'\colon\ \partial U \to \alpha$, and identifying the other ends of the two copies of $\partial\overline{U}$ by the identity. Then replace f with $f'\colon\ S' \to M$ by using f or d on the relevant pieces of S'. Then, we may expand $\mathcal{N}(k)$ slightly, and make f' transverse to ∂N. This has the effect of turning two dots into one, and decreases the number of dots. By induction, we may assume there are no such compressions.

(2) α connects the same dot in S. Take a maximal collection of disjoint, non-parallel ∂–compressions, and as in case 1, we push S along these ∂–compressions (see the previous case for a more precise description of this push operation), getting a new map which we will still call $f\colon\ S \to M$. Then $f^{-1}(\mathcal{N}(k))$ is a collection of planar surfaces such that each one separates S. Take an innermost planar surface. If there are no dots in the disks it separates off, or if there are disks whose boundary maps to a homotopically trivial loop in $\mathcal{N}(k)$, then we can homotope f in a neighborhood of these disks in S into $\mathcal{N}(k)$, keeping f fixed on the rest of S, since ∂N is incompressible in N and N is irreducible, decreasing the number of dots in S. Otherwise, one of these disks has boundary which is essential in $\mathcal{N}(k)$. We then take T to be this disk adjoined a collar of the outermost curve in $\mathcal{N}(k)$, and $g = f_{|T}$. □

The next lemma deals with the case in which one has a singular map of a disk into M, with boundary mapped into the complement of the knot. Then one can homotope the map of the disk to be essential in the knot complement, as long as there are no essential punctured disks in the knot complement. This will be used later for bounding the area of such a disk.

Lemma 3.2 (Essential punctured disks) *Let M^3 be a compact 3–manifold, and take a knot $k \subset M$ with $N = M \setminus \mathcal{N}(k)$, such that $\partial N \subset N$ is incompressible and N is irreducible. Also, assume that there are no maps of disks $a\colon\ A \to M$ transverse to k, with $a(\partial A) \subset \mathcal{N}(k)$, and $a^{-1}(N)$ essential in N. Then, if $f\colon\ D \to M$ is a disk whose boundary is in N, we may homotope f so that $f^{-1}(N)$ is essential in N.*

Proof The proof is similar to that of the previous lemma, but we need to observe that $\pi_2 M = 0$ by the previous lemma, so the disk surgeries can be done by homotopies. We may assume that $f_{|f^{-1}(N)}$ is incompressible and has no ∂–compression such that the arc α connnects different dots of $f^{-1}(\mathcal{N}(k))$. As before, take a maximal collection of non-parallel arcs which separate D and bound ∂–compressions, and push $f\colon\ D \to M$ along these ∂–compressions to obtain planar surfaces separating D (as in the previous lemma). None of the disks separated by an innermost planar surface can be essential, since this would contradict our assumption. Thus, as in the previous lemma, we may homotope f on the innermost disks into $\mathcal{N}(k)$, decreasing the number of dots. $\square$

4 Pleated Surfaces

The argument in this section is similar to that of Thurston [15], but the hypotheses are slightly different. This result will be used next section to compare the geometry of the hyperbolic metric on the pleated surface to the geometry of the manifold.

Lemma 4.1 (Pleated Surfaces) *Let N be a hyperbolic 3–manifold with a distinguished cusp, let S be a surface of finite type with $\chi(S) < 0$, and let $f\colon S \to N$ be a singular essential map, with cusps of S mapping properly to cusps of N, and ∂S mapping to geodesics in N. Then we can find a hyperbolic metric on S and a map $g\colon S \to N$, such that g is pleated, $g_{|\mathrm{int}S}$ is homotopic to $f_{|\mathrm{int}S}$, and $g_{|\partial S}$ is an isometry.*

