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Symplectic Lefschetz fibrations on $S^1 \times M^3$

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Abstract

In this paper we classify symplectic Lefschetz fibrations (with empty base locus) on a four-manifold which is the product of a three-manifold with a circle. This result provides further evidence in support of the following conjecture regarding symplectic structures on such a four-manifold: if the product of a three-manifold with a circle admits a symplectic structure, then the three-manifold must fiber over a circle, and up to a self-diffeomorphism of the four-manifold, the symplectic structure is deformation equivalent to the canonical symplectic structure determined by the fibration of the three-manifold over the circle.

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1 Introduction and statement of results

Suppose M^3 is a closed oriented 3–manifold. If M^3 fibers over S^1, then the 4–manifold $X = S^1 \times M^3$ has a symplectic structure canonical up to deformation equivalence.

An interesting question motivated by Taubes' fundamental research on symplectic 4–manifolds [12] asks whether the converse is true: *Does every symplectic structure (up to deformation equivalence) on $X = S^1 \times M^3$ come from a fibration of M^3 over S^1, in particular, is it true that M^3 must be fibered?*

In this paper we prove the following:

Theorem 1.1 *Let M^3 be a closed 3–manifold which contains no fake 3–cell. If ω is a symplectic structure on the 4–manifold $X = S^1 \times M^3$ defined through a Lefschetz fibration (with empty base locus), then:*

(1) *M^3 is fibered, with a fibration $p\colon M^3 \to S^1$.*

(2) *There is a self-diffeomorphism $h\colon X \to X$ such that $h^*\omega$ is deformation equivalent to the canonical symplectic structure on X associated to the fibration $p\colon M^3 \to S^1$.*

Remarks (1) A 3–manifold could admit essentially different fibrations over a circle (see eg [10]). The fibration $p\colon M^3 \to S^1$ in the theorem is specified by the Lefschetz fibration.

(2) The self-diffeomorphism $h\colon X \to X$ is homotopic to the identity.

(3) R Gompf and A Stipsicz have shown [5] that for any 4–manifold with the rational homology of $S^1 \times S^3$, any Lefschetz pencil or fibration (even allowing singularities with the wrong orientation) must be a locally trivial torus fibration over S^2, in particular, the manifold is $S^1 \times L(p,1)$ with the obvious fibration.

A stronger version of the theorem, in which we also classify symplectic Lefschetz fibrations on $S^1 \times M^3$, is given in section 4.

The proof of the theorem consists of two major steps. First, we show that any symplectic Lefschetz fibration on $X = S^1 \times M^3$ has no singular fibers, ie, it is a locally trivial fibration. This result is the content of lemma 3.4 (Lemma on Vanishing Cycles). Secondly, we show that every such fibration of $X = S^1 \times M^3$ is induced in a certain way from a fibration of M^3 over a circle.

The essential ingredient in the proof of the Lemma on Vanishing Cycles is a result of D Gabai, which says roughly that the minimal genus of an immersed surface representing a given homology class in a 3–manifold is equal to the minimal genus of an embedded surface representing the same homology class. This result is specific for dimension three and does not hold in dimension four.

This paper is organized as follows: Section 2 contains a brief review of some background results. Section 3 is devoted to a preparatory material for the proof of the main theorem, in particular, it contains the Lemma on Vanishing Cycles. In section 4, the full version of the main theorem is stated and proved.

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2 Recollections

2.1 Lefschetz pencils and fibrations

Let X be a closed, oriented, smooth 4–manifold. A Lefschetz pencil on X is a smooth map $P\colon X \setminus \mathcal{B} \to \mathbb{C}P^1$ defined on the complement of a finite subset $\mathcal{B}$ of X, called the base locus, such that each point in $\mathcal{B}$ has an orientation-preserving coordinate chart in which P is given by the projectivization map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$, and each critical point has an orientation-preserving chart on which $P(z_1, z_2) = z_1^2 + z_2^2$. Blowing up at each point of $\mathcal{B}$, we obtain a Lefschetz fibration on $X \# n\overline{\mathbb{C}P^2}$ $(n = \#\mathcal{B})$ over $\mathbb{C}P^1$ with fiber $F_t = P^{-1}(t) \cup \mathcal{B}$ for each $t \in \mathbb{C}P^1$.

More generally, a Lefschetz fibration on a closed oriented smooth 4–manifold X is a smooth map $P\colon X \to B$ where B is a Riemann surface, such that each critical point of P has an orientation-preserving chart on which $P(z_1, z_2) = z_1^2 + z_2^2$. We require that each fiber is connected and contains at most one critical point. Every Lefschetz fibration can be changed to satisfy these two conditions. A Lefschetz fibration is called symplectic if there is a symplectic structure ω on X whose restriction to each regular fiber is non-degenerate.

In an orientation-preserving chart at a critical point $x \in X$, the map P is given by $P(z_1, z_2) = z_1^2 + z_2^2$. Let $t \in \mathbb{R} \subset \mathbb{C}$ be a positive regular value. The fiber $F_t = P^{-1}(t)$ contains a simple closed loop γ which is the intersection of

F_t with the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$, ie, the boundary of the disc in X defined by $x_1^2 + x_2^2 \le t$. This simple closed loop γ on F_t is called the vanishing cycle associated to the critical point x. A regular neighborhood of the singular fiber F_0 can be described as the result of attaching a 2–handle along the vanishing cycle γ to a regular neighborhood of the regular fiber F_t with framing -1 relative to the product framing on $F_t \times S^1$. The monodromy around the critical value $P(x)$ is a right Dehn twist along the vanishing cycle γ.

We quote an observation of Gompf which roughly says that most of the Lefschetz fibrations are symplectic. For a proof, see [5] or [1].

Theorem 2.1 (Gompf) *A Lefschetz fibration on an oriented 4–manifold X is symplectic if the fiber class is non-zero in $H_2(X;\mathbb{R})$. Any Lefschetz pencil (with non-empty base locus) is symplectic.*

A remarkable theorem of Donaldson [3] says that any symplectic 4–manifold admits a Lefschetz pencil by symplectic surfaces.

The following lemma is known for general Lefschetz fibrations. For completeness, we give a short proof for the case of non-singular fibrations. This is sufficient for our purpose.

