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The Seiberg–Witten invariants and 4–manifolds with essential tori

Clifford Henry Taubes

Department of Mathematics
Harvard University
Cambridge, MA 02138, USA

Email: `chtaubes@math.harvard.edu`

Abstract

A formula is given for the Seiberg–Witten invariants of a 4–manifold that is cut along certain kinds of 3–dimensional tori. The formula involves a Seiberg–Witten invariant for each of the resulting pieces.

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1 Introduction

This article presents a proof of a cited but previously unpublished Mayer–Vietoris theorem for the Seiberg–Witten invariants of four dimensional manifolds which contain certain embedded 3–dimensional tori. This Mayer–Vietoris result is stated in a simple, but less than general form in Theorem 1.1, below. The more general statement is given in Theorem 2.7. Morgan, Mrowka and Szabó have a different (as yet unpublished) proof of this theorem. The various versions of the statement of Theorem 2.7 have been invoked by certain authors over the past few years (for example [16], [5, 6], [12]) and so it is past time for the appearance of its proof.

Note that when the 4–manifold in question is the product of the circle with a 3–manifold, then Theorem 2.7 implies a Mayer–Vietoris theorem for the 3–dimensional Seiberg–Witten invariants. A proof of the latter along different lines has been given by Lim [10].

The formulation given here of Theorems 1.1 and 2.7 is directly a consequence of conversations with Guowu Meng whose conceptual contributions deserve this special acknowledgment here at the very outset.

By way of background for the statement of Theorem 1.1, consider a compact, connected, oriented 4–manifold with $b^{2+} \geq 1$. Here, b^{2+} denotes the dimension of any maximal subspace of $H^2(X;\mathbb{R})$ on which the cup product pairing is positive definite. (Such a subspace will be denoted here by $H^{2+}(X;\mathbb{R})$.)

When $b^{2+} > 1$, then X has an unambiguous Seiberg–Witten invariant. In its simplest incarnation, the latter is a map from the set of $\mathrm{Spin}^{\mathbb{C}}$ structures on X to $\mathbb{Z}$ which is defined up to ± 1. Moreover, the sign is pinned with the choice of an orientation for the real line L_X which is the product of the top exterior power of $H^1(X;\mathbb{R})$ with that of $H^{2+}(X;\mathbb{R})$. The Seiberg–Witten invariants in the case where $b^{2+} = 1$ can also be defined, but with the extra choice of an orientation for $H^{2+}(X;\mathbb{R})$. In either case, the Seiberg–Witten invariants are defined via an algebraic count of solutions to a certain geometrically natural differential equation on X. (See [22], [11], [8].)

Now, imagine that $M \subset X$ is a compact, oriented 3–dimensional submanifold. Supposing that M splits X into two manifolds with boundary, X_+ and X_-, the problem at hand is to compute the Seiberg–Witten invariants for X in a Mayer–Vietoris like way in terms of certain invariants for X_+ , X_- and M. Such a formula exists in many cases (see, eg, [9], [15], [13], [17].) Theorem 1.1 addresses this problem in the case where:

- $M \subset X$ is a 3–dimensional torus.
- There is a class in $H^2(X;\mathbb{Z})$ with non-trivial restriction to $H^2(M;\mathbb{Z})$.

To consider the solution to this problem, invest a moment to discuss the structure of the set, $\mathbb{S}(X)$, of $\mathrm{Spin}^{\mathbb{C}}$ structures on X. In particular, remark that for any oriented 4–manifold X, this set $\mathbb{S}(X)$ is defined as the set of equivalence classes of pairs (Fr, F), where $Fr \to X$ is a principal $SO(4)$ reduction of the oriented, general linear frame bundle for TX, while F is a lift of Fr to a principal $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle. In this regard, remember that $SO(4)$ can be identified with $(SU(2) \times SU(2))/\{\pm 1\}$ in which case $\mathrm{Spin}^{\mathbb{C}}(4)$ appears as $(SU(2) \times SU(2) \times U(1))/\{\pm 1\}$. Here, $SU(2)$ is the group of 2×2, unitary matrices with determinant 1 and $U(1)$ is the circle, the group of unit length complex numbers. In any event, since $\mathrm{Spin}^{\mathbb{C}}(4)$ is an extension of $SO(4)$ by the circle, any lift, F, of Fr, projects back to Fr as a particularly homogeneous principal $U(1)$ bundle over Fr.

One can deduce from the preceding description of $\mathbb{S}(X)$ that the latter can be viewed in a canonical way as a principal $H^2(X;\mathbb{Z})$ bundle over a point. In particular, $\mathbb{S}(X)$ can be put in 1-1 correspondence with $H^2(X;\mathbb{Z})$, but no such correspondence is natural without choosing first a fiducial element in $\mathbb{S}(X)$. However, there is the canonical 'first Chern class' map

$$c\colon \mathbb{S}(X) \to H^2(X;\mathbb{Z}), \tag{1}$$

which is induced by the homomorphism from $\mathrm{Spin}^{C}(4)$ to $U(1)$ which forgets the $SU(2)$ factors. With respect to the $H^2(X;\mathbb{Z})$ action on $\mathbb{S}(X)$, the map c obeys

$$c(es) = e^2 c(s), \tag{2}$$

for any $e \in H^2(X;\mathbb{Z})$ and $s \in \mathbb{S}(X)$. Here, and below, the cohomology is viewed as a multiplicative group. Note that (2) implies that c is never onto, and not injective when there is 2–torsion in the second cohomology. By the way, c's image in the mod 2 cohomology is the second Stiefel–Whitney class of TX.

In any event, if X is a compact, oriented 4–manifold with $b^{2+} > 1$, then the Seiberg–Witten invariants define, via the map c in (1), a map

$$\underline{\mathrm{sw}}\colon H^2(X;\mathbb{Z}) \to \mathbb{Z},$$

which is defined up to ± 1 without any additional choices. That is, $\underline{\mathrm{sw}}(z) \equiv \sum_{s:c(s)=z} \mathrm{sw}(s)$, where $\mathrm{sw}(s)$ denotes the value of the Seiberg–Witten invariant on the class $s \in \mathbb{S}(X)$. Note that $\underline{\mathrm{sw}} = 0$ but for finitely many classes in $H^2(X;\mathbb{Z})$ if $b^{2+} > 1$.

Now, for a variety of reasons, it proves useful to package the map $\underline{\text{sw}}$ in a manner which will now be described. To start, introduce $\mathbb{Z}H^2(X;\mathbb{Z})$, the free $\mathbb{Z}$ module generated by the elements in the second cohomology. The notation in this regard is such that the abelian group structure on $H^2(X;\mathbb{Z})$ (as a vector space) is represented in a multiplicative fashion. For example, the identity element, 1, corresponds to the trivial class, and more generally, the vector space sum of two classes is represented as their product. With the preceding notation understood, a typical element in $\mathbb{Z}H^2(X;\mathbb{Z})$ consists of a formal sum $\sum a(z)z$, where the sum is over the classes $z \in H^2(X;\mathbb{Z})$ with $a(z) \in \mathbb{Z}$ being zero but for finitely many classes. Thus, a choice of basis over $\mathbb{Z}$ for $H^2(X;\mathbb{Z})$, makes elements of $\mathbb{Z}H^2(X;\mathbb{Z})$ into finite Laurent series.

With $\mathbb{Z}H^2(X;\mathbb{Z})$ understood, the invariant $\underline{\text{sw}}$ in the $b^{2+} > 1$ case can be packaged neatly as an element in $\mathbb{Z}H^2(X;\mathbb{Z})$, namely

$$\underline{\text{SW}}_X \equiv \sum_z \underline{\text{sw}}(z)z. \tag{3}$$

In the case where $b^{2+} = 1$, a choice of orientation for $H^{2+}(X;\mathbb{R})$ is needed to define $\underline{\text{sw}}$, and in this case the analog of (3) is a 'semi-infinite' power series rather than a finite Laurent series. In this regard, a power series such as $\sum_z a(z)z$ is termed semi-infinite with respect to a given generator of $H^{2+}(X;\mathbb{Z})$ when the following is true: For any real number m, only a finite set of classes $z \in H^2(X;\mathbb{Z})$ have both $a(z) \neq 0$ and cup product pairing less than m with the generator. In the case of $\underline{\text{SW}}_X$, the choice of a Riemannian metric and an orientation for $H^{2+}(X;\mathbb{Z})$ determines the generator in question. However, $\underline{\text{SW}}_X$ does not depend on the metric, it depends only on the chosen orientation of $H^{2+}(X;\mathbb{R})$. The associated, extended version of $\mathbb{Z}H^2(X;\mathbb{Z})$ which admits such power series will not be notationally distinguished from the original. In any event, when $b^{2+} = 1$, the extra choice of an orientation for $H^2(X;\mathbb{R})$ yields a natural definition of $\underline{\text{sw}}$ so that (3) makes good sense as an element in the extended $\mathbb{Z}H^2(X;\mathbb{Z})$.

Now, suppose that X is a compact, connected 4–manifold with boundary, ∂X, with each component of the latter being a 3–torus. Assume, in addition, that there is a fiducial class, ϖ, in $H^2(X;\mathbb{Z})$ whose pull-back is non-zero in the cohomology of each component of ∂X . Theorem 2.5 to come implies that such a manifold also has a Seiberg–Witten invariant, $\underline{\text{SW}}_X$, which lies either in $\mathbb{Z}H^2(X,\partial X;\mathbb{Z})$ or, in certain cases, a particular extension of this group ring which allows semi-infinite series. In this case, the extension in question consists of formal power series such as $\sum_z a(z)z$ where, for any given real number m, only a finite set of z's have both $a(z) \neq 0$ and cup product pairing less than m with ϖ. (This extension will not be notationally distinguished from $\mathbb{Z}H^2(X, \partial X; \mathbb{Z})$.) In any event, $\underline{\mathrm{SW}}$ is defined, as in the no boundary case, via an algebraic count of the solutions to a version of the Seiberg–Witten equations. This invariant is defined up to a sign with the choice of ϖ and the sign is fixed with the choice of an orientation for the line L_X which is the product of the top exterior power or $H^1(X, \partial X; \mathbb{R})$ with that of $H^{2+}(X, \partial X; \mathbb{R})$. Even in the non-empty boundary case, $\underline{\mathrm{SW}}$ is a diffeomorphism invariant.

By way of an example, the invariant $\underline{\mathrm{SW}}$ for the product, $D^2 \times T^2$, of the closed, 2–dimensional disk with the torus is $t(1-t^2)^{-1} = t + t^3 + \cdots$, where t is Poincaré dual to the class of the torus. For another example, take n to be a positive integer and let $E(n)$ denote the simply connected, minimal elliptic surface with no multiple fibers and holomorphic Euler characteristic n. The invariant $\underline{\mathrm{SW}}$ for the complement in $E(n)$ of an open, tubular neighborhood of a generic fiber is $(t - t^{-1})^{n-1}$, where t is the Poincaré dual of a fiber.

With the description of $\underline{\mathrm{SW}}$ in hand, a simple version of the promised Mayer–Vietoris formula can be stated. In this regard, mind that certain pairs of elements in $\mathbb{Z}H^2(X, \partial X; \mathbb{Z})$ can be multiplied together as formal power series. The multiplication rule used here is the evident one where $(\sum_z a(z)z)\cdot(\sum_z a'(z)z) \equiv \sum_z [\sum_{(w,x): wx = z} a(x)a'(w)]z$.

Theorem 1.1 *Let X be a compact, connected, oriented, 4–manifold with $b^{2+} = 1$ and with boundary consisting of a disjoint union of 3–dimensional tori. Let $M \subset X$ be an embedded, 3–dimensional torus and suppose that there is a fiducial class $\varpi \in H^2(X; \mathbb{R})$ whose pull-back is non-zero in the cohomology of M and in that of each component of the boundary of X.*

- *If M splits X as a pair, $X_+ \cup X_-$, of 4–manifolds with boundary, let $j_\pm$ denote the natural, $\mathbb{Z}$–linear extensions of the canonical homomorphisms from $H^2(X_\pm, \partial X_\pm; \mathbb{Z})$ to $H^2(X, \partial X; \mathbb{Z})$ which arise by coupling the excision isomorphism with those from the long exact cohomology sequences of the pairs $X_-, X_+ \subset X$. Then $j_-(\underline{\mathrm{SW}}_{X_-})$ and $j_+(\underline{\mathrm{SW}}_{X_+})$ can be multiplied together in $\mathbb{Z}H^2(X, \partial X; \mathbb{Z})$ and*
$$\underline{\mathrm{SW}}_X = j_-(\underline{\mathrm{SW}}_{X_-}) j_+(\underline{\mathrm{SW}}_{X_+}).$$
Here, the orientation for the line L_X is induced by chosen orientations for the analogous lines for X_+ and X_-. Also, if X is compact and $b^{2+} = 1$, then ϖ naturally defines the required orientation of $H^{2+}(X; \mathbb{R})$.
- *If M does not split X, introduce X_1 to denote the complement of a tubular neighborhood of M in X. In this case,*
$$\underline{\mathrm{SW}}_X = j(\underline{\mathrm{SW}}_{X_1}),$$

where j is the $\mathbb{Z}$–linear extension of the map from $H^2(X_1, \partial X_1; \mathbb{Z})$ to $H^2(X, \partial X; \mathbb{Z})$ which arises by coupling the excision isomorphism with the natural homomorphism from the long exact cohomology sequence of the pair $M \subset X$. In addition the orientation for the line L_X is induced by a chosen orientation of the analogous line for X_1. Finally, if X is compact and $b^{2+} = 1$, then ϖ naturally defines the needed orientation for $H^{2+}(X; \mathbb{R})$.

As remarked at the outset, this theorem has a somewhat more general version which is given as Theorem 2.7. The latter differs from Theorem 1.1 in that it discusses the Seiberg–Witten invariants proper rather than their averages over 2–torsion classes. In any event, Theorem 1.1 follows directly as a corollary to Theorem 2.7.

When X in Theorem 1.1 has the form $S^1 \times Y$, where Y is a 3–manifold, then the Seiberg–Witten invariants of X are the same as those that are defined for Y by counting solutions of a 3–dimensional version of the Seiberg–Witten equations. In this case, Theorems 1.1 and 2.7 imply Mayer–Vietoris theorems for the 3–dimensional Seiberg–Witten invariants. In particular, the 3–dimensional version of Theorem 1.1 is stated as Theorem 5.2 in [16]. As noted above, Lim [10] has a proof of the 3–dimensional version of Theorem 2.7.

Before ending this introduction, a two part elipsis is in order which may or may not (depending on the reader) put some perspective on the subsequent arguments which lead back to Theorem 1.1.

Part one of this elipsis addresses, in a sense, a raison d'etre for Theorem 1.1. To start, remark that the Seiberg–Witten invariants, like the Donaldson invariants [1], [23], follow the 'topological field theory' paradigm where Mayer–Vietoris like results are concerned. To elaborate: According to the topological field theory paradigm, the solutions to the 3–dimensional version of the Seiberg–Witten equations associate a vector space with inner product to each 3–manifold; and then the Seiberg–Witten equations on a 4–manifold with boundary are expected to supply a vector in the boundary vector space. Moreover, when two 4–manifolds with boundary are glued together across identical boundaries to make a compact, boundary free 4–manifold, the field theory paradigm has the Seiberg–Witten invariants of the latter equal to the inner product of the corresponding vectors in the boundary vector space.

Now, the fact is that the topological field theory paradigm is stretched somewhat when boundary 3–tori are present. However, in the situation at hand, which is to say when there is a 2–dimensional cohomology class with nonzero restriction in the cohomology of each boundary component, the paradigm is not unreasonable. In particular, the relevant boundary vector space is 1–dimensional and so the topological field theory paradigm predicts that the Seiberg–Witten invariants of the 4–manifolds with boundary under consideration here are simply numbers. And, when two such 4–manifolds are glued across identical boundaries, then the Seiberg–Witten invariants of the result should be the product of the invariants for the pair. This last conclusion is, more or less, exactly what Theorem 1.1 states.

By the way, the essentially multiplicative form of the Mayer–Vietoris gluing theorems in [15] have an identical topological field theoretic 'explanation.'

Part two of this elipsis concerns the just mentioned [15] paper. The latter produced a Mayer–Vietoris gluing theorem for certain Seiberg–Witten invariants of a 4–manifold cut along the product of a genus two or more surface and a circle. In particular, [15] considers only $Spin^{\mathbb{C}}$ structures s whose class $c(s)$ evaluates on the surface to give two less than twice the genus; and [15] states a gluing theorem which is the genus greater than one analog of those given here. The case of genus one was not treated in [15] because the genus one case requires some special arguments. This paper gives a part of the genus one story. Meanwhile, other aspects of the genus one cases, Dehn surgery like gluings in particular, can be handled using the results in [13].

By the way, a version of the gluing theorem in the surface genus greater than one context of [15] is used in [14].

The introduction ends with the list that follows of the section headings.

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2 The Seiberg–Witten invariants

This section provides a review of the definition of the Seiberg–Witten invariants for compact 4–manifolds, and then extends the definition in Theorem 2.5 to cover the cases which are described in the introduction. It ends with the statement of Theorem 2.7, which is the principle result of this article.

a) The differential equation

In what follows, X is an oriented, Riemannian 4–manifold which can be non-compact. But, if the latter is the case, assume that there is a compact 4–manifold with boundary $X_0 \subset X$ whose boundary, ∂X_0, is a disjoint union of 3–tori and whose complement is isometric to the half infinite cylinder $[0, \infty) \times \partial X_0$. To be precise, the metric on the $[0, \infty)$ factor should be the standard metric and the metric on ∂X_0 should be a flat metric. (Unless stated to the contrary, all metrics under consideration on T^3 will be flat.) The letter 's' is used below to denote a fixed function on X which restricts to $[0,\infty)\times\partial X_0$ as the standard Euclidean coordinate on the factor $[0,\infty)$.

With the metric given, a $\mathrm{Spin}^{\mathbb{C}}$ structure is simply a lift (up to obvious equivalences) to a $\mathrm{Spin}^{\mathbb{C}}(4)$ principal bundle of the bundle $Fr \to X$ of oriented, orthonormal frames in the tangent bundle to X. Let $\mathbb{S}_0(X_0)\subset\mathbb{S}(X)$ denote the subset of $\mathrm{Spin}^{\mathbb{C}}$ structures s with $c(s)=0$ on ∂X_0. Choose a $\mathrm{Spin}^{\mathbb{C}}$ structure $s\in\mathbb{S}_0(X_0)$.

Associated to s's principal $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle $F\to X$ are a pair of the $\mathbb{C}^2$ vector bundles $S_\pm\to X$ as well as the complex line bundle $K\to X$. Here, S_+ arises from the group homomorphism which sends $\mathrm{Spin}^{\mathbb{C}}(4)=(SU(2)\times SU(2)\times U(1))/\{\pm1\}$ to $U(2)=(SU(2)\times U(1))/\{\pm1\}$ by forgetting the first $SU(2)$ factor. Meanwhile, S_- arises from the homomorphism to $U(2)$ which forgets the second factor; and K arises by forgetting both factors. Note that $K=\det(S_+)=\det(S_-)$ and the first Chern class of K is the class $c(s)$.

Fix a self-dual 2–form ω on X which is non-zero and covariantly constant on each component of $[0,\infty)\times\partial X_0$. This is to say that the restriction of ω to such a component has the form $\omega=ds\wedge\theta+\omega_0$, where ω_0 is a non-zero, covariantly constant 2–form on T^3 and θ is the metric dual to ω_0.

Consider now the set of smooth configurations (A,ψ) consisting of a connection, A, on $\det(S_+)$ and a section ψ of S_+ which solve the equations

$$\begin{aligned}&\bullet\ P_+F_A=\tau(\psi\otimes\psi^\dagger)-i\cdot\omega;\\ &\bullet\ D_A\psi=0;\\ &\bullet\ \int_X|F_A|^2<\infty.\end{aligned}\tag{4}$$

The notation in (4) is as follows:

- F_A denotes the curvature 2–form of the connection A.
- P_+ denotes the metric's orthogonal projection from the bundle of 2–forms to the bundle, Λ_+, of self dual 2–forms. (The latter is associated to the bundle $Fr\to X$ via the representation from $SO(4)=(SU(2)\times SU(2))/\{\pm1\}$ to $SO(3)=SU(2)/\{\pm1\}$ which forgets the first $SU(2)$ factor. There is, of course, the bundle, Λ_-, of anti-self dual 2–forms that is obtained via the representation to $SO(3)$ which forgets the second $SU(2)$ factor.)
- τ denotes the homomorphism from $\mathrm{End}(S_+)=S_+\otimes S_+^*$ which is the hermitian adjoint to the Clifford multiplication homomorphism from Λ_+ into $\mathrm{End}(S_+)$.

- D_A denotes a version of the Dirac operator. In particular, D_A is the first order, elliptic operator which sends a section of S_+ to one of S_- by composing a certain A–dependent covariant derivative on S_+ with the Clifford multiplication endomorphism from $S_+ \otimes T^*X$ to S_-. Here, the covariant derivative is defined from the connection on F which is obtained by coupling the connection A with the pull-back from Fr of the metric's Levi–Civita connection.
- In the last point of (4), the norm and the implicit volume form are defined by the given Riemannian metric.

A certain algebraic count of the solutions to (4) gives the Seiberg–Witten invariants.

b) A topology on the set of solutions

The set of solutions to (4) is topologized as follows: Fix a base connection A_b on $\det(S_+)$ which is flat on $[0,\infty)\times\partial X_0$. With A_b fixed, the set of connections on $\det(S_+)$ can be identified with the space of smooth, imaginary valued 1–forms, $i{\cdot}\Omega^1$. With the preceding understood, the space of solutions to (4) is topologized by its embedding in the Fréchet space $i\cdot\Omega^1 \oplus C^\infty(S_+)\oplus\mathbb{R}$ which sends (A,ψ) to $(A-A_b, \psi, \int_X |F_A|^2)$. In this regard, the vector spaces Ω^1 and $C^\infty(S_+)$ are topologized by the weak C^∞ topology in which a typical neighborhood of 0 is the space of sections which are small in the C^k topology for some finite k on some compact subset of X.

Note that the group $C^\infty(X;S^1)$ acts continuously on the space of solutions to (4) if this group is given the weak C^∞ Fréchet structure in which a pair of maps are close if they are C^k close for some finite k on some compact subset of X. Here, $\varphi\in C^\infty(X;S^1)$ sends a pair (A,ψ) to $(A-2\varphi^{-1}d\varphi, \varphi\psi)$. The quotient of the space of solutions by this action (with the quotient topology) will be called the *moduli space* of solutions to (4). An orbit of $C^\infty(X;S^1)$ will be called a 'gauge orbit' and two solutions on the same gauge orbit will be deemed 'gauge equivalent.' Except where confusion appears likely, a pair (A,ψ) and its gauge orbit will not be notationally distinguished.

Before embarking on a detailed discussion of the structure of the moduli space of solutions to (4), some remarks are in order which concern an important consequence of the constraint given by the third point in (4). In particular, if A is any connection on K, then up to factors of 2π and $i=\sqrt{-1}$, the curvature, F_A, is a closed 2–form on X whose cohomology class gives $c(s)$, the first Chern class of K. Now, by assumption, $c(s)$ restricts to zero on the ends of X and thus lies in the image of the natural homomorphism from $H^2(X_0, \partial X_0; \mathbb{Z})$ in $H^2(X; \mathbb{Z})$. And, if F_A is square integrable on X, arguments to follow prove that F_A canonically defines a preimage, c_A, of $c(s)$ in $H^2(X_0, \partial X_0;\, \mathbb{Z})$.

The construction of c_A employs an application of the abelian version of Uhlenbeck's compactness theorem [21] as follows: Use s to denote the Euclidean coordinate on the half line factor of $[0, \infty) \times \partial X_0$. Then, for large $s_0 \in [0, \infty)$, Uhlenbeck's theorem insures that for any $s > s_0$, the connection A restricts to the cylinder $[s, s+1] \times \partial X_0$ as $A^s_f + a_s$, where A^s_f is a flat connection on ∂X_0 and where a_s is an imaginary 1–form on $[s, s+1] \times \partial X_0$. Moreover, according to Uhlenbeck's theorem the sequence, indexed by $s \in [s_0, \infty)$, of the L^2_1 norms of a_s over the defining domains, $[s, s+1] \times \partial X_0$, converges to zero as s tends to infinity. Now, with the preceding understood, fix a sequence, $\{\beta_s\}$ of 'cut-off' functions on $[0, \infty)$, indexed by $s \in [s_0, \infty)$, with $\beta_s = 1$ on $[0, s]$, $\beta_s = 0$ on $[s+1, \infty)$ and $|\beta_s'| < 2$ everywhere. Then, for $s > s_0$, introduce the connection A^s on K that equals $A^s_f + \beta_s a_s$ on $[s, \infty) \times \partial X_0$ and A everywhere else. By construction, the curvature 2–form of A^s is zero on $[s+1, \infty) \times \partial X_0$. Thus, this 2–form gives a bona fide class in the relative cohomology group $H^2(X, [s+1, \infty) \times \partial X_0; \mathbb{Z})$. And, as the latter group is canonically isomorphic to $H^2(X_0, \partial X_0; \mathbb{Z})$, the curvature 2–form of A^s defines a class in this last group as well. When s is sufficiently large, the latter class is the desired c_A.

Of course, this definition of c_A makes sense provided that the $s \in [s_0, \infty)$ indexed set of curvature 2–forms $\{F_{A^s}\}$ give identical classes in $H^2(X_0, \partial X_0; \mathbb{Z})$ when s is large. To prove that such is the case, consider the classes indexed by some pair s and $s+\delta$ with $\delta \in (0, 1)$. The corresponding curvatures both define integral classes in the relative group $H^2(X, [s+\delta+1) \times \partial X_0; \mathbb{Z})$ and it is sufficient to prove that these two classes agree. In particular, since A^s and $A^{s+\delta}$ agree on $X - ([s, \infty) \times \partial X_0)$ and since $H^*(T^3; \mathbb{Z})$ has no torsion, such will be the case if the composition of exterior product and then integration over X pairs both curvature 2–forms identically with a given set of closed 2–forms on X that generate $H^2(X; \mathbb{Z})$. And, for this purpose, it is enough to take a generating set of forms which are covariantly constant on $[0, \infty) \times \partial X_0$.

With these last points understood, it then follows that the relevant cohomology classes agree if the curvature forms in question are close in the L^2 sense on $[s, s+2] \times \partial X_0$ since these curvature forms agree on the complement of $[s, \infty) \times \partial X_0$. Of course, these forms are L^2 close (when s is large) since each separately has small L^2 norm on $[s, s+2] \times \partial X_0$ by virtue of the fact that the 1–forms a_s and $a_{s+\delta}$ have small L^2_1 norms on this same cylinder.

