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Vanishing theorems and conjectures for the ℓ^2–homology of right-angled Coxeter groups

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Abstract

Associated to any finite flag complex L there is a right-angled Coxeter group W_L and a cubical complex Σ_L on which W_L acts properly and cocompactly. Its two most salient features are that (1) the link of each vertex of Σ_L is L and (2) Σ_L is contractible. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ_L is a contractible n–manifold. We describe a program for proving the Singer Conjecture (on the vanishing of the reduced ℓ^2–homology except in the middle dimension) in the case of Σ_L where L is a triangulation of $\mathbb{S}^{n-1}$. The program succeeds when $n \leq 4$. This implies the Charney–Davis Conjecture on flag triangulations of $\mathbb{S}^3$. It also implies the following special case of the Hopf–Chern Conjecture: every closed 4–manifold with a nonpositively curved, piecewise Euclidean, cubical structure has nonnegative Euler characteristic. Our methods suggest the following generalization of the Singer Conjecture.

Conjecture: If a discrete group G acts properly on a contractible n–manifold, then its ℓ^2–Betti numbers $b_i^{(2)}(G)$ vanish for $i > n/2$.

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0 Introduction

The Euler Characteristic Conjecture 0.1 *If M^{2k} is a closed, aspherical manifold of dimension $2k$, then its Euler characteristic, $\chi(M^{2k})$, satisfies:*

$$(-1)^k\chi(M^{2k}) \geq 0.$$

In the special case of Riemannian manifolds of nonpositive sectional curvature, this conjecture is usually attributed to H Hopf. (In this special case, in dimensions 2 and 4, the conjecture follows from the Gauss–Bonnet Theorem. The proof in dimension 4 is given in Chern's 1956 paper [14], where it is attributed to Milnor.) In the early 1970's, Thurston suggested that the conjecture might hold for all closed aspherical manifolds.

In [10], R Charney and the first author discuss the Euler Characteristic Conjecture in the context of piecewise Euclidean manifolds which are nonpositively curved in the sense of Aleksandrov and Gromov [26]. The case where the manifold is cellulated by regular Euclidean cubes is particularly easy to discuss. In this case, by a lemma of Gromov [26], the nonpositive curvature condition becomes a combinatorial statement: the link of each vertex must be a "flag complex". (A simplicial complex L is a *flag complex* if any finite nonempty set of vertices, which are pairwise connected by edges, span a simplex of L.)

There is also a combinatorial version of the Gauss–Bonnet Theorem for a piecewise Euclidean space X (cf [13]). It states that $\chi(X)$ is the sum over the vertices of X of a local contribution coming from the link L of a vertex. In the cubical case, the formula for the local contribution $\kappa(L)$ coming from a link L is simply,

$$\kappa(L) = \sum_{i=-1}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i(L),$$

where $f_i(L)$ denotes the number of i–simplices in L and $f_{-1}(L) = 1$. Hence, for piecewise Euclidean cubical manifolds of nonpositive curvature, the Euler Characteristic Conjecture is implied by (and, in fact, is equivalent to) the following conjecture of [10].

The Flag Complex Conjecture 0.2 *If S is a flag triangulation of a $(2k-1)$–sphere, then*

$$(-1)^k\kappa(S) \geq 0,$$

where $\kappa(S)$ is defined by the above formula.

In 1976, in [4], Atiyah introduced the study of ℓ^2–homology (or cohomology) into topology. Here one is interested in the following situation: X is either a closed manifold or a finite CW–complex, $\widetilde{X}$ is its universal cover and π is its fundamental group. For each natural number i, one can then define a Hilbert space, $\mathcal{H}_i(\widetilde{X})$, the "reduced ℓ^2–homology" of $\widetilde{X}$. There are two methods for defining this. In the case where X is a Riemannian manifold, one lifts the metric to $\widetilde{X}$ and then defines (de Rham) ℓ^2–cohomology by using differential forms with square integrable norms. When X is a finite CW–complex, one lifts the cell structure to $\widetilde{X}$ and then defines $\mathcal{H}_i(\widetilde{X})$ by using infinite cellular chains with square summable coefficients. In either case, the Hilbert space $\mathcal{H}_i(\widetilde{X})$ comes equipped with an orthogonal π–action. When X is a triangulated Riemannian manifold, the equivalence of the two definitions was proved by Dodziuk in [20]. In this paper, we will deal only with the cellular version of ℓ^2–homology.

A key feature of the ℓ^2–theory is that, by using the π–action, it is possible to attach to the Hilbert space $\mathcal{H}_i(\widetilde{X})$ a nonnegative real number, called the "i^{th} ℓ^2–Betti number". (This is explained in Section 3.) A formula of Atiyah [4] states that the alternating sum of these ℓ^2–Betti numbers is the ordinary Euler characteristic $\chi(X)$. (The precise statement of Atiyah's Formula can be found in Section 3.3 of this paper.)

Shortly after this formula became known, Dodziuk and Singer pointed out that Atiyah's Formula shows that the Euler Characteristic Conjecture follows if one can prove that the reduced ℓ^2–homology of the universal cover of any even dimensional, closed, aspherical manifold vanishes except in the middle dimension. (This is explained in the introduction of [21].) This led to the following conjecture.

Singer's Conjecture 0.3 *If M^n is a closed aspherical manifold, then*

$$\mathcal{H}_i(\widetilde{M^n}) = 0 \quad \text{for all} \quad i \neq \frac{n}{2}.$$

Singer's Conjecture holds for elementary reasons in dimensions ≤ 2. In [32] Lott and Lück proved that it holds for those aspherical 3–manifolds for which Thurston's Geometrization Conjecture is true. It is also known to hold for (a) locally symmetric spaces, (b) negatively curved Kähler manifolds (by [27]), (c) Riemannian manifolds of sufficiently pinched negative sectional curvature (by [22]), (d) closed aspherical manifolds with fundamental group containing an infinite amenable normal subgroup (by [12]), and (e) manifolds which fiber over $\mathbb{S}^1$ (by [33]).

We note that the Euler Characteristic Conjecture and Singer's Conjecture both make sense for closed aspherical orbifolds or, for that matter, for virtual Poincaré duality groups.

In several earlier papers (eg, [10], [15], [16], [17], or [19]), the first author has described a construction which associates to any finite flag complex L, a "right-angled" Coxeter group W_L and a cubical cell complex Σ_L on which W_L acts properly and cocompactly. (The details of this construction will be given in Sections 5 and 6, below.) Its two most salient features are that (1) the link of each vertex of Σ_L is isomorphic to L and (2) Σ_L is contractible.

If Γ is a torsion-free subgroup of finite index in W_L, then Γ acts freely on Σ_L and Σ_L/Γ is a finite complex. By (2), Σ_L/Γ is aspherical. If L is homeomorphic to the $(n-1)$–sphere, then by (1), Σ_L is an n–manifold. Hence, this construction gives many examples of closed aspherical manifolds. Singer's Conjecture for such manifolds becomes the following.

Conjecture 0.4 *Suppose S is a triangulation of the $(n-1)$–sphere as a flag complex. Then*

$$\mathcal{H}_i(\Sigma_S) = 0 \quad \text{for all} \quad i \neq \frac{n}{2}.$$

The purpose of this paper is to describe a partially successful program for proving this conjecture by using standard techniques of algebraic topology and induction on the dimension n. Our main result, Theorem 9.3.1, is that the program succeeds in half the cases: if Conjecture 0.4 is true in some odd dimension n, then it is also true in dimension $n+1$. Moreover, in odd dimensions it is only necessary to establish a weak form of the conjecture.

As we shall see in Section 10, the Geometrization Conjecture is true for the 3–manifolds which we are considering. Hence, the Lott–Lück result implies that Conjecture 0.4 is true for $n=3$ and, therefore, also for $n=4$. This gives the following (Theorem 11.1.1 of Section 11).

Theorem *Conjecture 0.4 is true for $n \leq 4$.*

Hence, 4–manifolds of the form Σ_S/Γ have nonnegative Euler characteristic. As explained in [10] and 6.3.4, below, this implies the following (Theorem 11.2.1).

Theorem *The Flag Complex Conjecture is true in dimension 3. In other words, if S is a triangulation of a homology 3–sphere as a flag complex, then*

$$\kappa(S) \geq 0.$$

The combinatorial Gauss–Bonnet Theorem then implies the next result (Theorem 11.2.2).

Theorem *The Euler Characteristic Conjecture holds true for all nonpositively curved, piecewise Euclidean 4–manifolds which are cellulated by regular Euclidean cubes. In other words, for any such 4–manifold M^4,*

$$\chi(M^4) \geq 0.$$

A surprising aspect of our analysis is that it turns out that Conjecture 0.4 is equivalent to a statement about the vanishing of $\mathcal{H}_i(\Sigma_L)$ for an arbitrary finite flag complex L (not necessarily a sphere). More precisely, we will show in Section 9, that Conjecture 0.4 is equivalent to the following.

Conjecture 0.5 *Suppose L is a finite flag complex. If L can be embedded as a full subcomplex of some flag triangulation of the $2k$–sphere, then*

$$\mathcal{H}_i(\Sigma_L) = 0 \quad \textit{for all } i > k.$$

Let us say that an n–dimensional polyhedron X has *spherical links in codimensions* $\leq m$ if, for $i \leq m$, the link of any $(n-i)$–cell in X is an $(i-1)$–sphere. For example, if $m = 1$, then X is a pseudomanifold, while if $m = n$, then X is a manifold. The inductive arguments of Section 9 suggest the following generalization of Singer's Conjecture.

Conjecture 0.6 *Suppose an n–dimensional aspherical polyhedron X has spherical links in codimensions $\leq 2l+1$, where $2l+1 \leq n$. Then*

$$\mathcal{H}_{n-i}(\widetilde{X}) = 0 \quad \textit{for } i \leq l.$$

When $l = 0$ (so that X is a pseudomanifold), this conjecture holds for elementary reasons (as we explain in 2.6). For right-angled Coxeter groups the conjecture reads as follows.