Lemma 2.1 *Let $F \hookrightarrow X \xrightarrow{P} B$ be a fibration and ω_1, ω_2 symplectic forms on X with respect to which each fiber of P is symplectic. Then ω_1 and ω_2 are deformation equivalent if they induce the same orientation on the fiber.*

Proof We orient the base B so that the pull-back of the volume form ω_B of B, $P^*\omega_B$, has the property that $\omega_1 \wedge P^*\omega_B$ is positive on X with respect to the orientation induced by ω_1. Then

$$\omega(s) := \omega_1 + sP^*\omega_B$$

is a symplectic form on X for any $s \ge 0$. Let $\omega(s,t) = t\omega_2 + (1-t)\omega(s)$ for $0 \le t \le 1$, $s \ge 0$. Then

$$\begin{aligned}\omega(s,t) \wedge \omega(s,t) &= t^2\omega_2 \wedge \omega_2 + (1-t)^2\omega(s) \wedge \omega(s) \\ &\quad +2t(1-t)\omega_2 \wedge \omega(s)\end{aligned}$$

is positive for all $0 \le t \le 1$ when s is sufficiently large, since $\omega_2 \wedge \omega(s)$ is positive for large enough s. This implies that ω_1 and ω_2 are deformation equivalent. □

2.2 Seiberg–Witten theory

On an oriented Riemannian 4–manifold X, a $Spin^c$ structure $\mathcal{S}$ consists of a hermitian vector bundle W of rank 4, together with a Clifford multiplication $\rho\colon T^*X \to End(W)$. The bundle W decomposes into two bundles of rank 2, $W^+ \oplus W^-$, such that $\det W^+ = \det W^-$. Here W^- is characterized as the subspace annihilated by $\rho(\eta)$ for all self-dual 2–forms η. We write $c_1(\mathcal{S})$ for the first Chern class of W^+. The Levi–Civita connection on X coupled with a $U(1)$ connection A on $\det W^+$ defines a Dirac operator $D_A\colon \Gamma(W^+) \to \Gamma(W^-)$ from the space of smooth sections of W^+ into that of W^-.

The 4–dimensional Seiberg–Witten equations are the following pair of equations for a section ψ of W^+ and a $U(1)$ connection A on $\det W^+$:

$$\begin{aligned} \rho(F_A^+) - \{\psi \otimes \psi^*\} &= 0 \\ D_A\psi &= 0 \end{aligned}$$

Here F^+ is the projection of the curvature onto the self-dual forms, and the curly brackets denote the trace-free part of an endomorphism of W^+.

The moduli space $\mathcal{M}_\mathcal{S}$ is the space of solutions (A,ψ) modulo the action of the gauge group $\mathcal{G} = Map(X,S^1)$, which is compact with virtual dimension

$$d(\mathcal{S}) = \frac{1}{4}(c_1(\mathcal{S})^2[X] - 2\chi(X) - 3\sigma(X)),$$

where $\chi(X)$ and $\sigma(X)$ are the Euler characteristic and signature of X respectively. When $b^+(X) \geq 1$, for a generic perturbation of the Seiberg–Witten equations where η is a self-dual 2–form

$$\begin{aligned} \rho(F_A^+ + i\eta) - \{\psi \otimes \psi^*\} &= 0 \\ D_A\psi &= 0, \end{aligned}$$

the moduli space $\mathcal{M}_{\mathcal{S},\eta}$ is a compact, canonically oriented, smooth manifold of dimension $d(\mathcal{S})$, which contains no reducible solutions (ie, solutions with $\psi \equiv 0$). The fundamental class of $\mathcal{M}_{\mathcal{S},\eta}$ evaluated against some universal characteristic classes defines the Seiberg–Witten invariant $SW(\mathcal{S}) \in \mathbb{Z}$, which is independent of the Riemannian metric and the perturbation η when $b^+(X) > 1$. When $b^+(X) = 1$, $SW(\mathcal{S})$ is well-defined if $c_1(\mathcal{S})^2[X] \geq 0$ and $c_1(\mathcal{S})$ is not torsion, by choosing $||\eta||$ sufficiently small. The set of complex line bundles $\{E\}$ on X acts on the set of $Spin^c$ structures freely and transitively by $(E,\mathcal{S}) \to \mathcal{S} \otimes E$. We will call a cohomology class $\alpha \in H^2(X;\mathbb{Z})$ a Seiberg–Witten basic class if there exists a $Spin^c$ structure $\mathcal{S}$ such that $\alpha = c_1(\mathcal{S})$ and $SW(\mathcal{S}) \neq 0$. There is an involution I acting on the set of $Spin^c$ structures on X which has the property that $c_1(\mathcal{S}) = -c_1(I(\mathcal{S}))$ and $SW(\mathcal{S}) = \pm SW(I(\mathcal{S}))$. As a consequence, if a cohomology class $\alpha \in H^2(X;\mathbb{Z})$ is a Seiberg–Witten basic class, so is $-\alpha$.

We will use the following fundamental result:

Theorem 2.2 (Taubes) *Let (X,ω) be a symplectic 4–manifold with canonical line bundle K_ω. Then $c_1(K_\omega)$ is a Seiberg–Witten basic class if $b^+(X) > 1$ or $b^+(X) = 1$, $c_1(K_\omega)\cdot[\omega] > 0$ and $2\chi(X) + 3\sigma(X) \geq 0$.*

2.3 Gabai's Theorem

The following theorem of Gabai says that, given a singular oriented surface in a closed oriented 3–manifold, one can find an embedded surface (not necessarily connected) representing the same homology class and having the same topological complexity as the singular surface.

Let us recall the definition of Thurston norm and the singular norm on the second homology group of a compact 3–manifold. Let S be an orientable surface. The complexity of S is defined by $x(S) = \sum_{S_i} \max(-\chi(S_i), 0)$, where the summation is taken over connected components of S. For a closed, oriented 3–manifold M, the Thurston norm $x(z)$ and singular norm $x_s(z)$ of a homology class $z \in H_2(M;\mathbb{Z})$ are defined by

$$\begin{aligned} x(z) &= \min\{x(S) | S \text{ is an embedded surface representing } z\}, \\ x_s(z) &= \inf\{\frac{1}{n}x(S) | f\colon S \to M, f([S]) = nz\}. \end{aligned}$$

Theorem 2.3 (Gabai [6]) *Let M be a closed oriented 3–manifold. Then $x_s(z) = x(z)$ for all $z \in H_2(M;\mathbb{Z})$.*

Gabai's theorem is specific for dimension three and fails in dimension four in general. Precisely because our 4–manifold under consideration is the product of a 3–manifold with a circle, we are able to apply Gabai's theorem to yield a stronger estimate for the 4–manifold. This is the essential point in the proof of the Lemma on Vanishing Cycles.

3 Preparatory material for the proof of the theorem

This section is devoted to preparatory material necessary for the proof of the theorem.