With only minor modifications, the preceding argument that c_A is well defined yields the following:

Lemma 2.1 *Let $\mathcal{A}$ denote the space of smooth connections on K whose curvature 2–form is square integrable, here endowed with the smallest topology for which the assignment of A to $\int_X |F_A|^2$ is continuous and which allows C^∞ convergence on compact sets. Then, the assignment of c_A to $A \in \mathcal{A}$ defines a locally constant function on $\mathcal{A}$. In particular, the connected components of the moduli space $\mathcal{M}(s)$ are labeled, in part, by the set of elements in $H^2(X_0, \partial X_0; \mathbb{Z})$ which map to $c(s)$ under the natural homomorphism to $H^2(X; \mathbb{Z})$.*

With Lemma 2.1 in mind and supposing that $s \in \mathcal{S}_0(X)$ and a preimage, z, of $c(s)$ in $H^2(X_0, \partial X_0; \mathbb{Z})$ have been given, introduce $\mathcal{M} \equiv \mathcal{M}(s, z)$ to denote the subspace of pairs (A, ψ) in the moduli space of solutions to the s version of (4) for which $c_A = z$.

c) The structure of $\mathcal{M}$

The local structure of $\mathcal{M}$ is described in the next proposition. However, the statement of this proposition requires a preliminary digression to point out certain topological features of X. To start the digression, note that there is an integer valued, bilinear pairing on $H^2(X_0, \partial X_0; \mathbb{Z})$ which is obtained by composing the cup-product map to $H^4(X_0, \partial X_0; \mathbb{Z})$ with evaluation on the fundamental class. In contrast to the case where $\partial X_0 = \varnothing$, this form has a null space in the non-empty boundary case, that being the image of $H^1(\partial X_0)$ via the natural connecting homomorphism of the long exact sequence for the pair $(X_0, \partial X_0)$. Thus, the cup product pairing is both well defined and non-degenerate on the image in $H^2(X_0)$ of $H^2(X_0, \partial X_0)$. Use $z \bullet z'$ to denote the cup product pairing between classes z and z'. Meanwhile, use $H^{2+}(X_0, \partial X_0; \mathbb{R}) \subset H^2(X_0, \partial X_0; \mathbb{R})$ to denote a maximal dimensional vector subspace on which this cup product pairing is positive definite and use $b^{2+}(X_0)$ to denote the dimension of $H^{2+}(X_0, \partial X_0; \mathbb{R})$. Also, use τ to denote the signature of the cup product pairing. Thus, $\tau = b^{2+} - b^{2-}$, where b^{2-} is the dimension of the maximal vector subspace in $H^2(X_0, \partial X_0)$ where the cup product pairing is negative definite. Finally, the digression ends by introducing b^1_0 to denote the dimension of $H^1(X_0, \partial X_0; \mathbb{R})$.

Here is the promised local structure result:

Proposition 2.2 *Let $c \equiv (A, \psi) \in \mathcal{M}$. Then, there exists a Fredholm operator $\mathcal{D}_c$ of index $d \equiv b^1_0 - 1 - b^{2+} + 4^{-1}(c(s) \bullet c(s) - \tau)$; a real analytic map f, from a ball in the kernel of $\mathcal{D}_c$ to the cokernel of $\mathcal{D}_c$ mapping the origin in the ball to the origin in the cokernel of $\mathcal{D}_c$; and, provided that ψ is not identically zero, a homeomorphism, φ, from $f^{-1}(0)$ onto an open neighborhood of c in $\mathcal{M}$.*

Note that when $\partial X_0 \neq \varnothing$, there are no solutions to (4) where ψ is identically zero.

For $c = (A, \psi)$, the operator $\mathcal{D}_c$ is a differential operator which is initially defined to send $(b, \eta) \in iC^\infty(T^*X) \oplus C^\infty(S_+)$ to the element in $iC^\infty(\mathbb{R}) \oplus iC^\infty(\Lambda_+) \oplus C^\infty(S_-)$ whose components in the three summands are as follows:

$$\begin{aligned} &d^*b - 2(\psi^\dagger\eta - \eta^\dagger\psi); \\ &P_+db - \tau'(\eta \oplus \psi^\dagger + \psi \oplus \eta^\dagger); \\ &D_A\eta + \mathrm{cl}(b)\psi. \end{aligned} \tag{5}$$

Here, τ' denotes the polarization of the bilinear form τ which appears in (4) and $\mathrm{cl}(\cdot)$ denotes the Clifford multiplication endomorphism from T^*X to $\mathrm{Hom}(S_+, S_-)$. To make $\mathcal{D}_c$ Fredholm, a preliminary domain and range are defined to allow only sections with compact support. This preliminary domain is then completed using the Sobolev L^2_1 norm, while the preliminary range is completed using the Sobolev L^2 norm.

Under favorable conditions, the local neighborhoods described in Proposition 2.2 fit nicely together to give $\mathcal{M}$ the structure of a smooth d–dimensional manifold. The following proposition elaborates:

Proposition 2.3 *With reference to the previous proposition, the set of points in $\mathcal{M}$ where the cokernel of the operator $\mathcal{D}_c$ is $\{0\}$ has the structure of a smooth, d–dimensional manifold whereby the homeomorphism φ is a smooth coordinate chart. In addition, this last portion of $\mathcal{M}$ is orientable and canonically so with a choice of orientation for the line $\Lambda^{\mathrm{top}}H^1(X_0, \partial X_0; \mathbb{R}) \otimes \Lambda^{\mathrm{top}}H^{2+}(X_0, \partial X_0; \mathbb{R})$. Finally, if $\partial X_0 \neq \varnothing$ or if $b^{2+} > 0$, then there exists a Baire subset of choices for the 2–form ω in (4) for which the corresponding $\mathcal{M}$ is everywhere a smooth manifold. In fact, given an open set U in X_0 whose closure is disjoint from ∂X_0, and a self-dual form ω' on X_0 that has the required structure near ∂X_0, there is a Baire set of smooth, self-dual extensions, ω, of ω' from $X_0 - U$ to the whole of X_0 for which this same conclusion holds. For such ω, $\mathcal{M} = \varnothing$ when $d < 0$; and when $d \geq 0$, then the cokernel of $\mathcal{D}_c$ is trivial for every $c \in \mathcal{M}$.*

The proofs of these last two propositions are given in Section 3 of this paper.

In what follows, a point $c \in \mathcal{M}$ will be called a ‘smooth point’ when the cokernel of the operator $\mathcal{D}_c$ is trivial.

d) Compactness properties

In the case where X_0 has no boundary, the moduli space $\mathcal{M}$ is compact; this compactness is one of the remarkable features of the Seiberg–Witten equations. However, even the simplest example with non-empty boundary, $T^2 \times D^2$, can yield non-compact moduli spaces. Even so, certain zero and 1–dimensional subspaces of $\mathcal{M}$ are compact if the form ω is suitably chosen.

A two part digression follows as a preliminary to the specification of the constraints on ω.

Part 1 Remember that X_0 is assumed to have a class $\varpi \in H^2(X_0;\mathbb{R})$ which is non-zero in the cohomology of each component of ∂X_0. Meanwhile, as the chosen 2–form ω is constant on each component of ∂X_0, it defines a cohomology class, $[\omega] \in H^2(\partial X_0;\mathbb{R})$. With this understood, say that ω is *tamed* by ϖ when $[\omega] = \varpi$ in $H^2(\partial X_0;\mathbb{R})$.

Part 2 Introduce $\varsigma(s) \subset H^2(X_0, \partial X_0;\mathbb{Z})$ to denote the set of elements which map to $c(s)$ in $H^2(X_0;\mathbb{Z})$. Next, introduce the sets

$$
\begin{aligned}
\mathcal{M}_s &\equiv \bigcup_{z\in\varsigma(s)} \mathcal{M}(s,z) \text{ and}\\
\mathcal{M}_{s,m} &\equiv \bigcup_{z\in\varsigma(s):z\bullet\varpi\le m} \mathcal{M}(s,z) \subset \mathcal{M}_s.
\end{aligned}
\tag{6}
$$

Endow $\mathcal{M}_s$ with the topology of C^∞ convergence on compact subsets of X and give $\mathcal{M}_{s,m}$ the subspace topology. This is the topology which arises by embedding the space of solutions to (4) in the Fréchet space $i \cdot \Omega^1 \oplus C^\infty(S_+)$ with the latter given the C^∞ weak topology. Any $\mathcal{M}(s,z) \subset \mathcal{M}_s$ will be called a *stratum* of $\mathcal{M}_s$.

With $\mathcal{M}_s$ and each $\mathcal{M}_{s,m}$ understood, here is the most that can be said at this point about compactness:

Proposition 2.4 *Let $\varpi \in H^2(X_0;\mathbb{R})$ be a class with non-zero pull-back to the cohomology of each component of ∂X_0. With ϖ given, use a form ω in* (4) *that is tamed by ϖ. Then each $\mathcal{M}_{s,m} \subset \mathcal{M}_s$ is compact and contains only a finite number of strata. Moreover, fix a self-dual form ω' that is non-zero and covariantly constant on each component of $[0,\infty) \times \partial X_0$ and that is tamed by ϖ; and fix a non-empty, open set $U \subset X_0$. Then, there is a Baire set of smooth, self-dual forms ω that agree with ω' on $X - U$ and have the following properties:*

- *As in Proposition 2.3, each stratum of $\mathcal{M}_s$ is a smooth manifold of dimension d given in Proposition 2.2. Moreover, the cokernel of the operator $\mathcal{D}_c$ vanishes for each $c \in \mathcal{M}_s$.*
- *The boundary of the closure in $\mathcal{M}_s$ of any stratum intersects the remaining strata as a codimension 2 submanifold.*

Roughly said, Proposition 2.4 guarantees the compactness of the zero set in $\mathcal{M}(s,z)$ of a reasonably chosen section of a d or $(d-1)$–dimensional vector bundle over $\mathcal{M}_{s,m}$.

By the way, it turns out that an extra cohomology condition on the class ϖ guarantees the compactness of the whole of each $\mathcal{M}(s,z)$. Indeed, this pleasant situation arises when the restriction of ω to each component of ∂X_0 defines a cohomology class which is not a linear multiple of an integral class. Proposition 4.6 gives the formal statement.

Proposition 2.4 is proved in Section 6a.

e) The definition of the Seiberg–Witten invariants

The simplest version of the Seiberg–Witten invariant for X_0 associates an integer to a pair $s \in \mathcal{S}_0(X)$ and $z \in H^2(X_0, \partial X_0; \mathbb{Z})$ mapping to $c(s)$. This integer will be denoted by $\mathrm{sw}(s,z)$. As in the empty boundary case, it is obtained via an algebraic count of the elements in $\mathcal{M}$. However, there are some additional subtleties when $\partial X_0 \neq \varnothing$ because $\mathcal{M}$ need not be compact.

In what follows, X_0 is as described above except that positivity of b^{2+} will be implicitly assumed when $\partial X_0 = \varnothing$. Fix a $\mathrm{Spin}^{\mathbb{C}}$ structure $s \in \mathcal{S}_0(X_0)$ and a class $z \in \varsigma(s) \subset H^2(X_0, \partial X_0; \mathbb{Z})$. Note that s provides the integer d in Proposition 2.2. Also, fix a class $\varpi \in H^2(X_0; \mathbb{R})$ which is non-zero in the cohomology of each component of ∂X_0. Finally, orient $L \equiv \Lambda^{\mathrm{top}} H^1(X_0, \partial X_0; \mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X_0, \partial X_0; \mathbb{R})$ and, when $\partial X_0 = \varnothing$ and $b^{2+} = 1$, orient $H^{2+}(X_0; \mathbb{R})$.

What follows is the definition of sw.

Case 1 This case has either $d < 0$ or d odd. Set $\mathrm{sw}(s,z) = 0$ in this case.

Case 2 This case has $d = 0$. Choose a form ω in (4) which is tamed by ϖ and which is such that each stratum of $\mathcal{M}_s$ has the structure described in Propositions 2.3 and 2.4. The latter insure that $\mathcal{M} = \mathcal{M}(s,z)$ is a finite set of points. In addition, each point $c \in \mathcal{M}$ comes with a sign, $\varepsilon(c) \in \{\pm 1\}$, from the orientation. With these points understood, set

$$\mathrm{sw}(s,z) \equiv \sum_c \varepsilon(c), \tag{7}$$

where the sum is taken over all $c \in \mathcal{M}$.

Case 3 This case has $d > 0$ and even. Once again, choose a form ω in (4) which is tamed by ϖ and which is such that $\mathcal{M}_s$ has the structure described in Propositions 2.3 and 2.4. Thus, each stratum of $\mathcal{M}_s$ is an oriented, d–dimensional manifold.

Next, choose a set, $\Lambda \subset X$, of $d/2$ distinct points, and for each $x \in \Lambda$, specify a $\mathbb{C}$–linear surjection $\Phi_x\colon S_+|_x \to \mathbb{C}$. Use $\underline{\Lambda}$ to denote the resulting set of $d/2$ pairs (x, Φ_x). With $\underline{\Lambda}$ understood, set

$$\mathcal{M}^{\underline{\Lambda}} \equiv \{c = (A, \psi) \in \mathcal{M} : \Phi_z(\psi(x)) = 0 \text{ for each } x \in \underline{\Lambda}\}. \tag{8}$$

Note that $\mathcal{M}^{\underline{\Lambda}}$ can be viewed as the zero set of a smooth section of a $d/2$–dimensional complex vector bundle over $\mathcal{M}$. This understood, Sard's theorem guarantees that $\mathcal{M}^{\underline{\Lambda}}$ is discrete for a Baire set of data $\underline{\Lambda}$, and each $c \in \mathcal{M}^{\underline{\Lambda}}$ comes with a sign, $\varepsilon(c) \in \{\pm 1\}$. Moreover, Proposition 2.4 guarantees that this Baire set can be found so that the corresponding $\mathcal{M}^{\underline{\Lambda}}$ is a finite set.

With $\underline{\Lambda}$ now chosen from the afore mentioned Baire set of possibilities, define $\mathrm{sw}(s, z)$ by (7) but with the sum restricted to those c in the set $\mathcal{M}^{\underline{\Lambda}}$.

When X is compact, there also exists an extension of sw whose image is in $\Lambda^* H^1(X_0; \mathbb{Z}) \equiv \mathbb{Z} \oplus H^1 \oplus \Lambda^2 H^1 \oplus \cdots$. The latter is described in [20] and the definition there can be readily adapted to the non-compact setting described here. Theorems 2.5 and 2.7 below have reasonably self-evident analogs which apply to this extended sw. However, to prevent an already long paper from getting longer, the extended version of sw will not be discussed further here. Thus, the statements of the versions of Theorems 2.5 and 2.7 that apply to the extended sw are left to the reader to supply.

f) Invariance of the Seiberg–Witten invariants

With $\mathrm{sw}(\cdot)$ so defined, there is an obvious question to address: To what extent does $\mathrm{sw}(\cdot)$ depend on the various choices that enter its definition?

In the case where X_0 is compact, the following answer is well known (the arguments are given in [22], but see also [11], [8]):

- *If $b^{2+} > 1$, then* sw *is independent of the choice of Riemannian metric and form ω; its absolute value depends only on the* $\mathrm{Spin}^{\mathbb{C}}$ *structure, and the sign is determined by the orientation of the line $L \equiv \Lambda^{\mathrm{top}} H^1(X_0; \mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X_0; \mathbb{R})$. Moreover, $\mathrm{sw}(\varphi^* s) = \mathrm{sw}(s)$ when φ is a diffeomorphism of X_0 which preserves the orientation of the line L.*

- *If $b^{2+} = 1$, first specify an orientation of $H^{2+}(X_0;\mathbb{R})$. Then,* sw *is independent of the choice of Riemannian metric and form ω provided that the integral over X_0 of the wedge of ω with an oriented harmonic representative of $H^{2+}(X_0;\mathbb{R})$ is sufficiently large and positive. So defined, the absolute value of* sw *only depends on the* $\mathrm{Spin}^{\mathbb{C}}$ *structure and the orientation of $H^{2+}(X_0,\mathbb{R})$, and its sign is determined by the orientation of the line L. Furthermore,* $\mathrm{sw}(\varphi^* s) = \mathrm{sw}(s)$ *when φ is a diffeomorphism of X_0 which preserves the orientations of the line L and $H^{2+}(X_0;\mathbb{R})$.*

The next result provides an answer to the opening question in the case where ∂X_0 is not empty.

Theorem 2.5 *Suppose that $\partial X_0 \neq \varnothing$. First, choose a class $\varpi \in H^2(X_0;\mathbb{R})$ which is non-zero in the cohomology of each component of ∂X_0. Next, define* sw *on a pair (s, z) using ϖ as described in the preceding subsection. Then, the result is independent of the chosen metric and the form ω provided that the latter is tamed by ϖ. Here, the absolute value of* sw *is determined solely by the triple (s, z, ϖ) and the sign is determined by the chosen orientation for the line $L \equiv \Lambda^{\mathrm{top}} H^1(X_0, \partial X_0;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X_0, \partial X_0;\mathbb{R})$. Moreover, if φ is a diffeomorphism of X_0 which fixes the orientation of the line L, then the value of* sw *on $\varphi^*(s, z, \varpi)$ is the same as its value on (s, z, ϖ). Finally,* sw *is insensitive to continuous deformation of ϖ in $H^2(X_0;\mathbb{R})$ through classes with non-zero restriction in the cohomology of each component of ∂X_0.*

The proof of Theorem 2.5 is provided in Section 3d.

g) The Mayer–Vietoris gluing theorems

The purpose of this subsection is to state the advertised generalization of the Mayer–Vietoris gluing result given by Theorem 1.1. This generalization is summarized in Theorem 2.7, below, but a four-part digression comes first to set the stage.

Part 1 In what follows, X_0 is a compact, oriented 4–manifold with boundary such that each component of ∂X_0 is a 3–torus. Suppose next that there is an embedded 3–torus $M \subset X_0$ which separates X_0 so that $X_0 = X_+ \cup X_-$, where $X_\pm$ are 4–manifolds with boundary embedded in X which intersect in M.

With the preceding set up understood, introduce the lines L_0, L_+ and L_- via

$$L_\diamond \equiv \Lambda^{\mathrm{top}} H^1(X_\diamond, \partial X_\diamond;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X_\diamond, \partial X_\diamond;\mathbb{R}), \qquad (9)$$

where $\diamondsuit$ is a stand in for 0, $+$ or $-$. An argument from [15] can be adapted almost verbatim to establish the existence of a canonical isomorphism

$$L_0 \approx L_+ \otimes L_-. \tag{10}$$

Thus, orientations of L_+ and L_- canonically induce an orientation of L_0.

If $M \subset X_0$ is non-separating, introduce X_1 to denote the complement in X_0 of a tubular neighborhood of M. Then, the afore-mentioned argument from [15] adapts readily to establish the existence of a canonical isomorphism between L_0 from (9) and $L_1 \equiv \Lambda^{\mathrm{top}} H^1(X_1, \partial X_1; \mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X_1, \partial X_1; \mathbb{R})$.

Part 2 By assumption, there is a class $\varpi \in H^2(X_0; \mathbb{R})$ whose pull-back is not zero in the cohomology of each component of ∂X_0. Theorem 2.7, below, will assume that the pull-back of ϖ to the cohomology of M is also non-zero. With this understood, then the pull-back of ϖ will be non-zero in the cohomology of each component of ∂X_+ and each component of ∂X_- in the case when $M \subset X$ is separating. Likewise, when M is not separating, then the pull-back of ϖ in the cohomology of each component of ∂X_1 will be non-zero.

By the way, in the case when X_0 is compact and has $b^{2+} = 1$, the choice of a class $\varpi \in H^2(X_0; \mathbb{R})$ whose pull-back to the cohomology of M is non-zero supplies an orientation for $H^{2+}(X_0; \mathbb{Z})$. Indeed, because $\varpi \neq 0 \in H^2(M; \mathbb{Z})$, there is a class in $H_2(M; \mathbb{Z})$ whose push-forward in $H_2(X_0; \mathbb{Z})$ is non-zero and which pairs positively with ϖ. This homology class has self-intersection number zero, so its image in $H^2(X_0; \mathbb{Z})$ lies on the 'light cone.' Thus, the latter's direction specifies an orientation to any line in $H^2(X_0; \mathbb{R})$ on which the cup-product pairing is positive definite.

Part 3 This part of the digression contains the instructions for the construction of a $\mathrm{Spin}^{\mathbb{C}}$ structure on X from what is given on $X_\pm$ or X_1. To start, consider a somewhat abstract situation where Y is a smooth, oriented 4–manifold and $U \subset Y$ is any set. Having defined $\mathcal{S}(Y)$ as in the introduction, define $\mathcal{S}(U)$ to denote the equivalence class of pairs $(Fr|_U, F_U)$, where $Fr|_U$ is a principal $SO(4)$ reduction of the restriction of the oriented, general linear frame bundle of Y to U, and where F_U is a lift of $Fr|_U$ to a principal $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle. This definition provides a tautological pull-back map $\mathcal{S}(Y) \to \mathcal{S}(U)$ which intertwines the action of $H^2(Y; \mathbb{Z})$ with that of its image in $H^2(U; \mathbb{Z})$.

With the preceding understood, let $\mathcal{S}_{0M}(X_0) \subset \mathcal{S}_0(X_0)$ denote the set of $\mathrm{Spin}^{\mathbb{C}}$ structures whose image under c in (1) is zero under pull-back to the cohomology of M. When M separates X_0, then the pull-back map from the preceding paragraph defines a map $\wp^0 \colon \mathcal{S}_{0M}(X_0) \to \mathcal{S}_0(X_-) \times \mathcal{S}_0(X_+)$. Meanwhile, in the case where M is non-separating, there is the analogous $\wp^0 \colon \mathcal{S}_{0M}(X_0) \to \mathcal{S}_0(X_1)$.

In this regard, note that $\mathcal{S}_{0M}(X_0)$ is a principal homogeneous space for the image in $H^2(X_0;\mathbb{Z})$ of $H^2(X_0, M;\mathbb{Z})$ and the map $\wp$ intertwines the action of the latter group with its image in either $H^2(X_-;\mathbb{Z}) \times H^2(X_+;\mathbb{Z})$ or $H^2(X_1;\mathbb{Z})$ as the case may be.

With $\wp^0$ understood, the question arises as to the sense in which it can be inverted. The answer requires the introduction of some additional terminology. For this purpose, let Y be a compact, oriented 4–manifold with boundary which is a disjoint union of tori. Introduce $\mathcal{S}_0(Y, \partial Y)$ to denote the set of pairs (s, z) where $s \in \mathcal{S}_0(Y)$ and where $z \in H^2(Y, \partial Y;\mathbb{Z})$ maps to $c(s) \in H^2(Y;\mathbb{Z})$ under the long exact sequence homomorphism. Note that $\mathcal{S}_0(Y, \partial Y)$ is a principal homogeneous space for the group $H^2(Y, \partial Y;\mathbb{Z})$. Perhaps it is needless to say that there is a tautological 'forgetful' map from $\mathcal{S}_0(Y, \partial Y)$ to $\mathcal{S}_0(Y)$ which intertwines the action of $H^2(Y, \partial Y;\mathbb{Z})$ with that of its image in $H^2(Y;\mathbb{Z})$.

With the new terminology in hand, consider:

Lemma 2.6 *Depending on whether M does or does not separate X_0, there is a canonical map, $\wp$, from $\mathcal{S}_0(X_-, \partial X_-) \times \mathcal{S}_0(X_+, \partial X_+)$ or $\mathcal{S}_0(X_1, \partial X_1)$ into $\mathcal{S}_0(X_0, \partial X_0)$ respectively, which has the following properties:*

- *The image of $\wp$ lies in $\mathcal{S}_{0M}(X_0, \partial X_0)$*
- *$\wp$ either intertwines the action of $H^2(X_-, \partial X_-;\mathbb{Z}) \times H^2(X_+, \partial X_+;\mathbb{Z})$ or that of $H^2(X_1, \partial X_1;\mathbb{Z})$, as the case may be, with their images in $H^2(X_0, \partial X_0;\mathbb{Z})$.*
- *The composition of $\wp$ and then $\wp^0$ gives the canonical forgetful map.*

Proof of Lemma 2.6 What follows is the argument for the case when M separates X. The argument for the other case is analogous and is left to the reader.

To start, remark that the given $\mathrm{Spin}^{\mathbb{C}}$ structures $s_\pm$ can be patched together over M with the specification of an isomorphism over M between the corresponding lifts, $F_\pm \to Fr$. In this regard, note that the choice of a Riemannian metric on X which is a product flat metric on a tubular neighborhood, $U \approx I \times M$, of $M \subset X$ determines principal $SO(4)$ reductions of the general linear frame bundles of $X_\pm$ which are consistent with the inclusions of $X_\pm$ in X.

Having digested the preceding, note next that the space of isomorphisms between $F_+|_M$ and itself which cover the identity on $Fr|_M$ has a canonical identification with the space of maps from M to the circle; thus the space of isomorphisms $\varphi\colon F_+|_M \to F_-|_M$ which cover the identity on $Fr|_M$ has a non-canonical identification with $C^\infty(M; S^1)$. This implies that the set of homotopy classes of such maps is a principal bundle over a point for the the group $H^1(M;\mathbb{Z})$. In this regard, note that a pair of isomorphisms between $F_+|_M$ and $F_-|_M$ yield the same $\mathrm{Spin}^{\mathbb{C}}$ structures over X if and only if they differ by a map to S^1 which extends over either X_+ or X_-.

In any event, a choice of isomorphism from $F_+|_M$ to $F_-|_M$ covering $Fr|_M$ is canonically equivalent to a choice of isomorphism between the restrictions to M of the associated $U(1)$ line bundles $K_\pm$. Meanwhile, as $c(s_\pm) = 0$, the data $z_+ \in H^2(X_+, \partial X_+; \mathbb{Z})$ mapping to $c(s_+)$ canonically determines a homotopy class of isomorphisms from $K_+|_M$ to $M\times\mathbb{C}$. Likewise, z_- determines a homotopy class of isomorphisms from $K_-|_M$ to $M \times \mathbb{C}$. With the preceding understood, use the composition of an isomorphism $K_+|_M \approx M \times \mathbb{C}$ in the z_+ determined class with the inverse of one between $K_-|_M$ to $M \times \mathbb{C}$ from the z_- determined class to construct the required isomorphism between $F_+|_M$ and $F_-|_M$. □

Part 4 Lemma 2.6 makes the point that the image of $\wp$ contains only those $\mathrm{Spin}^{\mathbb{C}}$ structures on X whose image under the map c pulls back as zero to $H^2(M;\mathbb{Z})$. There may well be other $\mathrm{Spin}^{\mathbb{C}}$ structures on X. Even so, a case of the main theorem in [9] asserts that $\mathrm{sw}(s) = 0$ if $c(s)$ does not pull back as zero in $H^2(M;\mathbb{Z})$.

The digression is now over, and so the stage is set for the main theorem:

Theorem 2.7 *Let X_0 be a compact, connected, oriented 4–manifold with (possibly empty) boundary consisting of a disjoint union of 3–dimensional tori such that restriction to each boundary component induces a non-zero pull-back map on the second cohomology. If the boundary is empty, require that $b^{2+} \geq 1$. Let $M \subset X_0$ be an embedded 3–dimensional torus for which the restriction induced pull-back homomorphism from $H^2(X_0;\mathbb{R})$ to $H^2(M;\mathbb{R})$ is non-zero. Choose a class $\varpi \in H^2(X_0;\mathbb{R})$ whose pull-back in the cohomology of M and in that of every component of ∂X_0 is non-zero. If M splits X_0 as a pair, $X_- \cup X_+$, of 4–manifolds with boundary, then orient the lines $L_\pm$ and then orient the corresponding line L_0 via (10). Otherwise, orient the line L_1 and use the isomorphism $L_1 \approx L_0$ to orient the latter. If X_0 has empty boundary and $b^{2+} > 1$, use the orientation for L_0 to define the map* sw *on $\mathbb{S}(X_0)$. If X_0 has empty boundary and $b^{2+} = 1$, use the orientation for L_0 and that for $H^{2+}(X_0;\mathbb{R})$ as defined by ϖ to define* sw *on $\mathbb{S}(X_0)$. Finally, if X_0*

has non-trivial boundary, use the orientation for L_0 *and the class* ϖ *to define* sw *on* $\mathbb{S}(X_0, \partial X_0)$.

