ISSN 1364-0380

Geometry & Topology
Volume 4 (2000) 85–116
Published: 28 January 2000

Combing Euclidean buildings

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Abstract

For an arbitrary Euclidean building we define a certain combing, which satisfies the "fellow traveller property" and admits a recursive definition. Using this combing we prove that any group acting freely, cocompactly and by order preserving automorphisms on a Euclidean building of one of the types A_n, B_n, C_n admits a biautomatic structure.

AMS Classification numbers Primary: 20F32

Secondary: 20F10

Keywords: Euclidean building, automatic group, combing

Proposed: Walter Neumann — Received: 9 February 1999
Seconded: Joan Birman, Wolfgang Metzler — Revised: 10 November 1999

1 Introduction

Let G be a group that acts properly and cocompactly on a piecewise Euclidean simply connected $CAT(0)$–complex Δ (see eg [4] for definitions). (The action of course is supposed to be cellular, properness means that the isotropy group G_σ is finite for every cell σ and cocompactness means that Δ has only finitely many cells mod G.) It is still unknown whether G is (bi)automatic. Moreover, the question remains unanswered even in the case when Δ is a Euclidean building [6].

"It is reasonable to guess that the answer is 'yes' because of the work of Gersten and Short and because of the geometry and regularity present in buildings, but this is far from a trivial question" (John Meier's review [MR 96k:20071] of the paper [6]).

The first results in this direction are contained in the papers of S Gersten and H Short [8], [9] where it is proven that if G is given by a finite presentation satisfying the small cancellation conditions $C(p), T(q)$ $((p,q) = (6,3), (4,4), (3,6))$ then G is biautomatic. They showed in [8] that the fundamental group of a piecewise Euclidean 2–complex of nonpositive curvature of type $A_1 \times A_1$ or A_2 is automatic. ($A_1 \times A_1$ corresponds to the Euclidean planar tessellation by unit squares, and A_2 to the tessellation by equilateral triangles). In the subsequent paper [9] the authors prove an analogous result for 2–complexes of types B_2 and G_2 corresponding to the Euclidean tessellations by $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ and $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangles, respectively. It follows from this work that any torsion free group G which admits a proper cocompact action on a Euclidean building of type A_2 is biautomatic. W Ballmann and M Brin [1] have proven the automatic property for a group G which acts simply transitively on the vertices of a simply connected (3,6)–complex. D Cartwright and M Shapiro have proven the following theorem [6]: Let G act simply transitively on the vertices of a Euclidean A_n–building in a type rotating way. Then G admits a geodesic, symmetric automatic structure. In [13] some variation of this result is proven in a more geometric way in the case of $n = 2$. It is worth mentioning that in the case of nonpositively curved cube complexes the general result was obtained by G Niblo and L Reeves [12], namely, any group acting properly and cocompactly on such a complex is biautomatic.

In this paper we define a certain combing on an arbitrary Euclidean building, prove the "fellow traveller property" for this combing and the "recursiveness property". Our main result is the following.

Theorem

(1) *Let Δ be any Euclidean building of one of the types A_n, B_n, C_n, ordered in a standard way (see Section 3.10 for a definition). Then any group acting freely and cocompactly on Δ by order-preserving automorphisms admits a biautomatic structure.*

(2) *If Δ is any Euclidean building of one of the types A_n, B_n, C_n, then any group acting freely and cocompactly on Δ is virtually biautomatic (that is there is a finite index subgroup in it, possessing a biautomatic structure).*

In Section 2 we review some of the standard facts on Euclidean Coxeter complexes. In Section 3 we introduce the main notion of an ordering of a Euclidean building and prove that any Euclidean building can be ordered. In Section 4 we define a natural combing $\mathcal{C}$ on a Euclidean building . In Sections 5 and 6 we prove the "fellow traveller property" and the "recursiveness property" for a combing $\mathcal{C}$. The concluding Section 7 is devoted to the proof our main result.

Acknowledgements I am grateful to the SFB 343, University of Bielefeld for their hospitality in the falls of 1996–97 years while I carried out most of this work. I would like to thank Herbert Abels for his kind invitation, interest and support. Thanks to Sarah Rees for the help to make the language of the paper more regular. Thanks to referee for many improvements of the text. The work was supported in part by a RFFI grant N 96-01-01610.

2 Euclidean Coxeter complexes

For the convenience of the reader we recall the relevant material from [2], [10], thus making our exposition self-contained.

2.1 Roots and Weyl group

Let Φ to be a *root system*, which is supposed to be reduced, irreducible and crystallographic. That is Φ is a finite set of nonzero vectors, spanning a finite dimensional Euclidean space V and such that

1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

2) Φ is invariant under reflection s_α in the hyperplane H_α orthogonal to α for all $\alpha \in \Phi$,

3) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbf{Z}$ for all $\alpha, \beta \in \Phi$,

4) V does not admit an orthogonal decomposition $V = V' \oplus V''$ such that $\Phi = \Phi' \cup \Phi''$ with $\Phi \subset V', \Phi'' \subset V''$.

The *Weyl group* W of Φ is the group generated by all reflections s_α $(\alpha \in \Phi)$. In equal terms W is generated by all reflections s_H, where H ranges over the set $\mathcal{H}$ of all hyperplanes, orthogonal to the roots from Φ.

For any choice of the basis $e_1, \dots, e_n$ of V there is a lexicographic order on V, where $\sum a_i e_i < \sum b_i e_i$ means that $a_k < b_k$ if k is the least index i for which $a_i \neq b_i$. We call a subset $\Phi_+ \subset \Phi$ a *positive system* if it consists of all those roots which are positive relative to some ordering of V of the kind above.

If this is the case, then Φ must be the disjoint union of Φ_+ and $-\Phi_+$, the latter being called a *negative system.* When Φ_+ is fixed, we can write $\alpha > 0$ in place of $\alpha \in \Phi_+$. It is clear that positive systems exist.

Call a subset Π of Φ a *simple system* if Π is a vector space basis for V and if moreover each $\alpha \in \Phi$ is a linear combination of Π with coefficients all of the same sign (all nonnegative or all nonpositive). If Π is a simple system in Φ, then there is a unique positive system containing Π. Every positive system Π in Φ contains a unique simple system; in particular, simple systems exist.

Any two positive (resp. simple) systems in Φ are conjugate under W. Thus W permutes the the various positive (or simple) systems in a transitive fashion. This permutation action is indeed a simply transitive action, that is if $w \in W$ leaves the positive (or simple) system invariant, then $w = 1$.

2.2 Coroots, lattices

Setting $\alpha^\vee := 2\alpha/(\alpha,\alpha)$, the set $\Phi^\vee$ of all *coroots* $\alpha^\vee(\alpha \in \Phi)$ is also a root system in V, with simple system $\Pi^\vee := \{\alpha^\vee | \alpha \in \Pi\}$. The Weyl group of $\Phi^\vee$ is W, with $w\alpha^\vee = (w\alpha)^\vee$. The $\mathbf{Z}$–span $\mathbf{Z}\Phi$ of Φ in V is called the *root lattice*; it is a lattice in V. Similarly, we define the *coroot lattice* $\mathbf{Z}\Phi^\vee$. Define the *coweight lattice* $\mathbf{Z}\Phi^\#$ – it is just a dual lattice of the root lattice $\mathbf{Z}\Phi$, that is

$$\mathbf{Z}\Phi^\# = \{\lambda \in V | (\lambda, \alpha) \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}.$$

Since $(\Phi^\vee, \Phi) \subset \mathbf{Z}$ and both $\mathbf{Z}\Phi^\vee, \mathbf{Z}\Phi$ are the lattices, one can conclude that $\mathbf{Z}\Phi^\#$ contains $\mathbf{Z}\Phi^\vee$ as a subgroup of finite index.

2.3 Fundamental domain and spherical Coxeter complex

Let W be the Weyl group of a root system Φ. The hyperplanes H with $s_H \in W$ cut V into polyhedral pieces, which turn out to be cones over simplices. One obtains in this way a simplicial complex $\Sigma_{sph} = \Sigma_{sph}(W)$ which triangulates the unit sphere in V. This is a *spherical Coxeter complex.* More exactly let Φ_+ be a positive system, containing the simple system Π. Associated with each hyperplane H_α are the closed half-spaces H_α^+ and H_α^-, where $H_\alpha^+ = \{\lambda \in V | (\lambda, \alpha) \geq 0\}$ and $H_\alpha^- = \{\lambda \in V | (\lambda, \alpha) \leq 0\}$. Define a *sector* $S = S_\Pi := \cap_{\alpha \in \Pi} H_\alpha^+$ *associated to* Π. As an intersection of closed convex subsets, S is itself closed and convex. It is also a cone (closed under nonnegative scalar multiples). Sectors associated to W are always *simplicial cones*, by which we mean that, for some basis $e_1, \dots, e_n$ of V the sector S consists of the linear combinations $\sum a_i e_i$ with all a_i positive. (In other words, S is a cone over the closed simplex with vertices $e_1, \dots, e_n$). We call $\mathbf{R}_{\geq 0} e_i$ the *defining rays* of S. One can describe the defining rays of the sector S_Π more explicitly in terms of the basis of coroot lattice. Namely, let $\{\bar{\omega}_i^\vee\}$ be the dual basis of $\Pi = \{\alpha_1, \dots, \alpha_n\}$, that is $(\bar{\omega}_i^\vee, \alpha_j) = \delta_{ij}$ for all $i, j = 1, \dots, n$. Then

$$\sum_{1 \leq i \leq n} a_i \bar{\omega}_i^\vee \in S \Longleftrightarrow a_j = (\sum_{1 \leq i \leq n} a_i \bar{\omega}_i^\vee, \alpha_j) \geq 0, \ j = 1, \dots, n$$

We assert that the rays $\mathbf{R}_{\geq 0} \bar{\omega}_i^\vee$ are the defining rays for S_Π. Indeed, each line $\mathbf{R}\ \bar{\omega}_i^\vee$ is precisely the line obtained by intersecting all but one H_{α_i}, namely $\mathbf{R}\ \bar{\omega}_i^\vee = \cap_{j \neq i} H_{\alpha_j}$. Consequently one of the halflines of this line is a defining ray and calculating the scalar products we conclude that this is exactly $\mathbf{R}_{\geq 0} \bar{\omega}_i^\vee$.

W acts *simply transitively* on simple systems and this translates into a simply transitive action on the the sectors. This means that any two sectors are conjugate under the action of W and if $wS = S$ then $w = 1$. Moreover any sector S is a fundamental domain of the action of W on V, ie, each $\lambda \in V$ is conjugated under W to one and only one point in S. The sectors are characterized topologically as the closure of the connected components of the complement in V of $\cup H_\alpha$. They are in one one correspondence with the top-dimensional simplices (= *chambers*) of the corresponding spherical complex. Given a sector S_Π corresponding to a simple system Π, its *walls* are defined to be the hyperplanes H_α $(\alpha \in \Pi)$.