3.1 Map(F) and Diff(F)

We first list some results about the mapping class group $Map(F)$ and the diffeomorphism group $Diff(F)$ of an orientable surface F.

The reader is referred to [8] for the proof of the following proposition.

Proposition 3.1 (Nielsen Representation Theorem) *Every finite subgroup G of the mapping class group of a big (≥ 2) genus surface can be lifted to the diffeomorphism group of that surface. Moreover, there is a conformal structure on F such that G is realized by conformal isometries of F.*

We will also need an analogue of the Nielsen Representation Theorem in a slightly different setting. It states that a commutativity relation between an arbitrary element and a finite order element of the mapping class group of a big (≥ 2) genus surface can be lifted to the homeomorphism group of the surface.

Lemma 3.1 *Let F be a surface with* $\text{genus}(F) \geq 2$ *and ψ, φ be two mapping classes of F such that φ has finite order and $\psi \circ \varphi = \varphi \circ \psi$ in $Map(F)$. Then there are homeomorphisms $\Psi, \Phi\colon F \to F$ in the mapping classes ψ, φ respectively, such that Φ has finite order, and $\Psi \circ \Phi = \Phi \circ \Psi$ as homeomorphisms.*

Proof The proof is based on a theorem of Teichmüller [13, 14]. A very clear treatment of Teichmüller's theorem can be found in [2].

We first recall the notions of quasi-conformal mappings and total dilatation of homeomorphisms of a surface. Let F be a Riemann surface with a given conformal structure, and $f\colon F \to F$ be an orientation preserving homeomorphism. At a point $p \in F$ where f is C^1–smooth we can measure the deviation of f from being conformal by the ratio of the bigger axis to the smaller axis of an infinitesimal ellipse, which is the image of an infinitesimal circle around the point p under f. This ratio is called the local dilatation of f and is denoted by $K_p[f]$. A homeomorphism $f\colon F \to F$ is called a quasi-conformal mapping if $K_p[f]$ is defined for almost all $p \in F$ and $\sup K_p[f] < \infty$. The number $K[f] = \sup K_p[f]$ is called the total dilatation of the quasi-conformal mapping f.

Teichmüller's theorem states that *among all the homeomorphisms in a given mapping class of a Riemann surface F of* $\text{genus}(F) \geq 2$, *there is a unique one which minimizes the total dilatation.*

Now back to the proof of the lemma. By the Nielsen Representation Theorem (cf Proposition 3.1), the mapping class φ can be represented by an isometry Φ of F

with respect to some conformal structure α. By Teichmüller's theorem, there is a unique homeomorphism Ψ in the mapping class ψ which minimizes the total dilatation with respect to the conformal structure α. Since Φ is an isometry of α, $\Phi^{-1}\circ\Psi\circ\Phi$ has the same total dilatation as Ψ. On the other hand, $\Phi^{-1}\circ\Psi\circ\Phi$ and Ψ are in the same mapping class since $\psi\circ\varphi=\varphi\circ\psi$ in $Map(F)$. The uniqueness of the extremal homeomorphism in Teichmüller's theorem implies that they must coincide, and $\Psi\circ\Phi=\Phi\circ\Psi$ as homeomorphisms. □

Proposition 3.2 *Let F_g be a closed orientable surface of genus g, and denote by $Diff_0(F_g)$ the identity component of $Diff(F_g)$. Then for all $g\geq 2$, $Diff_0(F_g)$ is contractible, for $g=1$, $Diff_0(F_g)$ is homotopy equivalent to the identity component of the group of conformal automorphisms of the torus and, for $g=0$, $Diff_0(F_g)$ is homotopy equivalent to $SO(3)$.*

The reader is referred to [4] for this result.

The rest of this subsection concerns locally trivial fibrations of 4–manifolds over a Riemann surface.

Definition Two locally trivial fibrations $F\hookrightarrow X\xrightarrow{P}B$ and $F'\hookrightarrow X'\xrightarrow{P'}B'$ are said to be equivalent if there are diffeomorphisms $f\colon B\to B'$ and $\tilde f\colon X\to X'$, such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde f} & X' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ B & \xrightarrow{f} & B' \end{array}$$

commutes. When $X=X'$, we say that P and P' are strongly equivalent if $\tilde f$ is isotopic to the identity.

Each fibration P defines a monodromy homomorphism Mon_P from the fundamental group of the base to the mapping class group of the fiber.

Lemma 3.2 *When the fiber is high-genus (≥ 2), monodromy Mon_P determines the equivalence class of a fibration P.*

Proof We write the base B as $B_0\cup D$ where B_0 is a disc with several 1–handles attached and D is a disc. We choose a base point $b_0\in B_0\cap D$. The fibration P restricted to B_0 is determined by Mon_P as a representation of $\pi_1(B_0,b_0)$ in the mapping class group of the fiber F_{b_0} over the base point b_0. Over the disc D, P is trivial. We can recover P on the whole manifold X by gluing $P^{-1}(B_0)$ and $P^{-1}(D)$ along the boundary via some gluing map φ, and P depends only on the homotopy class of φ viewed as a map $S^1 \to \mathit{Diff}_0(F_{b_0})$. Now we recall the fact that $\mathit{Diff}_0(F_g)$ is contractible for all $g \geq 2$, so that $\varphi \sim \mathit{Id}$ when $\text{genus}(F_{b_0}) \geq 2$. The lemma follows. $\square$

3.2 Cyclic coverings and S^1–valued functions

Here we describe a construction, which, starting with a S^1–valued function on a CW–complex and a positive integer, gives a cyclic finite covering of the CW–complex and a generator of the deck transformation group. An inverse of this construction is then discussed.

For the rest of the paper we view S^1 as the unit circle in $\mathbb{C}$, oriented in the usual way, ie, in the counter-clockwise direction.

Consider a CW–complex Y and let $g\colon Y \to S^1$ be a function. It determines a cohomology class $[g] \in H^1(Y;\mathbb{Z})$. Denote by $d(g)$ the divisibility of $[g]$. For a positive integer n and a function $g\colon Y \to S^1$ such that $\gcd(n, d(g)) = 1$, define a subset $\widetilde{Y} \subset S^1 \times Y$ by

$$\widetilde{Y} = \{(t, y) \in S^1 \times Y \mid t^n = g(y)\}.$$

Obviously $\widetilde{Y}$ is invariant under the transformation $\varphi\colon S^1 \times Y \to S^1 \times Y$ induced from rotation of S^1 by angle $\frac{2\pi}{n}$ in the positive direction. Let $pr\colon \widetilde{Y} \to Y$ be the restriction of the projection from $S^1 \times Y$ onto the second factor. It is a cyclic n–fold covering of Y and φ generates the group of deck transformations of this covering.