Conjecture 0.7 *Suppose an $(n-1)$–dimensional flag complex L has spherical links in codimensions $\leq 2l+1$, where $2l+1 \leq n$. (If $n = 2l+1$, we take this to mean that L is an $(n-1)$–sphere.) Then, for $i \leq l$,*

$$\mathcal{H}_{n-i}(\Sigma_L) = 0.$$

In Theorem 9.3.3, we show that if Conjecture 0.4 is true for $n = 2k+1$, then Conjecture 0.7 holds for $l = k$ and any $n \geq 2k+1$. In particular, since Conjecture 0.4 holds for $n = 3$ we get the following (Theorem 11.3.2 in Section 11).

Theorem *Suppose S is a flag triangulation of an $(n-1)$–sphere, $n \geq 3$. Then*

$$\mathcal{H}_i(\Sigma_S) = \mathcal{H}_{n-i}(\Sigma_S) = 0 \quad \text{for } i = 0, 1.$$

Conjecture 0.5, taken together with recent work of Bestvina, Kapovich and Kleiner [5], suggests the following different generalization of Singer's Conjecture (Conjecture 8.9.1 of Section 8).

Conjecture 0.8 *Suppose that a discrete group G acts properly on a contractible n–manifold. Then*

$$b_i^{(2)}(G) = 0 \quad \text{for } i > \frac{n}{2}.$$

(See 3.3.7 for the definition of the ℓ^2–Betti numbers $b_i^{(2)}(G)$.)

In the last three sections (12, 13 and 14) we discuss some possible attacks on (a weak form of) Conjecture 0.4 in odd dimensions.

Some of the results of this paper appeared in a preliminary form in [19].

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1 Group actions on CW complexes

1.1 Geometric G–complexes Let G be a discrete group. A *G–complex* is a CW complex X together with a cellular action of G on X. All G–complexes in this paper will be *geometric*. By this we mean that the G–action is proper (ie, that each cell stabilizer is finite) and cocompact (ie, that X/G is compact).

1.2 Regular complexes and orbihedra A CW complex X is *regular* if the characteristic map of each cell is an embedding (so that the boundary of each cell is an embedded sphere). If X is a geometric G–complex and if it is regular, then X/G is an *orbihedron* in the sense of [30]. The structure of an orbihedron encodes not only the topological space X/G, but also the isomorphism types of the cell stabilizers for each G–orbit of cells. If H is a subgroup of G, then the natural projection $X/H \to X/G$ is an *orbihedral covering map*.

1.3 The orbihedral Euler characteristic Suppose X is a geometric G–complex. Then there are only a finite number of G–orbits of cells in X and the order of each cell stabilizer is finite. The orbihedral Euler characteristic of X/G, denoted $\chi^{\text{orb}}(X/G)$, is the rational number defined by

1.3.1 $\chi^{\mathrm{orb}}(X/G) = \sum_\sigma \frac{(-1)^{\dim \sigma}}{|G_\sigma|}$,

where the summation is over a set of representatives for the G–orbits of cells and where $|G_\sigma|$ denotes the order of the stabilizer G_σ of σ.

1.3.2 If G acts freely on X, then $\chi^{\mathrm{orb}}(X/G)$ is the ordinary Euler characteristic of the finite CW complex X/G.

1.3.3 If H is a subgroup of finite index m in G, then it follows immediately from the definition that

$$\chi^{\mathrm{orb}}(X/H) = m\chi^{\mathrm{orb}}(X/G).$$

1.4 Universal spaces for proper G–actions A G–complex X is a *universal space for proper G–actions* if the action is proper and if the fixed point set X^F is contractible for each finite subgroup F of G. (In particular, taking F to be the trivial subgroup, this means that X is contractible.) Such universal spaces always exist and are unique up to G–equivariant homotopy equivalence. It is often denoted by $\underline{E}G$. If, in addition, the action is cocompact, then $\chi^{\mathrm{orb}}(\underline{E}G/G)$ is defined and is an invariant of G. It is the *Euler characteristic of* G, $\chi(G)$, in the sense of [39].

2 ℓ^2–homology

We review some basic facts about the ℓ^2–homology of geometric G–complexes. References for this material include [12], [20], [28], and [23] (which is particularly easy to read).

2.1 Square summable functions Suppose G is a countable discrete group. Let $\ell^2(G)$ denote the vector space of real-valued, square-summable functions on G, ie, $\ell^2(G) = \{f\colon G \to \mathbb{R} \mid \sum f(g)^2 < \infty\}$. It is a Hilbert space: the inner product is given by

$$\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g) f_2(g).$$

The group ring $\mathbb{R}G$ can be identified with the dense subspace of $\ell^2(G)$ consisting of the functions with finite support.

The action of G on itself by left translation induces an orthogonal (left) G–action on $\ell^2(G)$. (There is also an orthogonal right G–action on $\ell^2(G)$ induced by right translation.)

2.2 Hilbert G–modules Given a natural number n, let $\ell^2(G)^n$ denote the direct sum of a n copies of $\ell^2(G)$, equipped with the diagonal (left) G–action. A Hilbert space V with orthogonal G–action is a *Hilbert G–module* if it is isomorphic to a closed, G–stable subspace of $\ell^2(G)^n$, for some $n \in \mathbb{N}$. (In the literature, this is sometimes called a "finitely generated" Hilbert G–module or a Hilbert G–module of "finite type".)

2.2.1 If F is a finite subgroup of G, then $\ell^2(G/F)$, the space of square summable functions on G/F, can be identified with the subspace of $\ell^2(G)$ consisting of the square summable functions on G which are constant on each coset. This subspace is clearly closed and G–stable; hence, $\ell^2(G/F)$ is a Hilbert G–module.

2.2.2 A *map* of Hilbert G–modules is a G–equivariant, bounded linear map.

The complication which arises at this point is that the image of such a map need not be a closed subspace. This leads to the notions of a "weakly" exact sequence and a "weak" isomorphism, defined below.

2.2.3 (*Weak exactness*) A sequence $U \xrightarrow{e} V \xrightarrow{f} W$ of maps of Hilbert G–modules is *weakly exact* at V if the closure of the image of e (denoted $\overline{\operatorname{Im} e}$) is the kernel of f (denoted $\operatorname{Ker} f$). Similarly, $e\colon U \to V$ is *weakly surjective* if $\overline{\operatorname{Im} e} = V$ and it is a *weak isomorphism* if it is injective and weakly surjective.

2.2.4 If two Hilbert G–modules are weakly isomorphic, then they are G–isometric (Lemma 2.5.3 in [23]).

2.2.5 (*Induced representations*) Suppose H is a subgroup of G and that W is a Hilbert H–module. The *induced representation*, $\operatorname{Ind}_H^G(W)$, can be defined as the ℓ^2–completion of $\mathbb{R}G \otimes_{\mathbb{R}H} W$. Alternatively, it is the vector space of all square summable sections of the vector bundle $G \times_H W \to G/H$. (Here G/H is discrete.) The induced representation is obviously a Hilbert space with orthogonal G–action. If W is a closed subspace of $\ell^2(H)^n$, then $\operatorname{Ind}_H^G(W)$ is a closed subspace of $\ell^2(G)^n$. (This follows from the observation that $\operatorname{Ind}_H^G(\ell^2(H))$ can be identified with $\ell^2(G)$.) Thus, $\operatorname{Ind}_H^G(W)$ is a Hilbert G–module. For example, if F is a finite subgroup of G and $\mathbb{R}$ denotes the trivial 1–dimensional representation of F, then $\operatorname{Ind}_F^G(\mathbb{R})$ can be identified with $\ell^2(G/F)$.

2.3 ℓ^2–homology and cohomology Given a geometric G–complex X, let $K_*(X)$ denote the usual cellular chain complex on X, regarded as a left $\mathbb{Z}(G)$–module. (We use this notation since we want to reserve $C_*(X)$ for the chain complex of ℓ^2–chains on X.)

2.3.1 (*ℓ^2–chains*) Set

$$C_i(X) = \ell^2(G) \otimes_{\mathbb{Z}G} K_i(X)$$

where $\ell^2(G)$ is regarded as a right $\mathbb{Z}G$–module. An element of $C_i(X)$ is an *ℓ^2–chain*; it is an infinite chain with square summable coefficients. The Hilbert space $C_i(X)$ can also be regarded as the space of ℓ^2–cochains on X.

2.3.2 If σ is an i–cell of X, then the space of ℓ^2–chains which are supported on the G–orbit of σ can be identified with $\ell^2(G/G_\sigma)$. Since there are a finite number of such orbits, $C_i(X)$ is the direct sum of a finite number of such subspaces. Hence, by 2.2.1, $C_i(X)$ is a Hilbert G–module.

2.3.3 (*Unreduced and reduced ℓ^2–homology*) We define the boundary map $d_i\colon C_i(X) \to C_{i-1}(X)$ and the coboundary map $\delta^i\colon C_i(X) \to C_{i+1}(X)$ by the usual formulae. Then the boundary and the coboundary maps are G–equivariant, bounded linear maps. The coboundary map δ^i can be identified with d^*_{i+1} (the adjoint of d_{i+1}). Define subspaces of $C_i(X)$:

$$\begin{aligned} Z_i(X) &= \operatorname{Ker} d_i & \qquad Z^i(X) &= \operatorname{Ker} \delta^i \\ B_i(X) &= \operatorname{Im} d_{i+1} & \qquad B^i(X) &= \operatorname{Im} \delta^{i-1} \end{aligned}$$

the *ℓ^2–cycles, –cocycles, –boundaries* and *–coboundaries*, respectively. The corresponding quotient spaces

$$H_i^{(2)}(X) = Z_i(X)/B_i(X)$$

and

$$H^i_{(2)}(X) = Z^i(X)/B^i(X)$$

are the *unreduced ℓ^2–homology* and *–cohomology groups*, respectively. (In other words, $H_i^{(2)}(X)$ is the ordinary equivariant homology of X with coefficients in $\ell^2(G)$, ie, $H_i^{(2)}(X) = H_i^G(X;\ell^2(G))$.) Since the subspaces $B_i(X)$ and $B^i(X)$ need not be closed, these quotient spaces need not be isomorphic to Hilbert spaces.