2.4 Euclidean reflections and Euclidean Weyl group

Let Φ be the root system in V as it was defined in Section 2.1. For each root α and each integer k, define a Euclidean hyperplane $H_{\alpha,k} := \{\lambda \in V | (\lambda, \alpha) = k\}$. Note that $H_{\alpha,k} = H_{-\alpha,-k}$ and that $H_{\alpha,0}$ coincides with the reflecting hyperplane H_α. Note too that $H_{\alpha,k}$ can be obtained by translating H_α by $\frac{k}{2}\alpha^\vee$. Define the corresponding Euclidean reflection as follows: $s_{\alpha,k}(\lambda) := \lambda - ((\lambda, \alpha) - k)\alpha^\vee$. We can also write $s_{\alpha,k}$ as $t(k\alpha^\vee) s_\alpha$, where $t(\lambda)$ denotes the translation by a vector λ. In particular, $s_{\alpha,0} = s_\alpha$. Denote by $\mathcal{H}_\mathbf{Z}$ the collection of all hyperplanes $H_{\alpha,k}$ $(\alpha \in \Phi, k \in \mathbf{Z})$ which we shall call the *walls*. The elements of $\mathcal{H}_{\mathbf{Z}}$ are permuted in a natural way by W_a as well as by translations $t(\lambda)$, where $\lambda \in V$ satisfies $(\lambda, \mu) \in \mathbf{Z}$ for all roots α (that is $\lambda \in \mathbf{Z}\Phi^{\#}$). In particular, $\mathbf{Z}\Phi^{\#}$ permutes the hyperplanes in $\mathcal{H}_{\mathbf{Z}}$, hence so does its subgroup $\mathbf{Z}\Phi^{\vee}$. Define the *affine Weyl group* W_a to be the subgroup of *Aff*(V) generated by all affine reflections $s_{\alpha,k}$ where $a \in \Phi, k \in \mathbf{Z}$. Another description of W_a is that it is the semidirect product $W_a = \mathbf{Z}\Phi^{\vee} \rtimes W$ of the finite Weyl group W and the translation group corresponding to the coroot lattice $\mathbf{Z}\Phi^{\vee}$, see [10], Section 4.2.

Since the translation group corresponding to $\mathbf{Z}\Phi^{\#}$ is also normalized by W, we can form the semidirect product $\widehat{W_a} = \mathbf{Z}\Phi^{\#} \rtimes W$, which contains W_a as a normal subgroup of finite index. Indeed, $\widehat{W_a}/W_a$ is isomorphic to $\mathbf{Z}\Phi^{\#}/\mathbf{Z}\Phi^{\vee}$. One can easily see from 1), 2) that $\widehat{W_a}$ also permutes the hyperplanes in $\mathcal{H}_{\mathbf{Z}}$. We call this group the *extended affine Weyl group.*

2.5 Euclidean Coxeter complexes

The hyperplanes $H \in \mathcal{H}_{\mathbf{Z}}$ triangulate the space V and the resulting piecewise Euclidean complex $\Sigma = \Sigma_{\Phi}$ is a *Euclidean Coxeter complex.* More generally we shall apply the same term to the Euclidean simplicial structure Σ on a Euclidean space V' such that that for some root system Φ in a Euclidean space V there is an affine isometry $\phi : V \to V'$ which induces simplicial isomorphism between Σ_{Φ} and Σ. In particular in Σ we have all the notions as in Σ_{Φ}. The extended Weyl group $\widehat{W_a}$ acts by simplicial isometries on Σ_{Φ} and this translates by ϕ to the action on Σ but not in a canonical way – if $\phi : V \to V'$ is another isometry then the actions are conjugate by a suitable isometry of V. The possible ambiguity is resolved by the following lemma.

2.6 Lemma *Both $\widehat{W_a}$ and W_a are invariant under the conjugation by any isometry ϕ of V, which preserves the simplicial structure Σ_{Φ}.*

In particular the images of $\widehat{W_a}$ and W_a in $\mathrm{Aut}(\Sigma)$ are canonically defined and we call them the *extended affine Weyl group of* Σ and by the *affine Weyl group of* Σ respectively.

Proof Since ϕ leaves invariant the family of hyperplanes $\mathcal{H}_{\mathbf{Z}}$, it also leaves invariant the family of reflections in the hyperplanes of this family, hence normalizes the affine Weyl group W_a. Next it leaves invariant the set of special vertices (see definition in Section 2.8 and lemma 2.9) hence normalizes the translation group $\mathbf{Z}\Phi^{\#}$. Since $\widehat{W_a}$ is generated by W_a and $\mathbf{Z}\Phi^{\#}$ it is also normalized by ϕ. $\square$

The collection $\mathcal{A}$ of top-dimensional closed simplices consists of the closures of the connected components of $V^\circ := V \setminus \cup_{H \in \mathcal{H}_{\mathbf{Z}}} H$. Each element of $\mathcal{A}$ is called an *alcove*. The group W_a acts simply transitively on $\mathcal{A}$, [10], Chapter 4, Theorem 4.5. Any alcove A is a fundamental domain of the action of W_a on V, ie, each $\lambda \in V$ is conjugated under W to one and only one point in A. In particular $V = \cup\{S_\Pi : \Pi \text{ is a simple system}\}$. Since $\widehat{W_a}$ permutes the hyperplanes in $\mathcal{H}_{\mathbf{Z}}$, it acts simplicially on Σ.

2.7 Standard alcove

There is an alcove with a particularly nice description (see [2], Corollary of Proposition 4 in Section 2, Chapter VI or [10] Section 4.9). Namely let $\Pi = \{\alpha_i\}$ be a simple root system for Φ. Let $\{\bar{\omega}_i^\vee\}$ be the dual basis for $\{\alpha_i\}$ – this is the basis of the coweight lattice $\mathbf{Z}\Phi^\#$. Let $\tilde{\alpha} = \sum_{1 \le i \le n} c_i \alpha_i$ be the corresponding highest root. Then the alcove $A = A_\Pi$, associated to Π is a closed simplex with the vertices 0 and $\frac{1}{c_i}\bar{\omega}_i^\vee, i = 1, \dots, n$. We call this alcove *a standard alcove associated to* Π.

Another description of $A = A_\Pi$ is given by the formula $A = \cap_{\alpha \in \Pi} H_\alpha^+ \cap H_{\tilde{\alpha},1}^-$, where $H_{\tilde{\alpha},1}^-$ is a closed negative half-space defined by the hyperplane $H_{\tilde{\alpha},1}$ and $\tilde{\alpha}$ is the highest root. Comparing this description with the definition of a sector S_Π given in Section 2.3 we found that A_Π sits on the top of the sector S_Π and the defining rays of S_Π correspond to the ordered edges of Σ having 0 as its origin. There is a one one correspondence $A_\Pi \leftrightarrow S_\Pi$ between the set $\mathcal{A}_0$ of alcoves having 0 as a vertex and the set of sectors of a spherical Coxeter complex Σ_{sph}. W acts on $\mathcal{A}_0$ and the fact that any S_Π is a fundamental domain for the action of W on Σ_{sph} translates to the fact that A_Π is a fundamental domain for the action of W on $\mathcal{A}_0$ in a sense that any directed edge of A_Π having 0 as its origin is W–conjugate to one and only one such a vertex of A_Π.

2.8 Special vertices

The vertex $x \in \Sigma_\Phi$ is called a *special vertex* if its stabilizer $S_{W_a}(x)$ in W_a maps isomorphically onto the associated finite Weyl group W. (Note that the stabilizers of any vertex in W_a and in $\widehat{W_a}$ coincide). Equivalently, for any hyperplane $H \in \mathcal{H}_{\mathbf{Z}}$ there is a parallel hyperplane in $\mathcal{H}_{\mathbf{Z}}$, passing through x. Yet another equivalent definition is that the maximal possible number of hyperplanes from $\mathcal{H}_{\mathbf{Z}}$ pass through x.

2.9 Lemma *The set of special vertices of the complex Σ coincide with the lattice $\mathbf{Z}\Phi^\#$ (see [2], Proposition 3 in Section 2, Chapter VI).*

Proof Since the zero vertex is special and the coweight lattice $\mathbf{Z}\Phi^{\#}$ acts simplicially on Σ, we conclude that $\mathbf{Z}\Phi^{\#}$ consists of the special vertices. Conversely, let x be a special vertex. Since W_a preserves the property of the vertex being special and since it acts transitively on the set of alcoves, we may assume that x is the vertex of the standard alcove

$$A =< 0, \frac{1}{c_1}\bar{\omega}_1^{\vee}, \ldots, \frac{1}{c_n}\bar{\omega}_n^{\vee} >$$

described above. If $x = 0$, then obviously $x \in \mathbf{Z}\Phi^{\#}$. If $x = \frac{\bar{\omega}_i^{\vee}}{c_i}$ and $c_i = 1$ then again $x \in \mathbf{Z}\Phi^{\#}$. Finally, if $x = \frac{1}{c_i}\bar{\omega}_i^{\vee}$ and $c_i > 1$ then x can't be special. Indeed $(\frac{1}{c_i}\bar{\omega}_i^{\vee}, \alpha_i) = 1/c_i < 1$, thus no member of the family of hyperplanes in $\mathcal{H}_{\mathbf{Z}}$ parallel to H_i pass through x. $\square$

2.10 Lemma *All the vertices of the complex Σ are special if and only if Σ is of type A_n.*

Proof Since W_a preserves the property of the vertex being special and since it acts transitively on the set of alcoves, all the the vertices of Σ are special if and only if all the vertices of the standard alcove $A =< 0, \frac{1}{c_1}\bar{\omega}_1^{\vee}, \ldots, \frac{1}{c_n}\bar{\omega}_n^{\vee} >$ are special. As we have already seen in the proof of the preceding lemma, the non-special points of this alcove are in one one correspondence with the numbers $c_1, \ldots, c_n$, that are strictly greater than 1. Thus all the vertexes are special if and only if all the numbers c_i in the expression $\tilde{\alpha} = \sum_{1 \leq i \leq n} c_i \alpha_i$ are equal to 1. Now inspecting the tables of the root systems in [2], we conclude that this happens only in the case of the root system of type A_n. $\square$

2.11 More subcomplexes

Note that an intersection of any family of hyperplanes from $\mathcal{H}_{\mathbf{Z}}$ or corresponding halfspaces is a subcomplex of a Euclidean Coxeter complex. In particular the line $\mathbf{R}\,\bar{\omega}_i^{\vee} = \cap_{j \neq i} H_{\alpha_j}$ is a subcomplex. Note that for any $m \in \mathbf{Z}, i = 1, \ldots, n$ the point $m\bar{\omega}_i^{\vee}/c_i$ is the vertex of Σ. Indeed $(m\bar{\omega}_i^{\vee}/c_i, \tilde{\alpha}) = m$ implies that $m\bar{\omega}_i^{\vee}/c_i \in H_{\tilde{\alpha},m}$ and $(m\bar{\omega}_i^{\vee}/c_i, \alpha_j) = 0$, $j \neq i$ implies that $m\bar{\omega}_i^{\vee}/c_i \in H_{\alpha_j,0}$, hence $m\bar{\omega}_i^{\vee}/c_i$ is an intersection of n hyperplanes $H_{\tilde{\alpha},m}$, $H_{\alpha_j,0}$, $j \neq i$. In particular the line segments $[0, \bar{\omega}_i^{\vee}] \subset \mathbf{R}\,\bar{\omega}_i^{\vee}$ are the subcomplexes of Σ.