Proof Choose an ideal triangulation T of $\mathrm{int}S$, such that no edges connect ∂S to itself and every edge is essential in S. Then spin the triangles of T around ∂S. We obtain a lamination $\mathcal{L}$ consisting of $T^{(1)} \cup \partial S$, which is a geodesic lamination, in the sense that it is isotopic to a geodesic lamination in any complete hyperbolic structure on S with geodesic boundary. We may assume $f\colon S \to N$ is C^2 near $\mathcal{L}$, since it can be assumed that the singularities of f are in the interior of S, so we can make $\mathcal{L}$ miss the singularities. So each end of a leaf of $\mathcal{L}$ which limits to ∂S must eventually have curvature close to 0, and is therefore a quasi-geodesic in N. Its other end must map properly into a cusp of N. Lifting to $\mathbb{H}^3 \cong \tilde{N}$, we see that the endpoints must be distinct on $\partial\mathbb{H}^3$, by discreteness. If both ends of a leaf L of $\mathcal{L}$ map into the same cusp when lifted to $\mathbb{H}^3$, then L bounds a ∂–compressing disk in $\mathbb{H}^3$, whose end maps into the same cusp. Pushing down to N, we find a ∂–compression for S, which contradicts that the edges of T are essential in S and S is ∂–incompressible. In either case, the endpoints of each leaf lifted to $\mathbb{H}^3$ map to different points in $\partial\mathbb{H}^3$, so each leaf is homotopic to a unique geodesic. Therefore, we may homotope f so that $T^{(1)}$ is geodesic in N. We can homotope f on each triangle of T to be totally geodesic by homotopy extension in $\mathbb{H}^3$ and pushing down to N, giving a homotopic pleated map $g\colon \mathrm{int}S \to N$. A pleated surface has an induced hyperbolic metric, which we give to int$S$. Then we can complete the metric on int$S$ to a metric on a surface $S' \cong S$. Choose a geodesic γ in $f(\partial S)$. Then since the ends of leaves of $\mathcal{L}$ are quasi-geodesic, each end of a geodesic leaf of $g(\mathcal{L})$ is in a bounded neighborhood of the end in $f(\mathcal{L})$. Therefore, the ends of $g(\mathcal{L})$ limit to γ. When part of a geodesic of $\mathcal{L}$ wraps closely once about $\partial S'$, then its image wraps closely about γ in N. In the limit, we see that the length of $\partial S'$ must be the same as the length of γ. So we may extend g to an isometry $g\colon S \to N$. □

5 Cusp area shrinks

The next theorem is based on the fact that there are disjointly embedded cusps in S which have longer boundary than the cusp lengths in the image. This would be easy to show if S were totally geodesic, and we would get equality. But since S is actually pleated, the folding makes S have longer cusp lengths.

Theorem 5.1 (Bounding cusp length by Euler characteristic) *Let N be a hyperbolic 3–manifold with a distinguished horocusp C. Let S be a surface of finite type with no boundary components, and $f\colon S \to M$ be an essential mapping, where the cusps of S map into C. For each puncture p_i of S, consider the length of the corresponding slope $l_C(p_i)$ in ∂C. Then $\Sigma_i\, l_C(p_i) \leq 6\,|\chi(S)|$.*

Proof First, homotope f to a pleated map by lemma 4.1, which we will also call f. $f(\mathrm{int}S)$ is a union of ideal geodesic triangles $T_1,\dots,T_{2|\chi(S)|}$. If a corner of T_j is in C, the opposite edge of T_j might intersect C in its interior. Lifting T_j to $\tilde{T}_j$ in $\tilde{M} = \mathbb{H}^3$, it looks like Figure 1, with a parabolic limit point of C lifted to ∞, and $\tilde{C}$ a lift of C, in the upper half-space model of $\mathbb{H}^3$. Shrink the cusp C to a cusp C' such that each edge of T_j intersects C in no compact intervals. Then $\tilde{C}'$ looks like Figure 1. $f^{-1}(C') = H' = \cup H_i'$ consists of disjoint horocusps H_i' in S, one for each puncture of S which maps into C. Let $l_{H'}(p_i) =$ the length of p_i along $\partial H_i'$ in S. Then $l_{H'}(p_i) \geq l_{C'}(f(p_i))$, with equality iff there is no bending along the pleats at p_i. Let $d = d(C, C')$. Then choose horocusps $H_i \supseteq H_i'$ in S, such that $d(H_i, H_i') = d$. Then $f(H_i) \subseteq C$, since f is piecewise an isometry, so it shrinks distances. Suppose $H_i \cap H_j \neq \emptyset$, for some $i \neq j$. Then there is a geodesic arc a in S connecting p_i to p_j: just look at intersecting lifts of H_i, H_j in $\tilde{S} = \mathbb{H}^2$, and take the geodesic $\tilde{a}$ connecting the centers of $\tilde{H}_i, \tilde{H}_j$ in $\partial\mathbb{H}^2$. See Figure 2. $f(H_i), f(H_j) \subseteq C$, so $f(a) \subseteq C$. Then there is a ∂–compressing disk D for $f(a)$ in C. Just cone off $f(a)$ to the end of C by