Let $\overline{B_i}(X)$ (respectively, $\overline{B^i}(X)$) denote the closure of $B_i(X)$ (respectively, $B^i(X)$). The *reduced ℓ^2–homology* and *–cohomology groups* are defined by:

$$\mathcal{H}_i(X) = Z_i(X)/\overline{B_i}(X)$$
$$\mathcal{H}^i(X) = Z^i(X)/\overline{B^i}(X).$$

They are Hilbert G–modules (since each can be identified with the orthogonal complement of a closed G–stable subspace in a closed G–stable subspace of $C_i(X)$).

2.3.4 (*Hodge decomposition*) Since $\langle \delta^{i-1}(x), y\rangle = \langle x, d_i(y)\rangle$ for all $x \in C_{i-1}(X)$ and $y \in C_i(X)$, we have orthogonal direct sum decompositions:

$$C_i(X) = \overline{B_i}(X) \oplus Z^i(X)$$

and

$$C_i(X) = \overline{B^i}(X) \oplus Z_i(X).$$

Since $\langle \delta^{i-1}(x), d_{i+1}(y)\rangle = \langle x, d_i d_{i+1}(y)\rangle = 0$, the subspaces $\overline{B_i}(X)$ and $\overline{B^i}(X)$ are orthogonal. Hence,

$$C_i(X) = \overline{B_i}(X) \oplus \overline{B^i}(X) \oplus (Z_i(X) \cap Z^i(X)).$$

It follows that the reduced ℓ^2–homology and –cohomology groups can both be identified with the subspace $Z_i(X) \cap Z^i(X)$. We denote this intersection again by $\mathcal{H}_i(X)$ and call it the subspace of *harmonic* i–cycles. Thus, an i–chain is harmonic if and only if it is simultaneously a cycle and a cocycle.

The *combinatorial Laplacian* $\Delta\colon C_i(X) \to C_i(X)$ is defined by $\Delta = \delta^{i-1} d_i + d_{i+1}\delta^i$. One checks that $\mathcal{H}_i(X) = \operatorname{Ker}\Delta$.

2.3.5 (*Relative groups*) If Y is a G–stable subcomplex of X, then (X, Y) is a *pair of geometric G–complexes.* The reduced ℓ^2–homology (or –cohomology) groups $\mathcal{H}_i(X, Y)$ are then defined in the usual manner.

2.4 Basic algebraic topology Suppose (X, Y) is a pair of geometric G–complexes. Versions of most of the Eilenberg–Steenrod homology theory Axioms hold for $\mathcal{H}_*(X, Y)$. We list some standard properties below. (Of course, similar results hold for the contravariant ℓ^2–cohomology functor.)

2.4.1 (*Functoriality*) For $i = 1, 2$, suppose (X_i, Y_i) is a pair of geometric G–complexes and that $f\colon (X_1, Y_1) \to (X_2, Y_2)$ is a G–equivariant map (a *G–map* for short). Then there is an induced map $f_*\colon \mathcal{H}_i(X_1, Y_1) \to \mathcal{H}_i(X_2, Y_2)$ and this gives a functor from pairs of G–complexes to Hilbert G–modules. Moreover, if $f'\colon (X_1, Y_1) \to (X_2, Y_2)$ is another G–map which is homotopic to f (not necessarily G–homotopic), then $f_* = f'_*$.

2.4.2 (*Exact sequence of a pair*) The sequence of a pair (X, Y),

$$\to \mathcal{H}_i(Y) \to \mathcal{H}_i(X) \to \mathcal{H}_i(X, Y) \to$$

is weakly exact.

2.4.3 (*Excision*) Suppose that (X, Y) is a pair of geometric G–complexes and that U is a G–stable subset of Y such that $Y-U$ is a subcomplex. Then the inclusion $(X-U, Y-U) \to (X, Y)$ induces an isomorphism:

$$\mathcal{H}_i(X-U, Y-U) \cong \mathcal{H}_i(X, Y).$$

A standard consequence of the last two properties is the following.

2.4.4 (*Mayer–Vietoris sequences*) Suppose $X = X_1 \cup X_2$, where X_1 and X_2 are G–stable subcomplexes of X. Then $X_1 \cap X_2$ is also G–stable and the Mayer–Vietoris sequence,

$$\to \mathcal{H}_i(X_1 \cap X_2) \to \mathcal{H}_i(X_1) \oplus \mathcal{H}_i(X_2) \to \mathcal{H}_i(X) \to$$

is weakly exact.

2.4.5 (*Twisted products and the induced representation*) Suppose that H is a subgroup of G and that Y is a space on which H acts. The *twisted product*, $G \times_H Y$, is the quotient space of $G \times Y$ by the H–action defined by $h(g, y) = (gh^{-1}, hy)$. It is a left G–space and a G–bundle over G/H. Since G/H is discrete, $G \times_H Y$ is a disjoint union of copies of Y, one for each element of G/H. If Y is a geometric H–complex, then $G \times_H Y$ is a geometric G–complex and the following formula obviously holds:

$$\mathcal{H}_i(G \times_H Y) \cong \mathrm{Ind}_H^G(\mathcal{H}_i(Y)).$$

2.4.6 (*Künneth Formula*) Suppose $G = G_1 \times G_2$ and that for $j = 1, 2$, X_j is a geometric G_j–complex. Then $X_1 \times X_2$ is a geometric G–complex and

$$\mathcal{H}_k(X_1 \times X_2) \cong \sum_{i+j=k} \mathcal{H}_i(X_1) \widehat{\otimes} \mathcal{H}_j(X_2),$$

where $\widehat{\otimes}$ denotes the completed tensor product.

2.5 Homology in dimension 0 An element of $C_0(X)$ is an ℓ^2 function on the set of vertices of X; it is a 0–cocycle if and only if it takes the same value on the endpoints of each edge. Hence, if X is connected, any 0–cocycle is constant. If, in addition, G is infinite (so that the 1–skeleton of X is infinite), then this constant must be 0. So, when X is connected and G is infinite, $H^0_{(2)}(X) = \mathcal{H}^0(X) = 0$. Hence,

2.5.1 $\mathcal{H}_0(X) = 0$.

2.5.2 On the other hand, the unreduced homology $H_0^{(2)}(X)$ need not be 0. For example, if $X = \mathbb{R}$, cellulated as the union of intervals $[n, n+1]$, and $G = \mathbb{Z}$, then any vertex of $\mathbb{R}$ is an ℓ^2–0–cycle which is not ℓ^2–boundary. (A vertex bounds a half-line which can be thought of as an infinite 1–chain but this 1–chain is not square summable.) In fact, if G is infinite, then a theorem of Kesten [31] implies that $H_0^{(2)}(X) = 0$ if and only if G is not amenable.

2.6 The top-dimensional homology of a pseudomanifold Suppose that an n–dimensional, regular G–complex X is a pseudomanifold. This means that each $(n-1)$–cell is contained in precisely two n–cells. If a component of the complement of the $(n-2)$–skeleton is not orientable, then it does not support a nonzero n–cycle (with coefficients in $\mathbb{R}$). If such a component is orientable, then any n–cycle supported on it is a constant multiple of the n–cycle with all coefficients equal to $+1$. If the component has an infinite number of n–cells, then this n–cycle does not have square summable coefficients. Hence, if each component of the complement of the $(n-2)$–skeleton is either infinite or nonorientable, then $H_n^{(2)}(X) = 0$. In particular, if the complement of the $(n-2)$–skeleton is connected and if G is infinite, then $H_n^{(2)}(X) = 0$.

2.7 Poincaré duality Suppose $(X, \partial X)$ is a pair of geometric G–complexes and that X is an n–dimensional manifold with boundary. Then

2.7.1 $\mathcal{H}_i(X, \partial X) \cong \mathcal{H}^{n-i}(X)$ and

2.7.2 $\mathcal{H}_i(X) \cong \mathcal{H}^{n-i}(X, \partial X)$.

In the case where X is cellulated as a PL manifold with boundary, these isomorphisms are induced by the bijective correspondence $\sigma \leftrightarrow D\sigma$ which associates to each i–cell σ its dual $(n-i)$–cell $D\sigma$. A slight elaboration of this argument also works in the case where $(X, \partial X)$ is a polyhedral homology manifold with boundary; the only complication being that the "dual cells" need not actually be cells, rather they are "generalized homology disks" as defined in Section 4.3, below.

2.7.3 In fact, as is shown in [23, Theorem 3.7.2], in order to have the Poincaré duality isomorphisms of 2.7.1, all one need assume is that $(X, \partial X)$ is a "virtual PD^n–pair". This means that there is a subgroup H of finite index in G so that the chain complexes $K_*(X, \partial X)$ and ${}^nDK_*(X)$ are chain homotopy equivalent, where ${}^nDK_i(X)$ is defined by ${}^nDK_i(X) = \mathrm{Hom}_{\mathbb{Z}H}(K_{n-i}(X), \mathbb{Z}H)$.

2.8 Extended ℓ^2–homology In [24], Farber defines an "extended ℓ^2–(co)-homology" theory and demonstrates that this is the correct categorical framework for ℓ^2–homology. An extended ℓ^2–homology object is isomorphic to the sum of its "projective part" and its "torsion part". The projective part is essentially the reduced ℓ^2–homology group while its torsion part contains information such as Novikov–Shubin invariants. Since we have nothing to say about this torsion part, we shall stick to the simpler reduced ℓ^2–homology groups.