Next, the sectors $S = S_{\Pi} := \cap_{\alpha \in \Pi} H_{\alpha}^{+}$, $-S = -S_{\Pi} := \cap_{\alpha \in \Pi} H_{\alpha}^{-}$ are subcomplexes as well as any of their faces (which are the cones) $F = F_{\Pi} := (\cap_{\alpha \in \Pi' \subseteq \Pi} H_{\alpha}) \cap (\cap_{\alpha \in \Pi \setminus \Pi'} H_{\alpha}^{+})$.

3 Ordering Euclidean buildings

3.1 Definitions We will consider *special edges* of a Euclidean Coxeter complex $\Sigma = \Sigma_\Phi$ of dimension n, that is the directed edges $e \in \Sigma^{(1)}$ such that the origin ιe of e is a special vertex. Let E_s be the set of all such edges. The typical examples of such edges are given by the standard alcove $A =< 0, \frac{1}{c_1}\bar{\omega}_1^\vee, \dots, \frac{1}{c_n}\bar{\omega}_n^\vee >$ constructed in Section 2.10. All the directed edges

$$[0, \frac{1}{c_1}\bar{\omega}_1^\vee], \dots, [0, \frac{1}{c_n}\bar{\omega}_n^\vee]$$

are special. In some sense any special edge e arrives in this way – indeed, let $\iota e = \alpha \in \mathbf{Z}\Phi^\#$, then $e - \alpha$ starts at 0 and there is some simple system Π such that $e - \alpha$ is an edge of the alcove A_Π, starting at 0.

More generally call a directed edge e *quasi-special* if it lies on a line segment $[x, y]$ in $\Sigma^{(1)}$ with special vertices x, y. An example will be any directed edge lying on the line segment $[0, \bar{\omega}_i^\vee]$ since $0, \bar{\omega}_i^\vee$ are special, see Section 2.9. This remark implies that any special edge is quasi-special. Note that $\widehat{W_a}$ leaves E_s invariant as well as the set E_{qs} of all quasi-special edges. (It might be that all the edges in any Coxeter complex, and hence in any Euclidean building, are quasi-special, but the proof of this is not in the author's possession.)

Since the set of all special vertices on the line L of $\Sigma^{(1)}$ is discrete in Euclidean topology, we conclude that for any quasi-special edge e there is a unique minimal (with respect to inclusion) line segment $[x, y]$ in $\Sigma^{(1)}$ with special vertices x, y, which contains e.

By an *ordering* of Σ we mean a function $\tau : E_{qs} \mapsto \{1, \dots, n\}, n = \dim \Sigma$, such that

1) for any alcove $A =< x, x_1, \dots, x_n >$ with a special vertex x the function τ is bijective on the set of special edges $\{[x, x_1], \dots, [x, x_n]\}$,

2) τ is $\widehat{W_a}$–equivariant,

3) for any line segment in $[x, y] \subset \Sigma^{(1)}$ with special vertices x, y the ordering function τ is constant on a set of directed quasi-special edges lying on $[x, y]$ and oriented from x to y.

3.2 Remarks This resembles the notion of a *labelling* of a Euclidean Coxeter complex, which means that it is possible to partition the vertices into $n = \dim \Delta + 1$ "types", in such a way that each alcove has exactly one vertex of each type. The labellability of a Euclidean Coxeter complex Σ follows from the fact that the W_a–action partitions the vertices into n orbits, and we can label Σ by associating one label $i = 0, 1, \ldots, n$ to each orbit. In particular the labelling is W_a–invariant. There is one obvious distinction between these two notions – "ordering orders the directed edges" and "labelling labels the vertices". For us it is important that the ordering is invariant under translations in the apartments. In general there are translations on Σ which preserve the structure of a Coxeter complex but does not belong to W_a.

3.3 Theorem *Any Euclidean Coxeter complex $\Sigma = \Sigma_\Phi$ can be ordered. Moreover an ordering is uniquely defined by an ordering of a set of all directed edges of a fixed alcove starting at some fixed special vertex of alcove.*

Proof Consider the set of all pairs (x, A) of *based alcoves* that is alcoves A with a fixed special vertex x of it.

We wish to prove that the extended Euclidean Weyl group $\widehat{W_a} = \mathbf{Z}\Phi^{\#} \rtimes W$ acts simply transitively on the set of all based alcoves, that is for any pair of based alcoves $(x, A), (x', A')$ there is exactly one element $w \in \widehat{W_a}$ which takes x to x' and A to A'. The set of all special vertices coincides with the coweight lattice $\mathbf{Z}\Phi^{\#}$, (Section 2.9), consequently there is a translation from $\mathbf{Z}\Phi^{\#}$ which takes the special vertex x to the special vertex x', hence we may assume that $x = x'$. Since x is special $S_W(x) = W$ and the family of hyperplanes $\mathcal{H}_x$ passing through x define a spherical complex canonically isomorphic to Σ_{sph}. The alcoves based at x are in one–one correspondence with sectors of this spherical complex, thus by transitivity there is $w \in S_W(x) = W$ taking A to A'.

Now let the element $w \in \widehat{W_a} = \mathbf{Z}\Phi^{\#} \rtimes W$ fix (x, A). The translation $t_x : v \mapsto v + x$ belongs to $\widehat{W_a}$ by lemma 2.9 and $t_x^{-1} w t_x$ fixes $(0, t_x^{-1} A)$. In particular $w' = t_x^{-1} w t_x \in W$ and since W acts simply transitively on the set of chambers of Σ_{sph} the element w' is the identity, hence w is the identity.

Fix a based alcove (x, A) then we assert that any special edge is $\widehat{W_a}$–conjugate to one and only one such an edge of (x, A) having x as an origin. Let e be a special edge. Since $\mathbf{Z}\Phi^{\#}$ acts simply transitively on the set $\Sigma^{(0)}_{spec}$ one may assume that $x = 0$. For the same reason one may assume that 0 is the origin of e. Now the sector corresponding to A is a fundamental domain for the action of W on the corresponding spherical complex and we can conclude that W is W–conjugate to one and only one such edge of (x, A) having x as an origin. A priori this does not mean that it is $\widehat{W_a}$–conjugate to one and only one such edge of (x, A) having x as an origin. But the stabilizer of 0 in $\widehat{W_a}$ and that of in W_a is the same $(=W)$.

The properties just proven allow us to order all special edges in the following way. Order the edges of any fixed based alcove (x, A) starting at x arbitrarily and extend the ordering in a $\widehat{W_a}$–equivariant way.

What is left is to extend the ordering to all quasi-special edges. Any such edge e lies on a line segment $[x, y] \subset \Sigma^{(1)}$ with special x, y and there is unique minimal such segment (it is fully defined by the condition that there are only two special vertices on it, namely x and y.) Let $[x, y]$ be such a segment and let e be oriented from x to y. If e starts at x then it is a special edge and already has got its label. If not then we assign to e the same order, which has the special edge on $[x, y]$, beginning at x. This assigning is a canonical one thus we have a well defined function τ on the set of all quasi-special vertices. Let us verify conditions 1)–3) in the definition of an ordering (Section 3.1). Condition 1) concerns only special edges and the bijection desired is given by the construction. $\widehat{W_a}$–equivariance for special edges again is given by the construction and for quasi-special vertices it follows from the observation that the minimal $[x, y]$ attached to any such an edge is obviously $\widehat{W_a}$–equivariant. Finally 3) is given by the construction in the case of minimal $[x, y]$ and clearly any non minimal such a segment is the union of minimal one. The only jumping of the order can be in the internal special vertex, but now we can use the fact that the ordering of special edges is $\widehat{W_a}$–equivariant and in particular $\mathbf{Z}\Phi^{\#}$–equivariant. □

3.4 Corollary *The ordering of any Euclidean Coxeter complex corresponding to a root system is uniquely defined by an (arbitrary) choice of type function τ from the set $\{\bar{\omega}_i^\vee\}$ of all fundamental coweights corresponding to some simple system Π to the set $\{1, \ldots, n\}$.*

Proof Indeed, there is an alcove of the form

$$A = < 0, \frac{1}{c_1}\bar{\omega}_1^\vee, \ldots, \frac{1}{c_n}\bar{\omega}_n^\vee >,$$

where $c_1, \ldots, c_n$ are defined by the expression of the highest root

$$\tilde{\alpha} = \sum_{1 \le i \le n} c_i \alpha_i,$$

see Section 2.7. The directed edges $[0, \frac{\bar{\omega}_i^\vee}{c_i}]$ of this alcove (based in 0) are in one one correspondence with the fundamental coweights $\{\bar{\omega}_i^\vee\}$. □

3.5 Definitions We adopt the following direct definition of a Euclidean building, see [5]. We call a simplicial piecewise Euclidean complex Δ a *Euclidean building* if it can expressed as the union of a family of subcomplexes Σ, called *apartments* or *flats*, satisfying the following conditions:

B0) There is a Euclidean Coxeter complex Σ_0, such that for each apartment Σ there is a simplicial isometry between Σ and Σ_0.

B1) Any two simplices of Δ are contained in an apartment.

B2) Given two apartments Σ, Σ' with a common top-dimensional simplex (=chamber or alcove), there is an isomorphism $\Sigma \overset{\approx}{\to} \Sigma'$ fixing $\Sigma \cap \Sigma'$ point wise.

Note that isomorphisms in B2) are uniquely defined. A Euclidean building has a canonical metric, consistent with the Euclidean structure on the apartments. So each apartment E of Δ is a Euclidean space with a metric $|x-y|_E, x, y \in E$. Moreover, the isomorphisms $\Sigma_0 \overset{\approx}{\to} \Sigma$ and isomorphisms between apartments given by the building axiom B2) are isometries. The metrics $|x-y|_E$ can be pieced together to make the entire building Δ a metric space. The resulting metric will be denoted by $x, y \mapsto |x-y|$. It is known (see [5], Theorem VI.3) that the metric space Δ is complete. Besides for any $x, y \in \Delta$ the line segment $[x, y]$ is independent of the choice of E and can be characterized by

$$[x, y] = \{z \in X; |x-y| = |x-z| + |z-y|\}.$$

Moreover $[x, y]$ is *geodesic*, that is, it is the shortest path joining x and y and there is no other geodesic joining x and y.

3.6 Local geodesics are geodesic

A *local geodesic* is defined as a finite union of line segments such that the angles between subsequent segments are equal π. The important fact is that in Euclidean buildings local geodesics are geodesics. This fact is valid for the much more general case of CAT(0)–spaces, [3], Proposition II.10.