- *Suppose that* M *splits* X_0 *as a pair,* $X_- \cup X_+$, *of* 4*–manifolds with boundary. Use the chosen orientations for the lines* $L_\pm$ *and the restriction of* ϖ *to* $X_\pm$ *to define the corresponding maps* $\mathrm{sw}\colon \mathbb{S}_0(X_\pm, \partial X_\pm) \to \mathbb{Z}$. *Then, for all* $(s, z) \in \mathbb{S}_0(X_0, \partial X_0)$, *there are just finitely many pairs* $((s_-, z_-), (s_+, z_+)) \in \wp^{-1}((s, z))$ *with either* $\mathrm{sw}((s_-, z_-))$ *or* $\mathrm{sw}((s_+, z_+))$ *non-zero; and with this last fact understood,*
$$\mathrm{sw}((s,z)) = \sum_{((s_-,z_-),(s_+,z_+))\in\wp^{-1}((s,z))} \mathrm{sw}((s_-,z_-))\,\mathrm{sw}((s_+,z_+)).$$
- *If* M *does not split* X, *use the chosen orientations for the line* L_1 *and the restriction of* ϖ *to* X_1 *to define the corresponding map* $\mathrm{sw}\colon \mathbb{S}_0(X_1, \partial X_1) \to \mathbb{Z}$. *Then, for each* $(s, z) \in \mathbb{S}_0(X_0, \partial X_0)$, *there are just a finite number of* $(s_1, z_1) \in \wp^{-1}((s, z))$ *with* $\mathrm{sw}((s_1, z_1)) \neq 0$*; and with this last point understood,*
$$\mathrm{sw}((s,z)) = \sum_{((s_1,z_1))\in\wp^{-1}((s,z))} \mathrm{sw}((s_1,z_1)).$$

The proof of Theorem 2.7 is given in Section 3f.

3 Preliminary analysis

The proofs of Theorems 2.5 and 2.7 use many of the ideas from [15], but new techniques are also involved. The new techniques enter into the proof of Proposition 2.4 and into the proofs of related compactness assertions which concern the moduli spaces for manifolds with long cylinders that are products of 3–tori with intervals. These related compactness assertions are summarized below in Propositions 3.7 and 3.9.

This section sees to the separation of these compactness aspects of the proofs of Theorems 2.5 and 2.7 from the more well known techniques. In so doing, it explains how these theorems follow from Propositions 2.4, 3.7 and 3.9 while leaving the proofs of the latter to the subsequent sections of this paper. The details start in Subsection 3a below. A guide to the analytical points that arise in this and the subsequent sections immediately follows.

Any attempt to define an 'invariant' via an integer weighted count of solutions to an equation must deal with the following two absolutely central issues:

First, there must be some guarantee of a finite count. Second, to insure the invariance of the count, solution appearance and disappearance with change of movable parameters must occur in groups with zero aggregate count. Given a suitable topology on the solution set, both of these issues are issues of compactness. Indeed the former concerns the compactness of the solution set for some fixed parameter value while the latter concerns the compactness of a family of solutions spaces as determined by a corresponding family of parameter values.

The investigation of this compactness issue starts with the next two points. The first is a standard application of elliptic regularity theory and the second follows from a Bochner–Weitzenböck formula for the Seiberg–Witten equations.

- *A bound on the L^2 norms of F_A and ψ on a ball implies uniform C^∞ estimates for some pair on the gauge orbit of (A, ψ) on the concentric, half-radius ball. Hence, the space of gauge orbits of solutions that satisfy an a priori L^2 norm on a ball is precompact in the concentric, half-radius ball.*
- *For the toroidal end manifolds under consideration, the* $\mathrm{Spin}^{\mathbb{C}}$ *structure determines a uniform L^2 bound for both F_A and ψ on any ball when the perturbing form ω in* (4) *has the properties stated in Proposition 2.4.*

Although quite powerful, the preceding points are not powerful enough to imply compactness for $\mathcal{M}$ since they do not foreclose leakage down the ends of X. The characterization of this leakage requires an investigation of solutions to (4) on finite and infinite cylinders. This investigation begins with the derivation of the following key result:

- *If (A, ψ) solves* (4) *on a given cylindrical portion of $\mathbb{R} \times T^3$ where ω is non-zero and constant, and if F_A has small L^2 norm on each subcylinder of length* 4 *in the given cylinder, then $|F_A|$ decays exponentially from both ends of the given cylinder.*

This last fact, as confirmed in the initial subsections below, is ultimately a consequence of the structure of a moduli space of a certain version of the Seiberg–Witten equations on T^3.

As argued in Section 4, the preceding point with the first two implies:

- *All limits of non-convergent sequences in $\mathcal{M}$ are described by data sets that consist of a second solution to* (4) *on X and also a finite set (with topological bound on its size) of solutions on $\mathbb{R} \times T^3$.*

Given an assertion that precludes the extra solutions on $\mathbb{R}\times T^3$, this last point implies that $\mathcal{M}$ is compact. Moreover, certain geometric assumptions about the tubular end 4–manifold and the form ω actually do imply the absence of the relevant $\mathbb{R}\times T^3$ solutions.

When these just mentioned geometric assumptions are not met, then the argument for compactness employs the following additional observations:

- *The relevant moduli space of solutions to* (4) *on* $\mathbb{R}\times T^3$ *is a smooth manifold with a vector bundle whose fiber at each point is the cokernel of a Fredholm operator associated to the solution in question. This vector bundle has positive dimension and comes with a canonical, nowhere zero section.*
- *This canonical section defines a canonical,* $\mathbb{R}^2$*–valued function on every moduli space of solutions to* (4) *over* X. *There is one such function for each end of* X.
- *One or more of these canonical functions vanishes at any solution on* X *that appears in the data set for a non-convergent sequence in* $\mathcal{M}$.
- *If* ω *is suitably generic on the interior of* X_0, *then the zero set in each moduli space of each of these canonical functions is a codimension* 2 *submanifold.*

These last points are used in Section 6 to establish the assertions of Proposition 2.4 and those of the forthcoming Propositions 3.7 and 3.9. These points are established in Section 5 using a detailed analysis of the first order operator that is obtained by linearizing the Seiberg–Witten equations about a solution on the cylinder $\mathbb{R}\times T^3$.

As illustrated by the preceding discussion, the moduli spaces on T^3 and $\mathbb{R}\times T^3$ are central to a significant chunk of the compactness story and so they are the focus of much of the subsequent discussion.

a) Moduli spaces for T^3

The story behind Theorems 2.5 and 2.7 starts here with a description of the moduli spaces of a version of the Seiberg–Witten equations on a flat, oriented 3–torus. These moduli spaces enter into the story because they are naturally identified with the moduli spaces of translationally invariant solutions to a version of (4) on $\mathbb{R}\times T^3$. In any event, the equations considered here require the choice of a flat metric on T^3 plus a lift of the resulting $SO(3)$ frame bundle to a principal $\mathrm{Spin}^{\mathbb{C}}(3) = U(2)$ bundle. Such lifts are classified up to isomorphism by the first Chern class of the complex line bundle, K, which is associated to the determinant representation of $U(2)$ on $\mathbb{C}$. Note that this class, c, is an even class in $H^2(T^3;\mathbb{Z})$. The equations will also require the choice of a covariantly constant 2–form, ω_0, on T^3.

It is worth digressing here momentarily to comment some on the relationship between the 3 and 4–dimensional stories. To start the digression, keep in mind that the tori that arise in this article come as constant 'time' slices of an oriented $\mathbb{R}\times T^3$. In addition, the $\mathbb{R}$ factor will come oriented and thus induce an orientation on T^3; this will be the implicit orientation of choice.

With compatible orientations for $\mathbb{R}$, T^3 and $\mathbb{R}\times T^3$ understood, the set of $\mathrm{Spin}^{\mathbb{C}}$ structures on $\mathbb{R}\times T^3$ has a canonical, 1-1 correspondence with the set of lifts of the $SO(3)$ principal frame bundle to a $U(2)$ bundle. Indeed, this correspondence comes about via a natural map from the set of isomorphism classes of $U(2)$ lifts of the $SO(3)$ frame bundle of T^3 to $\mathbb{S}(\mathbb{R}\times T^3)$. What follows is the definition of this map.

To describe the aforementioned map, note first that a $U(2)$ lift, P, of the $SO(3)$ frame bundle comes with an associated, principal $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle, F_P, which will be viewed both as a bundle over T^3 and, via pull-back, as one over $\mathbb{R}\times T^3$. In this regard, F_P is defined using the representation which sends a pair $(h,\lambda)\in U(2)=(SU(2)\times S^1)/\{\pm1\}$ to $(h,h,\lambda)\in \mathrm{Spin}^{\mathbb{C}}(4)$. This associated $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle is a lift of the pull-back to $\mathbb{R}\times T^3$ of the analogously defined, associated principal $SO(4)$ bundle to the frame bundle of T^3. Meanwhile this last $SO(4)$ bundle is canonically isomorphic to the frame bundle of $\mathbb{R}\times T^3$. Indeed, a choice of oriented, unit length tangent vector field to the $\mathbb{R}$ factor induces just this isomorphism.

Thus, in the manner just described, an isomorphism class of principal $U(2)$ lifts of the $SO(3)$ frame bundle of T^3 canonically determines an element $s_P\in\mathbb{S}(\mathbb{R}\times T^3)$. The inverse of this map starts with $s\in\mathbb{S}(\mathbb{R}\times T^3)$ and the corresponding $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle F. The oriented, unit length tangent vector to $\mathbb{R}$ defines the reduction of F to a principal $U(2)$ bundle, P, as follows: First, define $Fr^3\subset Fr$ as the subset of frames whose first basis element is metrically dual to the chosen vector on the $\mathbb{R}$ factor. Then, view F as a bundle over Fr (with fiber S^1) and use $P\subset F$ to denote restriction of this bundle to Fr^3. This P is a principal $U(2)$ bundle which covers the $SO(3)$ frame bundle of T^3.

In any event, let $P\to T^3$ denote a principal $U(2)$ lift of the $SO(3)$ frame bundle. Introduce S to denote the complex 2–plane bundle $P\times_{U(2)}\mathbb{C}^2$. The Seiberg–Witten equations on T^3 are equations for a pair (A,ψ), where A is a connection on the complex line bundle $\Lambda^2 S = P \times_{U(2)} \mathbb{C}$ and where ψ is a section of S. These equations read:

$$\begin{aligned} &\bullet\ F_A = \tau(\psi \otimes \psi^\dagger) - i \cdot \omega_0; \\ &\bullet\ D_A\psi = 0. \end{aligned} \tag{11}$$

In this last equation, τ denotes the homomorphism from $\mathrm{End}(S) = S \otimes S^*$ which is the hermitian adjoint to the Clifford multiplication homomorphism from $\Lambda^2(T^3)$ into $\mathrm{End}(S)$ while D_A denotes a version of the Dirac operator. In particular, D_A is the first order, elliptic operator which sends a section of S to another section of S by composing a certain A–dependent covariant derivative on S with the Clifford multiplication endomorphism from $S \otimes T^*X$ to S. Here, (and below) the covariant derivative is defined from the connection on F which is obtained by coupling the connection A with the pull-back from Fr of the metric's Levi–Civita connection.

A second digression is in order here concerning the relationship between the 3 and 4 dimensional versions of Clifford multiplication. To start, introduce the $\mathrm{Spin}^{\mathbb{C}}(4)$ principal bundle $F_P \to \mathbb{R} \times T^3$ which corresponds to P, and then introduce the associated C^2 bundles $S_\pm \to \mathbb{R} \times T^3$. Both are canonically isomorphic to $S = P \times_{U(2)} \mathbb{C}^2$. In this way, the 4–dimensional Clifford multiplication endomorphism from $T^*(\mathbb{R} \times T^3) \times S_+ \to S_-$ induces a Clifford multiplication map $T^*(T^3) \times S \to S$. Here, $T^*(T^3)$ is viewed as a summand in $T^*(\mathbb{R} \times T^3)$ via the pull-back monomorphism from the projection map to T^3. By the way, with S_+ and S_- identified as S, Clifford multiplication by the 1–form dt is just multiplication by i.

With the digression now over, introduce $\mathcal{M}_P$ to denote the moduli space of solutions to (11) for a given flat metric, covariantly constant form ω_0 and lift P of the $SO(3)$ frame bundle. Thus, $\mathcal{M}_P$ is the quotient of the space of smooth pairs (A, ψ) which solve (11) by the action of the group $C^\infty(T^3; S^1)$. Here, the action is the same as in the 4–dimensional case. Note that $\mathcal{M}_P$ is given the quotient topology.

The lemma that follows describes the salient features of $\mathcal{M}_P$:

Lemma 3.1 *Given the flat metric, there exists $\delta > 0$ such that if $|\omega_0| < \delta$, then the space $\mathcal{M}_P$ is empty unless $c_1(S) = 0$. In addition, given that $c_1(S) = 0$ and $\omega_0 \neq 0$, then $\mathcal{M}_P$ is a single point, the orbit of a pair (A_0, ψ_0), where A_0 is a trivial connection, ψ_0 is covariantly constant and $\tau(\psi_0 \otimes \psi_0^\dagger) = i\omega_0$. When $\omega_0 = 0$, then $\mathcal{M}_P$ consists of the orbits of those pairs $(A, 0)$, where A is a flat connection; thus $\mathcal{M}_P = H^1(T^3; \mathbb{R}^3)/H^1(T^3; \mathbb{Z}^3)$.*

Proof of Lemma 3.1 The proof sticks closely to a well worn trail initially blazed by Witten in [22] and translated to the 3–manifold context at the start of Section 5 of [15]. In fact, the argument follows almost verbatim the discussion in the proof of [15]'s Lemma 5.1. Here is a brief synopsis of the argument: First of all, in the case where $\omega_0 = 0$, use the Bochner–Weitzenböck formula $D_A^2 = \nabla_A^* \nabla_A \psi + 2^{-1}\,\mathrm{cl}(F_A)\cdot\psi$ on T^3, and the two Seiberg–Witten equations to conclude that $\nabla_A \psi = 0$. Here, ∇_A denotes the covariant derivative on sections of S which is defined by A and the Levi–Civita connection on the $SO(3)$ frame bundle. As ψ is covariantly constant, either $F_A = 0$ or $\psi = 0$, or both. In particular, the first point in (11) is only consistent with both vanishing.

In the case where $\omega_0 \neq 0$, Clifford multiplication on S by ω_0 defines a skew Hermitian, covariantly constant endomorphism of S. Decompose S into the eigenbundles for this endomorphism and write ψ with respect to this decomposition as (α, β). In so doing, follow the steps in [22] or at the beginning of Section 5 in [15]. Here, it may be useful to consider lifting the story to $S^1 \times T^3$ with S^1 viewed as $\mathbb{R}/\mathbb{Z}$ to make the connection with the 4–dimensional framework in these references. In any event, with ψ written as (α, β), the second equation in (11) becomes a coupled system of equations for the pair (α, β). Continuing the analysis in either [22] or [15] then leads directly to the conclusion that $\beta = 0$ and that when $|\omega|$ is small, then α is constant and $F_A = 0$. □

An orbit in $\mathcal{M}_P$ is termed either a smooth point or not. These are technical terms which correspond to whether or not there are non-trivial deformations of pairs on the given orbit which solve (11) to first order. A precise definition is given in [15] subsequent to Lemma 5.2 and continued in [15]'s section 5.1. As is standard in gauge theory problems, the notion of being smooth or not is based on whether or not a certain Fredholm operator has vanishing cokernel. Here, the operator in question is obtained by first linearizing the equations in (11) about a given solution, then restricting the domain to the L^2–orthogonal complement of the tangents to the orbit of the $C^\infty(T^3; S^1)$ action, and finally projecting the resulting expression onto an isomorphic image of this same orthogonal complement. In this regard, the tangents to the orbit here appear as the image of the operator denoted by D_x in Section 5 of [15]. Alternately, one can view the equations in (11) as an S^1 invariant version of (4) on $S^1 \times T^3$ in which case the operator under scrutiny here is the S^1 invariant version of the operator in (5). In any event, consider:

Lemma 3.2 *Suppose that P is a principal $U(2)$ lift of the frame bundle of T^3 for which $c_1(S) = 0$. If $\omega_0 \neq 0$, then the point in $\mathcal{M}_P$ is a smooth point.*

Proof of Lemma 3.2 This is a straightforward computation since the operators involved have constant coefficients. The details are left to the reader. □

Remark that $\mathcal{M}_P$ has no smooth points when $\omega_0 = 0$. Moreover, in this case, the solution which corresponds to the trivial connection has a larger space of first order deformations than do the others.

The remainder of this paper considers only the case where $P \to T^3$ is the 'trivial lift,' that is, the lift of the $SO(3)$ frame bundle for which the associated C^2 bundle S is topologically trivial. This assumption about P is made implicitly from here to the end of the paper.

b) Fundamental lemmas

The preceding lemmas on the structure of $\mathcal{M}_P$ can be used to deduce certain key facts about solutions to the Seiberg–Witten equations on the product of T^3 with an interval. The latter are summarized by the lemmas in this subsection. Here is the first:

Lemma 3.3 *Fix a flat metric on T^3 and then fix a non-zero, constant 2–form ω on $\mathbb{R} \times T^3$. The metric on T^3, the form ω, a non-negative integer k and a choice of $\varepsilon > 0$ determine $\delta > 0$ which has the following significance: Suppose that (A, ψ) is a solution to (4) on $Y \equiv (2, -2) \times T^3$ as defined by the product flat metric and ω. If $\int_Y |F_A|^2 < \delta$, then (A, ψ) has C^k distance ε or less on the subinterval $[-1, 1] \times T^3$ from a point in the gauge orbit on Y of $(A_0, \psi_0) \in \mathcal{M}_P$.*

By way of explaining terminology, the statement that some (A_1, ψ_1) is in the 'gauge orbit on Y of $(A_0, \psi_0) \in \mathcal{M}_P$' means only that (A, ψ) is gauge equivalent via some element in $C^\infty(Y; S^1)$ to a pair of connection and section of S_+ which are the pull-backs from T^3 of a pair which solves (11).

The second fundamental lemma can be viewed as a corollary to Lemmas 3.2 and 3.3. Here is this second lemma:

Lemma 3.4 *Fix a flat metric on T^3 and then fix a non-zero, self-dual, constant 2–form ω on $\mathbb{R}\times T^3$. The metric and form ω determine a constant $\delta > 0$ and a set of constants $\{\zeta_k\}_{k=0,1,\dots}$ with the following significance: Suppose that $R \geq 0$ and that (A, ψ) is a solution to (4) on $Y \equiv (-R-2, R+2) \times T^3$ as defined by the product metric and ω. Suppose as well that $\int_U |F_A|^2 < \delta$ for every length 2 cylinder $U = (t-1, t+1) \times T^3 \subset Y$. Then, there is a pair (A_1, ψ_1) in the gauge orbit on Y of $(A_0, \psi_0) \in \mathcal{M}_P$ such that*

$$|\nabla^k(A - A_1)| + |(\nabla_{A_1})^k(\psi - \psi_1)| \leq \zeta_k(e^{-\delta(R-t)} + e^{-\delta(R+t)}) \qquad (12)$$

at any point $(t, x) \in [-R, R] \times T^3$.

These two lemmas are proved shortly so accept them for now to consider one of their more immediate consequences, that elements in the moduli space $\mathcal{M}$ of Sections 2a–c decay exponentially fast along the cylindrical ends of X.

Lemma 3.5 *Let X_0 be the usual 4–manifold with toroidal boundary components, and let $X = X_0 \cup ([0,\infty) \times \partial X_0)$ denote the corresponding non-compact manifold with a metric which restricts to $[0,\infty) \times \partial X_0$ as a flat, product metric. Let ω denote a self-dual form on X which is constant and non-zero on each component of $[0,\infty) \times \partial X_0$. This data determines $\delta > 0$, a sequence of constants $\{\zeta_k\}_{k=0,1,2,\ldots}$ and, with the choice of $r \geq 1$, a constant R; and these constants have the following significance: Let $(s, z) \in \mathcal{S}_0(X_0, \partial X_0)$, let $\mathcal{M}$ denote the resulting moduli space of solutions to (4), and let $(A, \psi) \in \mathcal{M}$ obey $\int_{[r,\infty)\times\partial X_0} |F_A|^2 < \delta$. Then, on each component, $Y \subset [R,\infty) \times \partial X_0$ there is a point (A_1, ψ_1) on the gauge orbit on Y of $(A_0, \psi_0) \in \mathcal{M}_P$ for which*

$$|\nabla^k (A - A_1)| + |(\nabla_{A_1})^k (\psi - \psi_1)| \leq \zeta_k e^{-\delta(t-R)} \tag{13}$$

at any point $(t, x) \in Y$.

Proof of Lemma 3.5 Write t in (12) as $t' + R$ and then take R to infinity in the resulting equation to obtain bounds on $(A - A_1, \psi - \psi_1)$ at points $(t', x) \in [0,\infty) \times T^3$. The latter are identical to those in (13) after shifting t in (13) to $t' - R$. □

This subsection ends with the proofs of Lemmas 3.3 and 3.4.

Proof of Lemma 3.3 The L^2 bound on $P_+ F_A$ by any $\delta \geq 0$ immediately yields an L^2 bound on $|\psi|$ since

$$|\tau(\psi \otimes \psi^\dagger)| = z|\psi|^2 \tag{14}$$

with z a universal constant. Thus, since $|\omega|$ is a constant, the first point in (4) implies that there is a constant z_ω and a bound of the form

$$\int_Y ||\psi| - z_\omega|^4 \leq z_1 \delta, \tag{15}$$

with z_1 depending only on $|\omega|$.

The next step obtains bounds on the L^2 norm of $\nabla_A \psi$, and the Bochner–Weitzenböck formula for the Dirac operator is the principle tool for doing so. Without assumptions on the Riemannian metric, the connection A and the section ψ, this formula reads

$$D_A^* D_A \psi = \nabla_A^* \nabla_A \psi + 4^{-1} s\psi + 2^{-1} \operatorname{cl}_+(P_+ F_A) \cdot \psi. \tag{16}$$

Here, s denotes the metric's scalar curvature while D_A^* and ∇_A^* denote the formal, L^2 adjoints of the Dirac operator D_A and the covariant derivative ∇_A. Also, $\mathrm{cl}_+(\cdot)$ denotes the Clifford multiplication induced homomorphism from Λ_+ into $\mathrm{End}(S_+)$.

As indicated first by Witten in [22], this formula in conjunction with the Seiberg–Witten equations can be used with great effect to analyze the behavior of solutions to (4). In particular, the left hand side of (16) is zero when the second line in (4) holds, while the first line can be used to control the term with P_+F_A.

In any event, for the purposes at hand, take the inner product of both sides of (16) with ψ and use the equations in (4) to rewrite the result as

$$2^{-1}d^*d|\psi|^2 + |\nabla_A\psi|^2 + 2^{-1}|P_+F_A|^2 + 2^{-1}\langle P_+F_A, i\omega\rangle = 0. \tag{17}$$

Here, d^* denotes the formal L^2 adjoint of the exterior derivative d.

Equation 3.7 implies that the square of the L^2 norm of $|\nabla_A\psi|$ over $[-3/2,3/2]\times T^3$ is bounded by $z_2\delta^{1/2}$, with z_2 only dependent on $|\omega|$. Indeed, to obtain such a bound, first replace $|\psi|^2$ by $(|\psi^2| - z_\omega)$ in the first term on the left side. Then, multiply both sides of the resulting equation by a smooth, non-negative function which equals 1 on $[-3/2,3/2]\times T^3$ and vanishes near the boundary of Y. Next, integrate the result over Y and then integrate by parts to remove the derivatives in d^*d from $|\psi|^2 - z_\omega$. Finally, an appeal to (15) and a suitable application of the inequality $2|ab| \le \delta^{-1/2}|a|^2 + \delta^{1/2}|b|^2$ produces the asserted bound.

With the L^2 norms of ψ, $\nabla_A\psi$ and F_A bounded in terms of δ over the interior cylinder $[-3/2,3/2]\times T^3$, the next step proves that there is, given $\varepsilon > 0$, a value for δ which implies that the (A,ψ) has L^2_1 distance ε or less from a pair (A_1,ψ_1) in the gauge orbit on Y of $(A_0,\psi_0) \in \mathcal{M}_P$. This step is a straightforward argument by contradiction which invokes fairly standard elliptic techniques. In this regard, the only novelty is that the action of $C^\infty(Y;S^1)$ must be used to make (4) an elliptic system. Indeed, ellipticity can be achieved by using the $C^\infty(Y;S^1)$ action to write $(A = A_0 + b, \psi = \psi_0 + \eta)$, where (a,η) are constrained so that the expression in the top line of (5) holds. The details of all of this are left to the reader.

Finally, given that (A,ψ) is L^2_1 close to a point in the gauge orbit on Y of (A_0,ψ_0), the final step proves that (A,ψ) is C^k close to (A_0,ψ_0). This last part of the proof is also left to the reader as it constitutes a direct application of standard procedures in elliptic regularity theory. □

Proof of Lemma 3.4 The lemma can be proved by invoking, with only minor changes, the argument which [15] uses to prove its Corollary 6.17. However, a slightly more direct argument can made by filling out the sketch that follows. The sketch starts with the remark that $\varepsilon > 0$ provides a positive upper bound for the L^2 norm over U of F_A that has the following significance: When the L^2 norm of F_A is less than this bound, then (A, ψ) is gauge equivalent to $(A_0 + b, \psi_0 + \eta)$, where the C^1 norms of (b, η) are bounded by ε on the sub-cylinder $Y' \equiv [-R-1, R+1] \times T^3$; and where the Seiberg–Witten equations in terms of $\lambda \equiv (a, \eta)$ have the form

$$\partial_t \lambda + L_0 \lambda + r(\lambda) = 0. \tag{18}$$

Here, ∂_t is the tangent vector field to the line segment factor in Y, and L_0 is a linear, symmetric, first order differential operator. Meanwhile, $r(\lambda)$ in (18) is a 'remainder' term which is formally second order in λ. To be precise here, the L^2 norm of $r(\lambda)$ on each constant $t \in [-R-1, R+1]$ slice of Y is bounded by the product of ε and the L^2 norm of λ on the same constant t slice.

What follows are some important points about L_0. First, L_0 is determined solely by the metric on T^3, ω and (A_0, ψ_0), and thus only by the metric and ω. Second, the L^2 spectrum of L_0 is discrete, real, and lacks accumulation points. Third, 0 is not in the spectrum of L_0. This last conclusion is essentially the statement of Lemma 3.2 and plays the starring role in the subsequent part of the argument.

With L_0 introduced, let $\lambda_\pm$ denote the projections of λ onto the respective eigenspaces of L_0 with positive $(+)$ and negative $(-)$ eigenvalues. Next, introduce functions $f_\pm$ on the interval $[-R-1, R+1]$ whose values at a point t are the respective L^2 norms of $\lambda_\pm$ on the corresponding constant t slice of Y. Then (18) yields the differential inequalities

$$\begin{aligned} &\partial_t f_+ + (E-\varepsilon) f_+ - \varepsilon f_- \le 0; \\ &\partial_t f_- - (E-\varepsilon) f_- + \varepsilon f_+ \ge 0. \end{aligned} \tag{19}$$

Here, $E > 0$ is the distance between 0 and the spectrum of L_0.

With (19) understood, a simple comparison argument establishes the following: When $\varepsilon \ll E$, then the inequalities in (19) require both $f_\pm$ to decay exponentially from the ends of $[-R-1, R+1]$. This exponential decay for the L^2 norms of (a, η) on the constant t slices of Y' can then be bootstrapped to give (12) using standard elliptic regularity techniques. □

c) Immediate applications to the structure of $\mathcal{M}$

Lemmas 3.1–3.5 have certain automatic consequences with regard to the moduli spaces which are considered in Section 2c. In particular, Propositions 2.2 and 2.3 follow from these lemmas.