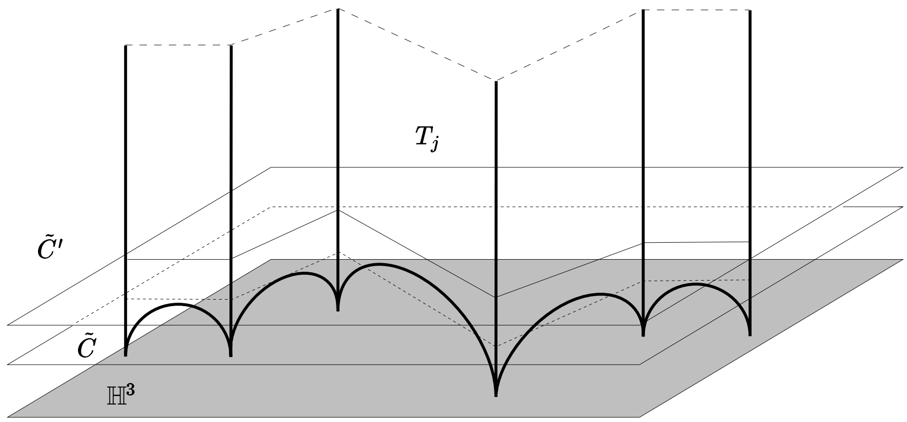

Figure 1: Shrinking the horocusp

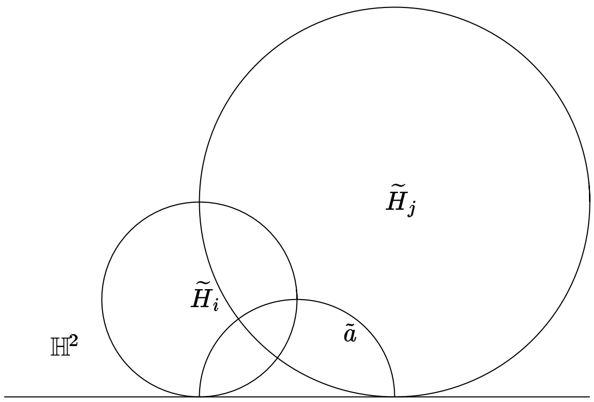

Figure 2: Intersecting horocusps give ∂–compression

geodesics. That is, in the universal cover of N, take a cover $\tilde{C}$ of C tangent to ∞, and $\widetilde{f(a)}$ of $f(a)$. Then over each point of $f(a)$, take a geodesic connecting the point to ∞. This describes a map of a half plane compressing $\widetilde{f(a)}$. Map down to N, to get a ∂–compression of the arc a in f.

So we have shown that since S is ∂–incompressible, $H_i \cap H_j = \emptyset, i \neq j$. Thus, we have disjoint horocusps in S. $l_H(p_i) = e^d l_{H'}(p_i)$ and $l_C(f(p_i)) = e^d l_{C'}(f(p_i))$. So

$$l_H(p_i) = e^d l_{H'}(p_i) \geq e^d l_{C'}(f(p_i)) = l_C(f(p_i)).$$

A theorem of Boroczky [5] implies that $\text{Area}(H) \le \frac{3}{\pi}\text{Area}(S)$ (one may also consult the argument of Lemma 3.1 in Marc Lackenby's paper [12], which can be easily modified to prove this inequality). A well-known computation implies that $\text{Area}(H) = l(\partial H)$. So we have

$$\Sigma_i l_C(f(p_i)) \le \Sigma_i l_H(p_i) = l(\partial H) =$$
$$\text{Area}(H) \le \frac{3}{\pi}\text{Area}(S) = \frac{3}{\pi} \cdot 2\pi|\chi(S)| = 6|\chi(S)|$$

by the Gauss–Bonnet theorem. □

6 Word-hyperbolic Dehn filling

Let N be a finite volume hyperbolic 3–manifold with a unique embedded horocusp C. For a slope α in ∂C, if $l_C(\alpha) > 2\pi$, then Gromov and Thurston proved that $N(\alpha)$ has a metric of negative curvature. Theorem 6.2 implies that if $l_C(\alpha) > 6$, then $N(\alpha)$ is hyperbolike. The intuition for why such an improvement is possible is that the 2π–theorem only makes use of the negative curvature of N in the cusp C, whereas this result takes account of negative curvature of N outside of C as well.

First, we need to state a theorem of Lackenby. Let N, C, and α be as above. Let k be the core of the Dehn filling in $N(\alpha)$. We will fix a Riemannian metric on $N(\alpha)$ which agrees with the hyperbolic metric on $N \backslash C$. For a homotopically trivial mapping $c\colon S^1 \to N(\alpha)\backslash\mathcal{N}(k)$, we define the *wrapping number*

$$\text{wr}(c,k) = \min\{|d^{-1}\mathcal{N}(k)|,\ d\colon D^2 \to M(\alpha), d \text{ is transverse to } \mathcal{N}(k),$$
$$\text{and } d_{|\partial D} = c\}.$$

It measures the minimal number of intersections with $\mathcal{N}(k)$ of maps of disks spanning c. The following theorem is due to Lackenby [12, Theorem 2.1]:

Theorem 6.1 (Ubiquity theorem) *In the situation above, there is a constant w such that for any least area disk $d\colon D^2 \to N(\alpha)$, we have $\text{area}(d) \le w(\text{wr}(\partial D, k) + \text{length}(\partial D))$.*

This theorem strengthens the ubiquity theorem of Gabai [8], in that it doesn't count the multiplicities of intersections of $d\colon D^2 \to N(\alpha)$ with k. The point of this theorem is that to obtain a linear isoperimetric inequality for $N(\alpha)$, we need only show that there is a constant v so that for maps $d\colon D^2 \to N(\alpha)$, $\text{wr}(\partial D, k) \le v\text{length}(\partial D))$.