3.7 Definitions We extend the definitions from Section 3.1 to the case of Euclidean buildings. The notion of a special vertex can be defined in the case of a Euclidean building just by putting the vertex into an apartment. This is well defined because any isometrical isomorphism ϕ between apartments takes special vertices to special ones (since the same number of walls pass through x and $\phi(x)$). Analogously the notion of a special edge is well defined.

Call a directed edge e of a building Δ *quasi-special* if it lies on a line segment $[x, y]$ in $\Delta^{(1)}$ with special vertices x, y. Let E_s, E_{qs} be the sets of all special and quasi-special edges respectively. Note the inclusion $E_s \subseteq E_{qs}$ which follows from the fact that any special edge e can be brought to the form $e = [0, \frac{1}{c_i}\bar{\omega}_i^\vee]$ by the extended Weyl group, see Section 3.1, and now it is contained in the line segment $[0, \bar{\omega}_i^\vee]$ in $\Delta^{(1)}$ which has the special end points.

By an *ordering of an Euclidean building* Δ we mean a function $\tau : E_{qs} \mapsto \{1, \ldots, n\}, n = \dim \Delta^{(1)}$, such that when restricting to the set of quasi-special edges of any apartment it becomes an ordering of a corresponding Coxeter complex.

3.8 Lemma *Let $\Sigma_\Phi, \Sigma'_\Phi$ be two realizations of a Euclidean Coxeter complex in Euclidean spaces V, V' respectively and let $\phi : V \to V'$ be an isometry taking Σ_Φ to Σ'_Φ. If τ is any ordering on Σ'_Φ, then $\tau\phi$ is an ordering on Σ_Φ.*

Proof The only non obvious condition is that $\tau\phi$ is $\widehat{W_a}$–equivariant that is $\tau\phi w = \tau\phi$ for any $w \in \widehat{W_a}$. But the last equation is equivalent to $\tau\phi w \phi^{-1} = \tau$ and the assertion follows from the fact that ϕ normalizes the extended Weyl group by the lemma 2.6. □

3.9 Theorem *Any Euclidean building can be ordered. Moreover an ordering is uniquely defined by an ordering of special edges of a fixed alcove of a building starting at some fixed special vertex. In particular, there are only finitely many orderings on any Euclidean building. Isomorphisms in (B2) in 3.5 can be taken to be order-preserving. If ϕ is an automorphism of a Euclidean building Δ and τ is its ordering, then $\tau\phi$ is again an ordering on Δ.*

Proof We follow the proof of a labellability of a building [5], Chapter IV, Proposition 1. Fix an arbitrary alcove $A =< x, x_1, \ldots, x_n >$ with special vertex and order the edges $[x, x_1], \ldots, [x, x_n]$ by $1, \ldots, n$ respectively. If Σ is any apartment containing A, then as was proved in Section 3.3, there is a unique ordering τ_Σ which agrees with the chosen ordering on A. For any two such apartments Σ, Σ' the orderings $\tau_\Sigma, \tau_{\Sigma'}$ agree on the special edges of $\Sigma \cap \Sigma'$; this follows from the fact that that $\tau_{\Sigma'}$ can be constructed as $\tau_\Sigma \phi$, where $\phi : \Sigma \to \Sigma'$ is the isomorphism fixing $\Sigma \cap \Sigma'$. Since by Section 3.8 $\tau\phi$ is again an ordering and since it coincide on the based alcove in $\Sigma \cap \Sigma'$ they coincide everywhere. The various orderings τ_Σ therefore fit together to give an ordering τ defined on the union of the apartments containing A. But this union is all of Δ.

To prove the second assertion, note that isomorphisms $\phi : \Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ also fixes some alcove in $\Sigma \cap \Sigma'$ pointwise, hence it preserves the order.

Finally, to prove that $\tau\phi$ is again an ordering, just note that ϕ leaves invariant the set of all (quasi)-special edges. The $\widehat{W_a}$–invariance follows from the lemma 2.6. □

3.10 Standard ordering

Let us order the fundamental coweights $\bar{\omega}_1^\vee, \ldots, \bar{\omega}_n^\vee$ of the corresponding root system as they are naturally ordered in the tables of root systems given in [2]. This gives the *standard ordering of the standard alcove*, see Section 2.7 and thereby the *standard ordering of the building.*

4 Definition of a combing

Let Δ be an ordered Euclidean building. We wish to construct a combing $\mathcal{C}$ on Δ which consists of edge paths in the 1–skeleton $\Delta^{(1)}$ and is $\widehat{W_a}$–equivariant when restricted to any apartment. By definition an *edge path* α in a graph $\Delta^{(1)}$ is a map of the interval $[0, N] \subset \mathbf{N}$ into $\Delta^{(0)}$ such that $\forall i \in [0, N-1]$ the vertices $\alpha(i), \alpha(i+1)$ are the end points of an edge in $\Delta^{(1)}$. It is convenient to consider α as an ultimately constant map by extending it to a map of $[0, \infty]$ making $\alpha(t)$ stop after $t = N$, ie, by setting $\alpha(t) = \alpha(N)$ for $t \geq N$.

4.1 Combing the building

Take any special vertices $x, y \in \Delta^{(1)}$ and put them into some apartment Σ. We consider Σ as a vector space, taking x as an origin. Since x is a special vertex, the walls $H \in \mathcal{H}_{\mathbf{Z}}$, passing through x, define the structure of a spherical Coxeter complex $\Sigma_{sph}(x)$. In particular y lies in some closed sector S of this spherical complex, based in x. To S one can canonically associate a based alcove (x, A_S) – this is a unique alcove in S with x as one of its vertices. As with any sector, S is a simplicial cone and the rays defining it are the rays spanned by the edges of the alcove A_S, started in x. More exactly, let $A_S = < x, x_1, \ldots, x_n >$ with the edges $[x, x_i] = e_i$ of type $i, i = 1, \ldots, n$. These edges constitute the basis of a vector space structure on Σ, corresponding to a choice x as an origin. For any e_i we define e_i–direction as the set of all rays of the form $z + \mathbf{R}e_i, z \in \Sigma$. We have $S = \sum_{1 \leq j \leq n} \mathbf{R}_+ e_j$ relative to our vector space structure. Now $y \in S$, hence $y = \sum_{1 \leq i \leq n} m_i e_i$ with $m_1, \ldots, m_n \geq 0$. We are able now to define the path α_{xy} of a combing $\mathcal{C}$ connecting x ($= 0$ relative to a vector space structure) and y. This is a concatenation of the line segments

$$[0, m_1 e_1], [m_1 e_1, m_1 e_1 + m_2 e_2], \ldots, [\sum_{1 \leq i \leq n-1} m_i e_i, \sum_{1 \leq i \leq n} m_i e_i], \tag{1}$$

passing in this order. Geometrically speaking, α_{xy} is a concatenation of an ordered sequence of n segments (degenerate segments are allowed), such that the i-th segment is parallel to the line $\mathbf{R}\, e_i$ (degenerate segment is considered as to be parallel to any line).

Define now a combing $\mathcal{C}$ by collecting all the paths of the form α_{xy} (for the special vertices x, y) as well as all their prefix subpaths in $\Delta^{(1)}$.

4.2 Graph structure $\Delta^{(1)}_{spec}$

Apart from the natural simplicial graph structure on $\Delta^{(1)}$ we wish to use another rougher simplicial graph structure $\Delta^{(1)}_{spec}$. The vertices of $\Delta^{(1)}_{spec}$ are the special vertices of $\Delta^{(1)}$. Two special vertices x, y are connected by the edge $[x, y]$ (which is a line segment joining these vertices) if $[x, y]$ lies in $\Delta^{(1)}$ and there are no other special vertices between x and y. The main example of an edge in $\Delta^{(1)}_{spec}$ will be the line segment $[x, x + \bar{\omega}_i^{\vee}], i = 1, \ldots, n$ in a Euclidean Coxeter complex, where x is a special vertex, see Section 2.9. One obtains other examples from the observation that the extended Weyl group acts preserving the structure $\Delta^{(1)}_{spec}$.

Let $\mathcal{C}_{spec}$ be a subcombing of $\mathcal{C}$ consisting of those paths that connect only the special vertices of Δ. Note that a combing $\mathcal{C}_{spec}$ gives rise naturally to a combing of a graph $\Delta^{(1)}_{spec}$ which we will denote by the same symbol.

4.3 Lemma *The natural embeddings $\Delta^{(1)}_{spec} \subseteq \Delta^{(1)} \subset \Delta$ induce quasi-isometry between the graphs $\Delta^{(1)}_{spec}$, $\Delta^{(1)}$ with their graph metrics and the building Δ with its piecewise Euclidean metric.*

Proof Recall some definitions (see [11]). Given $\lambda \geq 1$ and $\epsilon \geq 0$, a map $f : X \to Y$ of metric spaces is a (λ, ϵ)*–quasi-isometric map* if

$$\frac{1}{\lambda} d_Y(x, y) - \epsilon \leq d_X(f(x), f(y)) \leq \lambda d_Y(x, y) + \epsilon$$

for all $x, y \in X$. If X is an interval, we speak of a *quasigeodesic path* in Y. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometric map $f : X \to Y$ such that Y is a bounded neighborhood of the image of f. Then f is called a *quasi-isometry.* If the constant ϵ above is zero, then we speak about *Lipschitz maps* and *Lipschitz equivalence.*

Firstly we prove that the embedding $\Delta^{(1)}_{spec} \subset \Delta$ is a Lipschitz equivalence. If λ is the Euclidean length of the longest edge in $\Delta^{(1)}_{spec}$, then clearly $d_\Delta \leq \lambda d_{1,sp}$, where $d_{1,sp}$ is a graph metric on $\Delta^{(1)}_{spec}$. On the other hand, if x, y are the special vertices, then by definition α_{xy} is a concatenation of the line segments

$$[0, k_1\bar{\omega}_1^\vee], [k_1\bar{\omega}_1^\vee, k_1\bar{\omega}_1^\vee + k_2\bar{\omega}_2^\vee], \ldots, [\sum_{1\le i\le n-1} k_i\bar{\omega}_i^\vee, \sum_{1\le i\le n} k_i\bar{\omega}_i^\vee]$$

passing in this order. Its graph length in $\Delta^{(1)}_{spec}$ is equal to $\sum_{1\le i\le n} k_i$ and this is an ℓ^1–distance between x, y in ℓ^1–metric on Σ, given in coordinates attached to a basis $\bar{\omega}_1^\vee, \ldots, \bar{\omega}_n^\vee$. Since, up to translations in Σ, there are only finite number of bases of this type and since the ℓ^1–metric is Lipschitz equivalent to the Euclidean metric there is a constant $\lambda_2 > 0$, such that $d_{1,sp}(x,y) \le \lambda_2 d_\Delta(x,y)$ – this proves that an embedding $\Delta^{(1)}_{spec} \subset \Delta$ is a Lipschitz equivalence.