3 ℓ^2–Betti numbers

The feature which distinguishes ℓ^2–homology from its brothers, the ℓ^p–homology theories, is that one can associate to each Hilbert G–module a nonnegative real number called its "von Neumann dimension".

3.1 von Neumann algebra The *von Neumann algebra* $\mathcal{N}(G)$ associated to G is the algebra of all G–equivariant, bounded linear endomorphisms of $\ell^2(G)$. Since $\ell^2(G)$ is also a right $\mathbb{R}G$–module we see that $\mathbb{R}G \subset \mathcal{N}(G)$. In fact, $\mathcal{N}(G)$ is the weak closure of $\mathbb{R}G$ in the space $\mathrm{End}(\ell^2(G))$ of all bounded linear endomorphisms of $\ell^2(G)$.

For each $g \in G$, let e_g denote the characteristic function of $\{g\}$, ie, $e_g(h) = 0$ if $h \neq g$ and $e_g(h) = 1$ if $h = g$. Then $\{e_g\}_{g \in G}$ is a basis for $\mathbb{R}G$ and an orthonormal basis for the Hilbert space $\ell^2(G)$.

Define a linear functional $\mathrm{tr}_G \colon \mathcal{N}(G) \to \mathbb{R}$ by

3.1.1 $\mathrm{tr}_G(\varphi) = \langle \varphi(e_1), e_1 \rangle.$

(The restriction of tr_G to the subset $\mathbb{R}G$ is the classical Kaplansky trace.)

Next, suppose that φ is a G–equivariant, bounded linear endomorphism of $\ell^2(G)^n$, $n \in \mathbb{N}$. Then φ can be represented as an n by n matrix (φ_{ij}) with coefficients in $\mathcal{N}(G)$. Define

3.1.2 $\mathrm{tr}_G(\varphi) = \sum_{i=1}^{n} \mathrm{tr}_G(\varphi_{ii}).$

The standard argument shows that $\mathrm{tr}_G(\varphi)$ depends only on the conjugacy class of φ.

3.2 von Neumann dimension Let V be a Hilbert G–module. Choose an embedding of V as a closed G–stable subspace of $\ell^2(G)^n$ for some $n \in \mathbb{N}$. Let $p_V\colon \ell^2(G)^n \to \ell^2(G)^n$ denote orthogonal projection onto V. The *von Neumann dimension* of V, denoted by $\dim_G(V)$, is defined by

3.2.1 $\dim_G(V) = \mathrm{tr}_G(p_V)$.

Standard arguments (as in [23]) show that this definition is independent of the choice of embedding $V \to \ell^2(G)^n$.

We list some properties of $\dim_G(V)$. Proofs can be found in [23].

3.2.2 $\dim_G(V) \in [0, \infty)$.

3.2.3 $\dim_G(V) = 0$ if and only if $V = 0$.

3.2.4 If G is the trivial group (so that the Hilbert space V is finite dimensional), then $\dim_G(V) = \dim(V)$.

3.2.5 $\dim_G(\ell^2(G)) = 1$.

3.2.6 $\dim_G(V_1 \oplus V_2) = \dim_G(V_1) + \dim_G(V_2)$.

3.2.7 If $f\colon V \to W$ is a map of Hilbert G–modules, then by 2.2.2 and 3.2.6,

$$\dim_G(V) = \dim_G(\mathrm{Ker}\, f) + \dim_G(\overline{\mathrm{Im}\, f}).$$

3.2.8 If $f\colon V \to W$ is a map of Hilbert G–modules and $f^*\colon W \to V$ is its adjoint, then $\mathrm{Ker}\, f$ and $\overline{\mathrm{Im}\, f^*}$ are orthogonal complements in V. Hence,

$$\dim_G(V) = \dim_G(\mathrm{Ker}\, f) + \dim_G(\overline{\mathrm{Im}\, f^*}).$$

So, by 3.2.7

$$\dim_G(\overline{\mathrm{Im}\, f}) = \dim_G(\overline{\mathrm{Im}\, f^*}).$$

3.2.9 By 3.2.6 and 3.2.7, if $0 \to V_n \to \cdots \to V_0 \to 0$ is a weakly exact sequence of Hilbert G–modules, then

$$\sum_{i=0}^{n} (-1)^i \dim_G(V_i) = 0.$$

3.2.10 If H is a subgroup of finite index m in G, then

$$\dim_H(V) = m \dim_G(V).$$

Combining 3.2.10 with 3.2.4 we get the following.

3.2.11 If G is finite, then

$$\dim_G(V) = \frac{1}{|G|}\dim(V).$$

3.2.12 If H is a subgroup of G and W is a Hilbert H–module, then

$$\dim_G(\mathrm{Ind}_H^G(W)) = \dim_H(W).$$

3.2.13 If F is a finite subgroup of G, then by 2.2.5 and 3.2.12,

$$\dim_G(\ell^2(G/F)) = \frac{1}{|F|}.$$

3.2.14 Suppose $G = G_1 \times G_2$ and that for $j = 1, 2$, V_j is a Hilbert G_j–module. Then $V_1 \widehat{\otimes} V_2$ is a Hilbert G–module and

$$\dim_G(V_1 \widehat{\otimes} V_2) = \dim_{G_1}(V_1)\dim_{G_2}(V_2).$$

3.3 ℓ^2–Betti numbers Given a pair (X, Y) of geometric G–complexes, its i^{th} *ℓ^2–Betti number*, $b_i^{(2)}(X, Y; G)$, is defined by

3.3.1 $b_i^{(2)}(X, Y; G) = \dim_G(\mathcal{H}_i(X, Y))$.

From the properties of von Neumann dimension in 3.2 and the properties of reduced ℓ^2–homology in Section 2, we get properties of ℓ^2–Betti numbers. We list a few of these properties below.

3.3.2 $b_i^{(2)}(X, Y; G) = 0$ if and only if $\mathcal{H}_i(X, Y) = 0$ (by 3.2.3).

3.3.3 If H is a subgroup of finite index m in G, then, by 3.2.10,

$$b_i^{(2)}(X, Y; H) = m b_i^{(2)}(X, Y; G).$$

3.3.4 By 2.4.5 and 3.2.12, for any geometric H–complex Y, with $H \subset G$,

$$b_i^{(2)}(G \times_H Y; G) = b_i^{(2)}(Y; H).$$

3.3.5 (*Künneth Formula*) If $G = G_1 \times G_2$ and for $j = 1, 2$, X_j is a geometric G_j–complex, then by 2.4.6 and 3.2.14,

$$b_k^{(2)}(X_1 \times X_2; G) = \sum_{i+j=k} b_i^{(2)}(X_1; G_1) b_j(X_2; G_2).$$

3.3.6 (*Atiyah's Formula*) By 1.3.1, 2.2.1 and 3.2.13,

$$\chi^{\mathrm{orb}}(X/G) = \sum \frac{(-1)^{\dim \sigma}}{|G_\sigma|} = \sum_{i=0}^{\dim X} (-1)^i \dim_G(C_i(X)).$$

A standard argument (given in [23, Theorem 3.6.1]) then proves Atiyah's Formula:

$$\chi^{\mathrm{orb}}(X/G) = \sum_{i=0}^{\dim X} (-1)^i b_i^{(2)}(X;G).$$

3.3.7 (*ℓ^2–Betti numbers of a group*) As in 1.4, let $\underline{E}G$ denote the universal space for proper G–actions. Also, assume that $\underline{E}G/G$ is compact (so that $\underline{E}G$ is a geometric G–complex). Since any two realizations of $\underline{E}G$ as a geometric G–complex are G–equivariantly homotopy equivalent, the ℓ^2–Betti number $b_i^{(2)}(\underline{E}G;G)$ is an invariant of the group. We denote this number by $b_i^{(2)}(G)$.

3.3.8 (*Poincaré duality*) Suppose $(X, \partial X)$ is a pair of geometric G–complexes and also an n–dimensional polyhedral homology manifold with boundary. Then, by 2.7,

$$b_i^{(2)}(X;G) = b_{n-i}^{(2)}(X, \partial X; G).$$

4 Simplicial complexes and flag complexes

4.1 Definitions and notation Given a simplicial complex L, denote by $\mathcal{S}(L)$ the set of simplices in L together with the empty set $\emptyset$. It is partially ordered by inclusion. $\mathcal{S}_i(L)$ denotes the subset of $\mathcal{S}(L)$ consisting of the simplices of dimension i. (For notational purposes it will be convenient to regard $\emptyset$ as an element of dimension -1 in $\mathcal{S}(L)$.) $\mathcal{S}_0(L)$ is the *vertex set* of L.

4.1.1 (*Full subcomplexes*) A subcomplex A of L is a *full subcomplex* if whenever $\sigma \in \mathcal{S}(L)$ is such that the vertex set of σ is contained in $\mathcal{S}(A)$, then $\sigma \in \mathcal{S}(A)$.

4.1.2 (*Joins*) Suppose L_1 and L_2 are simplicial complexes. Define a partial order on $\mathcal{S}(L_1) \times \mathcal{S}(L_2)$ by $(\sigma, \tau) \leq (\sigma', \tau')$ if and only if $\sigma \leq \sigma'$ and $\tau \leq \tau'$. For example, if σ and τ are simplices of dimension i and j, respectively, then $\mathcal{S}(\sigma) \times \mathcal{S}(\tau)$ is isomorphic to the poset of faces of a simplex of dimension $i+j+1$. We denote this simplex by $\sigma * \tau$. It follows that there is a unique simplicial complex $L_1 * L_2$, called the *join* of L_1 and L_2, characterized by the property that $\mathcal{S}(L_1 * L_2)$ is isomorphic to $\mathcal{S}(L_1) \times \mathcal{S}(L_2)$. The empty element of $\mathcal{S}(L_1 * L_2)$

corresponds to $(\emptyset, \emptyset) \in \mathcal{S}(L_1) \times \mathcal{S}(L_2)$ and the vertex set of $L_1 * L_2$ corresponds to $(\mathcal{S}_0(L_1) \times \{\emptyset\}) \cup (\{\emptyset\} \times \mathcal{S}_0(L_2))$.