Theorem 6.2 (Hyperbolike fillings) *Let N be a finite volume hyperbolic 3–manifold with single embedded horocusp C. If α is a slope with $l_C(\alpha) > 6$, then $N(\alpha)$ is hyperbolike.*

Proof If ∂C is not embedded, replace C with a slightly smaller horocusp, retaining the property that $l_C(\alpha) > 6$. Define $M = N \backslash C$, and let k be the core of the Dehn filling $N(\alpha)$, and $\mathcal{N}(k)$ the open solid torus which is attached to M. Suppose $\pi_2 N(\alpha) \neq 0$ or $|\pi_1 N(\alpha)| < \infty$. M has incompressible boundary, so by 3.1, $N(\alpha)$ contains a mappping of a sphere or disk $f\colon S \to N(\alpha)$ such that $f_{|\hat{S}=f^{-1}(M)}$ is essential in M. Let $n = |f^{-1}(\mathcal{N}(k))|$. Then there are at least $n-1$ boundary components of $\hat{S}$ which map to multiples of α in ∂M. If $n = 0$, $\hat{S}$ would be an inessential sphere in M, since $\pi_2 M = 0$, and therefore is trivial in $\pi_2 N(\alpha)$, a contradiction. If $n = 1$ or 2, so $\hat{S}$ is a disk or annulus, then $f_{|\hat{S}}$ can be homotoped into ∂M, since ∂M is incompressible and M is acylindrical. So $n \geq 3$. Applying Lemma 5.1, we see

$$6(n-2) = 6|\chi(S)| \geq (n-1)\cdot l_C(\alpha) > 6(n-1)$$

a contradiction. So $N(\alpha)$ is irreducible with the core having infinite order in $\pi_1 N(\alpha)$.

Choose a metric on $N(\alpha)$ which agrees with the hyperbolic metric on M, and is any metric on $H = \overline{\mathcal{N}(k)}$. We want to show that $N(\alpha)$ has linear isoperimetric inequality with this metric.

Choose a map $c\colon S^1 \to N(\alpha)$ which is homotopically trivial. First, we will find a homotopy of c to a map c' in M, such that the length c' and the area of the homotopy are linearly bounded by the length of c. The second step is to show that the wrapping number of c' is linearly bounded by its length. We then apply the ubiquity theorem to conclude that $N(\alpha)$ has linear isoperimetric inequality.

Then $c^{-1}(\mathrm{int} H)$ consists of a collection of intervals. Let us consider one of these intervals δ. Lift $c_{|\delta}$ to a map $c_0\colon \delta \to \tilde{H}$, where $\tilde{H}$ is the universal cover of H. Change the metric on H to be isometric to a euclidean cylinder quotient a translation. Then this Riemannian metric is quasi-isometric to the original metric on H. We can homotope c_0 to a map $c_1\colon \delta \to \partial\tilde{H}$ keeping endpoints fixed, such that $\mathrm{length}(c_1) \leq \frac{\pi}{2}\,\mathrm{length}(c_0)$ (see Figure 3), where $c_1(\delta)$ is a shortest arc in $\partial\tilde{H}$ connecting the endpoints of $c_0(\delta)$. The extremal case occurs when $c_0(\delta)$ is a diameter of the cylinder. $c_0(\delta) \cup c_1(\delta)$ bounds a map of a disk whose area is linearly bounded by $\mathrm{length}(c_0) + \mathrm{length}(c_1) \leq C\,\mathrm{length}(c)$. For example, the disk which connects each point of $c_0(\delta) \cup c_1(\delta)$ by the shortest

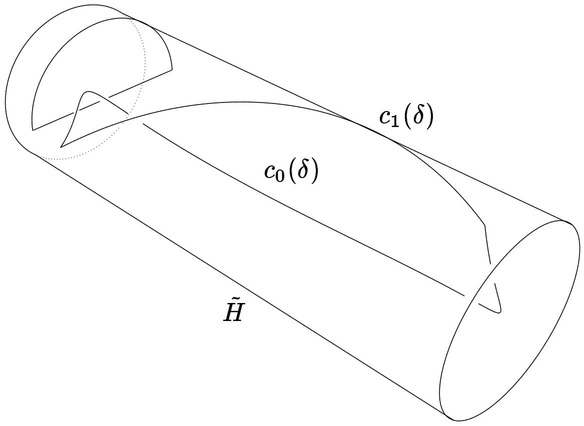

Figure 3: Comparing lengths

segment to the axis of the cylinder works, as can be seen by an elementary computation. Since the metric on H is quasi-isometric to the euclidean metric, we can find a homotopy of c to a map of a curve $c'\colon S^1 \to M$, whose length is linearly bounded by c, and such that the area of the homotopy is linearly bounded by c. Replace c with this map c'.