Now, any edge in $\Delta^{(1)}_{spec}$ consists of several edges of $\Delta^{(1)}$ and let λ_3 be the largest number of the edges in $\Delta^{(1)}$ lying on the edge of $\Delta^{(1)}_{spec}$. Then for the graph metric d_1 on $\Delta^{(1)}$ we have an inequality $d_1 \le \lambda_3 d_{1,sp}$. Conversely, if λ_4 is the Euclidean length of the longest edge of the graph $\Delta^{(1)}$, then $d_\Delta \le \lambda_4 d_1$ and since $d_{1,sp}$ and d_Δ are Lipschitz equivalent we get the inequality $d_{1,spec} \le \lambda_5 d_1$ for a suitable positive constant λ_5. □

4.4 Lemma *The induced metric on $\Delta^{(1)}_{spec}$ as well as on $\Delta^{(1)}$ (induced from the Euclidean metric on a building) is Lipschitz equivalent to the edge path metric on these graph.*

Proof This simple observation is indeed true in a more general situation of $\mathbf{R}$–*graphs*, that is simplicial graphs, in which any edge is endowed with a metric, making it to be isometric to a segment of a real line. Namely if the lengths of the edges in such a graph are bounded from above and from below by some positive constants, then the $\mathbf{R}$–metric on a graph is Lipschitz equivalent to the edge path metric on this graph. This is clear. Now in our situation of Euclidean Coxeter complexes there are only finitely many isometry types of the edges, so we have the bounds just mentioned. □

Properties of $\mathcal{C}$

4.5 *α_{xy} is quasigeodesic relative to the edge path metric on $\Delta^{(1)}$.*

Obviously the path α_{xy} is geodesic relative to an ℓ^1–metric on Σ, given in coordinates attached to a basis $e_1, \ldots, e_n$. This ℓ^1–metric is Lipschitz equivalent to a Euclidean metric on Σ, which is Lipschitz equivalent to a graph metric on Σ. These equivalences preserve the the quasigeodesicity of paths, hence our path is quasigeodesic. (It seems likely that indeed α_{xy} is geodesic).

4.6 *Support of the path*

The path α_{xy} uniquely defines a cone $C_\alpha = \sum\{\mathbf{R}_+ e_i; m_i > 0\}$ which is called a *support* of α. (It is uniquely defined in Σ, not in Δ!). C_α is a subcomplex of Σ, see Section 2.11. Note that α travels inside of C_α (indeed it is the least cone in which it travels, since it can be defined as the cone span of α). Note also that C_α is the smallest face of S, containing y in its interior. In particular, any sector, containing y, contains also C_α.

4.7 α_{xy} *is an edge path in the 1–skeleton* $\Delta^{(1)}$.

Let $S = S_\Pi$ for some simple system $\Pi = \{\alpha_i\}$ of an underlying root system. Then the corresponding alcove A_Π is a closed simplex with the vertices 0 and $\bar{\omega}_i^\vee / c_i, i = 1, \ldots, n$, see Section 2.7. Here $\{\bar{\omega}_i^\vee\}$ is the dual basis for $\{\alpha_i\}$ (the basis of coweight lattice $\mathbf{Z}\Phi^\#$) and $\tilde{\alpha} = \sum_{1 \le i \le n} c_i \alpha_i$ is the corresponding highest root. Hence, after appropriate re-ordering of α_i, we may assume that $e_i = \bar{\omega}_i^\vee / c_i, i = 1, \ldots, n$. The fundamental coweight $\bar{\omega}_i^\vee$ is a special vertex and lies on the line passing through e_i, thus the line segment $[0, \bar{\omega}_i^\vee]$ is an edge path in $\Sigma^{(1)}$. Since y is a special vertex it belongs to the coweight lattice $\mathbf{Z}\Phi^\#$ by Section 2.9, and thus, in the notations of Section 4.1, all the numbers $k_i = \frac{m_i}{c_i}$ are integers. By definition α_{xy} is a concatenation of the line segments

$$[0, k_1\bar{\omega}_1^\vee], [k_1\bar{\omega}_1^\vee, k_1\bar{\omega}_1^\vee + k_2\bar{\omega}_2^\vee], \ldots, [\sum_{1 \le i \le n-1} k_i\bar{\omega}_i^\vee, \sum_{1 \le i \le n} k_i\bar{\omega}_i^\vee]$$

passing in this order. We conclude from this formula that the path is a concatenation of the paths, obtained from the line segments $[0, \bar{\omega}_i^\vee]$ (which are edge paths) by the action of $\mathbf{Z}\Phi^\#$ and the last action preserves the simplicial structure on Σ.

4.8 α_{xy} *is contained in the convex hull of the set* $\{x, y\}$.

By a *convex hull* of an arbitrary set in the building Δ we mean the smallest convex subcomplex containing this set. For example apartments are convex, hence the convex hull of any set is contained in any apartment, which contains this set. In a case of a set consisting of two special vertices x, y one can describe the convex hull $\mathrm{ch}(x, y)$ as a parallelepiped, spanned by $\{x, y\}$. More exactly, in the notations of Section 4.7, we assert that

$$\mathrm{ch}(x, y) = \{z = \sum z_i e_i : \ 0 \le z_i \le m_i, \ i = 1, \ldots, n\}. \tag{2}$$

It immediately follows from this description that α_{xy} is contained in $\mathrm{ch}(x, y)$. To prove the equality (2), note firstly that the sector S is a convex subcomplex as well as $-S$ (Section 2.11). We see immediately that the parallelepiped P on the right hand side is the intersection of the sectors S and $-S+y$ and thus is a convex subcomplex. Hence $P \supseteq \mathrm{ch}(x,y)$. Suppose that $P \neq \mathrm{ch}(x,y)$, then $\mathrm{ch}(x,y)$ is a proper convex subcomplex of P, containing x,y. Because $\mathrm{ch}(x,y)$ is an intersection of closed half-spaces bounded by elements of $\mathcal{H}_{\mathbf{Z}}$, there is a half-space H^+ defined by some $H \in \mathcal{H}_{\mathbf{Z}}$, containing $\mathrm{ch}(x,y)$, but not P. (One can prove this fact by showing firstly that $\mathrm{ch}(x,y)$ is a convex hull of finite number of vertices and then follow the standard proof that the convex polygon is an intersection of half-subspaces, supported on codimension one faces.) Suppose, for instance, that y is not farther from H than x. Take the hyperplane $H_1 \in \mathcal{H}_{\mathbf{Z}}$ parallel to H and passing through y and let H_1^+ be a half-space bounded by H_1 and contained in H^+. Since H_1 pass through the special vertex y it belongs to the structure $\Sigma_{sph}(x)$ of spherical complex in y, see Section 4.1, and since H_1^+ contains x it contains also the support $C_{\alpha^{-1}}$ of the path α^{-1}, inverse to α. But $C_{\alpha^{-1}} = -C_\alpha + y$, hence it contains the parallelepiped P, contradiction.

This proof does not work in the case when y is not special, since the set of hyperplanes from $\mathcal{H}_{\mathbf{Z}}$ passing through y does not constitute the structure of spherical complex. But still we can prove that the parallelepiped P in (2) is contained in $\mathrm{ch}(x,y)$. Suppose the contrary, that P is not contained in $\mathrm{ch}(x,y)$, then again there is a half-space H^+ defined by some $H \in \mathcal{H}_{\mathbf{Z}}$, containing $\mathrm{ch}(x,y)$, but not P. Suppose that y is not farther from H than x (the opposite case was already treated above). Define the structure $\Sigma_{sph}(y)$ by translating such a structure from any special vertex. Relative to this structure x lies in the support $C_{\alpha^{-1}} = -C_\alpha + y$, where α^{-1} is the path inverse to α. Consider the hyperplane $H_1 \in \mathcal{H}_{\mathbf{Z}}$ parallel to H and passing through y and let H_1^+ be a half-space bounded by H_1 and contained in H^+. Since H_1^+ contains x it contains also $C_{\alpha^{-1}} = -C_\alpha + y$, hence it contains the parallelepiped P, contradiction.

4.9 *$\mathcal{C}$ is $\widehat{W}_a$–invariant.*

This immediately follows from the fact that the ordering is $\widehat{W}_a$–invariant and from the geometric interpretation of the paths in $\mathcal{C}$ just given.

4.10 *If x,y are special vertices, then α_{xy} is uniquely defined by x,y.*

Firstly, the convex hull $\mathrm{ch}(x,y)$ is uniquely defined. It is a parallelepiped and its 1–dimensional faces are ordered. Now α_{xy} is a unique edge path from x to y in $\mathrm{ch}(x,y)$ which is a concatenation of 1–faces of $\mathrm{ch}(x,y)$ passing in the increasing order.

5 Fellow traveller property

5.1 Theorem *The combing $\mathcal{C}$ of an ordered Euclidean building Δ constructed in Section 4 satisfies the "fellow traveller property", namely there is $k > 0$ such that if $\alpha, \beta \in \mathcal{C}$ begin and end at a distance at most one apart, then*

$$d_1(\alpha(t), \beta(t)) \leq k$$

for all $t \geq 0$. (The metric d_1 is the graph metric on $\Delta^{(1)}$.) The same is true for the combing $\mathcal{C}_{spec}$ on a graph $\Delta^{(1)}_{spec}$, see 4.2.

Proof A) Let us firstly consider the case where α, β begin at the same vertex and end at a distance one apart, that is $[\alpha(\infty), \beta(\infty)]$ is an edge. It is easily seen from a k–fellow traveller property that for any $c > 0$ if $\gamma, \gamma' \in \mathcal{C}$ begin and end at a distance at most c apart, then $|\gamma(t) - \gamma'(t)| \leq kc$ for all $t \geq 0$. Note also that we can work in one apartment since the initial vertex of the paths and the edge $[\alpha(\infty), \beta(\infty)]$ are contained in some apartment and thus the whole paths lie in this apartment, see Section 4.8. Thus we may assume that α, β are contained in the Coxeter complex Σ and start at 0. Associated to α, β are their supports C_α, C_β in which they travel respectively. The intersection $K = C_\alpha \cap C_\beta$ is a simplicial cone of the form $K = \sum \mathbf{R}_+ v_j$ for some set $\{v_j\}$ of special edges. Hence

$$C_\alpha = \sum \mathbf{R}_+ u_i + \sum \mathbf{R}_+ v_j$$

and

$$C_\beta = \sum \mathbf{R}_+ w_k + \sum \mathbf{R}_+ v_j$$

where $\{u_i\}, \{v_j\}, \{w_k\}$ are the sets of special edges (possibly empty) and the sets $\{u_i\}, \{w_k\}$ do not intersect. We will argue by induction on the sum $\dim C_\alpha + \dim C_\beta$. The least nontrivial case is when the sum is equal to 2 and both of C_α, C_β are one dimensional, that is they are simplicial rays. If α and β have the same direction then obviously they 1–fellow travel each other. If not then they diverge linearly with a speed bounded from below by a constant not depending on the paths (indeed, there only finitely many of possibilities for the angle between C_α and C_β). Thus they could end at a distance one apart only when they passed a bounded distance, thus they k–fellow travel for some $k > 0$. A similar argument applies when $C_\alpha \cap C_\beta = 0$. The main case is when $K = C_\alpha \cap C_\beta = \sum \mathbf{R}_+ v_j$ is nonzero. Again in this case the argument similar to the above shows that α (resp. β) can move in the u–direction (resp. in the w–direction) only for a bounded amount of time, c say. The rest of the proof is the reduction to the case when the paths lie in K – then we can apply inductive hypothesis. But we need a definition. Consider a path γ, consising of two linear subpaths γ', γ'' passing in this order and having the directions v, w correspondingly. Construct the path $\tilde{\gamma}$, which goes distance $|\gamma''|$ in the w–direction and distance $|\gamma'|$ in the v–direction. We call $\tilde{\gamma}$ an *elementary transformation of the 2–portion path* γ. Note now that if $|\gamma''| \leq c$, then $\tilde{\gamma}$ and γ $2c$–travel each other. Using the elementary transformations as above we can push out all the linear subpaths of α with u–directions to the end of the path. So we assume that α and β are such from the beginning. Of course this operation takes α from the combing $\mathcal{C}$, but note that the initial v–portion of α continues to lie in $\mathcal{C}$, since the elementary transformations do not change the ordering of v–directions. Cutting out the u–tails and w–tails of α and β correspondingly, we may assume that α and β lie in the cone K and end within at most distance c for some universal constant c. Now let v_{j_0} be of the smallest order in the set $\{v_j\}$. Then each of the paths α, β move some nonzero time in v_{j_0}–direction, hence they coincide during some nonzero time. Cutting out the longest coinciding part of α and β we may assume that one of them does not contain the v_{j_0}–direction at all. But now the dimension of the support either of α or β decreases and we may apply an induction hypothesis.