Proof of Proposition 2.2 The argument here for the structure of $\mathcal{M}$ is completely analogous to that derived in [18] for the $SU(2)$ self-dual moduli spaces on manifolds with cylindrical ends. In this regard, observe that the lower two components of the image of $\mathcal{D}_c$ in (5) are nothing more than the linearization of the equations in the first two points of (4). Meanwhile, the vanishing of the first component of the image of $\mathcal{D}_c$ in (5) only asserts that the given section of $i\cdot T^*X \oplus S_+$ is L^2–orthogonal to the space of tangents to the orbit of the gauge group. The fact that $\mathcal{D}_c$ is Fredholm with the L^2_1 domain and L^2 range follows from Lemma 3.2 by standard arguments; for example, by invoking Lemma 3.5 to control the behavior of (A,ψ) on $[0,\infty)\times\partial X_0$, the fact that $\mathcal{D}_c$ is Fredholm follows almost directly from results in [2].

The formula in Proposition 2.2 for the index of $\mathcal{D}_c$ can be derived with the help of the excision properties of the index from the following input: First, the formula in Proposition 2.2 holds when X is compact, see eg [22]. Second, take $X=\mathbb{R}\times T^3$ and ω to be a constant, non-zero self-dual 2–form. Then, take the solution c to be the pull-back via the projection to T^3 of a solution in Lemma 3.1's space $\mathcal{M}_P$. Here, the form ω_0 in (11) is the pull-back to $\{0\}\times T^3$ of ω. In this case, the kernel and the cokernel of $\mathcal{D}_c$ are both trivial. (The operator has constant coefficients, so is straightforward to analyze.) □

Proof of Proposition 2.3 The argument here is, modulo some notational changes, almost identical to that which proves the analogous assertion in the case where X is compact; see, for example the books [11] or [8]. The largest modifications to the compact case argument are needed to address the orientation assertion, and in this regard, the reader can refer to the proof of Corollary 9.2 in [15]. □

Lemmas 3.1–3.5 also have immediate applications to the subject of $\mathcal{M}$'s compactness. The particular applications here are summarized below in Proposition 3.6. Here is the background for this proposition: The proposition introduces the manifold with boundary X_0 as described in the beginning of Section 2, except that here, X_0 is assumed to have non-empty boundary. Put a Riemannian metric on X_0 which is a flat, product metric on some neighborhood of ∂X_0 and extend this metric in the usual way to obtain a metric on $X = X_0\cup([0,\infty)\times\partial X_0)$.

Also, fix a self-dual 2–form ω on X which is non-zero and covariantly constant on $[0,\infty)\times\partial X_0$. Having made these selections, choose an element $s\in\mathcal{S}_0(X)$ and then introduce, as in Section 2d, the set $\varsigma(s)\subset H^2(X_0,\partial X_0;\mathbb{Z})$ which consists of those elements z which map to $c(s)$ in $H^2(X_0;\mathbb{Z})$.

What follows might be called a partial compactness assertion.

Proposition 3.6 *The restrictions of the chosen metric and form ω to ∂X_0 determine a constant $\delta>0$ with the following significance: For each $z\in\varsigma(s)$, construct the moduli space $\mathcal{M}$ and then for each $r\geq 1$, introduce the subspace $\mathcal{M}(r)\subset\mathcal{M}$ of orbits of (A,ψ) for which*

$$\int_{[r,\infty)\times\partial X_0}|F_A|^2\leq\delta.$$

Then, $\mathcal{M}(r)$ is compact in all cases, and actually empty for all but a finite set of $z\in\varsigma(s)$.

Proof of Proposition 3.6 As Witten pointed out [22], the key to compactness theorems for the Seiberg–Witten moduli spaces is the Bochner–Weitzenböck formula in (16). Of course, if $D_A\psi=0$, then the left hand side of (16) is zero; thus contracting both sides with ψ using the hermitian metric on S_+ yields a differential inequality for the function $|\psi|^2$. Moreover, when the first point in (4) also holds, then, as noted in [9], the maximum principle applies to this differential inequality and provides a uniform upper bound for $|\psi|^2$ in terms of $|s|$, $|\omega|$ and an asymptotic bound for $|\psi|^2$ on the ends of X. Lemma 3.5 provides such an asymptotic bound for $|\psi|$, so

$$|\psi|^2\leq\zeta\sup_X(|s|+|\omega|)\tag{20}$$

on the whole of X. Here, ζ depends only on the Riemannian metric. Note, by the way, that this constant ζ is independent of both the Spin$^{\mathbb{C}}$ structure s and $z\in\varsigma(s)$.

This bound on $|\psi|$ together with Lemma 3.5 provide a uniform upper bound on the L^2 norm of P_+F_A. This upper bound is also independent of both s and $z\in\varsigma(s)$. This last bound on P_+F_A provides an L^2 bound on the anti-self dual part, P_-F_A, of A's curvature 2–form via the string of identities

$$c(s)\bullet c(s)=-(4\pi^2)^{-1}\int_X F_A\wedge F_A=(4\pi^2)^{-1}\int_X(|P_+F_A|^2-|P_-F_A|^2).\tag{21}$$

In particular, notice that the resulting upper bound for the L^2 norm of F_A is independent of the class $z\in\varsigma(s)$.

Standard elliptic regularity techniques can be invoked to bootstrap these upper bounds on F_A and $|\psi|$ into uniform and $z \in \varsigma(s)$ independent upper bounds for all C^k norms for a suitable point on the gauge orbit of (A, ψ). (See [11] or [8] to see how this is done.) In particular the latter imply that any sequence of gauge orbits in $\cup_{z\in\varsigma(s)}\mathcal{M}(r)$ is defined by a corresponding sequence $\{(A_k, \psi_k)\}$ of solutions to (11) which converges in the C^∞ topology on compact subsets of X. Meanwhile, the uniform bounds that are provided by Lemma 3.5 imply that a sequence of gauge orbits in $\cup_{z\in\varsigma(s)}\mathcal{M}(r)$ is actually defined by a sequence $\{(A_k, \psi_k)\}$ of solutions to (4) which converges in the strong C^∞ topology on the whole of X. This last fact implies convergence in each $\mathcal{M}(r)$ and it implies that there can be only finitely many $z \in \varsigma(s)$ for which the corresponding $\mathcal{M}(r)$ is not empty. □

d) The family version of Proposition 2.4

Though Proposition 3.6 asserts that each $\mathcal{M}(r)$ is compact, there is no reason for the whole of $\mathcal{M}$ to be compact. In fact, most probably, the statement in Proposition 2.4 is about as strong as can generally be made. As remarked in Section 2d, the compactness asserted in Proposition 2.4 is strong enough to provide a definition of $\mathrm{sw}(s, z)$ given a reasonable choice of ω and, in the $d > 0$ cases, a suitably generic choice of the set $\underline{\Lambda}$.

However, the compactness asserted by Proposition 2.4 is not strong enough for use in the proof of Theorem 2.5. Indeed, a comparison between the values of $\mathrm{sw}(\cdot)$ as defined by different, but still allowable choices for the triple of metric, ω and $\underline{\Lambda}$ involves an interpolating path in the space of such triples, and thus a corresponding 1–parameter family of moduli spaces. Meanwhile, Proposition 2.4 says nothing about compactness for families of moduli spaces. This weakness in Proposition 2.4 is addressed below with the statement of Proposition 2.4's family version for use in the proof of Theorem 2.5.

To set the stage for the family version of Proposition 2.4, fix $s \in \mathcal{S}_0(X_0)$ and consider two sets of triples, $\Gamma_0 \equiv (g_0, \omega_0, \underline{\Lambda}_0)$ and $\Gamma_1 \equiv (g_1, \omega_1, \underline{\Lambda}_1)$, for use in defining $\mathrm{sw}(s, z)$ for $z \in \varsigma(s)$. Thus, g_0 and g_1 are metrics on X which restrict to flat, product metrics on $[0, \infty) \times \partial X_0 \subset X$. Meanwhile, ω_0 and ω_1 are self-dual forms on X for the respective metrics g_0 and g_1 which restrict to each component of $[0, \infty) \times \partial X_0$ as non-zero, covariantly constant 2–forms. Finally, $\underline{\Lambda}_0$ and $\underline{\Lambda}_1$ are two sets of points and the associated data needed to define sw when the number d in Proposition 2.2 is positive.

The triple Γ_0 has its associated moduli space $\mathcal{M}(s, z)$. There is, of course, an analogous moduli space that is defined by Γ_1 and these two spaces will be distinguished as $\mathcal{M}^0$ and $\mathcal{M}^1$, respectively. With this notation understood, require now of ω_0 and ω_1 that their moduli spaces $\mathcal{M}^0$ and $\mathcal{M}^1$ consist only of smooth points. Meanwhile, require of $\underline{\Lambda}_0$ and $\underline{\Lambda}_1$ that the conditions which define their respective versions of the set $\mathcal{M}^{\underline{\Lambda}}$ in (8) cut the latter out of the corresponding $\mathcal{M}$ in a transversal fashion. These will be denoted respectively by $\mathcal{M}^{0\underline{\Lambda}}$ and $\mathcal{M}^{1\underline{\Lambda}}$.

Here is the final key assumption on Γ_0 and Γ_1. Assume that there exists a continuous, 1–parameter path $\{\varpi_t \in H^2(X_0;\mathbb{R}) : t \in [0,1]\}$ with each ϖ_t having non-zero pull-back in the cohomology of each component of ∂X_0, and with ϖ_0 and ϖ_1 taming ω_0 and ω_1, respectively.

With these assumptions in hand, consider now the following:

Proposition 3.7 *Under the assumptions just made, there exists a continuous, interpolating family, $\{\Gamma_t : t \in [0,1]\}$, of data triples for which the corresponding family of moduli spaces $\mathcal{W} \equiv \cup_{t\in[0,1]}\mathcal{M}^t$ has the following structure:*

- *$\mathcal{W}$ is a smooth, oriented, $d+1$ dimensional manifold with boundary for which the tautological map to $[0,1]$ is smooth and a product over a neighborhood of $\{0,1\}$. Moreover, the induced boundary orientation on $\mathcal{M}^1$ agrees with its orientation from Proposition 2.3, while that on $\mathcal{M}^0$ disagrees.*
- *Let $\mathcal{W}^{\underline{\Lambda}} \equiv \cup_{t\in[0,1]}\mathcal{M}^{t\underline{\Lambda}} \subset \mathcal{W}$. Then $\mathcal{W}^{\underline{\Lambda}}$ has the structure of a finite, disjoint set of embedded, oriented intervals each mapping to $[0,1]$ as a product over a neighborhood of $\{0,1\}$. Moreover, the induced boundary orientation on $\mathcal{M}^{1\underline{\Lambda}}$ agrees with its orientation from the definition of $\mathrm{sw}(s,z)$, while that on $\mathcal{M}^{0\underline{\Lambda}}$ disagrees with its $\mathrm{sw}(s,z)$ orientation.*

The proof of this proposition is discussed in Section 6b so accept its assertions for the time being.

With Proposition 3.7 in hand, consider the following:

Proof of Theorem 2.5 The invariance assertions in the theorem are a standard consequence of the third point in Proposition 3.7. Indeed, the intervals in $\mathcal{W}^{\underline{\Lambda}}$ pair each point in $\mathcal{M}^{0\underline{\Lambda}}$ either with another point in this space, but one with the opposite sign for the sw count, or else with a point in $\mathcal{M}^{1\underline{\Lambda}}$ which has the same sign for the sw count. Meanwhile, each point in the latter space which is not paired by a component of $\mathcal{W}^{\underline{\Lambda}}$ to one in $\mathcal{M}^{0\underline{\Lambda}}$ is paired by a component of $\mathcal{W}^{\underline{\Lambda}}$ with another such point, but one with the opposite sw count sign. $\square$

e) Gluing moduli spaces

In this subsection, X_0 denotes a compact, connected, oriented 4–manifold with boundary consisting of a disjoint union of 3–dimensional tori. Here, the boundary can be empty, but if so, require that $b^{2+} \geq 1$. Let $M \subset X_0$ be an embedded, 3–dimensional torus, and if M is separating, write $X_0 = X_- \cup X_+$, where $X_+ \cap X_- = M$. Otherwise, let $X_1 \subset X$ denote the complement of an open, tubular neighborhood of M. In the separating case, introduce the non-compact manifolds $\underline{X}_\pm \equiv X_\pm \cup ([0,\infty) \times \partial X_\pm)$, and in the non-separating case, $\underline{X}_1 \equiv X_1 \cup ([0,\infty) \times \partial X_1)$.

With the introduction of $\underline{X}_\pm$ and $\underline{X}_1$, the remainder of this subsection describes how moduli spaces on $\underline{X}_\pm$, in the separating case, or on $\underline{X}_1$ otherwise, can be glued together over the ends that contain M to produce portions of the moduli space for X.

To begin the presentation, choose a flat metric on M, and then choose a metric on X_0 which restricts as a flat product metric on an interval neighborhood of M. This is to say that the metric should allow for an isometric embedding of $(-\varepsilon,\varepsilon) \times M$ into X which sends $\{0\} \times M$ to M. The chosen metric for X_0 should also restrict as a product, flat metric on a neighborhood of ∂X_0.

With this metric fixed, select a self-dual 2–form ω which is non-zero and constant on a neighborhood of M and likewise on a neighborhood of each component of ∂X_0. The metric on X_0 and the form ω induce, in a presumably obvious way, a pair of metric and self-dual 2–form on $\underline{X}$ and also on $\underline{X}_\pm$ or $\underline{X}_1$, as the case may be. Here, the metric on these spaces is flat and a product on the ends, and the self-dual 2–form, still called ω, is non-zero and constant on each end component. Moreover, in the separating case, $\underline{X}_\pm$ have one special end, that which contains $M \subset \partial X_\pm$. In X_-, this end has an orientation preserving isometry with $[0,\infty) \times M$, while in X_+, the orientation preserving isometry sends the end to $(-\infty, 0] \times M$. In the non-separating case, $\underline{X}_1$ has two special ends, one with an orientation preserving isometry to $[0,\infty) \times M$ and the other to $(-\infty,0] \times M$. In all of these cases, the metric and the form ω restrict to these special ends as the constant extension of the given metric and form on $(-\varepsilon,\varepsilon) \times M \subset X$.

With regard to the choice of ω, Proposition 2.3 asserts the following: Fix neighborhoods of M and ∂X whose closure is not the whole of X and there is a Baire subset of choices for ω which have the given restriction near M and ∂X and are such that all moduli spaces of solutions to (4) on X, $\underline{X}_+$ and $\underline{X}_-$, or on $\underline{X}_1$, are smooth manifolds for which the operator $\mathcal{D}_c$ has trivial cokernel at all points. In particular, when choosing the form ω, be sure to take one for which this last conclusion holds.

The chosen metric on X, call it g, will now be used to construct a 1–parameter family, $\{g_R\}_{R\geq 0}$, of metrics on X. Here, $g_0 = g$, while X with the metric g_R admits an isometric embedding of the cylinder $[-R, R]\times M$ whose complement with the metric g_R is isometric to $X - M$ with the metric g. Alternately, the metric g_R makes X isometric to

$$(X - M) \cup ([-R, R] \times M), \tag{22}$$

where $X - M$ has the metric g and $[-R, R]\times M$ has the product, flat metric. With g_R understood, introduce X^R to denote X as a Riemannian manifold with the metric g_R.

With regard to (22), note that in the respective cases where M does and does not separate X, the Riemannian manifold $X^R - M$ admits a canonical, orientation preserving isometry

$$\begin{aligned}\Theta\colon X^R - M \to &\big(\underline{X}_- - ([R,\infty)\times M)\big)\\ &\cup \big(\underline{X}_+ - ((-\infty,-R]\times M)\big) \subset \underline{X}_- \cup \underline{X}_+ \text{ or}\\ \Theta\colon X^R - M \to &\underline{X}_1 - (([R,\infty)\times M)\cup((-\infty,-R]\times M)) \subset \underline{X}_1,\end{aligned} \tag{23}$$

respectively.

There is one last remark to make here about the metric g_R, which is that the form ω can be viewed as living on X^R in as much as its restriction to $X - M \subset X$ defines it on the isometric $X - M \subset X^R$ and then there is an evident extension as a self-dual 2–form on the whole of X^R which is non-zero and constant on the cylinder $[-R, R]\times M$.

With the geometric preliminaries complete, now choose $(s, z) \in \mathcal{S}_{0M}(X_0, \partial X_0)$. In the case where M is separating, each pair $((s_-, z_-), (s_+, z_+)) \in \wp^{-1}(s, z) \subset \mathcal{S}_0(X_-, \partial X_-) \times \mathcal{S}_0(X_+, \partial X_+)$ determines moduli spaces $\mathcal{M}_-$ and $\mathcal{M}_+$ on $\underline{X}_-$ and $\underline{X}_+$, respectively. In the non-separating case, each $(s_1, z_1) \in \wp^{-1}(s, z)$ determines the moduli space $\mathcal{M}_1$ on $\underline{X}_1$. With regard to these spaces, remember that the form ω has been chosen so that these moduli spaces are smooth manifolds with $\operatorname{cokernel}(\mathcal{D}_c) = \{0\}$ at all points. Note for use below that the specification of $r \geq 1$ defines the subsets $\mathcal{M}_\pm(r) \subset \mathcal{M}_\pm$ and $\mathcal{M}_1(r) \subset \mathcal{M}_1$ of pairs (A, ψ) which obey the curvature bound in Proposition 3.6.

Meanwhile, the choice of $R \geq 0$ determines the Riemannian manifold X^R, and then, with the help of ω, the pair (s, z) determine a corresponding moduli space, $\mathcal{M}^R$.

With all of this understood, what follows in Proposition 3.8 is a description of the fundamental fact that parts of $\mathcal{M}^R$ can be constructed from $\mathcal{M}_-(r)\times\mathcal{M}_+(r)$ or from $\mathcal{M}_1(r)$ as the case may be. Here, a lower bound for R is determined by r.

Proposition 3.8 *Each pair $r' \geq 0$ and $\delta > 0$ determines a lower bound for a choice of r and then a choice of $r \gg r'$ which is greater than this lower bound determines a lower bound for a choice of $R \gg r$. Choose $r' \geq 0$, then $r \gg r'$ consistent with the lower bound, and finally $R \gg r$ consistent with its lower bound. Then the following are true:*

- *In the case where M separates, there exists an embedding*
$$\Phi_r\colon \bigcup_{((s_-,z_-),(s_+,z_+))\in\wp^{-1}((s,z))} \mathcal{M}_-(r)\times\mathcal{M}_+(r)\to\mathcal{M}^R,$$
which maps the interior of its domain onto an open set of smooth points that contains the subspace of (A,ψ) with
$$\int_{[-R+r',R-r')\times M} |F_A|^2 \leq \delta/2. \tag{24}$$
In addition, if the lines L_-, L_+, and L_0 are oriented as in Theorem 2.7 and if the induced orientations on the respective moduli spaces are used, then Φ_r is orientation preserving. Moreover, if (c_-,c_+) is in the domain of Φ_r, there are points $(A_\pm,\psi_\pm)\in c_\pm$ and $(A,\psi)\in\Phi_r(c_-,c_+)$ such that
$$\sum_{0\leq k\leq 2}\Big(|\nabla^k(A-\Theta^*(A_-,A_+)|+|\nabla_A^k(\psi-\Theta^*(\psi_-,\psi_+)|\Big)\leq e^{-R/\zeta},$$
where Θ is the map in the first line of (23) and $\zeta\geq 1$ depends only on the restriction to M of the form ω and the metric g.
- *In the case where M does not separate, there exists an embedding*
$$\Phi_r\colon \bigcup_{((s_l,z_l))\in\wp^{-1}((s,z))} \mathcal{M}_1(r)\to\mathcal{M}^R,$$
which maps the interior of its domain onto an open set of smooth points that contains the subspace of (A,ψ) which obey (24). In addition, if the lines L_1 and L_0 are oriented as in Theorem 2.7 and if the induced orientations on the respective moduli spaces are used, then Φ_r is orientation preserving. Moreover, if c_1 is in the domain of Φ_r, there are points $(A_1,\psi_1)\in c_1$ and $(A,\psi)\in\Phi_r(c_1)$ such that
$$\sum_{0\leq k\leq 2}\Big(|\nabla^k(A-\Theta^*A_1)|+|\nabla_A^k(\psi-\Theta^*\psi_1)|\Big)\leq e^{-R/\zeta},$$

where Θ is the map in the second line of (23) *and $\zeta \geq 1$ depends only on the restrictions to M of ω and g.*

As the gluing result in Proposition 3.8 is now standard fair in gauge theories, the proof will be omitted. However, note that a proof can be produced via a straightforward application of the techniques introduced in [4] for gluing self-dual moduli spaces in $SU(2)$ gauge theory. In this regard, the choice of ω to make either each pair $\mathcal{M}_{\pm}$ or each $\mathcal{M}_1$ smooth insures the absence of obstruction bundles. As a last remark on this subject, note that reference to Section 9.1 of [15] can also be made to deal with the assertions in the proposition that concern orientations.

f) Implications from gluing moduli spaces

This subsection discusses the application of Proposition 3.8 to the proof of Theorem 2.7. For this purpose, suppose that X_0, M, $X_{\pm}$ and X_1 are as described at the outset of the preceding subsection.

Coupled with Proposition 2.4, the gluing result in Proposition 3.8 is almost enough to prove Theorem 2.7. Indeed, missing still is a guarantee that each point in $(\mathcal{M}^R)^{\underline{\Lambda}}$ satisfies (24) for some fixed r' and all R sufficiently large. The next proposition provides such a guarantee:

Proposition 3.9 *Continuing the discussion and notation from Proposition 3.8 and the preceding subsection, introduce d as defined in Proposition 2.2. If $d > 0$, choose the data set $\underline{\Lambda}$ so that the set of base points Λ is disjoint from M and use* (22) *to define the analogous data sets for each X^R. Choose a class $\varpi \in H^2(X_0;\mathbb{R})$ whose pull-back in the cohomology of M and in that of every component of ∂X_0 is non-zero. Next, choose a self-dual 2–form ω' on X_0 which is tamed by ϖ and whose restriction to the fiducial tubular neighborhood of M and to that of each component of ∂X_0 is constant and non-zero. Then there is a Baire set of choices for the self-dual forms ω which are tamed by ϖ, which agree with ω' on the complement of a fixed, tubular neighborhood of $M \cup \partial X_0$, and which have the following additional property: If r is sufficiently large, and R also, subject to its lower bound constraints, then the set Λ can be chosen to insure that $(\mathcal{M}^R)^{\underline{\Lambda}}$ lies in the image of Proposition 3.8's map Φ_r. In particular, there exists $r' \geq 1$ such that when R is sufficiently large, then each point in $(\mathcal{M}^R)^{\underline{\Lambda}}$ obeys* (24).

This proposition is proved in Section 6c.

The preceding two propositions play the key roles in the following:

Proof of Theorem 2.7 Except for two assertions, the theorem follows directly from Propositions 3.7 and 3.9 via arguments which are standard fair in gauge theories. (Arguments of this sort were first given by Donaldson in the context of $SU(2)$ gauge theories, see eg [4].) The assertions which do not follow immediately from Propositions 3.7 and 3.8 concern the vanishing of all but finitely many of the numbers $\mathrm{sw}(s_-, z_-)$ and $\mathrm{sw}(s_+, z_+)$ or $\mathrm{sw}(s_1, z_1)$ as the case may be. However, these last assertions follow from Proposition 2.4. □

4 Energy and compactness

The estimates provided here make the first steps towards the proof of Proposition 2.4. Although they are not strong enough to give Proposition 2.4 and its brethren in their entirety, they do yield the previously mentioned result that $\mathcal{M}$ is compact if the pull-back to each boundary component of ϖ is not a multiple of an integral cohomology class.

In the subsequent discussions of this section, X_0 is as described at the beginning of Section 2, a compact, connected, oriented 4–manifold with boundary such that each boundary component is a 3–torus. As before, X_0 is endowed with a metric which is a product flat metric on a tubular neighborhood of the boundary. Also, X_0 is endowed with a class $\varpi \in H^2(X_0;\mathbb{R})$ whose pull-back to the cohomology of each component of ∂X_0 is non-zero. As previously, let X denote $X_0 \cup ([0,\infty) \times \partial X_0)$ with the induced Riemannian metric, and let ω be a self-dual 2–form on X which is non-zero and covariantly constant on each component of $[0,\infty) \times \partial X_0$, and which is tamed by ϖ.

a) The first energy bound

The first of the important energy bounds is presented below as Proposition 4.1. Its proof then occupies the remainder of this subsection.

Proposition 4.1 *There exist universal, positive constants $\{\kappa_j\}_{j=1,\dots,4}$, and a positive constant, ζ, which depends on the Riemannian metric, ϖ and ω; and these constants have the following significance: As usual, let $\mathcal{M}$ denote the moduli space for a given $(s,z) \in \mathcal{S}_0(X_0, \partial X_0)$, and let $(A,\psi) \in \mathcal{M}$. Then,*

$$\int_X (|\nabla_A\psi|^2 + |P_+F_A|^2) \le \zeta + \kappa_1 z\bullet\varpi - \kappa_2 c(s)\bullet c(s) \text{ and}$$
$$\int_X |F_A|^2 \le \zeta + 2\kappa_1 z\bullet\varpi - \kappa_3 c(s)\bullet c(s).$$

Moreover, if ω is a closed form, then

$$\int_X (|\nabla_A \psi|^2 + |P_+ F_A|^2) \leq \zeta + \kappa_4 z \bullet \varpi.$$

Finally, in the case where $X = \mathbb{R} \times T^3$, these results hold with $\zeta = 0$.

Proof of Proposition 4.1 To obtain the proposition's first assertion, start with the Bochner–Weitzenböck formula in (16), take the inner product of both sides of the latter with ψ, and then integrate the result over X. Integration by parts (which (13) justifies) and input from the Seiberg–Witten equations can then be used to derive the equality

$$0 = \int_X \left(|\nabla_A \psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{2}|P_+ F_A|^2 - \frac{i}{2} F_A \wedge \omega \right).$$

Next, fix a closed 2–form μ which represents the class ϖ and equals ω on $[0, \infty) \times \partial X_0$. Because both s and $\omega - \mu$ are supported on ∂X_0, this last equation with (20) implies the inequality

$$\int_X (|\nabla_A \psi|^2 + |P_+ F_A|^2) \leq \zeta' + \kappa_0 \int_X i F_A \wedge \mu + \frac{1}{16} \int_{X_0} |F_A|^2. \tag{25}$$

Here, ζ' depends only on the metric, ω and μ, while κ_0 is a positive, universal constant. With regard to (25), note that when μ is self-dual, the integrand of the last term in (25) can be replaced by $|P_+ F_A|^2$.

The first integral on the right side of (25) is equal to $2\pi \cdot z \bullet \varpi$. Meanwhile, (21) relates the integral in (25) of $|F_A|^2$ to that of $|P_+ F_A|^2$. With these relations understood, the inequality in the first point of the proposition follows directly from (25). The second point's inequality follows from the first and (21). Meanwhile, the proposition's third point follows directly from (25) given that the integrand in the last term on the right is replaced by $|P_+ F_A|^2$. □

b) Uniform asymptotics of (A, ψ)

The bounds on $P_+ F_A$ and $\nabla_A \psi$ that Proposition 4.1 provides are crucial inputs to a key generalization of Lemma 3.5. The latter is stated next as Proposition 4.2. The proof of this proposition then occupies the remainder of this subsection.

Proposition 4.2 *There exist positive constants, ζ_0 and ζ_1, which depend on the Riemannian metric, ϖ and ω; and which have the following significance: Given $(s, z) \in \mathcal{S}_0(X_0, \partial X_0)$, set*

$$f \equiv \kappa_1 z \bullet \varpi - \kappa_2 c(s) \bullet c(s), \tag{26}$$

where κ_1 and κ_2 are the constants that are introduced in Proposition 4.1. Let $\mathcal{M}$ denote the moduli space for (s, z) and let $(A, \psi) \in \mathcal{M}$. Then:

- $|F_A| < \zeta_0 + \zeta_1 f$ *everywhere on X.*
- *Let $[0, \infty) \times T^3$ be an end of X. There exists some number $N \leq \zeta_0 + \zeta_1 f$ of points $\{t_i\} \in [0, \infty)$ (not necessarily distinct) such that*

$$|F_A|(t, \cdot) \leq \zeta_0 \left(e^{-t/\zeta_0} + \sum_i e^{-(t-t_i)/\zeta_0} \right) \tag{27}$$

at all points $(t, \cdot) \in [0, \infty) \times T^3$.