We want to estimate $\mathrm{wr}(c,k)$, for $im(c) \subset M$. By lemma 3.1 we may assume that c bounds a map of a punctured disk $d\colon S \to N$ such that $d_{|d^{-1}(M)}$ is incompressible and ∂–incompressible in M, with n boundary components of $d^{-1}(M)$ mapping to multiples of α in ∂M, and $d_{|d^{-1}(C)}$ consists of maps of annuli which can be assumed to be products with respect to the horotorus foliation of C. So $\mathrm{wr}(c,k) \leq n$. If c is homotopic to ∂M, then c would be homotopic in M to a multiple of α, since otherwise it would be homotopic to a multiple of k, and it would not be homotopically trivial in $N(\alpha)$. The area of the annulus realizing the homotopy into ∂M can be chosen to be linearly bounded by $\mathrm{length}(c)$, for example by coning off c to the cusp in N. Therefore c bounds a map of a disk in $N(\alpha)$ whose area is linearly bounded by $\mathrm{length}(c)$. If c is not homotopic into ∂M, then we may homotope c to be geodesic in N, and d to be pleated in N, by lemma 4.1. Consider $d^{-1}(C) \subset S$. Then as in lemma 5.1, we can find disjoint cusp neighborhoods H_i in S, some of which might intersect ∂S. Let us estimate how many horocusps can meet ∂S. We will assume the first j cusps meet ∂S. Shrink each cusp meeting ∂S until it is tangent to ∂S. Lifting to $\tilde{S} \subset \mathbb{H}^2$, so that a geodesic component γ of $\widetilde{\partial S}$ runs from 0 to ∞, we see a sequence of $j+1$ horodisks tangent to γ, such that the first and $j+1$st horodisks are identified by the covering translation of γ. So the length of ∂S is the distance between the tangent points of these two horodisks. Consider two sequential horodisks. Then we may move the horodisk of larger euclidean radius by a hyperbolic isometry, keeping it tangent to γ until it is tangent to the smaller one. A geometric calculation shows that

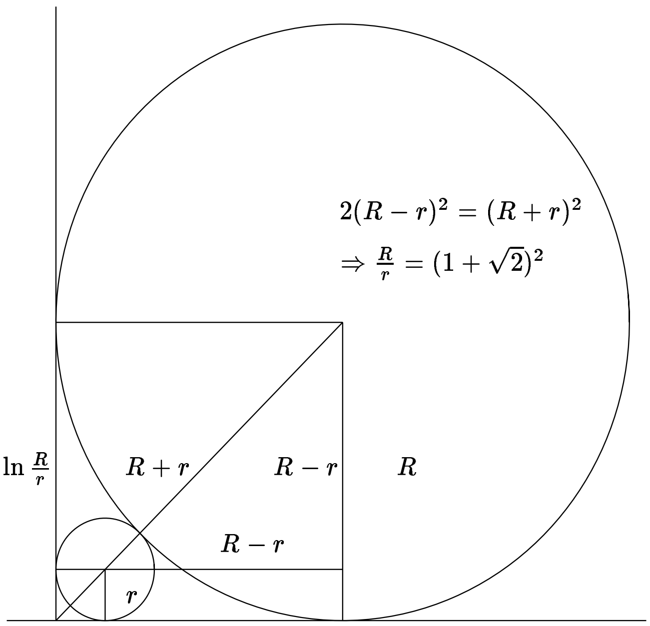

Figure 4: Bounding translation length

the hyperbolic length between the tangency points of the horodisks is $2\ln(1+\sqrt{2})$ (Figure 4). So $l(\partial S) \geq 2j\ln(1+\sqrt{2})$.

Take S and double it along its geodesic boundary ∂S to a hyperbolic surface DS. As in lemma 5.1, $l(\partial H_i) \geq l_C(\alpha)$, for $i > j$. So we take the collection of horocusps in DS consisting of H_i and its reflection, for $i > j$. Choose a number ϵ such that $l(\alpha) > 6+\epsilon$. Then we have

$$6(2n-2) = 6|\chi(DS)| \geq 2\sum_{i=j+1}^{n} l(\partial H_i) \geq 2\sum_{i=j+1}^{n} l_C(\alpha) \geq 2(n-j)(6+\epsilon).$$

Thus,

$$2\epsilon n \leq 2j(6+\epsilon) - 12 \leq \frac{(6+\epsilon)l(c)}{\ln(1+\sqrt{2})}.$$

So $\mathrm{wr}(c,k) \le n \le \frac{(6+\epsilon)l(c)}{2\epsilon\ln(1+\sqrt{2})}$. By the ubiquity theorem 6.1, $N(\alpha)$ has linear isoperimetric inequality. □

7 Essential surfaces and Dehn filling

The next theorem gives a condition for which a quasifuchsian surface in a hyperbolic knot complement remains π_1–injective under Dehn filling.