B) Consider now the general case where $\alpha, \beta \in \mathcal{C}$ begin and end at a distance at most one apart. Adding a bounded number of edges to α and β one may assume that α and β end at the special vertices which are distance at most c apart for some constant c, depending only on Δ. Drawing a path from $\mathcal{C}$ connecting $\alpha(0)$ and $\beta(\infty)$ and making use the part A) of the proof we reduce the problem to the case when α, β end at the same special vertex. Denote by $\mathcal{C}_{spec}$ the set of all paths from $\mathcal{C}$ connecting special vertices. Now consider the combing $-\mathcal{C}_{spec}$ consisting by definition of the paths which are inverse to the paths from $\mathcal{C}_{spec}$. Obviously $-\mathcal{C}_{spec}$ is obtained in the same way as $\mathcal{C}_{spec}$ but reversing the underlying ordering. Thus for some k' the combing $-\mathcal{C}_{spec}$ satisfies the k'–fellow traveller property for paths which begin at the same vertex. Now let α^{-1}, β^{-1} be the paths inverse to α, β so that $\alpha^{-1}, \beta^{-1} \in -\mathcal{C}_{spec}$. Consequently α^{-1}, β^{-1} k'–fellow travel each other. This doesn't imply that α and β are k'–fellow travellers since they arrive at $\alpha(0)$ and $\beta(0)$ at different times. Let N_α and N_β the the length of α and β respectively. Then $\alpha^{-1}(t) = \alpha(N_\alpha - t)$ (assuming that α extended to the negative times in an obvious way). We have $|\alpha(t) - \beta(t)| = |\alpha^{-1}(N_\alpha - t) - \beta^{-1}(N_\beta - t)| \leq |\alpha^{-1}(N_\alpha - t) - \beta^{-1}(N_\alpha - t)| + |\beta^{-1}(N_\alpha - t) - \beta^{-1}(N_\beta - t)|$. The first modulus is bounded since α^{-1}, β^{-1} fellow travel each other. The second modulus is bounded since the paths $\beta^{-1}(N_\alpha - t), \beta^{-1}(N_\beta - t)|$ differ by only a bounded time shift.

C) Let $\alpha, \beta \in \mathcal{C}$ be the paths from a combing $\mathcal{C}_{spec}$, beginning and ending at a distance at most one apart. There is a constant $c > 0$, depending only on Δ, such that any edge in $\Delta^{(1)}_{spec}$ is of a length $\leq c$. Then, applying part B) we prove that α, β fellow travel each other relative to $\Delta^{(1)}$–metric and since the metrics $d_1, d_{1,spec}$ are Lipschitz equivalent, we get the fellow traveller property for the combing $\mathcal{C}_{spec}$ on a graph $\Delta^{(1)}_{spec}$. □

6 Recursiveness of a combing $\mathcal{C}$

In this section Δ will be an ordered Euclidean building with a standard ordering, see Section 3.10.

The definition of $\mathcal{C}$ given above is "global" in the sense that a path from $\mathcal{C}$ "knows" where it goes to. In this section we show that a path from $\mathcal{C}$ can be defined by a simple local "direction set": namely any pair of consecutive directed quasi-special edges $\{e_1, e_2\}$ shall be one of the following two types:

1) $\{e_1, e_2\}$ is *straight*, that is the angle between e_1 and e_2 is equal π and hence the union $e_1 \cup e_2$ is the line segment of length 2 in the edge path metric,

or

2) the type i of e_1 is strictly less than the type j of e_2, the end of e_1 (= the origin of e_2) is a special vertex and $(e_1, e_2) = (\frac{1}{c_i}\bar{\omega}_i^\vee, \frac{1}{c_j}\bar{\omega}_j^\vee)$.

Define $\mathcal{C}'$ to be the family of all paths γ in $\Delta^{(1)}$, in which any pair of consecutive edges satisfy either 1) or 2).

6.1 Theorem *If Δ is a Euclidean building of one of the following three types A_n, B_n, C_n, then the combings $\mathcal{C}$ and $\mathcal{C}'$ coincide.*

Proof It follows immediately from the global definition of $\mathcal{C}$ that it is contained in $\mathcal{C}'$. The proof of the converse proceeds by induction on the number of line segments constituting the path $\gamma \in \mathcal{C}'$. Take $\gamma \in \mathcal{C}'$ and write it as $\gamma = \beta \cup e$, where e is the last edge of γ and β is the portion of α, preceding e . By the induction hypothesis $\beta \in \mathcal{C}$ thus $\beta \subset \mathrm{ch}(x, y)$, where $x = \beta(0), y = \beta(\infty))$ in view of Section 4.8. Let Σ be an apartment containing both $\beta(0)$ and e. Since Σ is convex it contains $\mathrm{ch}(x, y)$ and thereby β. Consequently all our path γ is contained in Σ. Take x as an origin and identify Σ with the standard Coxeter complex Σ_Φ. Thus we may assume that β lies in a standard sector $\sum \mathbf{R}_+\bar{\omega}_i^\vee$ and

$$y = m_1\frac{\bar{\omega}_{i_1}^\vee}{c_{i_1}} + \ldots + m_r\frac{\bar{\omega}_{i_r}^\vee}{c_{i_r}},$$

where all the coefficients m_i are natural numbers. Let e_1 be the last edge of β, then it is parallel to $\bar{\omega}_{i_r}^\vee$ and is of type i_r. With the notation $j = i_r$ and by definition of $\mathcal{C}'$ the type k of e is not smaller than j and the pair $\{e_1, e\}$ is one of the following two types:

1) $\{e_1, e\}$ is *straight*, that is the angle between these two vectors is zero and, hence, by 3.6 the union $e_1 \cup e$ is the line segment of the length 2 in edge path metric

or

2) the type j of e_1 is strictly less than the type k of e and $(e_1, e) = (\frac{1}{c_i}\bar{\omega}_i^\vee, \frac{1}{c_j}\bar{\omega}_j^\vee)$.

Since y is a special vertex its stabilizer $W(y)$ is conjugate to W – the Weyl group of a root system Φ. Since the set of all the edges of type k starting in $\beta(\infty) = y$ is an orbit $W\bar{\omega}_k^\vee$ we have $e = w\frac{\bar{\omega}_k^\vee}{c_k}$ for some $w \in W(y)$. If one could find this $w \in W$ in such a way that it fixes all $\bar{\omega}_i^\vee, i \leq j$, then such w fixes β, since β lies in a Euclidean subspace spanned by the vectors $\bar{\omega}_i^\vee, i \leq j$. Now applying w^{-1} to the path α, we get that $w^{-1}\alpha = \beta \cup \frac{1}{c_k}\bar{\omega}_k^\vee$, hence $w^{-1}\alpha \in \mathcal{C}$. Taking into account that W_a preserves $\mathcal{C}$, we get that $\alpha \in \mathcal{C}$. The problem now is to find $w \in W(y)$ with the properties as above. As was mentioned above the last edge of β is parallel to $\bar{\omega}_j^\vee$ and is of type j. Again by definition of $\mathcal{C}'$ we deduce that $(\frac{1}{c_j}\bar{\omega}_j^\vee, e) = (\frac{1}{c_j}\bar{\omega}_j^\vee, \frac{1}{c_k}\bar{\omega}_k^\vee)$.

Thus, to finish the proof we need the following technical lemma.

6.2 Lemma (a) *Let Φ be a root system of one of the types A_n, B_n, C_n, given by the tables in [2], pages 250–275. Order fundamental coweights by their indices as they are given in [2]. Let ω be a coweight of type k and*

$$(\bar{\omega}_j^\vee, \omega) = (\bar{\omega}_j^\vee, \bar{\omega}_k^\vee)$$

for some $j < k$. Then there is $w \in W$ fixing all the vectors $\bar{\omega}_i^\vee, i \leq j$ and such that $\omega = w\bar{\omega}_k^\vee$.

(b) *The assertion is not true for the remaining classical case when Φ is of type D_n.*

6.3 Proof of lemma 6.2

(a) Case $A_n, n \geq 1$

Denote by $\epsilon_0, \ldots, \epsilon_n$ the standard basis of $\mathbf{R}^{n+1}, n \geq 1$. Let V be the hyperplane in $\mathbf{R}^{n+1}$ consisting of vectors whose coordinates add up to 0. Define Φ to be the set of all vectors of squared length 2 in the intersection of V with the standard lattice $\mathbf{Z}\epsilon_0 + \ldots + \mathbf{Z}\epsilon_n$. Then Φ consists of the $n(n+1)$ vectors:

$$\epsilon_i - \epsilon_j, \quad 0 \leq i \neq j \leq n$$

and W acts as a permutation group S_{n+1} on basis $\epsilon_0, \ldots, \epsilon_n$.

For the simple system Π take

$$\alpha_1 = \epsilon_0 - \epsilon_1, \alpha_2 = \epsilon_1 - \epsilon_2, \ldots, \alpha_n = \epsilon_{n-1} - \epsilon_n.$$

Then the highest root is

$$\tilde{\alpha} = \epsilon_0 - \epsilon_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

The fundamental coweights are

$$\bar{\omega}_j^\vee = (\epsilon_0 + \ldots + \epsilon_{j-1}) - \frac{j}{n+1} \sum_{i=0}^{n} \epsilon_i, \quad 1 \leq j \leq n \ .$$

Let the coweight ω satisfies the hypotheses of the lemma. Then since the type is W–invariant $\omega = u\bar{\omega}_k^\vee$ for some $u \in W$. As W acts by permutations on the basis $\bar{\omega}_i^\vee$

$$\omega = \sum_{1 \leq r \leq k} \epsilon_{i_r} - \frac{k}{n+1} \sum_{i=0}^{n} \epsilon_i. \tag{3}$$

By the hypotheses of the lemma

$$(\bar{\omega}_j^\vee, \omega) = (\bar{\omega}_j^\vee, \bar{\omega}_k^\vee) \tag{4}$$

for some $j < k$.