Now we would like to invert this construction, ie, starting with a finite cyclic covering $pr\colon \widetilde{Y} \to Y$ and a generator $\varphi\colon \widetilde{Y} \to \widetilde{Y}$ of the structure group of pr, find a function $g\colon \widetilde{Y} \to S^1$ such that the construction above yields $pr\colon \widetilde{Y} \to Y$ and φ. However this is not always possible.

Fix a finite cyclic covering $pr\colon \widetilde{Y} \to Y$ with a generator $\varphi\colon \widetilde{Y} \to \widetilde{Y}$ of the structure group of pr. Let $G = \langle \varphi \rangle$ be the group of deck transformations of pr and $n = |G|$. Define an action of G on S^1 by $\varphi \cdot z = e^{2\pi i/n} z$. This gives rise to an S^1–bundle Z over Y and $\widetilde{Y}$ sits naturally in Z. The Euler class of this bundle is a torsion class, for the n^{th} power of the bundle (considered as $U(1)$ bundle) is trivial. Moreover, there is a choice of trivialization, canonical up to rotations of S^1, given by an n–valued section of pr, which becomes a section in the n^{th} power of the bundle. Suppose the Euler class vanishes, then $Z \cong S^1 \times Y$ and this diffeomorphism is canonical up to isotopy. Consider the map $\tilde{g}\colon \widetilde{Y} \to S^1$, which is the restriction to $\widetilde{Y}$ of the projection from $S^1 \times Y$ onto the S^1–factor. The map $\tilde{g}^n$ is φ invariant and therefore descends to a map $g\colon Y \to S^1$. It is an easy exercise to show that the construction above applied to the pair (g,n) yields (pr,φ).

Thus we have proved the following:

Lemma 3.3 *Let Y be a CW–complex such that $H^2(Y;\mathbb{Z})$ has no torsion. Then the construction above gives a 1–1 correspondence between two sets $A = \{(pr, \varphi)\}$ and $B = \{(n,[g])\}$, where $pr\colon \widetilde{Y} \to Y$ is a finite cyclic covering, $\varphi\colon \widetilde{Y} \to \widetilde{Y}$ is a generator of the deck transformation group of pr, n is a positive integer and $[g]$ is the homotopy class of a map $g\colon Y \to S^1$ with $\gcd(n,d(g)) = 1$.*

3.3 Surface fibrations over a torus

In this subsection we will explore a family of surface fibrations of $X = S^1 \times M^3$ over a torus, where M^3 is a closed orientable 3–manifold fibered over a circle with fiber Σ and fibration $p\colon M^3 \to S^1$. Denote by Σ_β the fiber over a point $\beta \in S^1$ and by $[p] \in H^1(M^3;\mathbb{Z})$ the homotopy class of p.

Consider a smooth function $g\colon M^3 \to S^1$ and denote by $[g] \in H^1(M;\mathbb{Z})$ its homotopy class and by $d(g)$ the divisibility of $[g]$ restricted to the fiber Σ. Let n be a positive integer such that $\gcd(n,d(g)) = 1$.

Define $P_{g,n}\colon X = S^1 \times M^3 \to S^1 \times S^1$ by $P_{g,n}(t,m) = (t^n\overline{g(m)}, p(m))$, where $\overline{g(m)}$ stands for the complex conjugate of $g(m)$ in S^1. The map $P_{g,n}$ is a locally trivial fibration of X over a torus such that the fiber $F_{(\alpha,\beta)}$ over a point $(\alpha,\beta) \in S^1 \times S^1$ is the graph of a multi-valued function $(\alpha g|_{\Sigma_\beta})^{\frac{1}{n}}$ on Σ_β, ie,

$$F_{(\alpha,\beta)} = \{(t,m) \in S^1 \times \Sigma_\beta \mid t^n = \alpha g(m)\} \subset S^1 \times \Sigma_\beta \subset S^1 \times M^3.$$

Fix $g\colon M^3 \to S^1$ and n as above. Let us find the monodromy of $P_{g,n}$. First of all, the projection $S^1 \times \Sigma_\beta \to \Sigma_\beta$ restricted to $F_{(\alpha,\beta)}$ is a cyclic n–fold covering $pr\colon F_{(\alpha,\beta)} \to \Sigma_\beta$. Denote by φ the self-diffeomorphism of $F_{(1,1)}$ induced by a rotation of the S^1–factor in $S^1 \times \Sigma_1$ by angle $2\pi/n$, then φ generates the group of deck transformations of $pr\colon F_{(1,1)} \to \Sigma_1$. Secondly, denote by $Mon_p \in Map(\Sigma_1)$ the monodromy of $p\colon M^3 \to S^1$, then Mon_p pulls back to an element in $Map(F_{(1,1)})$.

Let $q = S^1 \times \{1\}$ and $r = \{1\} \times S^1$ be the "coordinate" simple closed loops in $S^1 \times S^1$. Then it is easily seen that the monodromy $Mon_{P_{g,n}}(q)$ of $P_{g,n}$ along

q is equal to $[\varphi]$ and the monodromy $Mon_{P_{g,n}}(r)$ along r is the pull-back of Mon_p to $Map(F_{(1,1)})$.

We end this subsection with the following classification of $P_{g,n}$.

Proposition 3.3 *Two fibrations $P_{g,n}$ and $P_{g',n'}$ are strongly equivalent if and only if $n = n'$ and $[g] \equiv [g'] \bmod ([p])$, ie, when the two planes in $H^1(X;\mathbb{Z}) = H^1(S^1;\mathbb{Z}) \oplus H^1(M^3;\mathbb{Z})$ spanned by two pairs of vectors $(n[t] - [g], [p])$ and $(n'[t] - [g'], [p])$ coincide.*

Proof The necessity follows from homotopy theoretical considerations: the two said planes are pull-backs of $H^1(S^1 \times S^1;\mathbb{Z})$ by $P_{g,n}$ and $P_{g',n'}$ respectively, therefore they must coincide if $P_{g',n'}$ is homotopic to $P_{g,n}$ post-composed with a self-diffeomorphism of $S^1 \times S^1$.

On the other hand, by changing basis on one of the tori, we can arrange that the homotopy classes of g and g' are equal. Then a homotopy from g to g' leads to a strong equivalence between $P_{g,n}$ and $P_{g',n'}$. □

We observe that when the fiber Σ of $p\colon M^3 \to S^1$ has genus zero, there is a unique equivalence class of $P_{g,n}$, ie, the trivial one when $[g] = 0$ and $n = 1$.