Proof of Proposition 4.2 The first bound on $|F_A|$ follows from the L^2 bounds in Proposition 4.1 using standard elliptic regularity techniques. To obtain the bound in (27), let $\{t_j\} \subset [2, \infty)$ denote the distinct integer points where the bound in Lemma 3.3 is violated for the cylinder $Y = [t_j - 2, t_j + 2] \times T^3$. Proposition 4.1 guarantees that there are no more than $N = (\zeta + f)/\delta$ such points where ζ is given in Proposition 4.1. The estimate in (27) then follows from Lemma 3.4. □

c) Refinements for the cylinder

The following proposition describes a useful refinement of the inequality in (27) which holds when X is the cylinder $\mathbb{R} \times T^3$.

Proposition 4.3 *Let $R \geq 2$ and let $X = [-R, R] \times T^3$. Let ω be a non-zero, covariantly constant, self-dual 2–form on X. There exist constants $\delta > 0$ and $\zeta_0 \geq 1$ which are independent of R and which have the following significance: Let (A, ψ) be a solution to (4) on X which obeys $\int_Y |F_A|^2 < \delta$ when $Y = [-R, -R + 4] \times T^3$ and $Y = [R - 4, R] \times T^3$. Then:*

- $|F_A| < \zeta_0$ *everywhere.*
- *There exists some number N of distinct, integer valued points $\{t_i\} \in [-R, R]$ such that*

$$|F_A| \leq \zeta_0 \left(\delta e^{-(R-|t|)/\zeta_0} + \sum_i e^{-(t-t_i)/\zeta_0} \right) \tag{28}$$

at all points $(t, \cdot)$ where $t \in [-R + 2, R - 2]$.

Proof of Proposition 4.3 First, note that $|\psi|$ is bounded by some number B on $X' = [-R+1, R-1] \times T^3$ which depends only on the metric and ω. To find the bound B, first use the Seiberg–Witten equations to rewrite P_+F_A in (17) in terms of ψ and so derive the following differential inequality for $|\psi|^2$:

$$2^{-1}d^*d|\psi|^2 + m|\psi|^2(|\psi|^2 - |\omega|) \leq 0. \tag{29}$$

Here, $m > 0$ is a universal constant. The maximum principle applies to (29) and bounds $|\psi|^2$ in terms of $|\omega|$ and its size near the ends of X'. Meanwhile, Lemma 3.3 bounds the size of $|\psi|^2$ near the ends of X' in terms of $|\omega|$.

For the next step in the proof, take $t \in [-R+2, R-2] \times T^3$, set $T = [t-1, t+1] \times \mathbb{R}$ and then multiply both sides of (17) by a standard, smooth function on $[-R, R]$ which is 1 on T and 0 on $[-R, t-2]$ and on $[t+2, R]$. Integrate the result over X', and then integrate by parts to remove the operator d^*d from $|\psi|^2$. With the bound on $|\psi|$ by B, a simple manipulation gives the inequality

$$\int_T (|\nabla_A\psi|^2 + |P_+F_A|^2) \leq \zeta'(1 + B^2); \tag{30}$$

here, ζ' is another constant which depends only on $|\omega|$.

The next task is to control the L^2 norm of $|P_-F_A|$. Here, the vanishing of dF_A is employed to conclude that $d(P_-F_A) = -d(P_+F_A)$. Now, note that by differentiating (4), the form $d(P_+F_A)$ can be expressed in terms of ψ and $\nabla_A\psi$ and as a result, its norm is bounded by a uniform multiple of $\zeta' B|\nabla_A\psi|$, where ζ' is a universal constant. Thus, (30) provides an L^2 bound on $d(P_-F_A)$. With this bound in hand, it now proves useful to rewrite the equation $d(P_-F_A) = -d(P_+F_A) \equiv \sigma$ by separating out t–derivatives from derivatives along the tori T^3. There result two equations,

$$\begin{aligned} &\partial_t f - *\partial f = \sigma_\perp \text{ and} \\ &\partial * f = \sigma_0. \end{aligned} \tag{31}$$

Here, ∂ is the exterior derivative along the torus, and $*$ denotes the Hodge star along the torus. Also, f is the t–dependent 1–form on T^3 which is obtained from P_-F_A by contracting with the unit vector in the t–direction.

To analyze the preceding equations, write $f = g + \partial u$, where $\partial * g = 0$ and where u is a time dependent function on T^3 obeying $\partial * \partial u = \sigma_0$. Letting $\|\cdot\|_t$ denote the L^2 norm over $\{t\} \times T^3$, it follows by standard arguments that there is a solution u which obeys

$$\|\partial u\|_t \leq \zeta \cdot \|\sigma_0\|_t$$

for all $t \in [-R+2, R-2]$.

Next, consider the projection of the top line in (31) onto the kernel of $\partial *$. The result is an equation for g which reads

$$\partial_t g - *\partial g = \sigma', \tag{32}$$

where σ' is the L^2–orthogonal projection of $\sigma_\perp$ onto the kernel of $\partial *$. (Thus, $\|\sigma'\|_t \le \|\sigma\|_t$.) To analyze (32), consider the projections g_+, g_- and g_0 onto the respective positive, negative, and zero eigenspaces of $*\partial$ acting on the kernel of $\partial *$. Likewise, introduce the analogous projections of σ', namely $\sigma'_{+,-,0}$. Then (32) implies

$$\begin{aligned}
&(\partial_t - \lambda)\cdot \|g_+\|_t \ge -\|\sigma'_+\|_t;\\
&(\partial_t + \lambda)\cdot \|g_-\|_t \le \|\sigma'_-\|_t;\\
&\partial_t g_0 = \sigma'_0.
\end{aligned} \tag{33}$$

Here, λ is the smallest non-zero absolute value of an eigenvalue of $*\partial$ on kernel$(\partial *)$.

The first and second lines in (33) can be integrated to yield

$$\|g_+\|_t \le e^{-\lambda\cdot(R-1-t)}\|g_+\|_{R-1} + \sup_{s\in[-R+1,R-1]} \int_{[s-1,s+1]} \|\sigma'_+\|_s^2 ds \text{ and}$$

$$\|g_-\|_t \le e^{\lambda\cdot(t+1-R)}\|g_-\|_{-R+1} + \sup_{s\in[-R+1,R-1]} \int_{[s-1,s+1]} \|\sigma'_-\|_s^2 ds.$$

The final line can be integrated to find that

$$\|g_0\|_t \le \sup_{s\in[-R+1,R-1]} \|P_+F_A\|_s. \tag{34}$$

The explicit formula for σ'_0 must be used to derive (34). For this purpose, introduce the t–dependent 1–form h on T^3 by writing $P_+F_A = dt\wedge h + *h$. Then, up to a sign, σ'_0 is the time derivative of the projection of h onto the space of harmonic 1–forms on T^3.

These last two equations with (30), the equation in the first point of (4), the bound $|\psi| \le B$ and Lemma 3.3 have the following consequence: The L^2 norm of $|P_-F_A|$ is uniformly bounded on each t = constant slice of the cylinder $[-R+1, R-1]\times T^3$ in terms of $|\omega|$ and the assumed L^2 bound of F_A on $[-R, -R+4]\times T^3$ and on $[R-4, R]\times T^3$. With the latter understood, standard elliptic regularity techniques find a uniform pointwise bound for $|F_A|$ at points where $t \in [-R+2, R-2]$. Of course, the bound on the L^2 norm on t = constant slices provides one on all length 4 cylinders in X', and with this understood, the bound in (28) follows from Lemma 3.4. □

d) Vortices on the cylinder

This section serves as a digression of sorts to describe the space of solutions to the vortex equation on $\mathbb{R}\times S^1$. These solutions will be used to describe the Seiberg–Witten moduli space on $\mathbb{R}\times T^3$.

To describe the vortices, endow $\mathbb{R}\times S^1$ with its standard product metric and with the complex structure from the identification via the exponential map with $\mathbb{C}^* = \mathbb{C}-\{0\}$. Fix a constant $r>0$. A vortex solution is a pair (v,τ) of imaginary valued 1–form and complex function which obey the conditions

$$\begin{aligned}&\bullet\ *dv = -ir(1-|\tau|^2);\\ &\bullet\ \bar\partial\tau + v_{0,1}\tau = 0;\\ &\bullet\ (1-|\tau|^2) \text{ is integrable.}\end{aligned}\tag{35}$$

Here, $v_{0,1}$ is the $(0,1)$ component of v in $T(\mathbb{R}\times S^1)_{\mathbb{C}}$.

Let $\mathcal{C}$ denote the set of solutions to (35) modulo the equivalence relation that identifies (v,τ) with (v',τ') when $v' = v+\varphi d\varphi^{-1}$ and $\tau' = \varphi\tau$ whenever φ is a smooth map from $\mathbb{R}\times S^1$ to the unit circle $S^1\subset\mathbb{C}$. Topologize $\mathcal{C}$ as with $\mathcal{M}$. That is, first topologize the solution set to (35) with the subspace topology by embedding the latter in $i\cdot\Omega^1\times\Omega^0_{\mathbb{C}}\times[0,\infty)$, where the first two coordinates are v and τ, respectively, and the last is $r\int(1-|\tau|^2)*1$. Then, give $\mathcal{C}$ the quotient topology.

Here are some facts (without proofs) about $\mathcal{C}$ (see, eg Section 2 of [19] or [7] for the proofs of similar assertions about vortices on $\mathbb{C}$):

- *$\mathcal{C}$ is the disjoint union of components, $\{\mathcal{C}_n\}_{n=0,1,\dots}$. The component $\mathcal{C}_n$ is a manifold of complex dimension n and is diffeomorphic to $\mathcal{Z}_n = \{(y_1,\dots,y_n)\in\mathbb{C}^n : y_n\neq 0\}$ by the map $\Upsilon\colon \mathcal{Z}_n\to\mathcal{C}$ which sends $y\in\mathcal{Z}_n$ to a solution of the form*
$$(v,\tau) = (\bar\partial u - \partial u, e^{-u}p[y]),\tag{36}$$
where $p[y]$ sends $\eta\in\mathbb{C}^* = \mathbb{R}\times S^1$ to the polynomial $\eta^n + y_1\eta^{n-1}+\dots+y_n$. Meanwhile, u is the unique, real valued function on $\mathbb{R}\times S^1$ which obeys
$$\begin{aligned}&2i*\partial\bar\partial u = -r(1-e^{-2u}|p[y]|^2);\\ &u = n\ln t + o(1) \text{ at points } (t,\cdot)\in\mathbb{R}\times S^1 \text{ with } t\gg 1;\\ &u = \ln|y_n| + o(1) \text{ at points } (t,\cdot)\in\mathbb{R}\times S^1 \text{ with } t\ll -1.\end{aligned}$$
- *If $(v,\tau)\in\mathcal{C}_n$, then*
$$r\int(1-|\tau|^2) = 2\pi n.\tag{37}$$

- *$|\tau| \le 1$ with equality if and only if $|\tau| = 1$ everywhere and $n = 0$.*
- *There is a constant ξ which depends only on the vortex number and is such that*

$$(1 - |\tau|^2) + r^{-1/2}|\nabla_v \tau| \le \zeta \exp[-(2r)^{1/2}\,\mathrm{dist}(\cdot, \tau^{-1}(0))]. \tag{38}$$

- *$\mathcal{Z}_n$ is diffeomorphic to $\mathbb{C}^{n-1} \times \mathbb{C}^*$.*
- *There is a gluing map, $\mathcal{G}$, that sends any finite product $\mathcal{Z}_{n_1} \times \cdots \times \mathcal{Z}_{n_k}$ to the corresponding $\mathcal{Z}_{n_1+\cdots+n_k}$; it is defined by the requirement that $p[\mathcal{G}(y_{n_1}, \dots, y_{n_k})] = p[y_{n_1}] \cdots p[y_{n_k}]$, where $p[\cdot]$ is as described in the first point above.*
- *The group $\mathbb{C}^* = \mathbb{R} \times S^1$ acts on each $\mathcal{C}_n$ as pull-back via its natural action on itself. In terms of the parameterization of a vortex solution as a point $y = (y_1, \dots, y_n) \in \mathcal{Z}_n$, this action has $\lambda \in \mathbb{C}^*$ send y to $(\lambda^{-1}y_1, \dots, \lambda^{-n}y_n)$. The action of an element in $\mathbb{C}^*$ on a vortex will be called a translation.*
- *A vortex parametrized by $y \in \mathcal{Z}_n$ with $|y_n| = 1$ will be called centered. In this regard, note that $\ln|y_n|$ equals the average of the t coordinates of the zeros in $\mathbb{R} \times S^1$ of the corresponding τ. The value of $\ln|y_n|$ will be called the center of the vortex.*

Vortex solutions on $\mathbb{R} \times S^1$ give Seiberg–Witten solutions on $X = \mathbb{R} \times S^1 \times T^2$ as follows: Take ω on X to equal $r \cdot P_+\omega_1$, where ω_1 is the standard volume form on $\mathbb{R} \times S^1$. Now let (v, τ) be a vortex solution and set $A = A_0 + 2v$, $\psi = (\sqrt{r}\tau, 0)$, where A_0 is the product connection. Here, the bundle S_+ has been split into eigenspaces for Clifford multiplication by ω.

e) The moduli space for $\mathbb{R} \times T^3$

This section constitutes a second digression to consider the moduli space of solutions to (4) for the case $X = \mathbb{R} \times T^3$. For this purpose, take ω in (4) to be a non-zero, covariantly constant form on the whole of X. When considering the possibilities for $\mathcal{M}$ in this case, note that as $X_0 = [-1, 1] \times T^3$, there is just one element in $\mathcal{S}_0(X_0)$, the trivial $\mathrm{Spin}^{\mathbb{C}}$ structure. Meanwhile, $\mathcal{S}_0(X_0, \partial X_0) = H^2_{\mathrm{comp}}(\mathbb{R} \times T^3)$. Moreover, cup product with a generator of $H^1_{\mathrm{comp}}(R; \mathbb{Z}) = \mathbb{Z}$ provides an isomorphism, $\Xi\colon H^1(T^3; \mathbb{Z}) \approx H^2_{\mathrm{comp}}(\mathbb{R} \times T^3; \mathbb{Z})$ and so $\mathcal{S}_0(X_0, \partial X_0) = H^1(T^3; \mathbb{Z})$.

With the preceding understood, consider:

Proposition 4.4 *Fix $z \in H^1(T^3;\mathbb{Z})$ and then the corresponding Seiberg–Witten moduli space $\mathcal{M}$ on $\mathbb{R}\times T^3$ consists of a single point (the orbit of a solution with $F_A \equiv 0$) unless*

$$\omega = P_+(dt\wedge\theta),$$

where θ is a constant 1–form on T^3 whose cohomology class is a positive multiple of z in $H^1(T^3;\mathbb{Z})$. When the latter is true, then $\mathcal{M}$ is diffeomorphic to the vortex moduli space $\mathcal{C}_n$, where n is the divisibility of z in $H^1(T^3;\mathbb{Z})$. In particular, this diffeomorphism arises from the fact that each point in $\mathcal{M}$ is gauge equivalent to $(A = A_0 + 2a, \psi = \sqrt{r}(\alpha, 0))$, where the pair (a,α) is the pull-back from $\mathbb{R}\times S^1$ of a vortex in $\mathcal{C}_n$ via the map that identifies the $\mathbb{R}$ factors while it fibers T^3 over S^1 so that the constant 1–form that gives S^1 length 1 pulls back as the constant 1–form which represents $n^{-1}z$ in $H^1(T^3;\mathbb{Z})$.

The remainder of this subsection is occupied with the following:

Proof of Proposition 4.4 Use ω to decompose the bundle S_+ on $\mathbb{R}\times T^3$ as $E\oplus E^{-1}$, where $E\to\mathbb{R}\times T^3$ is a complex line bundle. Note that E must be topologically trivial since the restriction of its first Chern class to T^3 must vanish.

Now consider (A,ψ). Write $\psi = (\alpha,\beta)$ to correspond with the splitting of S_+. Then, Witten's arguments from [22] for compact Kähler manifolds (using Lemma 3.5 to justify integration by parts) can be employed here to prove that $\beta = 0$ and that α is holomorphic with respect to that complex structure on X that is defined by the flat metric and the self-dual form ω. In addition, $iP_+F_A = (1-|\alpha|^2)\omega$.

Note that the maximum principle insures that $|\alpha| < 1$ everywhere unless $\alpha\equiv 1$. In this case, the solution is gauge equivalent to the constant solution on T^3. With this understood, assume below that $|\alpha| < 1$.

To proceed, note that $dF_A = 0$ so $dP_-F_A = -dP_+F_A$. Differentiate this last identity to find an equation of the form $\nabla^*\nabla(iP_-F_A) + \zeta_1|\alpha|^2 iP_-F_A = i\zeta_2 P_-(d_A\alpha^*\wedge d_A\alpha)$, where $\zeta_{1,2} > 0$ are universal constants. A similar equation holds for P_+F_A and comparing these two equations with the help of the maximum principle gives the pointwise bound $|P_-F_A| \le |P_+F_A|$. This last inequality, with the condition $z\bullet z = 0$, implies that $|P_-F_A| = |P_+F_A|$ which is equivalent to $F_A\wedge F_A = 0$. In fact, it follows now that $iP_-F_A = (1-|\alpha|^2)\sigma$, where σ is a constant, anti-self dual 2–form with norm equal to that of ω. Furthermore, $d_A\alpha^*\wedge d_A\alpha$ is proportional to F_A everywhere. (See (4.28–30) in

[19].) It also follows that α is holomorphic with respect to the complex structure on $\mathbb{R}\times T^3$ that is defined by the given flat metric and the self-dual form ω.

Let $\omega_1 = \omega + \sigma$. This form has square zero, and its kernel defines a 2–dimensional distribution on $\mathbb{R}\times T^3$ on which F_A and $d_A\alpha$ both vanish. This distribution is also invariant under the complex structure on $\mathbb{R}\times T^3$ because $d_A\alpha$ has zero projection onto the corresponding $T^{0,1}X$. Furthermore, since ω_1 is constant, the resulting foliation is the image in $\mathbb{R}\times T^3$ of a linear foliation of the universal covering space $\mathbb{R}\times\mathbb{R}^3=\mathbb{C}^2$ by complex lines. Moreover, as $|F_A|^2$ has finite integral over $\mathbb{R}\times T^3$, the time coordinate (the $\mathbb{R}$ factor in $\mathbb{R}\times T^3$) is constant on each leaf of the foliation. Furthermore, as α is holomorphic, its zero set, a union of leaves of the foliation, is a smooth, compact, codimension 2, complex submanifold of X. This implies that each leaf of the foliation is a closed, linear torus in T^3.

The proposition follows directly from the preceding remarks. $\square$

f) Compactness in some special cases

This subsection returns now to consider the compactification of the Seiberg–Witten moduli spaces for those X_0 from Proposition 2.4. In this case, Propositions 4.2–4.4 can be employed to compactify the space $\mathcal{M}_{s,m}$ from (6) as a stratified space. The basic tool for this is Proposition 4.5, below. The statement of Proposition 4.5 reintroduces the function f on $H^2_{\mathrm{comp}}(X;\mathbb{Z})$ from (26).

Proposition 4.5 *Fix $(s,z)\in\mathcal{S}_0(X_0,\partial X_0)$, let $\mathcal{M}$ denote the corresponding moduli space and let $\{c_j\}_{j=1,2,\dots}\in\mathcal{M}$ be a sequence with no convergent subsequences. Then, the following exist:*

1. *$z'\in\varsigma(s)$ with $z'\bullet\varpi < z\bullet\varpi$.*
2. *A point c_∞ in the corresponding $\mathcal{M}'$.*
3. *For each component $T=[0,\infty)\times T^3\subset[0,\infty)\times\partial X_0$ an element $z_{\mathcal{E}}\in H^1(T^3;\mathbb{Z})$ and a sequence $\{o_j\}_{j=1,2,\dots}$ in the corresponding moduli space $\mathcal{M}^{\mathcal{E}}$ on $\mathbb{R}\times T^3$.*

This data has the following significance: There is a subsequence of $\{c_j\}$, hence relabled consecutively, such that:

- *$\{c_j\}$ converges to c_∞ on compact subsets of X in the C^∞ topology. This is to say that there exists a pair $(A_\infty,\psi_\infty)\in c_\infty$, and, for each index j,* *a pair (A_j, ψ_j) on c_j such that if $K \subset X$ is a compact set and k is a positive integer, then*

$$\sum_{0 \le p \le k} \left(|\nabla^p (A_j - A')| + |(\nabla_{A'})^p (\psi_j - \psi'_j)|\right) < \varepsilon \tag{39}$$

at all points in K.

- *Let $T = [0, \infty) \times T^3$ be a component of $[0, \infty) \times \partial X_0$. Given $\varepsilon > 0$ and a positive integer k, there exists $R \ge 2$ and, for each index j, there is a pair (A'_j, ψ'_j) on o_j such that*

$$\sum_{0 \le p \le k} \left(|\nabla^p (A_j - A')| + |(\nabla_{A'})^p (\psi_j - \psi'_j)|\right) < \varepsilon \tag{40}$$

at all points $(t, \cdot)$ with $t \ge R$.

- *The vortex number which corresponds to o_j is no greater than $\zeta \cdot f$, where ζ is independent of both z and the sequence $\{c_j\}$.*

Here are two immediate corollaries of Propositions 4.4 and 4.5:

Proposition 4.6 *Let X_0 and X be as in Proposition 2.4. Now, suppose that the restriction of ω to each component of $[0, \infty) \times \partial X_0$ has the form $P_+(dt \wedge \theta)$, where θ is a non-zero, covariantly constant 1–form whose cohomology class is not proportional to an integral class in $H^1(T^3; \mathbb{R})$. For each $(s, z) \in \mathcal{S}_0(X_0, \partial X_0)$, let $\mathcal{M}$ denote the resulting Seiberg–Witten moduli space. Then $\mathcal{M}$ is compact.*

Proposition 4.7 *Suppose that X is isometric to the product metric on $S^1 \times M$, where M is an oriented, Riemannian 3–manifold whose ends are isometric to $[0, \infty) \times T^2$. In addition, suppose that ω is invariant under the evident S^1 action and that on each end, $\omega = P_+(dt \wedge \theta)$, where θ is a covariantly constant 1–form on $S^1 \times T^2$ which is not pulled back via projection to the T^2 factor. For each $(s, z) \in \mathcal{S}_0(X_0, \partial X_0)$, let $\mathcal{M}$ denote the corresponding Seiberg–Witten moduli space and let $\mathcal{M}_S \subset \mathcal{M}$ denote the subset of S^1–invariant orbits. Then $\mathcal{M}_S$ is compact.*

The remainder of this subsection contains the proofs of these propositions.

Proof of Proposition 4.6 If the assertion were false, then Proposition 4.5 would find a solution on $\mathbb{R} \times T^3$ with $F_A \neq 0$ identically. The latter is outlawed by Proposition 4.4. □

Proof of Proposition 4.7 If this assertion were false, Proposition 4.5 would find a non-trivial, S^1–invariant solution on $\mathbb{R} \times S^1 \times T^2$ which is not obtained from a vortex solution via a map which factors through the projection to $\mathbb{R}\times T^2$. This is impossible, for if $(A, (\alpha, 0))$ has an S^1–invariant orbit under $C^\infty(\mathbb{R} \times T^3; S^1)$, then $\alpha^{-1}(0)$ must be a union of S^1 orbits in T^3. □

Proof of Proposition 4.5 Proposition 4.2 describes each $c = (A, \psi) \in \{c_j\}$ on the ends of X. In particular, for each component $T \subset [0, \infty) \times \partial X_0$ and each such c, there is the corresponding set $\{t_i \equiv t[T, c]_i\} \subset [0, \infty)$ that appears in (28). If there is no convergent subsequence, then Proposition 3.6 requires at least one end T for which the sequence of sets $\{\{t_i[T, c]\} : c \in \{c_j\}_{j=1,2,\ldots}\}$ is not uniformly bounded. Even so, a subsequence of $\{c_j\}$ can be found for which the sets $\{t_i[T, c]\}$ for each fixed end T all have the same number of elements as c ranges over the subsequence. (Henceforth, all subsequences will be implicitly relabled by consecutive integers starting from 1.) By passing again to a subsequence, these sets can be assumed to converge on compact domains in each end T. With this last point understood, then a limit, c_∞, of a subsequence of $\{c_j\}$ is obtained using relatively standard compactness arguments.

The sequence $\{o_j\}$ for a component $T \subset [0, \infty)\times\partial X_0$ is obtained by translating along $[0, \infty)$ to follow elements in $\{t[T, c]_i\}$ which do not stay bounded as c ranges through the sequence $\{c_j\}$. In this regard, Propositions 4.2, 4.3 and 4.4 together with the translation invariance of the equations in (4) on $\mathbb{R} \times T^3$ play the key role. Indeed, the construction of $\{o_j\}$ begins by obtaining a finite set of centered, limit vortices for each end by translating each $c \in \{c_j\}$ on the end and then, after passing to a subsequence, one follows each 'clump' of energy. Here, the centers of these 'clumps' on the end T for the element $c \in \{c_j\}$ are, by definition, obtained by first partitioning the set $\{t[T, c]_i\}$ which appears in (27) into subsets whose elements are much closer to each other than to the other subsets of the partition. Then, the center of the clump subset is declared to be the average of the t coordinates of the elements in the subset.

After passing again to a subsequence, these 'clump' partitions of the sets $\{t[T, c]_i\}$ can be assumed to produce the same number of subsets as c ranges through $\{c_j\}$ and to be labeled for each such c so that the following is true: First, the distance between subsets with different labels diverges as the index j on c_j tends to infinity. Second, the subsets with the same label have a fixed number of elements as c ranges through $\{c_j\}$, and these elements can be themselves labeled so that the resulting ordered sets converge as the index j tends to infinity.

One then translates the restriction of each $c \in \{c_j\}$ on an end T so that the average t–coordinate of a given, labeled clump subset of $\{t[T,c]_i\}$ is 0. After passing to a subsequence, the resulting translated sequence of Seiberg–Witten solutions will then converge strongly in the C^∞ topology on compact domains in $\mathbb{R} \times T^3$ to a solution to (4) on $\mathbb{R} \times T^3$. This limit is equivalent to a vortex solution as described by Proposition 4.4. No generality is lost by translating this vortex in $\mathbb{R}$ so it is centered.

By the way, the bound provided in Proposition 4.2 on the size of $\{t[T,c]_i\}$ explains the bound in Proposition 4.5 on the number of vortex solutions that arise.

In any event, a component, $T \subset [0,\infty) \times \partial X_0$ provides a well defined, finite set of centered vortex solutions, each corresponding to one of the labels of the clump partition just described. This labeled set of limit vortex solutions can be characterized by the corresponding data $\{y^{(\alpha)}\}$, with each $y^{(\alpha)} \in \mathcal{Z}_m$ for $m = m_\alpha$. This characterization of the vortices is then used to construct, for each $c \in \{c_j\}$, a vortex solution on $\mathbb{R} \times S^1$ which is obtained by gluing with the map $\mathcal{G}$ the translated (via the $\mathbb{R}$ factor in $\mathbb{C}^* = \mathbb{R} \times S^1$) versions of the vortices from the limit set. Here, the particular translation for a vortex depends on the particular $c \in \{c_j\}$ and the particular clump label. To be precise, the translation is chosen so that the center in $\mathbb{R}$ of the translated vortex agrees with the average t–coordinate of the corresponding clump subset in $\{t[T,c]_i\}$.