As usual, N is a hyperbolic 3–manifold with a horocusp C and S is a surface of finite type. Let $f\colon S \to N$ be a π_1–injective mapping, taking cusps of S to cusps of N. We will assume that the covering $\tilde{N}_f$ of N corresponding to $f_*(\pi_1 S)$ is geometrically finite. Let $Q(S)$ be the convex core of $\tilde{N}_f$. Suppose f has no accidental parabolics, that is $Q(S)$ is homeomorphic to $S\times[0,1]$ (or it is homeomorphic to S if $\pi_1(S)$ is fuchsian), and all cusps of S map to the same boundary slope in C (could be none). Let $\tilde{C}$ be the preimage of C in $\tilde{N}_f$. Suppose $Q(S)\cap \tilde{C} \cong \mathcal{N}(cusps(Q(S)))$, that is the only intersections with $\tilde{C}$ are the ones which must occur. Call such a mapping f *geometrically proper with respect to C*.

Theorem 7.1 (Quasifuchsian filling) *Assume we have N and $f\colon S\to N$ as above, so that f is geometrically proper with respect to C. Let α be the slope on ∂C corresponding to the image of the cusps of S under the mapping f, or any slope in C, if $Q(S)$ is compact. Suppose $l_C(\alpha)\ge 6$. Form the compact surface $S' \supseteq S\backslash f^{-1}(C)$ such that $K = S'\backslash(S\backslash f^{-1}(C))$ consists of disks, and a mapping $f'\colon S'\to N(\alpha)$, such that $f'_{|S'\backslash K} = f$ and $f'_{|K}\subset N(\alpha)\backslash(N\backslash C)$. Then f' is π_1–injective in $N(\alpha)$.*

Proof Suppose f' is not injective into $\pi_1 N(\alpha)$. Let $g\colon S^1\to S'\backslash K$ be a map which is homotopically non-trivial in S' and which bounds a map of a disk D into $N(\alpha)$, that is there is a map $d\colon D\to N(\alpha)$ with $d_{|\partial D} = f\circ g$. Choose $d^{-1}(\mathcal{N}(k))$ to have as few components as possible, where k is the core of the Dehn filling. Then $f\circ g$ is homotopic to a unique map with geodesic image γ in N, which will lie inside of $Q(S)$ when we lift to $\tilde{N}_f$, since g is homotopically non-trivial in S'. By lemmas 3.2 and 4.1 γ bounds an incompressible, ∂–incompressible pleated map of a punctured disk $d\colon F\to N$, with n punctures mapping to multiples of α in C, $d(\partial F)=\gamma$. Suppose $d^{-1}(C)\cap\partial F\neq\emptyset$. Then look at a component of $d^{-1}(C)$ which intersects ∂F, and suppose it is noncompact. Then there is an embedded arc β in $d^{-1}(C)$ connecting a point in ∂F to a cusp of F. There must also be such a geodesic arc β' in $Q(S)\cap\tilde{C}$

in the cover $\tilde{N}_f$ connecting the preimage of γ with the corresponding cusp in $\tilde{C}$, by the assumption that f is geometrically proper with respect to C. Since the lift of $d(\beta)$ to $\tilde{N}_f$ and β' lie entirely in the same component of $\tilde{C}$, d can be homotoped so that $d(\beta) = \beta'$. Take a neighborhood R of β in F which contains the cusp at one end of β, and consider the subsurface $F' = F \backslash R$. Then $d_{|\partial F'}$ lifts to a map into $Q(S)$, and there is a map $g'\colon S^1 \to S' \backslash K$ so that $f \circ g'$ is homotopic to $d_{|\partial F'}$. Moreover, g' is homotopic to g, since $\pi_1 Q(S) = \pi_1(S)$. Thus, we have found a map of a loop $g' \to S' \backslash K$ which bounds a map of a disk in $N(\alpha)$ with fewer intersections with $\mathcal{N}(k)$ Thus, every component of $d^{-1}(C)$ which intersects ∂F must be compact. If $n = 0$ or 1, then S would be compressible, or have an accidental parabolic. Otherwise we may apply the argument of theorem 5.1 to get

$$6(n-1) = 6|\chi(F)| \geq n \cdot l_C(\alpha) \geq 6n$$

a contradiction. The point is that since $d^{-1}(C)$ intersects ∂F only in compact pieces, we may find embedded horocusp neighborhoods of the punctures in F which miss ∂F, so that we may apply Boroczky's theorem to the double of F, as we did in theorem 6.2. $\square$

Here is an example which shows that the bound in theorem 6.2 is sharp. We construct a manifold which has a totally geodesic punctured torus with maximal possible cusp size. Take an ideal octahedron O in $\mathbb{H}^3$ which has all angles between faces $\pi/2$, as in Figure 5.