3.4 Lemma on Vanishing Cycles

Lemma 3.4 *Let M^3 be a closed 3–manifold and $P\colon X = S^1 \times M^3 \to B$ be a symplectic Lefschetz fibration with regular fiber F. Then P has no singular fibers.*

Proof We first observe that every vanishing cycle must be non-separating since X has an even intersection form. There are three cases to consider according to the genus of the fiber.

(1) The fiber F is a sphere. There are no vanishing cycles since every curve on a sphere is separating.

(2) The fiber F is a torus. Recall a formula for the Euler characteristic of the total space of a Lefschetz fibration

$$\chi(X) = \chi(F) \cdot \chi(B) + \#\{\text{vanishing cycles}\}.$$

There must be no vanishing cycles since $\chi(F) = 0$ and $\chi(X) = 0$.

(3) The fiber F is a high genus (≥ 2) surface. We first show that the canonical class K is Seiberg–Witten basic. By Theorem 2.2, we only need to show that, if $b^+(X)=1$, then $K\cdot[\omega]>0$, where ω is the symplectic form on X. This could be seen as follows. The 4–manifold X has a hyperbolic intersection form since $b^+(X)=1$ and $\sigma(X)=0$. Let $x,y\in H^2(X;\mathbb{R})$ form a hyperbolic basis, ie, $x\cdot x=y\cdot y=0$ and $x\cdot y=1$. Since $F\cdot F=0$, $K\cdot K=2\chi(X)+3\sigma(X)=0$ and $K\cdot F=2g_F-2>0$, we can assume without loss of generality that $PD[F]=ax$ and $K=by$ for some positive a and b. Let $[\omega]=\alpha x+\beta y$. Observe that $[\omega]\cdot[\omega]=\alpha\beta>0$ and $[\omega]\cdot F=a\beta>0$, therefore α and β are both positive. But $K\cdot[\omega]=b\alpha>0$. Thus, by Theorem 2.2, K is a Seiberg–Witten basic class.

Let $V\in H_2(M^3;\mathbb{Z})$ be the homology class of the projection of F into M^3. Suppose there is a singular fiber. Since the vanishing cycle for this singular fiber is non-separating, the class V can be represented by a map $f\colon F'\to M^3$ such that $g_{F'}=g_F-1$. The singular norm of V, $x_s(V)$, is less than or equal to $2g_{F'}-2$. By Theorem 2.3, there are embedded surfaces S_i in M^3 such that $\sum_i[S_i]=V$ and $\sum_i x(S_i)=x_s(V)\leq 2g_{F'}-2$, where x stands for the complexity of an orientable surface. On the other hand, since K is Seiberg–Witten basic, by the adjunction inequality (cf [9]), we have

$$x(S_i)\geq |K\cdot S_i|,\ \text{ for all } i.$$

Therefore,

$$2g_{F'}-2\geq\sum_i x(S_i)\geq |K\cdot V|=K\cdot F=2g_F-2,$$

which is a contradiction. We used the fact that $K\cdot V=K\cdot F$. This is because $H_1(S^1;\mathbb{Z})\otimes H_1(M^3;\mathbb{Z})\subset H_2(X;\mathbb{Z})$ consists of classes represented by embedded tori in X, thus, by the adjunction inequality, $K\cdot H=0$ for any $H\in H_1(S^1;\mathbb{Z})\otimes H_1(M^3;\mathbb{Z})$. This concludes the proof for the case of high genus fiber. □

4 The main theorem

Let us recall that if M is a closed oriented 3–manifold fibered over S^1, then the 4–manifold $X=S^1\times M$ carries a canonical (up to deformation) symplectic structure compatible with the orientation on X. Note that we have canonically oriented S^1 so that the orientation of M determines an orientation of the fiber of $M\to S^1$.

The Poincaré associate of a 3–manifold M, denoted by $\mathcal{P}(M)$, is defined by the condition that $\mathcal{P}(M)$ contains no fake 3–cell and $M = \mathcal{P}(M)\#A$ where A is a homotopy 3–sphere. The theorem about unique normal prime factorization of 3–manifolds implies that the Poincaré associate exists and is unique [7]. An orientation on M canonically determines an orientation on the Poincaré associate $\mathcal{P}(M)$.

Theorem 4.1 *Let M^3 be a closed oriented 3–manifold and $X = S^1 \times M^3$. Let $P\colon X \to B$ be a symplectic Lefschetz fibration with respect to some symplectic form ω on X compatible with the orientation. Denote by F the regular fiber of P. Then:*

(1) *The Poincaré associate $\mathcal{P}(M^3)$ of M^3 is fibered over S^1 with fibration $p\colon \mathcal{P}(M^3) \to S^1$.*

(2) *There is a diffeomorphism $h\colon S^1 \times \mathcal{P}(M^3) \to X$ such that $h^*\omega$ is deformation equivalent to the canonical symplectic structure on $S^1 \times \mathcal{P}(M^3)$ defined through the fibration $p\colon \mathcal{P}(M^3) \to S^1$.*

There are two possibilities for the Lefschetz fibration P:

(a) *If $P(S^1 \times \{pt\})$ is not null-homotopic in B, then $\mathrm{genus}(B) = 1$, and $P \circ h = P_{g,n}$ for some integer $n > 0$ and map $g\colon \mathcal{P}(M^3) \to S^1$ (see subsection 3.3 for the definition of $P_{g,n}$).*

(b) *If $P(S^1 \times \{pt\})$ is null-homotopic, then $\mathrm{genus}(F) = 1$. Moreover, $S^1 \times \mathcal{P}(M^3) = F \times B$ and $P \circ h = pr_2$ is the projection onto the second factor, and the fibration $p = pr_1\colon \mathcal{P}(M^3) = S^1 \times B \to S^1$ is the projection onto the first factor.*

Proof By lemma 3.4, the Lefschetz fibration P has no singular fibers.

Let's first consider the case when the base $B = S^2$. The fiber F must be T^2, and X is simply the product $F \times B$ and $P = pr_2$ is the projection onto the second factor. This belongs to the case when $P(S^1 \times \{pt\})$ is null-homotopic. It is easily seen that the Poincaré associate $\mathcal{P}(M^3)$ of M^3 is homeomorphic to $S^1 \times B$.

For the rest of the proof, we assume $\mathrm{genus}(B) \geq 1$. Denote x_0 a base point in X and let $b_0 = P(x_0)$. Consider the exact sequence induced by the fibration:

$$1 \to \pi_1(F_{b_0}, x_0) \hookrightarrow \pi_1(X, x_0) \xrightarrow{P_*} \pi_1(B, b_0) \to 1$$

Observe that $\pi_1(X, x_0) = \mathbb{Z} \oplus \pi_1(M^3)$. Denote the generator of the $\mathbb{Z}$–summand by $u = [S^1 \times \{pt\}] \in \pi_1(X, x_0)$.