After the application of the gluing map $\mathcal{G}$, the result is a vortex which gives a Seiberg–Witten solution on $\mathbb{R} \times T^3$ that is close to c, where the latter differs substantially from the limit c_∞ and which is close to the trivial solution elsewhere.

With the preceding understood, the convergence assertion in (39) and (40) follows with standard elliptic regularity arguments for compact domains together with (28), (38) and Lemma 3.4 to predict the form of each $c \in \{c_j\}$ on those parts of $[0,\infty) \times \partial X_0$ where the connection component of c has small curvature.

The proof of Proposition 4.5 ends with an explanation for the fact that the solution c_∞ sits in a moduli space defined by the original $\mathrm{Spin}^{\mathbb{C}}$ structure s but with a class $z' \in \varsigma(s)$ with $z' \bullet \varpi < z \bullet \varpi$. The explanation starts with the observation that the convergence behavior described by (39) and (40) insures that the original s is also the $\mathrm{Spin}^{\mathbb{C}}$ structure for c_∞. Keep in mind here that each $\{o_j\}$ gives solutions to (4) on $\mathbb{R} \times T^3$ whose $\mathrm{Spin}^{\mathbb{C}}$ structure is sent to zero by the map $c(\cdot)$ in (2). The explanation ends with the observation that the difference between the cup products of z and z' with ϖ is a consequence of the final line in Proposition 4.1 when the convergence behavior in (40) is noted. □

5 Refinements for the cylinder

The step from the compactness which is asserted in Proposition 4.6 to the assertions in Proposition 2.4 requires a substantial refinement of the asymptotic estimates from the preceding section. This section derives the required estimates for the behavior of solutions to (4) on subcylinders in $\mathbb{R} \times T^3$.

a) The operator $\mathcal{D}_c$ when $X = \mathbb{R} \times T^3$

The compactness study requires a more in depth study of the operator $\mathcal{D}_c$ in the case where $X = \mathbb{R} \times T^3$ and where ω in (4) is a non-zero, covariantly constant form. The discussion here is broken into five steps.

Step 1 Fix $z \in H^1(T^3;\mathbb{Z})$ and use the identification of the latter with $\mathcal{S}_0(X_0, \partial X_0)$ to specify a Seiberg–Witten moduli space $\mathcal{M}$. However, if $\mathcal{M}$ is to contain more than the $F_A = 0$ solutions, it is necessary to further assume that $\omega = P_+(dt \wedge \theta)$, where θ is a covariantly constant form whose cohomology class in $H^1(T^3;\mathbb{Z})$ is proportional to z. Use the metric on X and ω to define a complex structure, J. Note that $J \cdot dt = r\theta$, where $r \neq 0$, and note that $*\theta$ (a 2–form on T^3) is invariant under the induced action of J on $\Lambda^2 T^*X$. Write $T^{0,1}X = \varepsilon_0 \oplus \varepsilon_1$, where the first factor, ε_0, signifies the span of $\nu_0 = dt - ir\theta$, and where the second factor is the orthogonal complement. Thus, the factor ε_1 has a covariantly constant, unit length frame ν_1 with the property that the wedge of ν_1 with its conjugate is proportional to $*\theta$. Meanwhile, the anti-canonical bundle is spanned by $\nu_0 \wedge \nu_1$ and so is isomorphic to $\varepsilon_{01} \equiv \varepsilon_0 \varepsilon_1$.

Given the preceding, the bundle S_+ can be written as $S_+ = \varepsilon_{\mathbb{C}} \oplus \varepsilon_{01}$, where $\varepsilon_{\mathbb{C}} \to T^3$ is a topologically trivial complex line bundle. Of course, ε_{01} is also topologically trivial, but these lines in S_+ are distinguished by being the eigenbundles for Clifford multiplication by ω. The point is that when $c \in \mathcal{M}$, then the domain of $\mathcal{D}_c$ consists of the L^2_1 sections over $\mathbb{R} \times T^3$ of

$$(\varepsilon_0 \oplus \varepsilon_{\mathbb{C}}) \oplus (\varepsilon_1 \oplus \varepsilon_{01}). \tag{41}$$

Meanwhile, Clifford multiplication on the $\varepsilon_{\mathbb{C}}$ summand of S_+ identifies S_- with $\varepsilon_0 \oplus \varepsilon_1$ and so identifies the range of $\mathcal{D}_c$ with the space of L^2 sections over $\mathbb{R} \times T^3$ of

$$(\varepsilon_{\mathbb{C}} \oplus \varepsilon_0) \oplus (\varepsilon_{01} \oplus \varepsilon_1). \tag{42}$$

Here, the factor $\varepsilon_{\mathbb{C}} \oplus \varepsilon_{01}$ corresponds to the $i\mathbb{R} \oplus \Lambda_+$ portion of $\mathcal{D}_c(\cdot)$. In this regard, the real part of $\varepsilon_{\mathbb{C}}$ corresponds to the $i\mathbb{R}$ summand and the imaginary part to $i\mathbb{R}\omega \subset \Lambda_+$.

Step 2 The ordering of the factors and the placing of the parentheses in (41) and (42) have been chosen so as to decompose the domain and range of $\mathcal{D}_c$ into direct sums, and this decomposition induces the following 2×2 block decomposition of $\mathcal{D}_c$:

$$\mathcal{D}_c = \begin{pmatrix} \Theta & \delta_F^{\dagger} \\ \delta_F & -\Theta^{\dagger} \end{pmatrix}. \tag{43}$$

Here, δ_F is a differential operator along the foliation of M given by the kernel of θ which annihilates the defining constant sections of the first two summands in (41) and whose adjoint annihilates those of the last two in (42). In this regard, remember that c is defined from a vortex solution on $\mathbb{R} \times S^1$ via a projection of the form identity $\times\varphi$ from $\mathbb{R} \times T^3$ to $\mathbb{R} \times S^1$. Here, φ pulls back the standard 1–form on S^1 to a multiple of θ. The leaves of the foliation are the fibers of the map φ. Meanwhile, the operator Θ corresponds to a version of the operator which gives the linearized vortex equations. To be explicit here,

$$\Theta(a, \lambda) = (\partial a + 2^{-1} r\bar{\tau}\lambda, \bar{\partial}_v \lambda + \tau a). \tag{44}$$

Here, $r \equiv |\theta|$, $\partial_v \equiv \partial + v_{0,1}$ and $\partial = 2^{-1}(\partial/\partial t - i\theta^*)$, where $\partial/\partial t$ denotes the tangent vector field to the lines $\mathbb{R} \times$ point while θ^* denotes the vector field which is metrically dual to the 1–form θ. The adjoint, $\Theta^{\dagger}$, of Θ is given by

$$\Theta^{\dagger}(b, \eta) = (-\bar{\partial} b + \bar{\tau}\eta, -\partial_v \eta + 2^{-1} r\tau b). \tag{45}$$

Step 3 The following lemma describes key properties of the operator Θ.

Lemma 5.1 *Given a non-negative integer n, let (v, τ) be a vortex solution on $\mathbb{R} \times S^1$ which has vortex number n. Then:*

- *The operator Θ is Fredholm from L^2_1 to L^2.*
- *The L^2 kernel of Θ is a $2n$–dimensional vector space of smooth forms.*
- *The L^2 kernel of $\Theta^{\dagger}$ is empty. In fact, there exists a constant E which depends only on $|\theta|$ and m and is such that $\|\Theta^{\dagger} w\|_2^2 \geq E\|w\|_2^2$ for all L^2_1 sections w.*
- *By the same token, $\|\Theta w\|_2^2 \geq E\|w\|_2^2$ for all L^2_1 sections w which are L^2 orthogonal to the kernel of Θ.*
- *If $w \in \mathrm{kernel}(\Theta)$, then $|(\nabla_v)^p w| \leq \zeta_p \|w\|_2 \exp\left(-(2r)^{1/2}\,\mathrm{dist}(\cdot, \tau^{-1}(0))\right)$. Here, ζ_p is independent of w and (v, τ).*

Proof of Lemma 5.1 These assertions are all derived using the Weitzenböck formula

$$\Theta\Theta^{\dagger}(b,\lambda) = (-\partial\bar{\partial}b + 2^{-1}r|\tau|^2 b, -\bar{\partial}_v\partial_v\eta + 2^{-1}r|\tau|^2\eta), \tag{46}$$

or the corresponding formula for $\Theta^{\dagger}\Theta$. The latter switches ∂ with $\bar{\partial}$ everywhere and adds an extra term which is proportional to $(1-|\tau|^2)$. As this term is small at large distances from $\tau^{-1}(0)$ (see (38)), the formula for $\Theta^{\dagger}\Theta$ implies that Θ is Fredholm on the L^2_1 completion of its domain as a map into the L^2 completion of its range. With this understood, (46) implies that the cokernel of Θ (which is the kernel of $\Theta^{\dagger}$) is empty. Furthermore, (46) plus the last point in (38) implies the estimate in the third point of the lemma. The latter estimate plus the Fredholm alternative implies the estimate in the fourth point. In this regard, remark that when w is L^2–orthogonal to the kernel of Θ, then $w = \Theta^{\dagger}g$ for some element g. Thus,

$$\|\Theta w\|_2 = \|\Theta\Theta^{\dagger}g\|_2 \geq \|\Theta^t g\|_2^2/\|g\|_2 \geq E\|\Theta^t g\|_2 = E\|w\|_2.$$

The fact that the kernel dimension is $2n$ can be proved as follows: Differentiate the map which associates to $y \in \mathbb{C}^m$ a vortex (v,τ). (See (36).) Each such directional derivative gives an independent element in kernel(Θ). Conversely, every element in kernel(Θ) can be integrated to give an element in the tangent space to $\mathcal{Z}_n$. This follows from the vanishing of the kernel of $\Theta^{\dagger}$.

For the exponential decay estimate, consider first the C^0 case. If $w \in$ kernel(Θ), then (38) and the Weitzenböck formula for $\Theta^{\dagger}\Theta$ imply the following: Where the distance from $\tau^{-1}(0)$ is greater than 1, the norm of w obeys

$$d^*d|w| + 2r|w| \leq v \cdot |w|, \tag{47}$$

where v is a function on $\mathbb{R}\times S^1$ which obeys

$$v \leq \zeta \exp\left(-\sqrt{r}\,\mathrm{dist}(\cdot,\tau^{-1}(0))/\zeta\right), \tag{48}$$

with ζ a universal constant. The control of $|w|$ where the distance to $\tau^{-1}(0)$ is less than 1 comes via standard elliptic regularity which finds a constant ξ_1 such that

$$|w| \leq \xi_1\|w\|_2$$

at all points. Here, ξ_1 depends only on r. With the preceding understood, let $\{t_i\}$ denote the time coordinates of the n points in $\tau^{-1}(0)$. An application of the comparison principle to (47) now yields the following: Fix $\rho < (2r)^{1/2}$ and there exists ζ_ρ which is independent of (v,τ) and such that

$$|w| \leq \zeta_x\|w\|_2 \sum_i \exp(-\rho|t-t_i|). \tag{49}$$

To obtain the analog of (49) for the case where $\rho = (2r)^{1/2}$, introduce the Green's function $G(\cdot, z')$ for the operator $d^*d + 2r$ with a pole at $z' \in \mathbb{R} \times S^1$. An application of the comparison principle to the operator $d^*d + 2r$ finds a constant κ_0 which makes the following assertion true: If the $\mathbb{R}$ coordinate of z' is s, that of z is t, and also $|t - s| \geq \zeta \geq 1$, then

$$0 < G(z, z') \leq \kappa_0 \exp(-(2r)^{1/2}|t - s|). \tag{50}$$

Furthermore, the absolute values of the derivatives of $G(\cdot, z')$ at such z enjoy similar upper bounds.

With the Green's function in hand, multiply both sides of (47) by $G(z, \cdot)$ and integrate both sides of the result over the region, U, where the distance to the set $\tau^{-1}(0)$ is at least one. This operation produces an integral inequality which can be further manipulated to yield the following bound:

$$\begin{aligned} |w(z)| \leq \xi_2 \|w\|_2 \sum_i \exp(-(2r)^{1/2}|t - t_i|) \\ + \xi_2 \|w\|_2 \int_U ds \exp(-(2r)^{1/2}|t - s|) \sum_i \exp(-(2r)^{1/2}(1 + \delta)|s - t_i|). \end{aligned} \tag{51}$$

Here, $\delta > 0$ is a universal constant, while ξ_2 depends only on r. To explain, the first term in (51) is due to the integration by parts boundary term that arises when the operator d^*d in (47) is moved from w to $G(\cdot, z)$. Meanwhile, the second term in (51) comes from the integral of $G(z, \cdot)v \cdot |w|$. In this regard, (48) is used to bound v, (49) with ρ very close to $(2r)^{1/2}$ is used to bound $|w|$, and (50) is used to bound $G(z, \cdot)$.

The $\rho = (2r)^{1/2}$ version of (49) follows directly from (51) since the second term in this equation is not greater than $\xi_3 \|w\|_2 \sum_i \exp(-(2r)^{1/2}|t - t_i|)$, with ξ_3 depending only on the parameter r.

The proof for the asserted bounds on the higher derivatives of w is obtained by first differentiating the equation $\Theta w = 0$ say, p times, and then writing the latter as an equation of the form $\Theta(\nabla^p w) =$ lower order derivatives of w. The preceding argument for the C^0 bound can be copied to obtain the desired estimates. □

Step 4 It is important to note that there is one particularly canonical element in the kernel of Θ, this being

$$\pi_c = (2^{-1} r(1 - |\tau|^2), \partial_v \tau). \tag{52}$$

Viewed as a section over $\mathcal{C}_n$ of the complex tangent space, the vector π_c generates the translation induced $\mathbb{C}^*$ action.

Note that the pointwise bound in the last assertion of Lemma 5.1 is sharp for π_c. The following lemma makes a precise statement:

Lemma 5.2 *Given the form $r \equiv |\theta|$ and a positive integer n, there exists a constant $\zeta \geq 1$ such that when $(v,\tau) \in \mathcal{C}_n$ then*

$$(1-|\tau|^2) \geq \zeta^{-1} \sum_j \exp(-(2r)^{1/2}|t-t_j|), \tag{53}$$

where $\{t_j\}$ denote the t–coordinates of the zeros of τ.

Proof of Lemma 5.2 Let $x = (1-|\tau|^2)$. Then, by virtue of the vortex equations in (35), this function obeys

$$d^*dx + 2rx = 4|\partial_v\tau|^2 + 2r|x|^2;$$

thus

$$d^*dx + 2rx \geq 0 \tag{54}$$

everywhere. The first consequence of (54) comes via the maximum principle, this being the previously mentioned fact that $(1-|\tau|^2) > 0$ as long as $n \geq 0$. This last point can be parlayed to give a universal lower bound for $(1-|\tau|^2)$ at points at fixed distance from $\tau^{-1}(0)$. In particular, the following is true:

$$\begin{aligned}&\textit{Given } r = |\theta|, \textit{ the vortex number } n \textit{ and also } \rho > 0, \textit{ there exists}\\ &\xi > 0 \textit{ such that when } (v,\tau) \in \mathcal{C}_n \textit{ then } (1-|\tau|^2) \geq \xi \textit{ at points}\\ &\textit{with distance } \rho \textit{ or less from } \tau^{-1}(0).\end{aligned} \tag{55}$$

Accept (55) for the moment to see its application first. In this regard, (55) is applied here to supply a uniform, positive lower bound, ξ, for $(1-|\tau|^2)$ on the constant t circles where $|t-t_j| \leq 1$ for at least one t_j. With such a bound in place, reapply the maximum principle to (54) but use $x = (1-|\tau|^2) - n^{-1}\xi\sum_j \exp(-(2r)^{1/2}|t-t_j|)$ and restrict to points in $\mathbb{R}\times S^1$ where $|t-t_j| \geq 1$ for all t_j. The resulting conclusion (that $x \geq 0$) and (55) together give (53).

To justify (55), consider the ramifications were the claim false. In particular, there would exist a sequence $\{(v_i,\tau_i)\} \in \mathcal{C}_n$, and a corresponding sequence of pairs of points $\{(z_j,z_j')\} \subset \mathbb{R}\times S^1$, where $\tau_j(z_j) = 0$, $\operatorname{dist}(z_j,z_j') \leq \rho$ and $\lim_{j\to\infty}|\tau_j(z_j')| = 1$. After translating each (v_j,τ_j) appropriately, all of the points $\{z_j\}$ can be taken to be a fixed point $z \in \mathbb{R}\times S^1$. Meanwhile, the sequence $\{z_j'\}$, has a convergent subsequence with limit z' whose distance is ρ or less from z. Also, a subsequence $\{(v_j,\tau_j)\}$ converges strongly in the C^∞

topology on compact domains in $\mathbb{R} \times S^1$ to a vortex solution, (v, τ), although the latter may lie in some $\mathcal{C}_{n'}$ for $n' \leq n$. Indeed, the C^1 convergence of τ and C^0 convergence of v follows directly from (37) and (38) by appeal to the Arzela–Ascoli theorem; convergence in C^k can then be deduced by differentiating the vortex equations. In particular, the C^1 convergence here implies that $\tau(z) = 0$ and $|\tau(z')| = 1$. However, these two conclusions are not compatible; as argued previously, $|\tau| < 1$ everywhere if $|\tau|$ is less than 1 at any point. □

Step 5 This last step uses the results of Lemma 5.1 to draw conclusions about $\mathcal{D}_c$. The latter are summarized by:

Lemma 5.3 *Suppose $c = (A, \psi) \in \mathcal{M}$ comes from a vortex solution, $(v, \tau) \in \mathcal{C}_n$ on $\mathbb{C}^*$ via a fibration map φ, from $\mathbb{R} \times (S^1 \times S^2)$. Then:*

- *The kernel of $\mathcal{D}_c$ is the $2n$–dimensional vector space which consists of configurations $((a, \lambda), (0, 0))$ which are annihilated by both Θ and differentiation along the fibers F of φ. In particular, (a, λ) comes from the kernel of Θ on $\mathbb{C}^* = \mathbb{R} \times S^1$ via pull-back by the map φ.*
- *The kernel of $\mathcal{D}_c^\dagger$ is the $2n$–dimensional vector space which consists of configurations $((0, 0), (a, \lambda))$ which are in the kernel of Θ and differentiation along the fibers of φ. These also come from the kernel of Θ on $\mathbb{R} \times S^1$ via pull-back by the map φ.*
- *If w is an L^2_1 section of (41) which is L^2–orthogonal to the kernel of $\mathcal{D}_c$, then*
$$\|\mathcal{D}_c w\|_2 \geq E\|w\|_2, \tag{56}$$
where E depends only on θ and n and is, in particular, independent of (v, τ).
- *Likewise, if w is an L^2_1 section of (42) which is L^2–orthogonal to the kernel of $\mathcal{D}_c^\dagger$, then $\|\mathcal{D}_c^\dagger w\|_2 \geq E\|w\|_2$.*

Proof of Lemma 5.3 To consider the assertion about the kernel of $\mathcal{D}_c$, write w in 2–component form with respect to the splitting in (41) as (α, β), where each of α and β also have two components. Then,

$$\|\mathcal{D}_c w\|_2^2 = \|\Theta\alpha\|_2^2 + \|\bar{\partial}_F \alpha\|_2^2 + \|\Theta^\dagger \beta\|_2^2 + \|\bar{\partial}_F^\dagger \beta\|_2^2.$$

Integration by parts along the fibers of φ equates $\|\bar{\partial}_F \alpha\|_2^2 = 4^{-1}\|d_F \alpha\|_2^2$, and $\|\bar{\partial}_F^\dagger \beta\|_2^2 = 4^{-1}\|d_F \beta\|_2^2$. Now, write $\alpha = \alpha_0 + \alpha_1$, where α_0 is constant along each fiber of φ, and where α_1 is L^2–orthogonal to the constants along each fiber of φ. It then follows that

$$\|\mathcal{D}_c w\|_2^2 \geq E'(\|\alpha_1\|_2^2 + \|\beta_1\|_2^2) + \|\Theta\alpha_0\|_2^2 + \|\Theta^\dagger\beta_0\|_2^2. \tag{57}$$

The assertions that concern the kernel of $\mathcal{D}_c$ follow from (57) using Lemma 5.1.

An analogous argument proves the assertions in the lemma that concern the kernel of $\mathcal{D}_c^\dagger$. □

b) Decay bounds for kernel($\mathcal{D}_c$) when $c \in \mathcal{M}_P$

The proof of Proposition 2.4 requires a refinement of Proposition 4.3's decay estimates along a finite cylinder $[-R, R] \times T^3$. In particular, (28) establishes that a solution (A, ψ) to (4) on $[-R, R] \times T^3$ with $|F_A|$ small at all points is exponentially close in the middle of the cylinder to a T^3 solution, $(A_0, \psi_0) \in \mathcal{M}_P$. However, the bound in (28) was not concerned with the size of the decay constant in the exponential. This subsection supplies a precise estimate for the constant ζ_0 which appears in (28).

For this purpose, consider the operator $\mathcal{D}_c$ from (43) when $c = (A_0, \psi_0)$ is the pull-back to $[-R, R] \times T^3$ of a solution on T^3 which defines $\mathcal{M}_P$. In this case, write the operator $\mathcal{D}_c$ as

$$\mathcal{D}_c = 2^{-1} \cdot (\partial/\partial t + \mathcal{O}). \tag{58}$$

Here, $\mathcal{O}$ is a t–independent, symmetric, first order operator which differentiates along the T^3 directions. Moreover, there are natural trivializations of the summands of $S_+ = \varepsilon_{\mathbb{C}} \oplus \varepsilon_{01}$ and $T^{0,1}X$ which make $\mathcal{O}$ a constant coefficient operator. In this regard, the trivialization of $\varepsilon_{\mathbb{C}}$ makes ψ_0 the constant section with vanishing imaginary part and positive real part, while that of $T^{0,1}X$ is as described prior to (41).

With the afore-mentioned trivialization understood, then Fourier transforms can be employed to investigate the spectrum of $\mathcal{O}$. In particular, this spectrum is a nowhere accumulating subset of $(-\infty, -(2r)^{1/2}] \cup [(2r)^{1/2}, \infty)$ which is unbounded in both directions, and invariant under multiplication by -1. Furthermore, the minimal, positive eigenvalue $E_0 = (2r)^{-1/2}$ is degenerate, with two eigenvectors over $\mathbb{C}$, these being the sections

$$s_+^+ = (\sqrt{r}, \sqrt{2}, 0, 0) \text{ and } s_-^+ = (0, 0, \sqrt{2}, -\sqrt{r}) \tag{59}$$

of (41). Meanwhile, the eigenvalue $-E_0$ also has two eigenvectors over $\mathbb{C}$, the sections

$$s_+^- = (\sqrt{r}, -\sqrt{2}, 0, 0) \text{ and } s_-^- = (0, 0, \sqrt{2}, \sqrt{r}) \tag{60}$$

of (41).

Any section w of (41) over $[-R,R]\times T^3$ which is annihilated by $\mathcal{D}_c$ has the form

$$w = \sum_{E>0} e^{-E\cdot(t+R)}\cdot s_E^+ + \sum_{E>0} e^{-E\cdot(R-t)}\cdot s_E^-, \tag{61}$$

where s_E^+ is an eigenvector of $\mathcal{O}$ with eigenvalue E, and where s_E^- is likewise an eigenvector, but with eigenvalue $-E$. Note that $\sum_{E>0}\|s_E^+\|_{2,T}^2 + \sum_{E>0}\|s_E^-\|_{2,T}^2 \le \zeta\cdot\|w\|_2^2$. Here, $\|\cdot\|_{2,T}$ denotes the L^2 norm on T^3 and $\|\cdot\|_2$ denotes the L^2 norm over $[-R,R]\times T^3$.

Equation (61) is a linear version of the following:

Lemma 5.4 *Suppose that v is a homomorphism over $[-R,R]\times T^3$ from the bundle in (41) to that in (42) which obeys the bound $|v|\le\zeta\cdot e^{-(R-|t|)/\zeta}$, where $\zeta>0$ is a constant. Then, there are constants $z>(2r)^{1/2}$ and ζ' which depend only on ζ and which have the following significance: Let w be a section over $[-R,R]\times T^3$ of (41) which obeys*

$$\mathcal{D}_c w + vw = 0. \tag{62}$$

Then $w = w_0 + w_1$ with

$$\begin{aligned} &\|w_1\|_{2,T}\big|_t \le \zeta' K e^{-z\cdot(R-|t|)}, \text{ where } K = \sup_{t\in[-R,-R+1]\cup[R-1,R]}|w| \text{ and} \\ &w_0 = \exp(-(2r)^{1/2}(R+t))(u_+^+s_+^+ + u_-^+s_-^+) + \\ &\qquad \exp(-(2r)^{1/2}(R-t))(u_+^-s_+^- + u_-^-s_-^-), \text{ where } u_\pm^\pm \text{ are constants.}\end{aligned} \tag{63}$$

Proof of Lemma 5.4 Let Π_+ denote the L^2–orthogonal projection (on T^3) onto the span of the eigenvectors of $\mathcal{O}$ with eigenvalue $E>(2r)^{1/2}$. Meanwhile, let Π_- denote the corresponding projection onto those eigenvectors with eigenvalue $E<-(2r)^{1/2}$. Let $f_+(t)$ denote the L^2 norm of the time t version of Π_+w, and likewise define $f_-(t)$. Then $f_\pm$ obey a pair of coupled differential inequalities of the form

$$\begin{aligned}(\partial/\partial t + E_2)f_+ &\le \zeta e^{-(R-|t|)/\zeta}\left(f_+ + f_- + K\exp(-(2r)^{1/2}(R-|t|))\right); \\ (\partial/\partial t - E_2)f_- &\ge -\zeta e^{-(R-|t|)/\zeta}\left(f_+ + f_- + K\exp(-(2r)^{1/2}(R-|t|))\right).\end{aligned} \tag{64}$$

Here, $E_2>(2r)^{1/2}$ is the second smallest positive eigenvalue of $\mathcal{O}$. This last equation can be integrated (after some algebraic manipulations) to obtain the bound

$$(f_+ + f_-)\big|_t \le \zeta'(f_+\big|_{-R} + f_-\big|_R + K)e^{-z'\cdot(R-|t|)},$$

where $z' > (2r)^{1/2}$ and $\zeta' > 0$ depends only on ζ.