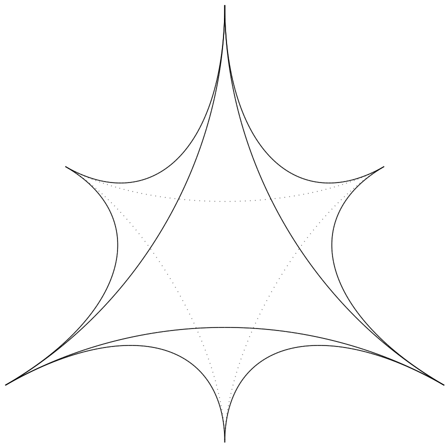

Figure 5: An ideal octahedron O in the conformal model

Then we take two copies of O, and glue the top six side faces together in pairs as indicated in Figure 6. The edges of the front faces get glued up in such a way

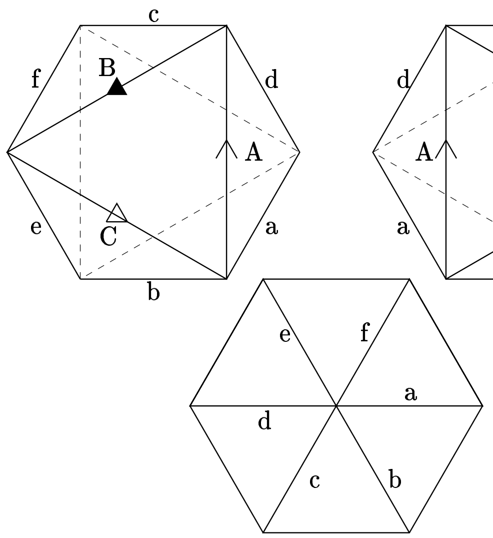

Figure 6: How to glue up the manifold

that we get a punctured torus made of two ideal triangles. The six side faces get glued cyclically, to form a punctured disk, as in the bottom diagram of figure 6. Double the manifold obtained so far along this punctured disk, then there are two punctured tori, and the back faces of the two octahedra double to form two 3–punctured spheres. We then glue the punctured tori and punctured spheres together to get a manifold N of finite volume, with 4 cusps (we can get two cusps by gluing the punctured spheres with a twist). The cusp C corresponding to the punctured torus has an embedded horoball neighborhood with boundary slope length $= 6$. The punctured torus remains incompressible after Dehn filling along this slope, by theorem 7.1 (this can also be shown using the fact that the torus is homologically non-trivial, and the filling is irreducible). This shows that the bound given in 5.1 is sharp. By Dehn filling the other cusps of N, we can get manifolds with an embedded punctured torus and a cusp corresponding to C, such that the boundary slope is as close to 6 as we like. This shows that the theorem 6.2 is sharp as well.

Here is an example of hyperbolic knots in S^3 with meridian slope length in a maximal horocusp approaching 4. Take the 5 component link L which is the 2–fold branch cover over one component of the Borromean rings. It is well known that the meridian slope for a maximal horocusp in the Borromean rings is 2, so the link L has one component with meridian slope length 4. Then we may do arbitrarily high Dehn fillings on the other components to obtain knots in S^3 with meridian slope length approaching 4 (see figure 7 for the Dehn filling description). One may see that the Dehn fillings in diagram 7 on each pair of unlinked components cancel each other by opposite Dehn twists on an annulus connecting up each pair in the complement of the other pair, so that the manifold obtained by the Dehn filling is still a knot in S^3. It would be interesting to find knots with longer meridian slope lengths.

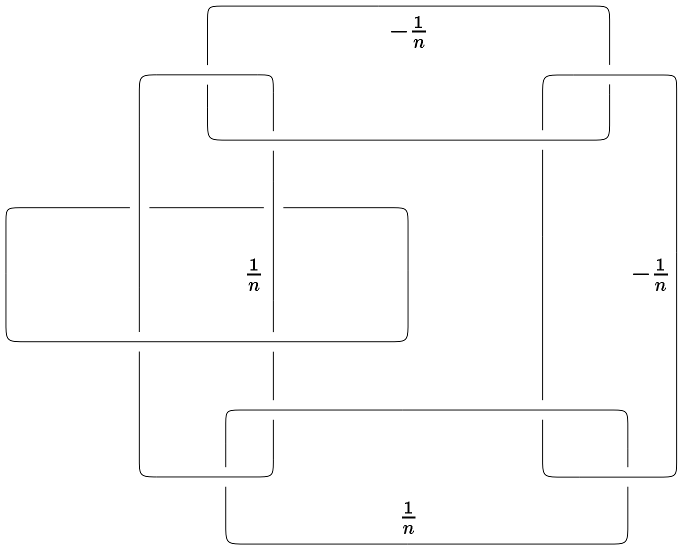

Figure 7: Knots with meridian length $\to 4$ as $|n| \to \infty$

8 Bounds on exceptional slopes

For a pair of slopes α, β on a torus, call their intersection number $\Delta(\alpha,\beta)$. If we choose a basis for the homology on the torus, such that $\alpha = (a,b)$, $\beta = (c,d)$, then $\Delta(\alpha,\beta) = |ad - bc|$. If α is a slope, then $\gcd(a,b) = 1$, since α represents a primitive homology class.