Case 1 The image $P_*(u) \neq 1$ in $\pi_1(B, b_0)$.

The element u is central in $\pi_1(X, x_0)$, therefore $P_*(u)$ is central in $\pi_1(B, b_0)$. It follows immediately that B is a torus. Let n be the divisibility of $P_*(u)$, $v = \frac{1}{n} P_*(u)$ and q be a simple closed loop in B containing b_0 and representing v. The monodromy $Mon_P(P_*(u))$ is trivial in the mapping class group of F_{b_0}. This is because the mapping class group of F_{b_0} is isomorphic to $\mathrm{Out}(\pi_1(F_{b_0}, x_0))$ and $Mon_P(P_*(u))(\gamma) = u\gamma u^{-1} = \gamma$ for every γ in $\pi_1(F_{b_0}, x_0) \subset \pi_1(X, x_0)$ (note that u is central in $\pi_1(X, x_0)$).

The cases when the fiber F is a torus or a sphere need special treatment, and will be considered after the general case.

We assume for now that $\mathrm{genus}(F) \geq 2$. Observe that $Mon_P(v)$ has finite order in $Map(F_{b_0})$.

Claim *$Mon_P(v) = Mon_P(q)$ has a representative $\varphi\colon F_{b_0} \to F_{b_0}$ which has order n and is periodic-point-free.*

Proof Let φ be a finite order representative of $Mon_P(q)$ provided by proposition 3.1. Suppose it has periodic points. We can go back to the beginning of the proof and make sure that $x_0 \in F_{b_0}$ is a periodic point of φ, ie, $\varphi^{n'}(x_0) = x_0$ for some n' such that $0 < n' < n$. The minimal period n' divides n and we set $k = n/n'$.

The idea of the proof, which follows, is that if $\varphi^{n'}$ has a fixed point, then $\varphi_*^{n'}$ acts (periodically) on $\pi_1(F_{b_0}, x_0)$ "on a nose", not up to inner automorphisms of $\pi_1(F_{b_0}, x_0)$ as in the case of periodic-point-free actions. This will contradict the structure of $\pi_1(X, x_0)$ seen from the exact sequence induced by the fibration.

Let $Z = F_{b_0} \times I/[_{(x,1)\sim(\varphi(x),0)}]$ be the mapping torus of φ and $\widetilde{Z} = F_{b_0} \times I/[_{(x,1)\sim(\varphi^{n'}(x),0)}]$ be the mapping torus of $\varphi^{n'}$. There are a natural n'–fold covering $cov\colon \widetilde{Z} \to Z$ and an embedding $emb\colon Z \hookrightarrow X$, such that $cov(x_0, 0) = (x_0, 0)$ and $emb(x_0, 0) = x_0$. Their composition induces a monomorphism in the fundamental groups $(emb \circ cov)_*\colon \pi_1(\widetilde{Z}, (x_0, 0)) \hookrightarrow \pi_1(X, x_0)$. Let δ be the image of $[\{x_0\} \times I] \in \pi_1(\widetilde{Z}, (x_0, 0))$ under the above monomorphism. For any $\gamma \in \pi_1(F_{b_0}, x_0) \subset \pi_1(X, x_0)$, the following relations hold in $\pi_1(X, x_0)$:

$$\delta\gamma\delta^{-1} = \varphi_*^{n'}(\gamma)$$
$$\delta^k\gamma\delta^{-k} = \gamma$$

On the other hand, we have $\delta^k = \xi u$ for some element ξ in $\pi_1(F_{b_0}, x_0)$, since $P_*(\delta^k) = P_*(u)$. The relations above then imply that $\xi\gamma\xi^{-1} = \gamma$ for any $\gamma \in \pi_1(F_{b_0}, x_0)$, thus $\xi = 1$. Thus we have $\delta^k = u$ for $k > 1$. This contradicts the fact that u generates a direct summand in $\pi_1(X, x_0)$. $\square$

Denote $\Sigma = F_{b_0}/\varphi$. Let r be an oriented simple closed loop in B such that $q \cap r = b_0$ and $\langle q, r\rangle = 1$. The monodromy $Mon_P(r)$ commutes with $Mon_P(q)$. Hence it has a representative $\tilde{\psi}\colon F \to F$, which commutes with φ by lemma 3.1, and therefore descends to a map $\psi\colon \Sigma \to \Sigma$. Let $\tilde{g}\colon \Sigma \to S^1$ be the S^1–valued function corresponding to the cyclic covering $pr\colon F_{b_0} \to \Sigma = F_{b_0}/\varphi$ and the generator φ of the deck transformations of pr (cf lemma 3.3). Since φ commutes with ψ, the function $\tilde{g}$ extends to a function $g\colon \widetilde{M}^3 \to S^1$, where $\widetilde{M}^3$ is the mapping torus of ψ. In general g is only a continuous function. We can always deform it into a smooth function, which is still denoted by g for simplicity. Identify B with $S^1 \times S^1$ so that (q, r) becomes $(S^1 \times \{1\}, \{1\} \times S^1)$, then the fibrations $P_{g,n}\colon S^1 \times \widetilde{M}^3 \to B$ and $P\colon X \to B$ have the same monodromy and therefore $X = S^1 \times M^3$ is diffeomorphic to $S^1 \times \widetilde{M}^3$ by lemma 3.2.

Now we consider the case when $\text{genus}(F) = 1$.