The left hand side of (4) is

$$\Big(\epsilon_0 + \ldots + \epsilon_{j-1} - \frac{j}{n+1} \sum_{i=0}^{n} \epsilon_i, \ \sum_{1 \leq r \leq k} \epsilon_{i_r} - \frac{k}{n+1} \sum_{i=0}^{n} \epsilon_i\Big) =$$

$$\operatorname{card}\{r | i_r \leq j-1\} - \frac{jk}{n+1}. \tag{5}$$

The right hand side of (4) is

$$(\bar{\omega}_j^\vee, \bar{\omega}_k^\vee) = \Big(\epsilon_0 + \ldots + \epsilon_{j-1} - \frac{j}{n+1} \sum_{i=0}^{n} \epsilon_i, \ \epsilon_0 + \ldots + \epsilon_{k-1} - \frac{k}{n+1} \sum_{i=0}^{n} \epsilon_i\Big) =$$

$$\frac{j(n+1-k)}{n+1} = j - \frac{jk}{n+1}. \tag{6}$$

Now comparing (5) and (6), we conclude that

$$\mathrm{card}\{r | i_r \leq j-1\} = j,$$

hence from (3)

$$e = (\epsilon_0 + \ldots + \epsilon_{j-1}) + \epsilon_{i_j} + \ldots + \epsilon_{i_k} - \frac{k}{n+1} \sum_{i=0}^{n} \epsilon_i .$$

But $\omega = u\ \bar{\omega}_k^\vee$ and

$$\bar{\omega}_k^\vee = (\epsilon_0 + \ldots + \epsilon_k) - \frac{k}{n+1} \sum_{i=0}^{n} \epsilon_i ,$$

consequently $\omega = w\bar{\omega}_k^\vee$ for some $w \in W$ fixing all the vectors $\epsilon_0, \epsilon_1, \ldots, \epsilon_{j-1}$. Since $\bar{\omega}_i^\vee, i \leq j$ are the linear combinations of the vectors

$$\epsilon_0, \epsilon_1, \ldots, \epsilon_{j-1}, \frac{j}{n+1} \sum_{i=0}^{n} \epsilon_i$$

(the last one is fixed by W), we get that $\bar{\omega}_i^\vee, i \leq j$ are also fixed by w.

Case $B_n, n \geq 2$

Denote by $\epsilon_1, \ldots, \epsilon_n$ the standard basis of $\mathbf{R}^n, n \geq 2$. Define Φ to be the set of all vectors of squared length 1 or 2 in the standard lattice $\mathbf{Z}\epsilon_1 + \ldots + \mathbf{Z}\epsilon_n$. So Φ consists of the $2n$ short roots $\pm\epsilon_i$ and the $2n(n-1)$ long roots $\pm\epsilon_i \pm \epsilon_j,\ (i < j)$, totalling $2n^2$. For the simple system Π take

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_n .$$

Then the highest root

$$\tilde{\alpha} = \epsilon_1 + \epsilon_2 .$$

The Weyl group W is the semidirect product of S_n (which permutes ϵ_i) and $(\mathbf{Z}/2)^n$ (acting by sign changes on the ϵ_i), the latter normal in W.

The fundamental coweights are

$$\bar{\omega}_i^\vee = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i,\quad 1 \leq i \leq n\ .$$

Let the coweight e satisfies the hypotheses of the lemma. Then since $W\bar{\omega}_k^\vee$ is the set of all edges of type k we have $\omega = u\bar{\omega}_k^\vee$ for some $u \in W$. As W acts by "sign" permutations on the basis $\bar{\omega}_i$ we have that

$$\omega = \sum_{1 \leq r \leq k} \pm\epsilon_{i_r} . \tag{7}$$

By hypotheses of the lemma

$$\bar{\omega}_j^{\vee} \omega = \bar{\omega}_j^{\vee} \bar{\omega}_k^{\vee} \tag{8}$$

for some $j < k$.

The left hand side of (8) is equal to

$$(\epsilon_1 + \ldots + \epsilon_j)(\sum_{1 \le r \le k} \pm \epsilon_{i_r}) \le \text{card}\{r | i_r \le j\}. \tag{9}$$

The right hand side of (8) is equal

$$\bar{\omega}_j^{\vee} \bar{\omega}_k^{\vee} = j. \tag{10}$$

Now comparing (9) and (10), we conclude that

$$\text{card}\{r | i_r \le j\} = j,$$

hence from (7)

$$\omega = (\pm \epsilon_0 \pm \ldots \pm \epsilon_j) + \pm \epsilon_{i_j} \pm \ldots \pm \epsilon_{i_k}.$$

But $\omega = u\ \bar{\omega}_k^{\vee}$ and

$$\bar{\omega}_k^{\vee} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k, \quad 1 \le k \le n\ .$$

Since $\bar{\omega}_i^{\vee}, i \le j$ are linear combinations of the vectors $\epsilon_0, \epsilon_1, \ldots, \epsilon_{j-1}$, we get that $\bar{\omega}_i^{\vee}, i \le j$ are also fixed by w.

But $\omega = w\ \bar{\omega}_j^{\vee}$, consequently w can be chosen in such a way that it fixes the vectors $\epsilon_1, \ldots, \epsilon_j$.

Case $C_n, n \ge 2$

Starting with B_n, one can define C_n to be the inverse root system. It consists of the $2n$ long roots $\pm 2\epsilon_i$ and the $2n(n-1)$ short roots $\pm \epsilon_i \pm \epsilon_j$, $(i < j)$, totalling $2n^2$. For the simple system Π take

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n.$$

Then the highest root

$$\tilde{\alpha} = 2\epsilon_1.$$

The Weyl group W is the semidirect product of S_n (which permutes ϵ_i) and $(\mathbf{Z}/2)^n$ (acting by sign changes on the ϵ_i), the latter normal in W.

The fundamental coweights are

$$\bar{\omega}_i^{\vee} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i, \quad 1 \le i < n\ .$$

Now one can repeat word by word the case of B_n. This completes the proof of part (a).

Proof of (b) Let now Φ be of the type $D_n, n \geq 4$. Denote by $\epsilon_1, \ldots, \epsilon_n$ the standard basis of $\mathbf{R}^n$. Define Φ to be the set of all vectors of squared length 2 in the standard lattice $\mathbf{Z}\epsilon_1 + \ldots + \mathbf{Z}\epsilon_n$. So Φ consists of the $2n(n-1)$ roots $\pm\epsilon_i \pm \epsilon_j (1 \leq i < j \leq n)$. For the simple system Π take

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_{n-1} + \epsilon_n.$$

Then the highest root

$$\tilde{\alpha} = \epsilon_1 + \epsilon_2.$$

The Weyl group W is the semidirect product of S_n (which permutes ϵ_i) and $(\mathbf{Z}/2)^{n-1}$ (acting by an even number of sign changes on the ϵ_i), the latter normal in W.

The fundamental coweights are

$$\bar{\omega}_i^\vee = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i, \quad 1 \leq i < n-2 \ ,$$

$$\bar{\omega}_{n-1}^\vee = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{n-2} + \epsilon_{n-1} - \epsilon_n),$$

$$\bar{\omega}_n^\vee = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{n-2} + \epsilon_{n-1} + \epsilon_n).$$

Now take the vector

$$\omega = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{n-2} - \epsilon_{n-1} - \epsilon_n),$$

which is a coweight since $\omega = u\bar{\omega}_n^\vee$ where $u \in W$ acts by signs changes on $\epsilon_{n-1}, \epsilon_n$. Take $j = n-1$, then $\bar{\omega}_{n-1}^\vee \omega = \bar{\omega}_{n-1}^\vee \bar{\omega}_n^\vee$ and it is impossible to replace u by some $w \in W$ so that $\omega = w\bar{\omega}_n^\vee$ and w fixes $\bar{\omega}_i^\vee, i < n$. Indeed, let $\omega = w\bar{\omega}_n^\vee$ and let w be the identity on $\bar{\omega}_1^\vee, \ldots, \bar{\omega}_{n-1}^\vee$. Then w fixes $\epsilon_{n-1} - \epsilon_n$, from which it follows that either $w = 1$ or w changes the signs both of $\epsilon_{n-1}, \epsilon_n$. Contradiction. □

7 Automatic structure for groups acting on Euclidean buildings of type A_n, B_n, C_n

7.1 Theorem (1) *Let Δ be any Euclidean building of one of the types A_n, B_n, C_n, ordered in a standard way (see Section 3.10 for a definition). Then any group acting freely and cocompactly on Δ by type preserving automorphisms admits a biautomatic structure.*

(2) *If Δ is any Euclidean building of one of the types A_n, B_n, C_n, then any group acting freely and cocompactly on Δ possesses a finite index subgroup which admits a biautomatic structure.*

Let G be a group satisfying the conditions of the theorem. We shall proceed in several steps. Firstly we recall the definitions related to an automatic group theory, then we establish an isomorphism between the complex $\Delta^{(1)}_{spec}$ and the Cayley graph of a fundamental groupoid $\mathcal{G} = \pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$. Making use the combing $\mathcal{C}_{spec}$ and its properties we prove the biautomaticity of the groupoid above. Finally, we apply a result from [7], [12], asserting that any automorphism group of a biautomatic groupoid $\mathcal{G}$ (which is isomorphic to G) is biautomatic.