As is well known, the geometric realization of $L_1 * L_2$ is homeomorphic to the space formed from $L_1 \times L_2 \times [-1,1]$ by identifying points of the form $(x_1, x_2, -1)$ with $(x_1', x_2, -1)$ and those of the form $(x_1, x_2, +1)$ with $(x_1, x_2', +1)$.

4.1.3 (*Cones*) The *cone* on a simplicial complex L is the join of L with a single point, say v. We will denote it by CL (or by C_vL when we wish to distinguish the *cone point* v).

4.1.4 (*Suspensions*) The *suspension* of L, denoted by SL, is the join of L with a 0–sphere $\mathbb{S}^0$.

4.1.5 (*Incidence relations and flags*) A symmetric and reflexive relation is an *incidence relation.* Suppose Q is a set equipped with an incidence relation. A *flag* in Q is a nonempty finite subset of pairwise related elements. There is an associated simplicial complex, $\mathrm{Flag}(Q)$, the i–simplices of which are flags of cardinality $i+1$. (The vertex set of $\mathrm{Flag}(Q)$ is Q and two vertices are connected by an edge if and only if they are incident.)

An important special case is where the incidence relation is given by symmetrizing the partial order on a poset P. A flag in P is then a nonempty finite totally ordered subset. In this case, $\mathrm{Flag}(P)$ is called the *derived complex* of P. When P is the poset of cells of a regular CW complex X, then $\mathrm{Flag}(P)$ can be identified with the barycentric subdivision of X. As another example, if L is a simplicial complex, then $\mathrm{Flag}(\mathcal{S}(L))$ is the cone on the barycentric subdivision of L. (The vertex corresponding to $\emptyset$ is the cone point.)

4.1.6 Given a poset P and an element $x \in P$, define a subposet by $P_{\leq x} = \{y \in P \mid y \leq x\}$. Subposets $P_{\geq x}$, $P_{<x}$ and $P_{>x}$ are defined similarly.

4.2 Links If τ is a simplex of L, then $\mathrm{Link}(\tau, L)$, the *link* of τ in L, is the union of all simplices σ such that

(a) intersection of σ and τ is empty and

(b) σ and τ span a simplex of L.

The subcomplex $\mathrm{Link}(\tau, L)$ is characterized by the condition that

$$\mathcal{S}(L_\tau) \cong \mathcal{S}(L)_{\geq \tau}.$$

The *star* of τ in L, denoted $\mathrm{St}(\tau, L)$, is the union of all simplices which intersect τ.

If v is a vertex of L, then we will denote its link $\text{Link}(v, L)$ by L_v. We have $\text{St}(v, L) = C_v L_v$. The *open star* of v is the complement of L_v in $\text{St}(v, L)$. It is an open subset of L.

4.3 Generalized homology spheres and disks A space X is a *homology n–manifold* over a ring R if it has the same local homology groups, with coefficients in R, as does an n–manifold, ie, for all $x \in X$,

$$H_i(X, X - x; R) = \begin{cases} 0 & \text{if } i \neq n, \\ R & \text{if } i = n. \end{cases}$$

The definition of when a pair $(X, \partial X)$ is a *homology n–manifold with boundary* over R is similar. It is well-known (cf [7]) that a homology n–manifold over R satisfies Poincaré duality over R. (In non-orientable case one have to use twisted coefficients. Also in general, for a finite dimensional locally compact space, possibly with a pathological topology, it is necessary to use Steenrod homology and Čhech cohomology in order for this to be true.)

For the remainder of this paper it can be always assumed that the coefficients $R = \mathbb{Q}$, the field of rational numbers.

4.3.1 A simplicial complex X is a homology n–manifold if and only if it is n–dimensional and for each k–simplex σ in X, its link $\text{Link}(\sigma, X)$ in X has the same homology as $\mathbb{S}^{n-k-1}$.

4.3.2 A simplicial complex S is a *generalized homology n–sphere* (abbreviated a GHS^n or simply a GHS) if it is a homology n–manifold with the same homology as $\mathbb{S}^n$. A simplicial pair $(D, \partial D)$ is a *generalized homology n–disk* (abbreviated GHD^n) if it is a homology n–manifold with boundary and if

$$H_i(D, \partial D) = \begin{cases} 0 & \text{if } i \neq n, \\ \mathbb{Z} & \text{if } i = n. \end{cases}$$

4.3.3 It follows from 4.3.1 that an n–dimensional simplicial complex X is a homology n–manifold if and only if for each vertex v of X, its link X_v is a GHS^{n-1}. Similarly, $(X, \partial X)$ is a homology n–manifold with boundary if and only if for each vertex v in $X - \partial X$, its link X_v is a GHS^{n-1} and for each $v \in \partial X$, the pair $(X_v, X_v \cap \partial X)$ is a GHD^{n-1}.

4.3.4 In particular, if S is a GHS^n and v is a vertex of S, then its link S_v is a GHS^{n-1}.

4.3.5 If $(D, \partial D)$ is a GHD^n, then it follows from Poincaré duality and the exact sequence of the pair that D is acyclic and that ∂D has the same homology as does $\mathbb{S}^{n-1}$.

4.3.6 We see from 4.3.3 and 4.3.5 that if a simplicial pair $(X, \partial X)$ is a homology n–manifold with boundary, then ∂X is a homology $(n-1)$–manifold.

4.3.7 If, for $i = 1, 2$, S_i is a GHS^{n_i}, then it follows from the Künneth Theorem and induction on dimension that the join $S_1 * S_2$ is a $GHS^{n_1+n_2+1}$. Similarly, if S is a GHS^n and $(D, \partial D)$ is a GHD^m, then $(S * D, S * \partial D)$ is a GHD^{n+m+1}.

4.3.8 In particular, the suspension of a GHS^n is a GHS^{n+1} and the suspension of a GHD^n is a GHD^{n+1}.

4.4 Flag complexes Recall from the Introduction that a simplicial complex L is a *flag complex* if any nonempty finite set of vertices which are pairwise connected by edges span a simplex in L. In other words, L is a flag complex if and only if whenever a subcomplex isomorphic to the 1–skeleton of a simplex is in L, then the entire simplex lies in L. (In [26] Gromov used the terminology that L satisfies the "no Δ condition" for this property.)

4.4.1 If Q is a set with an incidence relation, then $\mathrm{Flag}(Q)$ (defined in 4.1.5) is a flag complex. Conversely, any flag complex arises from this construction. (Indeed, given a flag complex L, define two vertices in $\mathcal{S}_0(L)$ to be *incident* if they are connected by an edge. Then $L \cong \mathrm{Flag}(\mathcal{S}_0(L))$.)

4.4.2 In particular, the barycentric subdivision of any regular CW complex is a flag complex. Hence, the condition that L be a flag complex imposes no restriction on its topological type: it can be any polyhedron.

4.4.3 An m–gon (ie, a triangulation of a circle into m edges) is a flag complex if and only if $m \geq 4$.

4.4.4 Any full subcomplex of a flag complex is a flag complex.

4.4.5 If v is a vertex of a flag complex L, then its link L_v and its star $\mathrm{St}(v, L)$ are both full subcomplexes. Hence, by 4.4.4, they are both flag complexes.

4.4.6 (*Joins of flag complexes*) The join of two flag complexes is again a flag complex. In particular, the cone on a flag complex is a flag complex and the suspension of a flag complex is a flag complex.

4.4.7 (*Notation*) For any set of vertices T of L, let $N(T)$ be the union of all open stars of vertices in T. We will use $L-T$ to denote the complement of $N(T)$ in L. In other words, $L-T$ is the full subcomplex of L spanned by $\mathcal{S}_0(L)-T$. For example, for any vertex s of L, $L-s$ denotes the complement of the open star of s in L. Similarly, if A is any subcomplex of L, then we will write $L-A$ for $L-\mathcal{S}_0(A)$.

5 Right-angled Coxeter groups

5.1 Definition of W_L Suppose L is a flag complex. The 1–skeleton of L gives the data for the presentation of a group W_L. The set of generators in the presentation is the vertex set $\mathcal{S}_0(L)$. The edges of L give relations, as follows:

$$\begin{aligned} s^2 &= 1, && \text{for all } s \in \mathcal{S}_0(L), \\ (st)^2 &= 1, && \text{whenever } \{s,t\} \text{ spans an edge in } L. \end{aligned}$$

The group W_L is the *right-angled Coxeter group* associated to L. $\mathcal{S}_0(L)$, regarded as a subset of W_L, is the *fundamental set of generators*. The flag complex L is called the *nerve* of $(W_L, \mathcal{S}_0(L))$.

5.2 Examples We give some examples of this construction for various flag complexes L.

5.2.1 (*The empty set*) If $L=\emptyset$, then $W_\emptyset$ is the trivial group.

5.2.2 (*A 0–simplex*) If L is a single point s, then $W_s \cong \mathbb{Z}_2$, the cyclic group of order 2.

5.2.3 (*Joins*) By 4.1.2, $W_{L_1 * L_2} = W_{L_1} \times W_{L_2}$.

5.2.4 (*Cones*) By 5.2.2 and 5.2.3, $W_{CL} = \mathbb{Z}_2 \times W_L$.

5.2.5 (*A k–simplex*) If σ is a k–simplex with vertex set $\{s_0, \dots, s_k\}$, then by 5.2.2 and 5.2.3, $W_\sigma = W_{s_0} \times \dots \times W_{s_k} \cong (\mathbb{Z}_2)^{k+1}$.

5.2.6 (*Disjoint unions*) If L is the disjoint union of two flag complexes L_1 and L_2, then W_L is the free product of W_{L_1} and W_{L_2}, ie, $W_{L_1 \cup L_2} = W_{L_1} * W_{L_2}$.