Claim *$Mon_P(v) = Mon_P(q)$ is trivial.*

Proof Suppose $Mon_P(v)$ is not trivial in $Map(F_{b_0})$. Then we must have $n \neq 1$. We identify $Map(F_{b_0})$ with $SL(2, \mathbb{Z})$ by the action of $Map(F_{b_0})$ on $H_1(F_{b_0}; \mathbb{Z})$, and denote by $A \in SL(2, \mathbb{Z})$ the element corresponding to $Mon_P(v)$. It has two complex eigenvalues, neither of which is equal to one. This implies, in particular, that $Id - A$ has non-zero determinant and $Q = H_1(F_{b_0}; \mathbb{Z})/Im(Id - A)$ is a finite group. Consider the 3–manifold $Z = P^{-1}(q)$. It is diffeomorphic to the mapping torus of $Mon_P(v)$. The natural embedding of Z into X induces a monomorphism of fundamental groups and u is in the image of this monomorphism; thus we can regard u as an element of $\pi_1(Z, x_0)$. Let us recall that $\pi_1(X, x_0)$ splits into a direct sum, $\pi_1(X, x_0) = \mathbb{Z}\langle u\rangle \oplus \pi_1(M^3)$. Since $\mathbb{Z}\langle u\rangle \subset \pi_1(Z, x_0)$, the fundamental group of Z also splits into a direct sum, $\pi_1(Z, x_0) = \mathbb{Z}\langle u\rangle \oplus [\pi_1(Z, x_0) \cap \pi_1(M^3)]$. In particular, the image $[u]$ of u in $H_1(Z; \mathbb{Z})$ has infinite order. Choose a loop $\delta \in \pi_1(Z, x_0)$ such that $P_*(\delta) = v$. Then u and δ are related by equation $\delta^n = u\gamma$ for some $\gamma \in \pi_1(F_{b_0}, x_0)$. Denote by $[\delta]$ the image of δ in $H_1(Z; \mathbb{Z})$. Calculating homology of Z we have

$$H_1(Z; \mathbb{Z}) = \mathbb{Z}\langle[\delta]\rangle \oplus Q \quad \text{and} \quad [u] = n[\delta] - [\gamma]$$

where $Q = H_1(F_{b_0}; \mathbb{Z})/Im(Id - A)$, and $[\gamma] \in Q$ is the image of γ. On the other hand, we have

$$H_1(Z; \mathbb{Z}) = \mathbb{Z}\langle[u]\rangle \oplus Q' \quad \text{and} \quad [\delta] = m[u] + [\gamma']$$

for some integer $m \neq 0$ and $[\gamma'] \in Q'$, where Q' is the abelianization of $\pi_1(Z, x_0) \cap \pi_1(M^3)$. Note that Q' is a finite group because the rank of $H_1(Z;\mathbb{Z})$ is 1. Putting the two equations together, we have

$$(nm-1)[u] = [\gamma] - n[\gamma'],\ n > 1.$$

This is a contradiction, since the left-hand-side has infinite order in $H_1(Z;\mathbb{Z})$, but the right-hand-side has finite order. □

Thus we have established that $Mon_P(v)$ is trivial and, in fact, the manifold Z is diffeomorphic to $S^1 \times F_{b_0}$. It will be convenient to fix a product structure $S^1 \times F_{b_0}$ on Z and a flat metric on F_{b_0}. These give rise to a flat metric on $Z = S^1 \times F_{b_0}$.

Let us recall that $Diff_0(F_{b_0})$ is homotopy equivalent to the set of conformal isometries of F_{b_0}; hence every element in $\pi_1(Diff_0(F_{b_0}), Id)$ could be represented by a linear family of parallel transforms of F_{b_0}.

Let $\psi\colon F_{b_0} \to F_{b_0}$ be a representation of the monodromy of P along r, where r is an oriented simple closed loop in B such that $q \cap r = b_0$ and $\langle q, r\rangle = 1$. We may assume that ψ is linear with respect to the chosen flat metric on F_{b_0}.

Since X fibers over a circle with fiber Z, it is diffeomorphic to the mapping torus of some self-diffeomorphism $\Psi\colon Z \to Z$, which could be chosen so that it preserves each fiber $\{pt\} \times F_{b_0}$. Such a Ψ is the composition of $Id \times \psi$ with a "Dehn twist" of Z along F_{b_0}, which is defined as follows: Choose an element in $\pi_1(Diff_0(F_{b_0}), Id)$ and represent it by a loop $\alpha\colon S^1 \to Diff_0(F_{b_0})$. Define a Dehn twist $t_{F_{b_0},\alpha}\colon Z = S^1 \times F_{b_0} \to Z$ by $t_{F_{b_0},\alpha}(s,x) = (s, \alpha(s)(x))$. If we choose α to be a linear loop in the space of conformal automorphisms of F_{b_0}, then Ψ will be a linear self-diffeomorphism of Z.

Recall that $X = S^1 \times M^3$ and denote by $pr_1\colon X \to S^1$ the projection onto the first factor. Let $\lambda\colon Z \to S^1$ be a linear map representing the homotopy class of $pr_1|_Z$, which defines a trivial fibration on Z with fiber $\Sigma \cong T^2$. Then λ and $\lambda \circ \Psi$ are homotopic, because pr_1 is defined on the whole X, which is the mapping torus of Ψ. This implies that $\lambda = \lambda \circ \Psi$ since both are linear maps.

Thus Ψ preserves the fibration structure of Z by λ, so that λ extends to a fibration $\Lambda\colon X \to S^1$. Let $\varphi\colon \Sigma \to \Sigma$ be the self-homeomorphism induced by Ψ, and denote the mapping torus of φ by $\widetilde{M}^3$ and the corresponding fibration by $p\colon \widetilde{M}^3 \to S^1$. We claim that Λ is a trivial fibration over S^1 with fiber $\widetilde{M}^3$, therefore $X \cong S^1 \times \widetilde{M}^3$. This can be seen as follows: The monodromy Ψ of Λ is a composition of $Id \times \varphi$ with a Dehn twist of Z along Σ since Ψ preserves the trivial fibration on Z by λ. The Dehn twist must be trivial since otherwise we would have $\Psi_*(u) \neq u$, which contradicts the fact that u is central in $\pi_1(X, x_0)$. Therefore Λ is trivial and $X = S^1 \times \widetilde{M}^3$, where $\widetilde{M}^3$ is fibered with fibration $p\colon \widetilde{M}^3 \to S^1$ and fiber $\Sigma \cong T^2$. Note that the product structure $h\colon X \to S^1 \times \widetilde{M}^3$ could be chosen to be homotopic to the given product structure $X = S^1 \times M^3$ in the sense that h is homotopic to a product of homotopy equivalences from S^1 to S^1 and from M^3 to $\widetilde{M}^3$. Specifically, this could be done as follows: Choose a product structure $S^1 \times \Sigma$ on Z by choosing a projection from Z to Σ so that the induced map on fundamental groups sends u to zero. Since both u and Σ are preserved by Ψ, the product structure on Z extends to a product structure on X, which has the required property.