7.2 Automatic structures on groups and groupoids

We shall use the groupoids technique and since the groups are a special case of groupoids, we give all the definitions for groupoids. We summarize here without proofs the relevant material on groupoids from [7], [12]. A *groupoid* $\mathcal{G}$ is a category such that the morphism set $\mathrm{Hom}_{\mathcal{G}}(v,w)$ is nonempty for any two objects (=vertices) and such that each morphism is invertible. In particular for any $v \in \mathrm{Ob}\ \mathcal{G}$ the morphism set $\mathcal{G}_v = \mathrm{Hom}_{\mathcal{G}}(v,v)$ is a group and any group G can be considered as a groupoid with one object, whose automorphism group is G. The group $\mathcal{G}_v$ does not depend on v, up to an isomorphism. A groupoid is said to be *generated* by a set A of morphisms, if every morphism is a composite of morphisms in $A \cup A^{-1}$. Fix a vertex $v_0 \in \mathcal{G}$ as a base point and assume A is a generating set of morphisms with $A = A^{-1}$. The *Cayley graph* $C\mathcal{G} = C(\mathcal{G}, A, v_0)$ of $\mathcal{G}$ with respect to a base point v_0 and generating set A is the directed graph with vertices corresponding to morphisms in $\mathcal{G}$ with domain (=source) v_0, that is

$$\mathrm{Vert}\ C\mathcal{G} = \mathrm{Hom}(v_0, *) = \cup\{\mathrm{Hom}(v_0, v) | v \in \mathrm{Ob}\ \mathcal{G}\}.$$

There is a directed edge $f \xrightarrow{a} fa$ from the morphism f to the morphism fa, $a \in A$ whenever fa is defined (we write the morphisms on the right); we give this edge a label a. In the case of a group the vertex set of a Cayley graph is just a group itself and we have a usual notion of a Cayley graph of a group. In particular each edge path in $C\mathcal{G}$ spells out a word in A^*, where as usual, A^* denotes the free monoid on A. And vice versa, for any word $w \in A^*$ and any vertex f there is a unique path in $C\mathcal{G}$, beginning in f and having w as its label. We put a path-metric on $C\mathcal{G}$ by deeming every edge to have unit length. Note that the group $\mathcal{G}_{v_0}$ acts on $C\mathcal{G}$ by left translations. The *generating graph* $G\mathcal{G} = G\mathcal{G}_A$ is a graph with the same set of vertices as $\mathcal{G}$ and with edges corresponding to morphisms in A and labeled by them.

There is a natural projection p from $C\mathcal{G}$ to $G\mathcal{G}$, defined as follows. Let the morphism $f : v_0 \to w$ be a vertex of a Cayley graph $C\mathcal{G}$ then $p(f : v_0 \to w) = w$. If $f \xrightarrow{a} fa$ is an edge of $C\mathcal{G}$ and $f : v_0 \to w$, $a : w \to u$ then p sends it to the edge $a : w \to u$ of $G\mathcal{G}$. The group $\mathcal{G}_{v_0}$ gives the group of covering transformations for p and $\mathcal{G}_{v_0} \backslash C(\mathcal{G}, A, v) \simeq G\mathcal{G}$. Indeed, the isomorphism is induced by p and if $f \xrightarrow{a} fa$, $f_1 \xrightarrow{a_1} f_1 a_1$ are in the fiber, then $g = f_1 f^{-1} \in \mathcal{G}_{v_0}$.

Automatic structures on groupoids

Let $\mathcal{G}$ be a finitely generated groupoid and A a finite set and $a \mapsto \bar{a}$ is a map of A to a monoid generating set $\bar{A} \subset \mathcal{G}$. A *normal form* for $\mathcal{G}$ is a subset L of A^* satisfying the following

(i) L consists of words labelling the paths in $C(\mathcal{G}, A, v_0)$ (that is only composable strings of morphisms are considered, starting at the base point $\mathrm{id} \in \mathrm{Hom}(v_0, v_0)$)

(ii) The natural map $L \to \mathrm{Hom}(v_0, *)$ which takes the word $w = a_1 a_2 \cdots a_n$ to the morphism $\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n \in \mathrm{Hom}(v_0, *)$ is onto.

A *rational structure* is a normal form that is a regular language ie, the set of accepted words for some finite state automaton. Recall that a *finite state automaton* $\mathcal{M}$ with alphabet A is a finite directed graph on a vertex set S (called the set of *states*) with each edge labeled by an element of S (maybe empty). Moreover, a subset of *start states* $S_0 \subset S$ and a subset of *accepted states* $S_1 \subset S$ are given. By definition, a word w in the alphabet A is in the *language* L *accepted by* $\mathcal{M}$ iff it defines a path starting from S_0 and ending in an accepted state in this graph. A language is *regular* if it is accepted by some finite state automaton.

We will say that a normal form L has the "*fellow traveller property*" if there is a constant k such that given any normal form words $v, w \in L$ labelling the paths α_v, α_w in Cayley graph $C(\mathcal{G}, A, v_0)$ which begin and end at a distance at most one apart, the distance $d(w(t), v(t)), t = 0, 1, \ldots$ never exceeds k. A *biautomatic structure* for a groupoid $\mathcal{G}$ is a regular normal form with the fellow traveller property.

7.3 Theorem ([7], 13.1.5, [12], 4.1, 4.2) *Let $\mathcal{G}$ be a groupoid and v_0 an arbitrary vertex of $\mathcal{G}$. Then $\mathcal{G}$ admits a biautomatic structure if and only if the automorphism group $\mathcal{G}_{v_0}$ of v_0 admits such a structure.*

7.4 Groupoid $\pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$

In this section G is a group acting freely and cocompactly on a Euclidean building Δ of of the following types A_n, B_n, C_n by automorphisms preserving the standard ordering.

Fundamental groupoid

The prime example of a groupoid will be the *fundamental groupoid* $\pi_1(X)$ of the path-connected topological space X . The set of objects(=vertices) of $\pi_1(X)$ is the set of points of X and the morphisms from x to y are homotopy classes of paths beginning at x and ending at y . The multiplication in $\pi_1(X)$ is induced by compositions of paths. Given a subset $Y \subset X$ we obtain a subgroupoid $\pi_1(X, Y)$ whose vertices are the points of Y and the morphisms are the same as before. In particular if Y consists of a single point then we get the fundamental group of X based at that point.

Generating set of groupoid $\pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$

7.5 Lemma *Let A be the set of homotopy classes of the images in $G\backslash\Delta$ of directed edges of the graph $\Delta^{(1)}_{spec}$. Then A is a finite set, generating groupoid $\mathcal{G} = \pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$.*

Proof This set is finite since by condition Δ has only finitely many cells under the action of G. To prove that A generates $\mathcal{G}$ take a path α from Gv to Gv', then since Δ is contractible , see [5] and G acts freely on Δ the projection $\Delta \to G\backslash\Delta$ is a universal cover and there is a unique lift $\widetilde{\alpha}$ of α into Δ which begins at v. This lift ends at a translate gv' of v' where g is determined by the homotopy class of α. Moreover, any path in Δ from v to gv' will project to a path in $G\backslash\Delta$ which is homotopic to α. In particular the path from $\mathcal{C}_{spec}$ which crawls from v to gv' is homotopic to $\widetilde{\alpha}$. Since this path is the edge path in

$\Delta^{(1)}_{spec}$ it projects into $G\backslash\Delta$ as a composition of homotopy classes of the images in $G\backslash\Delta$ of directed edges of the graph $\Delta^{(1)}_{spec}$, which is a product of morphisms from A. This means that A is a generating set for $\mathcal{G}$. □

Labelling the graph $\Delta^{(1)}_{spec}$

Consider $\Delta^{(1)}_{spec}$ as a directed graph and label the edge a by an element $Ga \in A$.

7.6 Lemma *$\Delta^{(1)}_{spec}$ as a labeled graph is isomorphic to a Cayley graph $C\mathcal{G}$ of a fundamental groupoid* $\mathcal{G} = \pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$.

Proof Fix a base vertex v_0 in $\Delta^{(1)}_{spec}$ and consider Gv_0 as a base point in $G\backslash\Delta$. A vertex in $\mathcal{G}$ is a homotopy class $[f]$ of paths from Gv_0 to some Gv. There is a unique lift $\widetilde{f}$ of f into Δ which begins at v_0. We send the vertex $[f]$ to the end of the path $\widetilde{f}$. Now if $[f] \overset{[a]}{\to} [f][a]$ is an edge in $C\mathcal{G}$ then there are unique lifts $\widetilde{f}, \widetilde{a}$ of $[f], [a]$ to Δ such that $\widetilde{a}$ starts at the end of $\widetilde{f}$. We map the edge $[f] \overset{[a]}{\to} [f][a]$ to the edge $\widetilde{a}$, labeled by $G\widetilde{a}$. This defines an isomorphism as required. □

7.7 Language

Recall that we label directed edges in $\Delta^{(1)}_{spec}$ in a G–equivariant way by A, so each path from $\mathcal{C}_{spec}$ spells out a word in A^*. Define a language L to be the subset of A^* which is given by all words which label the paths from combing $\mathcal{C}_{spec}$ starting at the basepoint v_0. It follows from the above discussion that we have a bijective map from L to morphisms in $\mathcal{G} = \pi_1\left(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec}\right)$.

Lemma *The language L over A determined by the combing $\mathcal{C}_{spec}$ is regular.*

Proof (cf [12], 6.1) We shall construct a non-deterministic finite state automaton M over A which has L as the set of acceptable words. The set of states of M is A; all states are initial states and all states are acceptable states. There is a transition labelled by Ge_1 from Ge_1 to Ge_2 if there are ordered edges $e_1', e_2' \in \Delta^{(1)}_{spec}$ in Ge_1, Ge_2 respectively, such that e_2' starts at the tail of e_1' and one of the following conditions holds:

1) both e_1', e_2' constitute a geodesic linear path of the length 2 in $\Delta^{(1)}_{spec}$, that is a local geodesicity condition is satisfied in the common vertex, see 3.6.

2) If e_1' is of type i and e_2' is of type j, and if e_1'' is the last edge in $\Delta^{(1)}$, lying on the segment e_1' and e_2'' is the first edge in $\Delta^{(1)}$ lying on e_2', then $(e_1'', e_2'') = (\frac{\bar\omega_i^\vee}{c_i}, \frac{\bar\omega_j^\vee}{c_j})$.

Since the condition defining the transitions is the same as in the local description of $\mathcal{C}$ in Section 6, the language L is precisely the language accepted by the finite state automaton.

7.8 Finishing the proof of the theorem 7.1

By Theorem 7.3 it is enough to show that the fundamental groupoid $\mathcal{G} = \pi_1(G\backslash\Delta, G\backslash\Delta^{(0)}_{spec})$ is automatic. Fix a base vertex Gv_0 for $G\backslash\Delta$, where $v_0 \in \Delta^{(0)}_{spec}$. Let A be an alphabet which is in one one correspondence with a finite generating set $G\backslash\Delta^{(1)}_{spec}$ of $\mathcal{G}$, see 7.5. Let $L \subseteq A^*$ be a language consisting of words which are spelled out from paths of $\mathcal{G}$. By the construction of a combing, see 4.1, L surjects onto $\mathcal{G}(v_0, *)$. By Section 7.7, L is regular. By Section 5 it satisfies k–fellow traveller property. Hence, in view of an isomorphism of A–labelled graphs $\Delta^{(1)}_{spec} \simeq C\mathcal{G}$ we get that $\mathcal{G}$ is biautomatic.

To prove the second assertion just note that the set of all orderings of a Euclidean building is finite and any group acting simplicially on a building, acts also on this finite set of orderings, hence it contains a subgroup of finite index which preserves any ordering, that is acts by a type preserving automorphisms. □

7.9 Remarks Actually, one can derive easily from Sections 4.3,4.4,4.5 that the structures we have built are quasigeodesic ones. On the other hand, as Prof W Neumann has pointed out to us, every (synchronous) automatic structure contains a sublanguage which is an (synchronous) automatic structure with the uniqueness property ([7] 2.5.1) and it follows ([7] 3.3.4) that (synchronous) automatic structures with uniqueness are always quasigeodesic.

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