Meanwhile, write $w = b_+^+ s_+^+ + b_-^+ s_-^+ + b_+^- s_+^- + b_-^- s_-^- + \Pi_+ w + \Pi_- w$ and consider the equations for $\mathbf{b}^+ = (b_+^+, b_-^+)$ and for the corresponding $\mathbf{b}^-$. In particular, these equations have the form

$$\begin{aligned}(\partial/\partial t + 2r)\mathbf{b}^+ &= g^+ \cdot \mathbf{b}^+ + g^- \cdot \mathbf{b}^- + v^+ \text{ and}\\ (\partial/\partial t - 2r)\mathbf{b}^- &= h^+ \cdot \mathbf{b}^+ + h^- \cdot \mathbf{b}^- + v^-,\end{aligned}$$

where $|g^\pm| + |h^\pm| \le \zeta e^{-(R-|t|)/\zeta}$ and $|v^\pm| \le \zeta e^{-z\cdot(R-|t|)}$ with $z > (2r)^{1/2}$. Integrating these last two equations gives

$$\begin{aligned}\mathbf{b}^+\big|_t &= \mathbf{b}^+\big|_{-R} \cdot \exp(-(2r)^{1/2}(t+R)) + \mathbf{c}^+ \text{ and}\\ \mathbf{b}^-\big|_t &= \mathbf{b}^-|_R \cdot \exp(-(2r)^{1/2}(R-t)) + \mathbf{c}^-,\end{aligned}$$

where $|\mathbf{c}^\pm| \le \zeta' K e^{-z'\cdot(R-|t|)}$ with $z' > (2r)^{1/2}$ and ζ' depending only on ζ. □

By way of an application, choose a vortex (v, τ) on $\mathbb{R} \times S^1$ and use the latter to define the gauge orbit, c, of a solution to (4) on $\mathbb{R} \times T^3$. The element π_c in (52) can be viewed as either an element, π_{c+}, in the kernel of $\mathcal{D}_c$ or as an element, π_{c-}, in cokernel$(\mathcal{D}_c)$. Here, π_{c+} is the section of (41) whose first two components are those of π_c and whose second two are zero. Meanwhile, π_{c-} is the section of (42) whose first two components are zero and whose second two are the components of π_c. (The applications below only use π_{c-}.) As there are points on the gauge orbit c which are asymptotic as $t \to \pm\infty$ to a solution (A_0, ψ_0) which defines $\mathcal{M}_P$, Lemma 5.4 can be applied to $\pi_{c\pm}$ with the following effect:

Lemma 5.5 *Let $(v, \tau) \in \mathcal{C}_n$ and use the latter, as instructed in Proposition 4.4, to define the gauge orbit, c, of a solution to (4) on $\mathbb{R} \times T^3$. Define $\pi_{c\pm}$ as in the preceding paragraph.*

- *Let (A_+, ψ_+) denote a point on the orbit c such that $|A_+ - A_0| + |\psi_+ - \psi_0| \le \zeta e^{-t/\zeta}$ at all points $(t, \cdot)$ with $t > 0$. Here, $\zeta > 0$. Use ψ_0 to define the constant real section of the first summand of $S_+ = \varepsilon_{\mathbb{C}} \oplus \varepsilon_0$. Then, as $t \to \infty$,*

$$\begin{aligned}\pi_{c+} &= \exp(-(2r)^{1/2}t)u^+ s_+^+ + \mathcal{O}(e^{-zt}) \text{ and}\\ \pi_{c-} &= \exp(-(2r)^{1/2}t)u^+ s_-^- + \mathcal{O}(e^{-zt}).\end{aligned} \tag{65}$$

- *Let (A_-, ψ_-) denote a point on the orbit c such that $|A_- - A_0| + |\psi_- - \psi_0| \le \zeta e^{t/\zeta}$ at all points $(t, \cdot)$ with $t < 0$. Here, $\zeta > 0$ also. Use ψ_0*

to define the constant real section of the first summand of $S_+ = \varepsilon_{\mathbb{C}} \oplus \varepsilon_0$. *Then, as* $t \to -\infty$,

$$\begin{aligned} \pi_{c+} &= \exp((2r)^{1/2}t)u^- s_+^- + \mathcal{O}(e^{zt}) \text{ and} \\ \pi_{c-} &= \exp((2r)^{1/2}t)u^- s_-^+ + \mathcal{O}(e^{zt}). \end{aligned} \tag{66}$$

In these equations, $s_\pm^\pm$ *are given in* (59) *and* (60), *and* $z > (2r)^{1/2}$ *is a constant which depends only on* r *and the constant* ζ, *but otherwise is independent of the chosen vortex. Meanwhile,* $u^\pm > 0$ *are constants which depend on the chosen vortex* $c = (v, \tau)$ *and which obey*

$$\begin{aligned} u^+ &\geq \xi^{-1} \cdot \sum_j \exp((2r)^{1/2}t_j) \text{ and} \\ u^- &\geq \xi^{-1} \cdot \sum_j \exp(-(2r)^{1/2}t_j), \end{aligned} \tag{67}$$

where $\{t_j\}$ *are the* t*–coordinates of the zeros of* τ *and where* ξ *depends only on the vortex number.*

Proof of Lemma 5.5 The assertions of (65) and (66) about π_{c+} follow directly from Lemma 5.4. Meanwhile, the assertions about π_{c-} follow from Lemma 5.4 after changing t to $-t$ in (58). Then, given these assertions, the bounds in (67) follow from Lemma 5.2. □

c) More asymptotics for solutions on a cylinder

Proposition 5.6, below, constitutes a second application of Lemma 5.4, here to the asymptotics of a solution to (4) on the cylinder $X = [-R-2, R+2] \times T^3$ with the form $\omega = P_+(dt \wedge d\theta)$, where θ is non-zero and covariantly constant. As in previous subsections, the lemma below takes (A_0, ψ_0) to be the pull-back to X of a solution to (11) which defines $\mathcal{M}_P$ and it takes $r \equiv |\theta|$.

Proposition 5.6 *The metric on* T^3 *and the form* ω *determine a constant* $\zeta \geq 1$ *that has the following significance: Let* $R \geq 4$ *and suppose that* (A, ψ) *obeys* (4) *and the assumptions of Lemma 3.4 on* $[-R-2, R+2] \times T^3$. *Then, there is a gauge equivalent pair* $(A_0 + b, \psi_0 + \eta)$, *where* (b, η) *defines a section of* (41) *that obeys*

$$\begin{aligned} (b, \eta) = {} & (u_+^+ s_+^+ + u_-^+ s_-^+) \exp(-(2r)^{1/2}(t+R)) \\ & + (u_+^- s_+^- + u_-^- s_-^-) \exp(-(2r)^{1/2}(R-t)) + w_1 \end{aligned} \tag{68}$$

at all $(t,\cdot) \in [-R+\zeta, R-\zeta] \times T^3$. *Here,*

$$|w_1| \le \zeta \cdot \exp\Big(-((2r)^{1/2} + \zeta^{-1}) \cdot (R - |t|)\Big),$$

$\{u^{\pm}_{\pm}\}$ *are constant, and*

$$\sum_{0\le k\le 3} |\nabla^k(b,\eta)| \le \zeta \cdot \exp(-(2r)^{1/2}(R-|t|)) \tag{69}$$

at all points $(t,\cdot) \in [-R,R] \times T^3$.

Proof of Proposition 5.6 According to Lemma 3.4, there exists a pair $(A_0 + b, \psi_0 + \eta)$ on the gauge orbit of (A,ψ) which obeys

$$\sum_{0\le k\le 3} |\nabla^k(b,\eta)| \le \zeta \cdot e^{-(R-|t|)/\zeta} \tag{70}$$

at all points $(t,\cdot)$ where $t \in [-R,R]$. Thus, on some smaller length cylinder, this pair differs very little from (A_0,ψ_0) and so can be analyzed by treating (4) as a perturbation of a linear equation. Moreover, as explained below, there is a fiducial choice of such a gauge equivalent pair $(A_0+b, \psi_0+\eta)$, where (b,η) obeys (69) and also

$$d^*b - 2i\,\mathrm{Im}(\psi_0^\dagger \eta) = 0 \tag{71}$$

on a subcylinder of the form $[-R+\zeta_1, R-\zeta_1] \times T^3$, where ζ_1 depends only on the metric and ω.

With the preceding understood, view $w = (b,\eta)$ as a section of (41). By virtue of (4) and (64), this section obeys an equation of the form in (62), where v is a linear function of the components of w. In particular, Lemma 5.4 is applicable here with R replaced by $R' \equiv R - \zeta_1$ because the condition in (70) insures that v obeys the requisite bounds on $[-R+\zeta_1, R-\zeta_1] \times T^3$. Thus, (68) can be seen to follow from (63).

The refined derivative bounds for (b,η) in (69) are obtained from the C^0 bounds via standard elliptic techniques via (62). In particular, to obtain the C^1 bounds, simply differentiate (62) and, remembering that v is a linear functional of the components of w, observe that the result has the same schematic form as (62). Thus, a second appeal to Lemma 5.4 provides the C^1 bounds on (b,η). The derivation of the C^3 bound is only slightly more complicated.

It remains now to justify the asserted existence of the point $(A_0+b, \psi_0+\eta)$ on the gauge orbit of (A,ψ) for which both (70) and (71) hold. For this purpose, use Lemma 3.4 to conclude that (A, ψ) is gauge equivalent to $(A_0 + b', \psi_0 + \eta')$, where (b', η') obeys

$$\sum_{0 \leq k \leq 4} |\nabla^k (b', \eta')| \leq \zeta' e^{-(R-|t|)/\zeta'} \tag{72}$$

at all points $(t, \cdot)$ where $t \in [-R, R]$. Here, $\zeta' > 0$ is a constant which is independent of R. The pair $(A_0 + b, \psi_0 + \eta)$ is guaranteed to come from the same gauge orbit as $(A_0 + b', \psi_0 + \eta')$ if (b, η) and (b', η') are related via the identity

$$(b, \eta) = (b' - 2i \cdot du, e^{i \cdot u} \cdot \eta' + (e^{i \cdot u} - 1) \cdot \psi_0), \tag{73}$$

where u is a smooth function on $[-R-2, R+2] \times T^3$. Thus, the goal is to find u in (73) so that (71) holds on an appropriate subcylinder. In this regard, (71) should be viewed as an equation for the function u. In particular, if u has a suitably small C^2 norm, then this equation has the schematic form

$$-2i(d^* du + 2ru) + d^* b' - 2i \operatorname{Im}(\psi_0^\dagger \eta') + \Re(u) = 0, \tag{74}$$

where $|\Re| \leq \zeta_1(|u|^2 + |u|(|b'| + |\eta'|))$ with ζ_1 a constant which is independent of (b', η') and R.

The analysis of (74) is facilitated by the following observation: Because of (72), the pair (b', η') is uniformly small on uniformly large subcylinders of $[-R, R] \times T^3$. That is, given $\varepsilon > 0$, there exists $\xi > 2$ which is independent of (b', η') and R such that at all points $(t, \cdot) \in [-R + \xi, R - \xi] \times T^3$, all derivatives from orders 0 through 4 of (b', η') are bounded in size by $\varepsilon e^{-(R-|t|)/\xi}$. Meanwhile, the Green's function, G, for $d^*d + 2r$ defines a bounded operator from $L^2(\mathbb{R} \times T^3)$ to $L^2_2(\mathbb{R} \times T^3)$ and satisfies the pointwise bound in (50).

The preceding observations suggest a contraction mapping construction of a solution u to (74) on a uniformly large subcylinder of $[-R-2, R+2] \times T^3$. For this purpose, fix $\xi > 2$ and introduce a smooth, non-negative function β on $\mathbb{R}$ which equals 1 on $[-R + \xi, R - \xi]$, vanishes where $|t| > R - \xi + 1$ and has first and second derivatives bounded by 10. Then, consider the map from $C^0(\mathbb{R} \times T^3)$ to itself which sends u to

$$T(u) \equiv -i2^{-1} G(\beta[d^* b' - 2i \operatorname{Im}(\psi_0^\dagger \eta') + \Re(u)]).$$

Here, the fact that T defines a self map on $C^0(\mathbb{R} \times T^3)$ is insured by the right hand inequality in (50). Moreover, (50) and (72) imply the following: There exist constants $\xi \geq 2$ and $\xi' > 0$ which are independent of R and (b', η') and such that T is a contraction mapping on the radius ξ' ball in $C^0(\mathbb{R} \times T^3)$. For such ξ, the map T has a unique fixed point, u, in this ball.

Of course, (50) insures that this u decays to zero exponentially fast as $|t| \to \infty$ on $\mathbb{R} \times T^3$. Moreover, (50) in conjunction with (72) can be used to prove that u and its derivatives obey the required norm bounds throughout in $[-R-2, R+2] \times T^3$. These last derivations are straightforward and omitted. □

d) The distance to a non-trivial vortex

Now, consider a half infinite tube of the form $Y = [-2R, \infty) \times T^3$, where $R \geq 4$. Fix $B_0 \geq 0$ and $B_1 \geq 1$ and suppose that (A, ψ) is a solution to (4) on Y with the following properties:

$$\begin{aligned} &\int_Y |F_A|^2 \leq B_0; \\ &|F_A| \leq B_1(\exp(t/B_1) + \exp(-(2R+t)/B_1)) \text{ at points with } t \in [-2R, 0]. \end{aligned} \tag{75}$$

With Propositions 4.5 and 5.6 in mind, it can be expected that when R is large, then (A, ψ) is close on $[-R, \infty) \times T^3$ to the restriction of a solution of (4) on the whole of $\mathbb{R} \times T^3$. Indeed, such is the case, as the subsequent proposition attests.

Proposition 5.7 *Given $B_0 \geq 0$ and $B_1 \geq 1$ as above, there exists $R_0 \geq 4$ and $\zeta \geq 1$ with the following significance: Suppose that $R \geq R_0$ and that (A, ψ) obeys (4) and (75) on the half infinite tube $Y = [-2R, \infty) \times T^3$. Then, there is a solution (A_1, ψ_1) on $\mathbb{R} \times T^3$ to (4) and a gauge transform of (A, ψ) on Y such that $(A, \psi) = (A_1, \psi_1) + w$ on Y, where*

$$\begin{aligned} &|w| \leq \zeta \exp(-(2r)^{1/2} R) \exp(-(2r)^{1/2}(R+t)) \text{ if } t \in [-R, -R/2]; \\ &\int_{t \geq -R/2} |w|^2 \leq \zeta \cdot \exp(-3(2r)^{1/2} R). \end{aligned} \tag{76}$$

The remainder of this subsection contains the following:

Proof of Proposition 5.7 Use Lemma 5.6 to find $\zeta_0 \geq 1$ and a gauge for (A, ψ) on the cylinder $[-2R+\zeta_0, -\zeta_0] \times T^3$ of the form $(A, \psi) = (A_0 + b, \psi_0 + \eta)$, where (b, η) come from Proposition 5.6 and obey (68), (69) and (71), while (A_0, ψ_0) is the pull-back from T^3 of a pair that defines $\mathcal{M}_P$. Now, choose a smooth, non-negative function β on $\mathbb{R}$ which equals 1 where $t \geq 1$ and 0 where

$t \le 0$. Use (b, η) and the function β to define a configuration (A', ψ') on $\mathbb{R}\times T^3$ as follows:

$(A', \psi') = (A_0, \psi_0)$ where $t < -R - 2$;
$(A', \psi') = (A_0, \psi_0) + \beta(t + R)(b, \eta)$ where $t \in [-R - 2, -R + 2]$;
$(A', \psi') = (A, \psi)$ where $t \ge -R + 2$.

Note that $\mathbf{H} = (P_+ F_{A'} - \tau(\psi' \otimes (\psi')^\dagger) + i \cdot \omega, D_{A'}\psi')$ vanishes except when the coordinate $t \in [-R + 1, -R + 2]$. Moreover, when $t \in [-R + 1, -R + 2]$, then (69) guarantees that

$$\sum_{0\le k\le 2} |\nabla^k \mathbf{H}| \le \zeta_1 \exp(-(2r)^{1/2} R), \tag{77}$$

where ζ_1 is independent of R; it depends only on the constants B_0 and B_1. Thus, for large R, the pair (A', ψ') is very close to solving (4) on $\mathbb{R} \times T^3$. The following lemma makes this notion precise:

Lemma 5.8 *Under the assumptions of Proposition 5.7, there exists $m \ge 0$, $\varepsilon_0 > 0$ and, given $\varepsilon \in (0, \varepsilon_0)$, there exists R_ε and these have the following properties: First, an upper bound for m, the numbers ε_0, and a lower bound for R_ε depend only on B_0 and B_1. Second, when $R \ge R_\varepsilon$, then the pair (A', ψ') has C^2 distance less than ε from gauge orbits of solutions to (4) on $\mathbb{R} \times T^3$ that come from vortex solutions with vortex number m. Third, such an orbit contains a unique pair (A_1, ψ_1) for which $(b_1, \eta_1) \equiv (A' - A_1, \psi' - \psi_1)$ obeys*

- $d^* b_1 - 2 \cdot i \cdot \mathrm{Im}(\psi_1^\dagger \eta_1) = 0$;
- $\int (|b_1|^2 + |\eta_1|^2) \le \xi \varepsilon^2$;
- $\sum_{0\le k\le 2} |\nabla^k (b_1, \eta_1)| \le \xi\varepsilon$ *everywhere*; (78)
- $|(b_1, \eta_1)| \le \xi\varepsilon \left(\exp(-(2r)^{1/2}(2R - \xi + t)) + \exp((2r)^{1/2}(t + \xi))\right)$, *where* $t \in [-2R + \xi, -\xi]$.

Here, ξ depends only on the vortex number m; in particular, it is independent of ε, R, and the original pair (A, ψ).

The proof of Lemma 5.8 is given below.

Lemma 5.8 enters the proof of Proposition 5.7 in the following way: Fix some positive $\varepsilon \ll \varepsilon_0$ and then R_ε as in Lemma 5.8. An upper bound for ε is derived in the subsequent arguments. In any event, suppose that $R > R_\varepsilon$. Let $\mathcal{U} \subset \mathcal{C}_m$ denote the open set of elements which provide gauge orbits of solutions to (4) on $\mathbb{R} \times T^3$ that have C^2 distance less than ε from the pair (A', ψ'). Given a vortex solution from $\mathcal{U}$, let (A_1, ψ_1) denote the corresponding solution to (4) that is provided by Lemma 5.8. Introduce the resulting (b_1, η_1) and observe that $w_1 = (b_1, \eta_1)$ obeys an equation of the form

$$\mathcal{D}_1 w_1 + \mathcal{Q}(w_1) + \mathbf{H} = 0. \tag{79}$$

Here, $\mathcal{D}_1$ denotes the operator $\mathcal{D}_c$ from (43) as defined using $c = (A_1, \psi_1)$, $\mathcal{Q}(\cdot)$ is a universal, quadratic, fiber preserving map from (41) to (42) and $\mathbf{H}$ is interpreted as a section of (42).

With w_1 and $\mathcal{D}_1$ understood, consider:

Lemma 5.9 *Under the assumptions of Proposition 5.7, there exists $\varepsilon_1 > 0$ which depends only on B_0 and B_1 and which has the following significance: If $\varepsilon < \varepsilon_1$, then there exists (A_1, ψ_1) as described in the preceding paragraph for which the corresponding w_1 is L^2–orthogonal to the kernel of the operator $\mathcal{D}_1$.*

The proof of this lemma is also given below.

Given the statement of Lemma 5.9, the proof of Proposition 5.7 continues with the following observation: There exists $\varepsilon_2 > 0$ and ζ_2 which depend only on B_0 and B_1 and are such that when $\varepsilon < \varepsilon_2$, then the L^2 norm of Lemma 5.9's section w_1 satisfies

$$\|w_1\|_2 \leq \zeta_2 \exp(-(2r)^{1/2} R). \tag{80}$$

Indeed, the existence of such a pair (ε_2, ζ_2) is guaranteed by (54), (77) and the third point in (78). As is explained below, this upper bound on the L^2 norm of w_1 implies the pointwise bounds:

- $|w_1| \leq \zeta_3 \exp(-(2r)^{1/2} R)$ everywhere;
- $|w_1| \leq \zeta_3 \exp(-(2r)^{1/2} R) \left[\exp(-(2r)^{1/2}(R + t)) + \exp((2r)^{1/2}(t + \zeta_3))\right]$ when $t \in [-R, -\zeta_3]$.

(81)

Here, ζ_3 depends only on B_0 and B_1; in particular, it is independent of R. Indeed, the first line in (81) follows from (77), (79) and (80) using standard elliptic estimates, while the second follows from the first after an appeal to Lemma 5.4.

Notice that the first point of (81) implies the first point in (76). The second point in (76) is obtained as follows: Start with the second line of (81) and so conclude that

$$|w_1| \leq \zeta_4 \exp(-3(2r)^{1/2}R/2) \text{ where } t \in \left[-\frac{R}{2}-2, -\frac{R}{2}+2\right]. \tag{82}$$

Here, ζ_4 is independent of R and is determined solely by B_0 and B_1. This last bound is used to derive an upper bound for the L^2 norm of the section w_2 of (41) that is defined by the rule

- $w_2 = 0$ where $t \leq -R/2$,
- $w_2 = \beta(-\frac{R}{2}+t)w_1$ where $t \geq -R/2$.

Note that a suitable upper bound for the L^2 norm of w_2 will yield, with (82), the second point in (76).

To obtain such a bound, use (79) to conclude that w_2 obeys an equation of the form

$$\mathcal{D}_1 w_2 + \mathcal{Q}(w_2) + \mathbf{H}_2 = 0, \tag{83}$$

where $\mathbf{H}_2 = 0$ except when $t \in [-\frac{R}{2}, -\frac{R}{2}+2]$. Moreover, where $\mathbf{H}_2$ is not zero, it obeys the bound $|\mathbf{H}_2| \leq \zeta_4 \exp(-3(2r)^{1/2}R/2)$ by virtue of (82); here ζ_4 is independent of R as it is determined solely by B_0 and B_1.

With (83) understood, write $w_2 = w_{20} + w_{21}$, where w_{20} is the L^2–orthogonal projection of w_2 onto the kernel of $\mathcal{D}_1$. In this regard, w_{20} enjoys the following upper bound:

$$|w_{20}| + \|w_{20}\|_2 \leq \zeta_5 \exp(-3(2r)^{1/2}R/2), \tag{84}$$

where ζ_5 is independent of R, being determined solely by B_0 and B_1. Indeed, remember that w_1 is orthogonal to the kernel of $\mathcal{D}_1$ and so the projection of w_2 onto this kernel is the same as that of $[1-\beta(-\frac{R}{2}+(\cdot))]w_1$. In particular, the size of the latter projection obeys (84) for the following reasons: First, w_1 enjoys the bound in (82). Second, Lemma 5.4 guarantees that any $\varpi \in \text{kernel}(\mathcal{D}_1)$ is bounded by $\zeta_6\|\varpi\|_2 \exp((2r)^{1/2}t)$ at the points where $t \leq -\zeta_6$. Finally, any $\varpi \in \text{kernel}(\mathcal{D}_1)$ obeys $|\varpi| \leq \zeta_6\|\varpi\|_2$ at all points. Here, again, ζ_6 is independent of R; it depends only on the vortex number m and hence only on B_0 and B_1.

Given (84), the previously mentioned bound on $\mathbf{H}_2$, and the fact that (83) can be written as $\mathcal{D}_1 w_{21} + \mathcal{Q}(w_{21}) + \mathcal{Q}'(w_{20}, w_{21}) + \mathcal{Q}(w_{20}) + \mathbf{H}_2 = 0$, another appeal to (54) finds $\varepsilon_3 > 0$ and ζ_7 which are independent of R, depend only on B_0 and B_1 and are such that when $\varepsilon < \varepsilon_3$, then $\|w_{21}\|_2 \leq \zeta_7 \exp(-3(2r)^{1/2}R/2)$. This last bound, (82) and (84) directly yield the final point in (76). □

Proof of Lemma 5.8 All of the arguments for Lemma 5.8 closely follow arguments previously given, and so, except for the outline that follows, the details are left to the reader. The outline starts with the observation that a slightly modified version of the argument for Proposition 4.5 establishes the existence of an upper bound for m and R_ε such that when $R > R_\varepsilon$, then (A', ψ') has C^2 distance ε or less from a solution to (4) on $\mathbb{R} \times T^3$. The bound on the vortex number comes from the L^2 constraint in the statement of Proposition 5.7. Given that (A', ψ') is close to a solution to (4), then the points in (78) are proved by arguments which are essentially the same as those used above to prove Proposition 5.6. Note that these may require an increase of the lower bound for R. The arguments for the lemma's uniqueness assertion are basically those used to prove the slice theorem for the action of the gauge group on the space of solutions to (4). □

Proof of Lemma 5.9 As explained previously, Lemma 5.8 provides the non-empty, open set $\mathcal{U} \subset \mathcal{C}_m$ of elements that determine solutions to (4) on $\mathbb{R} \times T^3$ whose C^2 distance is less than ε from (A', ψ'). The assignment to a point in $\mathcal{U}$ of the corresponding w_1 defines a map from $\mathcal{U}$ into the space of L^2 sections of (41). This map is smooth; the proof is straightforward so its details are left to the reader. The assignment to a point in $\mathcal{U}$ of the square of the L^2_1 norm of the corresponding w_1 then defines a smooth function, f, on $\mathcal{U}$. As is explained momentarily, the differential of f vanishes at precisely the points where w_1 is L^2–orthogonal to the kernel of $\mathcal{D}_1$. Indeed, let b denote a tangent vector to $\mathcal{U}$ at the point that corresponds to (A_1, ψ_1). Then, the differential of w_1 in the direction defined by b has the form $w_{1b} + \delta_b$, where $w_{1b} \in \mathrm{kernel}(\mathcal{D}_1)$ and δ_b is tangent to the orbit through (A_1, ψ_1) of the gauge group $C^\infty(\mathbb{R} \times T^3, S^1)$. This is a consequence of the fact that (A', ψ') is fixed and only (A_1, ψ_1) is moved by b. Meanwhile, w_1 is L^2–orthogonal to δ_b by virtue of the first point in (78) and so the differential of f vanishes in the direction of b if and only if w_1 is L^2–orthogonal to w_{1b}. As Proposition 4.4 guarantees that the kernel of $\mathcal{D}_1$ is the span of the $\{w_{1b}\}$, so w_1 is orthogonal to $\mathrm{kernel}(\mathcal{D}_1)$ if and only if (A_1, ψ_1) comes from a critical point of f.

With the preceding understood, it remains only to demonstrate that f has critical points. For this purpose, a return to (79) is in order. In particular, with (54), (69) and in conjunction with standard elliptic regularity arguments, (79) leads to the following conclusion: There exist constants $\xi \geq 1$ and $\delta > 0$ which are independent of ε and R and such that if $|w_1| < \delta$, then the C^1 norm of w_1 is bounded by $\xi(\|w_1\|_2 + \exp(-R/\xi))$. Since the C^0 norm of w_1 is bounded by its C^1 norm, this last point and the second point in (78) guarantee that f is proper when ε is small and R is large. Said precisely, $\varepsilon_1 > 0$ and $R_1 \geq 1$ must exist such that when $\varepsilon < \varepsilon_1$ and $R > R_1$, then the map f is a proper map on the set $\mathcal{U}$. As a proper function has at least one critical point so there exists at least one (A_1, ψ_1) where the corresponding w_1 and kernel($\mathcal{D}_1$) are orthogonal. □

6 Compactness

This last section contains the final arguments for Propositions 2.4, 3.7 and 3.9.

a) Proof of Proposition 2.4

The assertion that $\mathcal{M}_{s,m}$ is both compact and consists of a finite number of strata follows from Proposition 4.5. Meanwhile, the assertion in the first point of the proposition is a restatement of the conclusions of Proposition 2.3. Thus, the only remaining issue is that of the second point. The latter's assertion is proved in the subsequent seven steps.

Step 1 To begin, consider some $z \in \varsigma(s)$ and a sequence $\{c_j\} \subset \mathcal{M}(s, z)$ with no convergent subsequences. After passage to a subsequence, as always renumbered consecutively from 1, the sequence $\{c_j\}$ can be assumed to determine data c_∞ and, for each component of ∂X_0, a sequence $\{o_j\}$ of solutions on $\mathbb{R} \times T^3$ to (4), all as described in Proposition 4.5. In this regard, remember that each o_j is determined by a solution, (τ_j, v_j), of the vortex equations in (35).

Fix a component $[0, \infty) \times T^3$ of $[0, \infty) \times \partial X_0$, let $\{o_j\}$ denote the corresponding sequence of solutions on $\mathbb{R} \times T^3$, and for each j, let t_j denote the smallest of the time coordinates of the zeros of the corresponding τ_j. These $\{t_j\}$ can be assumed to define an increasing and unbounded sequence.