Now we will find $g\colon \widetilde{M}^3 \to S^1$ and n such that the fibration $P_{g,n}$ is equivalent to the original fibration P on X. Let $pr\colon F_{b_0} \to \Sigma$ be the restriction to F_{b_0} of the projection from Z to Σ. Recall that both F_{b_0} and Σ are linear subspaces in Z. It follows that pr is a cyclic covering with deck transformations being parallel transforms on F_{b_0}. Set n to be equal to the degree of pr. Now let $\tilde{g} = (\lambda|_{F_{b_0}})^n$, where the power is taken point-wise in S^1. The function $\tilde{g}$ descends to a function $\tilde{\tilde{g}}\colon \Sigma \to S^1$, which is linear and φ–invariant, therefore extends to a map $g\colon \widetilde{M}^3 \to S^1$. It is left as an exercise for the reader to show that $P_{g,n}$ is equivalent to $P\colon X \to B$.

It remains to show that in both cases $\widetilde{M}^3$ is homeomorphic to the Poincaré associate $\mathcal{P}(M^3)$ of M^3. This follows from the fact that the diffeomorphism between $S^1 \times \widetilde{M}^3$ and $X = S^1 \times M^3$ induces a homotopy equivalence $\widetilde{M}^3 \to M^3$, which, by a theorem of Stallings [11], implies $\widetilde{M}^3 \cong \mathcal{P}(M^3)$.

When $\mathrm{genus}(F) = 0$, X is diffeomorphic to $F \times B$ (since X is spin) with $P = pr_2$ the projection onto the second factor. The Poincaré associate $\mathcal{P}(M^3)$ is homeomorphic to $S^1 \times S^2$.

Case 2 The image $P_*(u) = 1$ in $\pi_1(B, b_0)$.

Thus u lies in $\pi_1(F_{b_0}, x_0)$ and generates a direct summand in $\pi_1(F_{b_0}, x_0)$. Hence the fiber F must be a torus, and u is primitive in $\pi_1(F_{b_0}, x_0)$.

Identify F with $S^1 \times S^1$ such that the loop $q = S^1 \times \{pt\}$ represents the class u in $\pi_1(F_{b_0}, x_0)$ and the loop $r = \{pt\} \times S^1$ represents a class in $\pi_1(M^3)$, which we denote by $[r]$. Then we have a reduced exact sequence:

$$1 \to \mathbb{Z}\langle [r] \rangle \hookrightarrow \pi_1(M^3) \xrightarrow{P_*|_{\pi_1(M^3)}} \pi_1(B, b_0) \to 1$$

Let $f\colon \mathcal{P}(M^3) \to M^3$ be a homotopy equivalence between the Poincaré associate $\mathcal{P}(M^3)$ and M^3. It is easily seen that there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \stackrel{j_*}{\to} & \pi_1(\mathcal{P}(M^3)) & \stackrel{\pi_*}{\longrightarrow} & \pi_1(B,b_0) & \to & 1 \\ & & \| & & \downarrow f_* & & \| & & \\ 1 & \to & \mathbb{Z}\langle [r]\rangle & \hookrightarrow & \pi_1(M^3) & \stackrel{P_*|_{\pi_1(M^3)}}{\longrightarrow} & \pi_1(B,b_0) & \to & 1. \end{array}$$

By Theorem 11.10 in [7], the Poincaré associate $\mathcal{P}(M^3)$ is an S^1–fibration over the Riemann surface B, $S^1 \stackrel{j}{\hookrightarrow} \mathcal{P}(M^3) \stackrel{\pi}{\to} B$, from which the exact sequence

$$1 \to \mathbb{Z} \stackrel{j_*}{\to} \pi_1(\mathcal{P}(M^3)) \stackrel{\pi_*}{\to} \pi_1(B,b_0) \to 1$$

is induced. We claim that $\mathcal{P}(M^3)$ must be the trivial fibration $S^1 \times B$. Suppose it is not a trivial fibration, then the homology class of the fiber $j_*[S^1]$ is torsion in $H_1(\mathcal{P}(M^3);\mathbb{Z})$, so is the fiber of the associated T^2–fibration of $S^1 \times \mathcal{P}(M^3)$ obtained by taking the product with S^1, $S^1 \times S^1 \stackrel{Id\times j}{\to} S^1 \times \mathcal{P}(M^3) \stackrel{\pi}{\to} B$. But $((Id \times f)\circ(Id \times j))_*[S^1 \times S^1]$ is homologous to the fiber $[F]$ in $H_2(X;\mathbb{Z})$, which is not torsion since $F \hookrightarrow X \stackrel{P}{\to} B$ is a symplectic Lefschetz fibration. Hence $\mathcal{P}(M^3)$ is the trivial fibration $S^1 \times B$.

We then have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}\oplus\mathbb{Z} & \stackrel{(Id\times j)_*}{\to} & \mathbb{Z}\oplus\mathbb{Z}\oplus\pi_1(B,b_0) & \stackrel{\pi_*}{\longrightarrow} & \pi_1(B,b_0) & \to & 1 \\ & & \| & & \downarrow (Id\times f)_* & & \| & & \\ 1 & \to & \pi_1(F_{b_0},x_0) & \hookrightarrow & \pi_1(X,x_0) & \stackrel{P_*}{\longrightarrow} & \pi_1(B,b_0) & \to & 1, \end{array}$$

from which it follows that P has trivial monodromy. Let B_0 be a surface with boundary obtained by removing from B a small disc D_0 disjoint from the base point $b_0 \in \partial B_0$. The restrictions $P|_{B_0}$ and $P|_{D_0}$ are trivial fibrations, and P is determined by the homotopy class $[\Theta]$ of a gluing map $\Theta\colon S^1 \times F_{b_0} \to S^1 \times F_{b_0}$ viewed as an element in $Map(S^1, Diff_0(F_{b_0}))$. By proposition 3.2, we have $\pi_1(Diff_0(F_{b_0}), Id) = \mathbb{Z}\oplus\mathbb{Z}$. So P is equivalent to a fibration which is the product of an S^1–fibration over B, say ξ, with S^1, where the first Chern number of ξ is the divisibility of $[\Theta]$ in $\pi_1(Diff_0(F_{b_0}), Id) = \mathbb{Z}\oplus\mathbb{Z}$. On the other hand, $[F] \neq 0$ in $H_2(X;\mathbb{R})$, so we must have $c_1(\xi) = 0$ so that $\Theta \sim Id$. Thus we have proved that X is diffeomorphic to $F \times B$ with $P = pr_2$ the projection onto the second factor.

We finish the proof of the theorem by showing the uniqueness of the symplectic structure: The existence of diffeomorphism $h\colon S^1 \times \mathcal{P}(M^3) \to X$ is clear from the above classification of the Lefschetz fibration P. Moreover, the canonical symplectic form on $S^1 \times \mathcal{P}(M^3)$ is positive on each fiber of the pull-back fibration $P \circ h$, hence by lemma 2.1, it is deformation equivalent to $h^*\omega$. $\square$

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