5.2.7 (*Amalgamated products*) More generally, if $L = L_1 \cup L_2$, $L_1 \cap L_2 = A$, where L_1 and L_2 (and therefore, A) are full subcomplexes, then W_L is the amalgamated product:

$$W_L = W_{L_1} *_{W_A} W_{L_2}.$$

5.2.8 (*k points*) If L is the disjoint union of k points $s_1, \dots, s_k$, then W_L is the free product $W_{s_1} * \cdots * W_{s_k}$ ($\cong \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$). In particular, $W_{\mathbb{S}^0}$ is the infinite dihedral group D_∞.

5.2.9 (*Suspensions*) By 5.2.3 and 5.2.8, $W_{SL} = D_\infty \times W_L$.

5.3 Special subgroups Let A be a full subcomplex of L. By [6, Théorème 2, p. 20], W_A can be identified with the subgroup of W_L generated by $\mathcal{S}_0(A)$. (N. B. Here it is important that A be a full subcomplex; for if two vertices of A were connected by an edge in L which was not in A, then there would be a relation in W_L not satisfied in W_A.) Such a subgroup W_A is called a *special subgroup* of W_L.

5.3.1 We note that a special subgroup W_A is finite if and only if A is a simplex of L (or if $A = \emptyset$). The special subgroups of W_L corresponding to the elements of $\mathcal{S}(L)$ are sometimes called the *spherical* special subgroups.

5.4 The poset of spherical cosets A *spherical coset* in W_L is a coset of the form wW_σ for some $\sigma \in \mathcal{S}(L)$ and $w \in W_L$. The set of all spherical cosets will be denoted by $W_L\mathcal{S}(L)$, ie,

$$W_L\mathcal{S}(L) = \bigcup_{\sigma \in \mathcal{S}(L)} W_L/W_\sigma.$$

It is partially ordered by inclusion of one coset in another. The group W_L acts in an obvious way on the poset $W_L\mathcal{S}(L)$. The quotient poset is $\mathcal{S}(L)$.

6 The complex Σ_L

We retain the notation of the previous section: L is a finite flag complex, W_L is the associated right-angled Coxeter group and $W_L\mathcal{S}(L)$ is the poset of spherical cosets.

6.1 Definitions and basic properties The space Σ_L is defined as the geometric realization of the poset $W_L\mathcal{S}(L)$. (In other words, it is the simplicial complex $\mathrm{Flag}(W_L\mathcal{S}(L))$.) Let K_L denote the geometric realization of $\mathcal{S}(L)$. (By 4.1.5, K_L is the cone on the barycentric subdivision L.) The inclusion $\mathcal{S}(L) \hookrightarrow W_L\mathcal{S}(L)$, defined by $\sigma \mapsto W_\sigma$, induces an inclusion $K_L \subset \Sigma_L$. When regarded in this way as a subset of Σ_L, K_L is called the *fundamental chamber.*

6.1.1 (*The W_L–action*) The natural W_L–action on $W_L\mathcal{S}(L)$ induces a simplicial action on Σ_L. The orbit space is K_L. The action is proper (since each cell stabilizer is a conjugate of a spherical special subgroup) and cocompact (since $\mathcal{S}(L)$ is finite).

6.1.2 (*Contractibility*) It is proved in [15] that Σ_L is contractible. In fact, Σ_L is the universal space for proper W_L–actions, in the sense of 1.4.

6.1.3 (*Special subcomplexes*) Suppose A is a full subcomplex of L. The inclusion $W_A \to W_L$ induces an inclusion of posets $W_A\mathcal{S}(A) \to W_L\mathcal{S}(L)$ and hence, an inclusion of Σ_A as a subcomplex of Σ_L. Such a Σ_A will be called a *special subcomplex* of Σ_L. If $w \in W_L - W_A$, then Σ_A and $w\Sigma_A$ are disjoint subcomplexes. It follows that the stabilizer of Σ_A in Σ_L is W_A and that

6.1.4 $W_L\Sigma_A \cong W_L \times_{W_A} \Sigma_A,$

where $W_L\Sigma_A$ denotes the union of all translates of Σ_A in Σ_L.

6.2 Examples We consider the above construction for the same flag complexes L as in 5.2.

6.2.1 (*The empty set*) $\Sigma_\emptyset$ is a point.

6.2.2 (*A 0–simplex*) If L is a single point s, then Σ_s can be identified with the interval $[-1,1]$. The nontrivial element $s \in W_s$ ($W_s \cong \mathbb{Z}_2$) acts as the reflection $t \to -t$.

6.2.3 (*Joins*) By 4.1.2, $\mathcal{S}(L_1 * L_2) \cong \mathcal{S}(L_1) \times \mathcal{S}(L_2)$ and by 5.2.3, $W_{L_1 * L_2} = W_{L_1} \times W_{L_2}$. It follows that

$$\Sigma_{L_1 * L_2} = \Sigma_{L_1} \times \Sigma_{L_2}$$

with the product action.

6.2.4 (*Cones*) $\Sigma_{CL} = [-1,1] \times \Sigma_L$.

6.2.5 (*A k–simplex*) If σ is a k–simplex with vertex set $\{s_0, \dots, s_k\}$, then by 5.2.5, $W_\sigma = W_{s_0} \times \cdots \times W_{s_k}$ ($\cong (\mathbb{Z}_s)^{k+1}$) and by 6.2.2 and 6.2.3,

$$\Sigma_\sigma = \Sigma_{s_0} \times \cdots \times \Sigma_{s_k} \quad (\cong [-1,1]^{k+1}).$$

6.2.6 (*Disjoint unions*) If L is the disjoint union of L_1 and L_2, then K_L is the one point union $K_{L_1} \vee K_{L_2}$ (the common point corresponding to $\emptyset \in \mathcal{S}(L_1) \cap \mathcal{S}(L_2)$).

6.2.7 (*k points*) Suppose $L = P_k$, the disjoint union of k points. Then K_{P_k} is the cone on k points and, if $k > 1$, Σ_{P_k} is the regular infinite tree where each vertex has valence k.

6.2.8 (*The 0–sphere*) In particular, $\Sigma_{\mathbb{S}^0}$ can be identified with the real line $\mathbb{R}$ cellulated as the union of intervals of the form $[2m-1, 2m+1]$, $m \in \mathbb{Z}$. The action of the infinite dihedral group $W_{\mathbb{S}^0}$ is the standard one, generated by the reflections across 0 and 2.

6.2.9 (*Suspensions*) By 6.2.3 and 6.2.8, $\Sigma_{SL} = \mathbb{R} \times \Sigma_L$.

6.3 The cubical structure on Σ_L

6.3.1 (*The case where L is a simplex*) Suppose σ is a k–simplex. Then by 5.2.5, $W_\sigma \cong (\mathbb{Z}_2)^{k+1}$ and by 6.2.5, $\Sigma_\sigma = [-1,1]^{k+1}$. The group W_σ acts simply transitively on the set of 0–dimensional faces (= "vertices") of $[-1,1]^{k+1}$. Moreover, a set of such vertices is the vertex set of a face of $[-1,1]^{k+1}$ if and only if it corresponds to the set of elements in a coset of the form wW_τ, for some $w \in W_\sigma$ and $\tau \in \mathcal{S}(\sigma)$. Hence, the poset of nonempty faces of $[-1,1]^{k+1}$ ($= \Sigma_\sigma$) is naturally identified with the poset $W_\sigma\mathcal{S}(\sigma)$.

6.3.2 (*The general case*) Now suppose that L is an arbitrary flag complex. For each $\sigma \in \mathcal{S}_k(L)$ and $w \in W_L$, the subcomplex $w\Sigma_\sigma$ of Σ_L is homeomorphic to $[-1,1]^{k+1}$. This gives a decomposition of Σ_L into a family of subcomplexes, $\{w\Sigma_\sigma\}_{wW_\sigma \in W_L\mathcal{S}(L)}$. The family is indexed by the poset of spherical cosets $W_L\mathcal{S}(L)$. Each subcomplex is homeomorphic to a cube. Thus, Σ_L has the structure of a regular CW complex in which (a) the poset of cells is identified with $W_L\mathcal{S}(L)$ and (b) the cell corresponding to wW_σ is a $(k+1)$–dimensional cube, where $k = \dim\sigma$. As before, there is a 0–dimensional cube (vertex) for each element of W_L ($= W_L/W_\emptyset$) and a set of such 0–cubes is the vertex set of a $(k+1)$–cube, $w\Sigma_\sigma$, if and only if it is the set of elements in the spherical coset wW_σ.

6.3.3 (*The link of a vertex in Σ_L*) With respect to this cubical structure, the link of each vertex of Σ_L is L. In other words, the poset of cubes of Σ_L which properly contain a given vertex is canonically identified with $\mathcal{S}(L)_{>\emptyset}$.

6.3.4 (*The orbihedral Euler characteristic of* Σ_L/W_L) The W_L–orbits of cubical cells in Σ_L are bijective with $\mathcal{S}(L)$. The dimension of a cube in an orbit corresponding to $\sigma \in \mathcal{S}_k(L)$ is $k+1$ and the order of its stabilizer is 2^{k+1}. Hence, by 1.3.1, the orbihedral Euler characteristic is given by

$$\chi^{\mathrm{orb}}(\Sigma_L/W_L) = \sum_{\sigma\in\mathcal{S}(L)} \left(-\frac{1}{2}\right)^{\dim\sigma+1}$$

or

$$\chi^{\mathrm{orb}}(\Sigma_L/W_L) = \sum_{k=-1}^{\dim L} \left(-\frac{1}{2}\right)^{k+1} f_k(L),$$

where $f_k(L)$ is the number of elements in $\mathcal{S}_k(L)$. We note that the right hand side of the last equation is precisely the quantity $\kappa(L)$, mentioned in the Introduction, in connection with the Combinatorial Gauss–Bonnet Theorem. It is the local contribution to the Euler characteristic coming from the link of a vertex in a piecewise Euclidean, cubical cell complex. (See [10].)