Propositions 5.6 and 5.7 determine certain data with certain special properties. Here is the data: A constant $\zeta \geq 1$; a pair (A_∞, ψ_∞) on the gauge orbit c_∞ over $[0, \infty) \times T^3$; for each sufficiently large index j, a pair (A_j, ψ_j) on the gauge orbit c_j over $[0, \infty) \times T^3$; and, for each such j, a pair (A_{1j}, ψ_{1j}) on o_j over $\mathbb{R} \times T^3$. This data enjoys the special properties that are listed (85)–(87) below. Note that in these equations, (A_0, ψ_0) denotes the solution to (4) on $\mathbb{R} \times T^3$ which is given by the vortex equation solution with vortex number zero. Moreover, the bundle S_+ is implicitly written as in (41), $S_+ = \varepsilon_{\mathbb{C}} \oplus \varepsilon_{01}$ and ψ_0 is used to trivialize the $\varepsilon_{\mathbb{C}}$ summand and thus defines the section with vanishing imaginary part and positive real part. Finally, the sections $s^+_\pm$ of (41) are defined in (59).

Here is the promised list of properties:

$$\bullet\ (A_\infty, \psi_\infty) = (A_0, \psi_0) + (u^+_+ s^+_+ + u^+_- s^+_-)\exp(-(2r)^{1/2}(t-\zeta)) + w_\infty, \text{ where } t \geq \zeta. \text{ Here, } |w_\infty| \leq \zeta \cdot \exp(-zt) \text{ and } z \geq (2r)^{1/2} + \zeta^{-1}. \tag{85}$$

This is by virtue of Proposition 5.6.

$$\begin{aligned} &\bullet\ (A_j, \psi_j) = (A_0, \psi_0) + (u^+_{j+} s^+_+ + u^+_{j-} s^+_-)\exp(-(2r)^{1/2}(t-\zeta)) + w_j, \\ &\quad \text{where } t \in [\zeta, t_j - \zeta]. \\ &\quad \text{Here, } |w_j| \leq \zeta[\exp(-zt) + \exp(-z(t_j - t))] \text{ and } z \geq (2r)^{1/2} + \zeta^{-1}. \\ &\bullet\ |u^+_{j+} - u^+_+| + |u^+_{j-} - u^+_-| \to 0 \text{ as } j \to \infty. \\ &\bullet\ t_j \to \infty \text{ as } j \to \infty. \end{aligned} \tag{86}$$

The points in (86) also follow from Proposition 5.6.

$$\begin{aligned} &\bullet\ (A_j, \psi_j) = (A_{1j}, \psi_{1j}) + w_{1j} \text{ where } t \geq t_j/2. \\ &\quad \text{Here, } |w_{1j}| \leq \zeta \exp(-(2r)^{1/2} t_j/2) \exp(-(2r)^{1/2}(t - t_j/2)), \\ &\quad \text{where } t \in [t_j/2, 3t_j/4]. \\ &\bullet\ \int_{t \geq 3t_j/4} |w_j|^2 \leq \zeta \exp(-3(2r)^{1/2} t_j/2). \\ &\bullet\ w_j \text{ satisfies an equation of the form } \mathcal{D}_j w_{1j} + \mathcal{Q}(w_{1j}) = 0, \text{ where} \\ &\quad t \geq t_j/2. \text{ Here, } w_{1j} \text{ is viewed as a section of } (41),\ \mathcal{D}_j \equiv \mathcal{D}_c \text{ with} \\ &\quad c = o_j \text{ and } \mathcal{Q}(\cdot) \text{ is a universal, quadratic, fiber preserving map} \\ &\quad \text{from } (41) \text{ to } (42). \end{aligned} \tag{87}$$

The points in (87) follow from Proposition 5.7 after translating the origin so that the new origin corresponds to t_j and hence $-t_j$ corresponds to the old 0. Then, take $R = t_j/2$. Note that t_j is the most negative of the t–coordinates of the zeros of the o_j version τ in (36).

Step 2 The solution (A_{1j}, ψ_{1j}) on $\mathbb{R} \times T^3$ to (4) is defined by a solution (τ_j, v_j) to the vortex equation in (35) on $\mathbb{R} \times S^1$. As long as the latter is not the trivial vortex (gauge equivalent to the pair $(0, 1)$), then the operator Θ in (44) has complex multiples of the element in (52) in its kernel. To underscore the index dependence, use π_j to denote this element. Then $\mathcal{D}_j$ has the element $\pi_{j-} = (0, \pi_j)$ in its cokernel. Here, π_{j-} is defined as in Lemma 5.4 using $c = o_j$.

Note that (66) and Lemma 5.4 assert that at $t \in [\frac{t_j}{2} - 2, \frac{t_j}{2} + 2]$, this π_{j-} has the form

$$\pi_{j-} = \exp(-(2r)^{1/2} t_j/2) h_j s^+_- + \sigma'_j. \tag{88}$$

Here, h_j is a positive constant, $h_j \geq \xi^{-1}$, while $|\sigma'_j| \leq \xi \cdot \exp(-z \cdot t_j/2)$ with $z > (2r)^{1/2} + \xi^{-1}$. In this regard, $\xi \geq 1$ is a constant which is independent of the index j. (The vector s^+_- which appears in (88) is given in (59).)

Step 3 Take the inner product of both sides of the equality $\mathcal{D}_j w_{1j} + \mathcal{Q}(w_{1j}) = 0$ with π_{j-} and then integrate over the region where $t \geq t_j/2$. With (58) in mind, apply integration by parts to the resulting expression and so find that

$$\int_{t \geq t_j/2} \langle \pi_{j-}, w_{1j} \rangle = 2 \int_{t \geq t_j/2} \langle \pi_{j-}, \mathcal{Q}(w_{1j}) \rangle. \tag{89}$$

It follows from (86) that the left hand side of (88) is equal to

$$h_j u^+_{j-} \exp(-(2r)^{1/2} t_j) + \xi \exp(-z t_j), \tag{90}$$

where $z > (2r)^{1/2}$ and ξ are independent of j. Furthermore, for large j, the number u^+_{j-} is determined also by c_∞ up to a small error because of the last point in (86). Indeed, (90) can be rewritten as

$$h_j (u^+_- + \varepsilon_j) \exp(-(2r)^{1/2} t_j), \tag{91}$$

where $|\varepsilon_j| \to 0$ as $j \to \infty$ and u^+_- is given in (85).

Meanwhile, the second and third points of (87) and (61) imply that the absolute value of the right hand side of (89) is no greater than

$$\xi \exp(-3(2r)^{1/2} t_j/2), \tag{92}$$

where ξ is, once again, independent of the index j.

Step 4 It follows from (91), (92) and the fact that $h_j > \zeta^{-1}$ that the sequence $\{c_j\}$ can exist as described only if the configuration c_∞ is such that the number u^+_-, which arises in (85), is zero.

Associate to each element $c \in \mathcal{M}_s$ and each component of ∂X_0 the complex number u^+_- using (85). The proof of Proposition 2.4 is completed with a demonstration of the fact that for a suitably generic choice of ω in (4), the set of those c in $\mathcal{M}_s$ where any given u^+_- is zero has codimension at least 2.

To see that such is the case, first fix a fiducial choice ω_0 of self-dual 2–form which is non-zero and covariantly constant on each end of X. Next, fix an open set $K \subset X_0$ with compact closure in the interior of X_0, and consider the space $\underline{\mathcal{M}}_s$ that is defined as follows: As a set, this space consists of pairs (c, ω) where

ω is a self-dual 2–form on X_0 that agrees with ω_0 on the complement of K, and where c is the orbit under the action of $C^\infty(X; S^1)$ of a solution to the version of (4) which is defined by the given form ω. The topology on $\underline{\mathcal{M}}_s$ is the minimal topology so that the assignment to a pair $(c, \omega) \in \underline{\mathcal{M}}_s$ of the form $\omega - \omega_0$ provides a continuous map to the space of smooth, compactly supported sections over K of Λ_+ whose fibers are topologized as before. The strata of $\underline{\mathcal{M}}_s$ are labeled by the elements of $\varsigma(s)$ and are defined in the obvious way. They are all smooth manifolds for which the assignment to (c, ω) of $\omega - \omega_0$ is a smooth map into the Fréchet space of compactly supported, smooth sections over K of Λ_+. (Given Proposition 2.2, these last assertions about the strata of $\underline{\mathcal{M}}_s$ are proved with virtually the same arguments that establish the analogous assertion for compact 4–manifolds.)

Fix attention on a stratum, $\underline{\mathcal{M}}(s, z) \subset \underline{\mathcal{M}}_s$. Given a component of ∂X_0, define a map, $\varphi\colon \underline{\mathcal{M}}(s, z) \to \mathbb{C}$, by associating the complex number u_-^+ from (85) to each (c, ω). An argument is provided below for the following assertion:

Lemma 6.1 *The map φ is a smooth map and it has no critical points.*

Given Lemma 6.1, an appeal to the Sard–Smale theorem finds the Baire subset which makes the second point of Proposition 2.4 true.

With regard to this Sard–Smale appeal, remember that the latter considers maps between Banach manifolds and so an extra step is required for its use here. This extra step requires the introduction of a sequence of Banach space versions of $\underline{\mathcal{M}}_s$ whose intersection is the smooth version given above. However, the use of Banach spaces modeled on C^p for $p \geq 2$ or L^2_k for $k \gg 1$ makes no essential difference in any of the previous or subsequent arguments.

Step 5 This step, and the subsequent steps contain the following:

Proof of Lemma 6.1 The proof that φ is smooth is straightforward and will be omitted. The subsequent discussion addresses the question of whether φ has critical points. For this purpose, it proves convenient to identify the tangent space to a given stratum $\underline{\mathcal{M}}$ of $\underline{\mathcal{M}}_s$ at a point of interest, $(c = (A, \psi), \omega)$, with a Fréchet space of smooth, square integrable pairs $(w = (a, \eta), \kappa)$ that obey the equation $\mathcal{D}_c w + \kappa = 0$. Here, a is an imaginary valued section of T^*X, η is a section of S_+, and κ is an imaginary valued section of Λ_+ with compact support on K. It is a consequence of Lemma 5.4 and Proposition 5.6 that the restriction of w to each component of $[0, \infty) \times \partial X_0$ has the form

$$w = \exp(-(2r)^{1/2} t)(w_+^+ s_+^+ + w_-^+ s_-^+) + w', \tag{93}$$

where $w_{\pm}^{+}$ are complex numbers and $|w'| \leq \zeta e^{-zt}$ with z a universal constant which is greater than $(2r)^{1/2}$.

With this last equation understood, then the statement of Lemma 6.1 follows with the verification that any value for the complex number w_{-}^{+} can be obtained by a suitable choice of pairs (w, κ) which obey $\mathcal{D}_c w + \kappa = 0$. This verification is the next order of business.

Step 6 The constant w_{-}^{+} in (93) is given by

$$w_{-}^{+} = (2r)^{1/2} \lim_{R\to\infty} \exp(2(2r)^{1/2}R) \int_{t>R} (s_{-}^{+})^{\dagger} w \exp(-(2r)^{1/2}t). \tag{94}$$

With (94) understood, let α be a non-zero complex number and let x_R denote the section of $i \cdot T^*X \oplus S_+$ which is zero except where $t \geq R$ on the given component of $[0, \infty) \times \partial X_0$ in which case

$$x_R \equiv (2r)^{1/2} \alpha s_{-}^{+} \exp((2r)^{1/2}(2R - t)). \tag{95}$$

Thus,

$$\mathrm{Re}(\bar{\alpha} w_{-}^{+}) = \lim_{R\to\infty} \langle x_R, w \rangle, \tag{96}$$

where $\langle\ ,\ \rangle$ denotes the L^2 inner product over X and Re denotes the real part. This section x_R is introduced for reasons which should be clear momentarily.

Now, as $\mathcal{D}_c w + \kappa = 0$, the section w can also be written as

$$w = w_0 - \mathcal{D}_c^{-1}\kappa,$$

where $\mathcal{D}_c w_0 = 0$ holds and $\mathcal{D}_c^{-1}$ maps the L^2–orthogonal complement in $L^2(i \cdot (\mathbb{R} \oplus \Lambda_+) \oplus S_-)$ of cokernel$(\mathcal{D}_c)$ to the L^2–orthogonal complement of kernel$(\mathcal{D}_c)$ in $L_1^2(iT^*X \oplus S_+)$. With the help of this decomposition, (96) becomes

$$\mathrm{Re}(\bar{\alpha} w_{-}^{+}) = -\lim_{R\to\infty} (\langle x_R, \mathcal{D}_c^{-1}\kappa \rangle - \langle x_R, w_0 \rangle). \tag{97}$$

To proceed, introduce x_R^0 to denote the L^2–orthogonal projection of x_R onto the kernel of $\mathcal{D}_c$ and then introduce $y_R \equiv (\mathcal{D}_c^{\dagger})^{-1}(x_R - x_R^0)$. Equation (97) can be rewritten with the help of y_R as

$$\mathrm{Re}(\bar{\alpha} w_{-}^{+}) = -\lim_{R\to\infty} (\langle y_R, \kappa \rangle - \langle x_R^0, w_0 \rangle). \tag{98}$$

In the meantime, it follows from Lemma 5.4 (after changing t to $-t$) that there exists $\zeta > 1$ which is independent of R and such that $|y_R| < \zeta$ on X_0. Moreover, there exists $L \geq 0$ which is independent of R and such that

$$|y_R| \geq \zeta^{-1} \exp((2r)^{1/2}t),$$

where $t \in [L, R]$ on the given component of $[0, \infty) \times \partial X_0$. In this regard, note that Lemma 5.4 insures that $|x_R^0|$ enjoys an R and α independent bound on X when $|\alpha| \leq 1$.

Now, there are two possibilities to consider. The first is that there exists a complex number $\alpha \neq 0$ such that

$$\lim_{R\to 0} \sup_X |x_R^0| = 0. \tag{99}$$

The second possibility is that there is an unbounded subsequence of values for R such that the corresponding sequence $\{\sup |x_R^0|\}$ has a non-zero limit for each unit length $\alpha \in \mathbb{C}$. Now, in this second case, it follows from (98) that all complex numbers can be realized by w_-^+ in (93) by tangent vectors at (c, ω) of the form $(w, 0)$ with $w \in \text{kernel}(\mathcal{D}_c)$.

With the preceding understood, suppose that (99) holds for some unit length complex number α. Then, as $\mathcal{D}_c^\dagger y_R = -x_R^0$ except where $t \geq R$ on the given component of $[0, \infty) \times \partial X_0$, the sequence $\{y_R\}_{R\gg 1}$ converges as $R \to \infty$ to a non-zero section, y, of the bundle $i \cdot (\mathbb{R} \oplus \Lambda_+) \oplus S_-$ which obeys

$$\begin{aligned} &\mathcal{D}_c^\dagger y = 0; \\ &\mathrm{Re}(\bar{\alpha} w_-^+) = -\langle y, \kappa\rangle. \end{aligned} \tag{100}$$

(As κ has compact support, the integral in (100) is well defined.)

Step 7 According to the preceding discussion, if (c, ω) is a critical point of φ, then there exists such unit length $\alpha \in \mathbb{C}$ such that

$$0 = \langle y, \kappa\rangle \tag{101}$$

for all sections κ of $i\Lambda_+$ which have compact support on U.

To make use of (101), it is also important to realize that y is also L^2–orthogonal to all compactly supported sections of the $i\mathbb{R}$ summand in $i \cdot (\mathbb{R} \oplus \Lambda_+) \oplus S_-$. To see that such is the case, let q denote a compactly supported section of the summand in question. Given q, there exists a unique, L^2 function u on X which obeys $d^*du + 2|\psi|^2 u = q$. With u in hand, then $\mathcal{D}_c \underline{u} = q$ where $\underline{u} \equiv (du, -2^{-1}u\psi)$. Thus,

$$\langle y_R, q\rangle = \langle x_R, \underline{u}\rangle. \tag{102}$$

To see that the left hand side of (102) vanishes in the limit as $R \to \infty$, note that x_R lies in the last two summands of (41). Meanwhile, Lemma 5.4 implies that the projection of $\underline{u}$ into these summands is $\mathcal{O}(e^{-zs})$ where $z > (2r)^{1/2}$ is independent of s. Given this last bound, the vanishing as R tends to infinity of the right hand side of (102) follows directly from (95).

To summarize: If (c, ω) is a critical point of the map φ, then there exists a non-zero section, y, over X of $i \cdot (\mathbb{R} \oplus \Lambda_+) \oplus S_-$ with the following properties:

$$\begin{aligned} &\bullet\ \textit{y is L^2–orthogonal to sections with compact} \\ &\quad \textit{support on U of the $i\cdot(\mathbb{R}\oplus\Lambda_+)$ summand.} \\ &\bullet\ \mathcal{D}_c^\dagger y = 0. \end{aligned} \tag{103}$$

Now, the first point here implies that y restricts to U as a section of S_- only. With this understood, then the projection of the equation in the second point of (103) onto the iT^*X summand of $iT^*X \oplus S_+$ asserts that

$$\operatorname{Im}(\psi^\dagger \operatorname{cl}(e) y) = 0$$

for all sections e of T^*X with compact support on U. This last condition can hold only if y is identically zero on U.

Having established that y vanishes identically on U, it then follows that $y \equiv 0$ on the whole of X since there is a version of Aronszajn's unique continuation principle [3] which holds for elements in the kernel of $\mathcal{D}_c^\dagger$. The preceding conclusion establishes that φ has no critical points as claimed. □

b) Proof of Proposition 3.7

Given the details of the preceding proof of Proposition 2.4, the assertions here follow via standard applications of the Sard–Smale theorem. The details for the application to this particular case are left to the reader.

c) Proof of Proposition 3.9

The argument given below considers only the case where M separates X. As the discussion in the case where $X - M$ is connected is identical at all essential points to that given below, the latter discussion is omitted.

To begin, suppose, for the sake of argument, that there exists an increasing and unbounded sequence, $\{R_j\}_{j=1,2,\ldots}$ and a corresponding sequence $\{c_j\} \subset \mathcal{M}^{R_j}$ with the property that for each fixed r', the inequality in (24) holds when $(A, \psi) = c_j$ for only finitely many j. Arguing as in Step 1 of the proof of Proposition 2.4 finds a subsequence of $\{c_j\}$ (hence renumbered consecutively), and data $c_{\infty-}$, $c_{\infty+}$ and $\{o_j\}_{j=1,2,\ldots}$, where now $c_{\infty\pm}$ are orbits of solutions to (4) on the respective $X_\pm$. In this regard, each $X_\pm$ may have other ends besides the end where M lived, and each such end has an associated map φ

as described in the preceding subsection. In this regard, assume that for each such end, $\varphi(c) \neq 0$ for $c = c_{\infty\pm}$.

To return to the data $\{c_j\}$, $c_{\infty\pm}$, $\{o_j\}$, remark that an evident analog of (85)–(87) exists here. In particular, the orbit $c_{\infty-}$ supplies the complex number u_-^+ as defined in (85) for the end $[0,\infty) \times M$ of X_-, while $c_{\infty+}$ supplies the analogous u_-^- which comes from the 'time reversed' version of (85) on the end $(-\infty, 0] \times M$ of X_+. There is also an analog of (91) and (92) here:

$$\begin{aligned} &h_{j-}(u_-^+ + \varepsilon_{j-}) \exp(-(2r)^{1/2}(R_j - t_{j-})) \\ &+h_{j+}(u_-^- + \varepsilon_{j+}) \exp(-(2r)^{1/2}(R_j - t_{j+})) = 0, \end{aligned} \tag{104}$$

where

$$\begin{aligned} &|\varepsilon_{j\pm}| \to 0 \text{ as } j \to \infty; \\ &t_{j\pm} \to \infty \text{ and } |t_{j\pm}|/R_j \to 0 \text{ as } j \to \infty. \end{aligned}$$

In (104), the data $h_{j\pm}$ and $t_{j\pm}$ are supplied by the vortex o_j. In particular, $h_{j\pm}$ are real numbers, both greater than a j–independent, positive ζ^{-1}, while $-t_{j-}$ and t_{j+} are, respectively, the most negative and most positive of the t–coordinates of the zeros of the o_j version τ in (36). With the preceding understood, it now proves useful to package (104) as

$$(u_-^+ + \varepsilon_{j-})\Delta_j + (u_-^- + \varepsilon_{j+})(1 - \Delta_j) = 0,$$

where $\Delta_j \in [0,1]$ is supplied by the vortex o_j.

As $|\varepsilon_{j\pm}| \to 0$ as $j \to \infty$, the sequence $\{\Delta_j\}$ has a unique limit, $\Delta \in [0,1]$ with

$$u_-^+\Delta + u_-^-(1 - \Delta) = 0. \tag{105}$$

Now, the set of non-zero pairs (u_-^+, u_-^-) which obey a relation as in (105) for some Δ determines a codimension 1 subvariety in $\mathbb{C}\times\mathbb{C}$. In particular, coupled with Lemma 6.1 and the Sard–Smale theorem, this last observation implies the next lemma.

Lemma 6.2 *Given a form ω' as described in Proposition 3.9 and open sets $U_\pm$ with respective compact closures in the $\pm$ components of $X_0 - M$, there exists a Baire subset of choices for ω in (4) which agree with ω' on $X - (U_- \cup U_+)$, and which have the following additional property: Let $(s,z) \in \mathcal{S}_0(X_0, \partial X_0)$, let $((s_-, z_-), (s_+, z_+)) \in \wp^{-1}((s,z))$ and use $\mathcal{M}_-$ and $\mathcal{M}_+$ to denote the corresponding moduli space of solutions to (4) on X_- and X_+, respectively. Then, there is a codimension one subvariety in the product, $\mathcal{M}_- \times \mathcal{M}_+$, which contains the only pairs $(c_-, c_+) \in \mathcal{M}_- \times \mathcal{M}_+$ for which the corresponding pair of complex numbers (u_-^+, u_-^-) satisfies (105) for some choice of $\Delta \in [0,1]$.*

Lemma 6.2 directly implies the assertions of Proposition 3.9 in the $d = 0$ case. Indeed, it is a consequence of Lemma 6.2 that the relevant subvariety in $\mathcal{M}_- \times \mathcal{M}_+$ must be empty as each $\mathcal{M}_\pm$ has dimension zero. Thus, no solution to (105) will exist and so for R large, each point in $\mathcal{M}^R$ is in the image of the map Φ from Proposition 3.8.

To argue the $d > 0$ assertion of the proposition, choose all of the points in Λ to lie on the X_- side of X. According to Lemma 6.2, the conclusion that $(\mathcal{M}^R)^{\underline{\Lambda}}$ lies in Φ's image for large R fails only if the following is true: There exist $\mathcal{M}_\pm$ with $\mathcal{M}_-$ having dimension $2d$, $\mathcal{M}_+$ having dimension 0, and $\mathcal{M}_-^{\underline{\Lambda}} \times \mathcal{M}_+ \subset \mathcal{M}_- \times \mathcal{M}_+$ intersecting the subvariety from Lemma 6.2. However, as Lemma 6.2's subvariety has codimension 1 and $\mathcal{M}_-^{\underline{\Lambda}} \times \mathcal{M}_+$ is a finite set, such an intersection is precluded by a suitably generic choice for the points in Λ.

7 3–dimensional implications

The purpose of this final section is to discuss the implications of the theorems and propositions of the preceding sections in the special case where $X_0 = S^1 \times Y_0$ and Y_0 is either a compact 3–manifold with positive first Betti number or else a compact 3–manifold with boundary whose boundary components are all tori.

Here are the key points to note with regard to such X_0: An argument from [9] readily adapts to prove that the first Chern class, $c(s)$, for any pair $(s, z) \in \mathcal{S}_0(X_0, \partial X_0)$ with non-zero Seiberg–Witten invariant is pulled up from Y_0. Moreover, as is explained below, the Seiberg–Witten invariants of X_0 are identical to invariants that are defined for Y_0 by counting solutions on $Y \equiv Y_0 \cup ([0, \infty) \times \partial Y_0)$ of a version of (11).

To describe this version of (11), a Riemannian metric, a $\mathrm{Spin}^{\mathbb{C}}$ structure and a closed 2–form ω_0 must first be chosen. Having made these choices, the two equations in (11) make perfectly good sense on any oriented 3–manifold. However, for Y as described in the preceding paragraph, the metric should be a product, flat metric on $[0, \infty) \times \partial Y_0$, and the form ω_0 must be non-zero and constant on each component of $[0, \infty) \times \partial Y_0$. With this understood, the equations in (11) should be augmented with the extra 'boundary condition'

$$\int_Y |F_A|^2 < \infty. \tag{106}$$

The invariants for Y_0 that the preceding paragraph mentions are then obtained via a count with appropriate algebraic weights of the orbits under the action of $C^\infty(Y; S^1)$ of the solutions to (11) which satisfy (106).

These 3–manifold equations on Y can be viewed as versions of (4) on $X = S^1 \times Y$ which is how the equivalence between the Seiberg–Witten invariants for Y_0 and X_0 arise. For this purpose, consider the version of (4) on $X = Y \times S^1$ when X has its product metric and when $\omega = d\tau \wedge \theta + \omega_0$. Here, θ is the metric dual on Y to ω_0 and $d\tau$ is an oriented, unit length, constant 1–form on S^1. In this case, note that solutions to (11) on Y which obey (106) provide solutions to (4) on X. Moreover, in the case where $c(s)$ is pulled up from Y_0, an integration by parts argument, much like that used to prove Proposition 4.1, proves that all solutions to (4) on X are pull-backs of solutions to (11) and (106) on Y. Thus, Theorems 1.1 and 2.7 directly imply Mayer–Vietoris like theorems for the 3–dimensional Seiberg–Witten equations. (In particular, Theorem 1.1 implies Theorem 5.2 in [16].)

The question arises as to whether the implications of Theorems 1.1 and 2.7 for the 3–dimensional Seiberg–Witten equations can be proved with a strictly 3–dimensional version of the arguments of the previous sections. The short answer is no as there are additional complications that arise and make for a somewhat more involved story. And, as the story here is already long enough, the additional discussion will not be provided, save for the brief comments of the next paragraph.

Because the solutions to (11) on Y provide solutions to (4) on X, the analysis in Sections 3–5 can be directly employed to characterize the behavior of the solutions to (11). However, there is one caveat: Properties that hold for solutions to (4) on X when the form ω is chosen from a Baire set may not hold for the solutions on Y because a Baire set may be devoid of 2–forms given by $d\tau \wedge \theta + \omega_0$ where ω_0 is a closed form on Y. In particular, the following analog of Proposition 2.4 holds in the purely 3–dimensional context:

Proposition 7.1 *Let θ_0 denote a closed 1–form on Y which is non-zero and constant on each component of $[0, \infty) \times \partial Y_0$. With θ_0 given, use a form ω_0 in (11) whose metric dual agrees with θ_0 on $[0, \infty) \times \partial Y_0$. Then each $\mathcal{M}_{s,m} \subset \mathcal{M}_s$ is compact and contains only a finite number of strata. Moreover, fix a closed 2–form ω' whose metric dual agrees with θ_0 on $[0, \infty) \times \partial Y_0$; and fix a non-empty, open set $U \subset Y_0$. Then, there is a Baire set of smooth, closed 2–forms ω that agree with ω' on $X - U$ and have the following properties:*

- *Each stratum of $\mathcal{M}_s$ is a smooth manifold of dimension 0. Moreover, the cokernel of the operator $\mathcal{D}_c$ vanishes for each $c \in \mathcal{M}_s$.*
- *The boundary of the closure in $\mathcal{M}_s$ of any stratum intersects the remaining strata as a codimension 1 submanifold.*

Note that this last proposition implies the third assertion in Equation (4) of [16].

The appearance in Proposition 7.1 of codimension 1 submanifolds as opposed to codimension 2 causes the added complications in the proof of the purely 3–dimensional version of Theorem 2.7. In particular, the purely 3–dimensional versions of Propositions 3.7 and 3.9 may not hold. Even so, somewhat more complicated analogs of these propositions can be established that are sufficient to provide the purely 3–dimensional proof of Theorem 2.7.

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