Theorem 8.1 *Let N be a hyperbolic 3–manifold, and C a distinguished embedded torus cusp. The intersection number between exceptional boundary slopes on C is ≤ 10, and there are at most 12 exceptional boundary slopes.*

Proof Given two exceptional slopes α, β, $l_C(\alpha) \leq 6$, and $l_C(\beta) \leq 6$ by theorem 6.2. By a result of Cao and Meyerhoff, theorem 5.9 in [6], area$(\partial C) \geq 3.35$. Let $\theta_{\alpha\beta}$ be the angle between the geodesics α and β on ∂C. Computing area, we have $l_C(\alpha) \cdot l_C(\beta) \sin(\theta_{\alpha\beta}) = \Delta(\alpha, \beta)\text{area}(\partial C)$. So

$$\Delta(\alpha, \beta) = \frac{l_C(\alpha) \cdot l_C(\beta) \sin(\theta_{\alpha\beta})}{\text{area}(\partial C)} \leq \frac{6^2}{3.35} = 10.75,$$

so $\Delta(\alpha, \beta) \leq 10$.

For the second part of the claim, we need the following lemma:

Lemma 8.2 (Bound on number of slopes) *If a collection of slopes on a torus have pairwise intersection numbers $\leq R$, then for any prime number $p > R$, the number of such slopes is bounded by $p+1$.*

Proof Denote the projective plane over the finite field of order p by $\mathbb{F}_p\mathbb{P}^1$. Then there is a map $\mathbb{QP}^1 \to \mathbb{F}_p\mathbb{P}^1$, where $\frac{a}{b} \mapsto (a \bmod p, b \bmod p)$. This map is well-defined, since if $\frac{a}{b} \mapsto (0,0)$, then $p | \gcd(a,b) = 1$. Suppose a pair of slopes $\frac{a}{b}$ and $\frac{c}{d}$ in the given collection map to the same point in $\mathbb{F}_p\mathbb{P}^1$, then $(a,b) \equiv k(c,d)(\text{mod } p)$, so $|ad - bc| \equiv |kcd - kcd| \equiv 0 \pmod p$. If $|ad - bc| = 0$, then $\frac{a}{b} = \frac{c}{d}$. Otherwise,

$$p \leq |ad - bc| = \Delta(\frac{a}{b}, \frac{c}{d}) \leq R < p,$$

a contradiction. So for each point of $\mathbb{F}_p\mathbb{P}^1$, there is at most one slope in the collection mapped to it. Thus, there are at most $|\mathbb{F}_p\mathbb{P}^1| = p+1$ slopes in the collection. □

In the case at hand, we have $R = 10 < 11$, so we compute that the number of exceptional fillings is ≤ 12. □

It is conjectured that the maximal intersection number between exceptional slopes is 8, realized by the figure 8 knot complement [9]. Moreover, we expect that the figure 8 knot has the fewest number of exceptional slopes, 10. When applied to the figure 8 knot, theorem 6.2 gives exactly the set of exceptional slopes for the maximal cusp. On the other hand, the figure eight knot sister has a regular torus cusp, with 12 slopes of length ≤ 6, but there are only 8 exceptional fillings [9] (see Figure 8). In the figure, the view is from ∞ in the cusp of the figure eight sister, and the circles correspond to other horoball copies of the cusp from our viewpoint. Signed pairs of lattice points correspond to slopes, where a segment from the center of the picture to the lattice point maps down to a boundary slope in the manifold. The exceptional slopes are shown in the box.

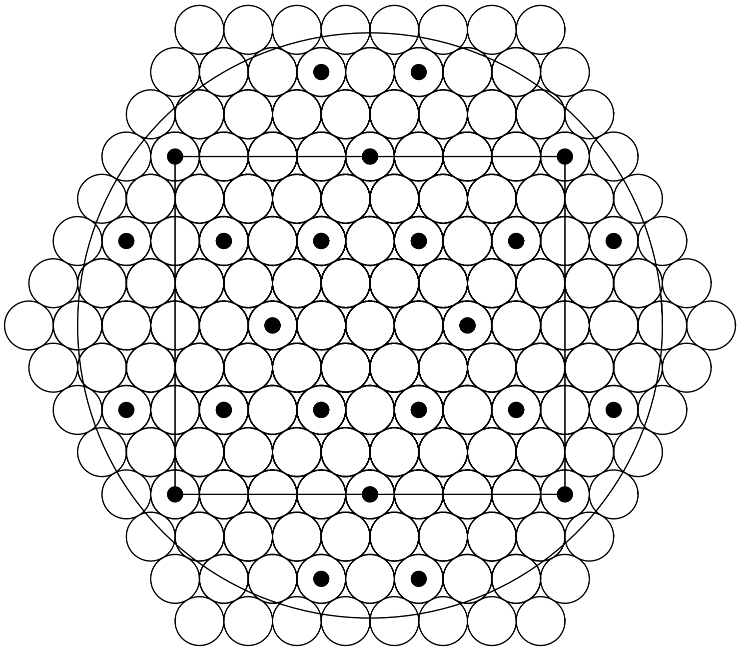

Figure 8: Primitive lattice points in the figure eight knot sister

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