Since Σ_L is the universal space for proper W_L–actions, $\chi^{\mathrm{orb}}(\Sigma_L/W_L)$ is the Euler characteristic of W_L.

6.3.5 Each special subcomplex of Σ_L is also a subcomplex in the cubical structure.

6.3.6 For any $s \in \mathcal{S}_0(L)$, Σ_s is an edge of Σ_L. Let $O(s)$ denote the union of the interiors of all cubes of Σ_L which have Σ_s as a face (ie, $O(s)$ is the open star of the interior of Σ_s). For any subset T of $\mathcal{S}_0(L)$, set

$$R(T) = \bigcup_{s\in T} W_L O(s).$$

Thus, $R(T)$ is an open, W_L–stable subset of Σ_L. Moreover, with notation as in 4.4.7 and 6.1.4, we have that

6.3.7 $\Sigma_L - R(T) = W_L \Sigma_{L-T}$.

6.4 The commutator cover of Σ_L/W_L

In this section we will describe a finite cubical complex P_L as a subcomplex of a Euclidean cube. It turns out that the universal cover of P_L can be identified with Σ_L. This gives an alternative, and perhaps more easily understandable method of describing the cubical structure on Σ_L.

6.4.1 (*The commutator subgroup*) The abelianization of W_L, denoted W_L^{ab}, is obviously $(\mathbb{Z}_2)^{\mathcal{S}_0(L)}$, the direct product of cyclic groups of order two. Let $\varphi\colon W_L \to W_L^{\mathrm{ab}}$ be the natural epimorphism. Its kernel, denoted by Γ_L, is the commutator subgroup. Since any finite subgroup of W_L is contained in a conjugate of a finite special subgroup and since the restriction of ϕ to any finite special subgroup is injective, Γ_L is torsion-free. Hence, it acts freely on Σ_L. The natural projection $\Sigma_L/\Gamma_L \to \Sigma_L/W_L$ is an orbihedral covering in the sense of 1.2; we call Σ_L/Γ_L the *commutator cover* of Σ_L/W_L.

6.4.2 (*The complex P_L*) Let $\square$ denote the Euclidean cube $[-1,1]^{\mathcal{S}_0(L)}$. For each $\sigma \in \mathcal{S}(L)$, let $\square_\sigma$ be the face of $\square$ defined by

$$\square_\sigma = e \times [-1,1]^{\mathcal{S}_0(\sigma)}$$

where e is the vertex of $[-1,1]^{\mathcal{S}_0(L)-\mathcal{S}_0(\sigma)}$ with all coordinates equal to 1. The faces of $\square$ which are parallel to $\square_\sigma$ have the form $f \times [-1,1]^{\mathcal{S}_0(\sigma)}$, where f is some vertex of $[-1,1]^{\mathcal{S}_0(L)-\mathcal{S}_0(\sigma)}$.

Define P_L to be the union of all faces of $\square$ which are parallel to $\square_\sigma$, for some $\sigma \in \mathcal{S}(L)$. Thus, P_L is a subcomplex of $\square$.

Each generator s of W_L^{ab} acts on $\square$ as reflection in the s^{th} coordinate. Thus, W_L^{ab} acts on $\square$ as a finite reflection group. The orbit space is $[0,1]^{\mathcal{S}_0(L)}$ and P_L/W_L^{ab} is the subcomplex consisting of all faces of the form $e \times [0,1]^{\mathcal{S}_0(\sigma)}$, for some $\sigma \in \mathcal{S}(L)$. (Moreover, this subcomplex can be canonically identified with $K_L = \Sigma_L/W_L$.)

6.4.3 (*Identification of P_L with Σ_L/Γ_L*) There is a natural φ–equivariant map $p\colon \Sigma_L \to P_L$ which sends the cube $w\Sigma_\sigma$ to $\varphi(w)\square_\sigma$. It is obvious that p is a covering projection and that it induces an isomorphism from Σ_L/Γ_L onto P_L. Henceforth, we identify these two cubical complexes.

6.5 The piecewise Euclidean metric on Σ_L

We review some material from [26] (which can also be found in [8], [16], [17], or [35]).

Identify each k–dimensional cube in Σ_L with the regular Euclidean cube of edge length 2. The length of a piecewise linear curve in Σ_L is then unambiguously defined. The *distance* $d(x,y)$ between two points x and y in Σ_L is then defined to be the infimum of the lengths of piecewise linear paths connecting them. With this metric, Σ_L becomes a *geodesic space*, that is, for any two points x and y there is a path of length $d(x,y)$ between them. Such a path is called a *geodesic segment*.

6.5.1 (*Nonpositive curvature*) For a geodesic space X the concept of "non-positive curvature" can be defined by comparing distances on small triangles in X (ie, configurations of three geodesic segments in X) with distances on comparison triangles in the Euclidean plane. X is *nonpositively curved* if Gromov's CAT(0)–inequality (page 106 of [26]) holds for all sufficiently small triangles in X.

Lemma 6.5.2 (Gromov) *A cubical cell complex X with piecewise Euclidean metric defined as above is nonpositively curved if and only if the link of each vertex is a flag complex.*

This is proved on page 123 of [26]. The proof can also be found in [8], [17], or [35].

6.5.3 It follows from 6.3.3 and Gromov's Lemma that Σ_L is nonpositively curved. Since, by 6.1.2, Σ_L is contractible, this implies that it is a CAT(0)–space (ie, that the CAT(0) inequality holds for all triangles).

6.5.4 It is not difficult to show that any special subcomplex Σ_A is a geodesically convex subspace of Σ_L. See Proposition 1.7.1, page 514 of [18] for details.

6.6 Reflection groups on manifolds

6.6.1 (*Classical reflection groups*) Let $\mathbb{X}^n$ stand for either Euclidean n–space $\mathbb{E}^n$, hyperbolic n–space $\mathbb{H}^n$ or the n–sphere $\mathbb{S}^n$. A *classical reflection group* W is a discrete, cocompact group of isometries of $\mathbb{X}^n$ generated by reflections. Then W is a Coxeter group. (The theory of general Coxeter groups arose from the study of this classical situation.)

Suppose W is a classical reflection group on $\mathbb{X}^n$. Choose a component of the complement of the union of the reflecting hyperplanes and call its closure K. Then K is a convex polytope. Moreover, it is a fundamental domain for the W–action and the set of reflections across the codimension-one faces of K is a fundamental set of generators for W.

Let S be the simplicial complex dual to the boundary of K. In the spherical case, S is the boundary of a simplex (and hence, not a flag complex when the dimension of the simplex is greater than 1). In the Euclidean case, S is the join of boundaries of simplices.

The condition that W be right-angled means that the codimension-one faces of K are orthogonal whenever they intersect. In the right-angled Euclidean case, $\mathbb{X}^n = \mathbb{E}^n$, the only possibility is that K is a product of intervals, S is the boundary of an n–dimensional octahedron (an n–fold join of 0–spheres) and $W = W_S$ is an n–fold product of infinite dihedral groups. If K is the regular n–cube $[0,2]^n$ (which we may assume after conjugating by an affine automorphism) then Σ_S is isometric with $\mathbb{E}^n$. In the right-angled hyperbolic case, Σ_S is equivariantly homeomorphic to $\mathbb{H}^n$ but not isometric to it. (They are quasi-isometric.) The cubical structure on Σ_S is dual to the tessellation of $\mathbb{H}^n$ by the translates of K.

6.6.2 (*An m–gon*) Suppose S is an m–gon, ie, a subdivision of the circle into m edges. To insure that S is a flag complex, we also assume $m \geq 4$. Then W_S is isomorphic to a classical reflection group and Σ_S is combinatorially dual to a tessellation of the Euclidean plane (when $m = 4$) by squares or to a tessellation of the hyperbolic plane (when $m > 4$) by right-angled m–gons.

6.6.3 (*Spheres*) Suppose that S is a triangulation of $\mathbb{S}^{n-1}$ as a flag complex. Then, by 6.3.3, Σ_S is an n–dimensional manifold (since a neighborhood of each vertex is homeomorphic to the cone on S). If $n > 3$, then very few of these triangulations correspond to classical reflection groups. The situation in dimension 3 will be explained in Section 10.

6.6.4 (*Generalized homology spheres*) Similarly, if S is a GHS^{n-1}, as defined in 4.3, the, by 6.3.3, Σ_S is a polyhedral homology n–manifold.

6.6.5 (*Generalized homology disks*) If D is a triangulation of an $(n-1)$–disk as a flag complex and ∂D is a full subcomplex, then, by 6.3.3, Σ_D is an n–manifold with boundary. Its boundary is $W_D \Sigma_{\partial D}$. Similarly, if $(D, \partial D)$ is a GHD^{n-1}, as defined in 4.3, then Σ_D is a polyhedral homology n–manifold with boundary.

7 Properties of the ℓ^2–homology of Σ_L

From now on, *all simplicial complexes will be flag complexes and all subcomplexes will be full subcomplexes.* Given a finite flag complex L, we have associated a group W_L, a geometric W_L–complex Σ_L and then, for each $i \in \mathbb{N}$, a Hilbert W_L–module, $\mathcal{H}_i(\Sigma_L)$. Similarly, to each pair (L, A) we can associate the Hilbert W_L–module, $\mathcal{H}_i(\Sigma_L, W_L \Sigma_A)$ (where, by 6.1.4, $W_L \Sigma_A \cong W_L \times_{W_A} \Sigma_A$).

We introduce some useful notation which reflects this situation.

7.1 Notation

7.1.1 $\mathfrak{h}_i(L) = \mathcal{H}_i(\Sigma_L)$

7.1.2 $\mathfrak{h}_i(A) = \mathcal{H}_i(W_L\Sigma_A)$

7.1.3 $\mathfrak{h}_i(L, A) = \mathcal{H}_i(\Sigma_L, W_L\Sigma_A)$

7.1.4 $\beta_i(A) = \dim_{W_L}(\mathfrak{h}_i(A))$

7.1.5 $\beta_i(L, A) = \dim_{W_L}(\mathfrak{h}_i(L, A))$

7.1.6 $\chi^{(2)}(L) = \sum(-1)^i\beta_i(L)$

The notation in 7.1.2 and 7.1.4 will not lead to confusion, since, by 2.4.5 and 6.1.4, $\mathcal{H}_i(W_L\Sigma_A)$ is the induced representation from $\mathcal{H}_i(\Sigma_A)$ and, therefore, by 3.3.4,

$$b_i^{(2)}(W_L\Sigma_A; W_L) = b_i^{(2)}(\Sigma_A; W_A).$$

7.2 Basic algebraic topology For the case at hand, we rewrite some of the basic properties of reduced ℓ^2–homology in our new notation. From 2.4.2, we get the following.

Lemma 7.2.1 (Exact sequence of the pair) *The sequence*

$$\to \mathfrak{h}_i(A) \to \mathfrak{h}_i(L) \to \mathfrak{h}_i(L, A) \to$$

is weakly exact.

Lemma 7.2.2 (Excision) *Given (L, A) as above, let T be a set of vertices of A such that the open star of any vertex in T is contained in the interior of A. Then, with notation as in 4.4.7,*

$$\mathfrak{h}_i(L, A) \cong \mathfrak{h}_i(L - T, A - T).$$

Proof This is immediate from 2.4.3 and 6.3.7. □

Lemma 7.2.3 (Mayer–Vietoris sequences) *Suppose $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where L_1 and L_2 (and therefore, A) are full subcomplexes of L.*

(1) *The Mayer–Vietoris sequence*

$$\to \mathfrak{h}_i(A) \to \mathfrak{h}_i(L_1) \oplus \mathfrak{h}_i(L_2) \to \mathfrak{h}_i(L) \to$$

is weakly exact.

(2) $\mathfrak{h}_i(L, A) \cong \mathfrak{h}_i(L_1, A) \oplus \mathfrak{h}_i(L_2, A)$.

Proof Statement (1) follows from 2.4.4. For (2), use the following relative version of the Mayer–Vietoris sequence,

$$\to \mathfrak{h}_i(A, A) \to \mathfrak{h}_i(L_1, A) \oplus \mathfrak{h}_i(L_2, A) \to \mathfrak{h}_i(L, A) \to \mathfrak{h}_{i-1}(A, A)$$

and the fact that $\mathfrak{h}_*(A, A) = 0$. □

Using 5.2.3 and 6.2.3 the Künneth Formula, 3.3.5, translates to the following.

Lemma 7.2.4 (The Betti numbers of a join)

$$\beta_k(L_1 * L_2) = \sum_{i+j=k} \beta_i(L_1)\beta_j(L_2).$$

Using 6.3.4, Atiyah's Formula, 3.3.6, translates as follows.

Lemma 7.2.5 (Atiyah's Formula)

$$\chi^{(2)}(L) = \sum_{k=-1}^{\dim L} \left(-\frac{1}{2}\right)^{k+1} f_k(L).$$

7.2.6 (0*–dimensional homology*) If L is nonempty and not a simplex, then, by 2.5.1,

$$\beta_0(L) = 0.$$

7.2.7 Similarly, suppose L is a pseudomanifold of dimension $n-1$, as in 2.6. It then follows from 6.3.3 that Σ_L is an n-dimensional pseudomanifold and it can be seen that each component of the complement of the codimension 2 skeleton is infinite. Hence, by 2.6,

$$\beta_n(L) = 0.$$

7.3 Examples Next we calculate the Betti numbers, $\beta_i(L)$, for some of the examples in 5.2 and 6.2.

7.3.1 (*The empty set*) Since $W_\emptyset$ is trivial and $\Sigma_\emptyset$ is a point,

$$\beta_i(\emptyset) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

7.3.2 (*A k–simplex*) Given a k–simplex σ, $W_\sigma \cong (\mathbb{Z}_2)^{k+1}$ and $\Sigma_\sigma = [-1,1]^{k+1}$. Hence,

$$\beta_i(\sigma) = \begin{cases} \left(\frac{1}{2}\right)^{k+1} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Lemma 7.3.3 (The Betti numbers of a disjoint union) *Suppose that L is the disjoint union of L_1 and L_2. Then, for $i \geq 2$,*

$$\beta_i(L) = \beta_i(L_1) + \beta_i(L_2).$$

If neither L_1 nor L_2 is a simplex, then

$$\beta_1(L) = \beta_1(L_1) + \beta_1(L_2) + 1.$$

Proof This follows from the Mayer–Vietoris sequence, Lemma 7.2.3 (1), after noting that $L_1 \cap L_2 = \emptyset$ has nonzero Betti number, $\beta_0(\emptyset) = 1$. The final sentence follows since if W_{L_1} and W_{L_2} are both infinite, then, by 7.2.6, $\beta_0(L_1) = \beta_0(L_2) = 0$. □

Lemma 7.3.4 (The Betti numbers of k points) *Let P_k denote the disjoint union of k points. If $k \geq 2$, then*

$$\beta_i(P_k) = \begin{cases} \frac{k}{2} - 1 & \textit{if } i = 1, \\ 0 & \textit{if } i \neq 1. \end{cases}$$

In particular,

$$\beta_i(\mathbb{S}^0) = \beta_i(P_2) = 0 \quad \textit{for all } i.$$

Proof Since Σ_{P_k} is 1–dimensional, $\beta_i(P_k) = 0$ for $i > 1$. Since $k \geq 2$, $\beta_0(P_k) = 0$, by 7.2.6. By 6.3.4, $\chi^{(2)}(P_k) = 1 - \frac{k}{2}$. Hence, by Atiyah's Formula 7.2.5, $\beta_1(L) = -\chi^{(2)}(L) = \frac{k}{2} - 1$. □

Lemma 7.3.5 (The Betti numbers of a suspension) $\beta_i(SL) = 0$ *for all i.*

Proof This follows from Lemma 7.2.4 and 7.3.4. □

7.3.6 Suppose L is the m–fold join, $L = P_{k_1} * \cdots * P_{k_m}$ where each $k_j \geq 2$. Then, by Lemmas 7.2.4 and 7.3.4,

$$\beta_i(L) = \begin{cases} \prod \left(\frac{k_j}{2} - 1\right) & \text{if } i = m, \\ 0 & \text{if } i \neq m. \end{cases}$$

Lemma 7.3.7 (The Betti numbers of a cone)

(1) $\beta_i(CL) = \frac{1}{2}\beta_i(L)$.

(2) $\beta_{i+1}(CL, L) = \frac{1}{2}\beta_i(L)$.

(3) *The sequence of the pair (CL, L) breaks up into short exact sequences:*

$$0 \to \mathfrak{h}_{i+1}(CL, L) \to \mathfrak{h}_i(L) \to \mathfrak{h}_i(CL) \to 0.$$

Proof Although formulas (1) and (2) follow from the proof of (3), we first give simple alternative arguments for them which illustrate the above methods. Since $\Sigma_{CL} = [-1,1] \times \Sigma_L$, the complexes Σ_{CL} and Σ_L are W_L–equivariantly homotopy equivalent; hence, $\mathcal{H}_i(\Sigma_{CL}) \cong \mathcal{H}_i(\Sigma_L)$. Since $W_{CL} = \mathbb{Z}_2 \times W_L$, we have, by 3.2.9, that $\beta_i(CL) = \frac{1}{2}\beta_i(L)$, proving (1). To prove (2), let -1 and $+1$ denote the two points of $\mathbb{S}^0$. Then $SL = C_{-1}L \cup C_{+1}L$ is the union of two copies of the cone on L along L. By excision, Lemma 7.2.2, $\mathfrak{h}_{i+1}(C_{+1}L, L) \cong \mathfrak{h}_{i+1}(SL, C_{-1}L)$ and by the exact sequence of the pair, $\mathfrak{h}_{i+1}(SL, C_{-1}L) \cong \mathfrak{h}_i(C_{-1}L)$. Hence, $\beta_{i+1}(CL, L) = \beta_{i+1}(SL, CL) = \beta_i(CL) = \frac{1}{2}\beta_i(L)$, which proves (2).

In the exact sequence which we are considering in (3), Σ_{CL} is the ambient space and $\mathfrak{h}_i(L)$ means the reduced ℓ^2–homology of the subcomplex $\{\pm 1\} \times \Sigma_L$ in Σ_{CL} $(= [-1,1] \times \Sigma_L)$. Thus, $\mathfrak{h}_i(L) = \mathcal{H}_i(\Sigma_L) \oplus \mathcal{H}_i(\Sigma_L)$. Let $j \colon \{\pm 1\} \times \Sigma_L \to \Sigma_{CL}$ be the inclusion. A class of the form $(\alpha, -\alpha)$ in the direct sum obviously goes to 0 in $\mathcal{H}_i(CL)$, while j_* maps the diagonal subspace of elements of the form (α, α) isomorphically onto $\mathcal{H}_i(CL)$. Statement (3) follows (as do formulas (1) and (2)). □

7.4 Poincaré duality If a flag complex S is a generalized homology sphere with rational coefficients, then Σ_S is a polyhedral homology manifold with rational coefficients. Hence, Σ_S satisfies Poincaré duality, 2.7.1. Similarly, if a pair $(D, \partial D)$ of flag complexes is a generalized homology disk with rational coefficients, then Σ_D is a polyhedral homology manifold with boundary with rational coefficients (its boundary being $W_D \Sigma_{\partial D}$) and hence, it satisfies the relative version of Poincaré duality.

7.4.1 If S is a GHS^{n-1}, then $\beta_i(S) = \beta_{n-i}(S)$.

7.4.2 If $(D, \partial D)$ is a GHD^{n-1}, then $\beta_i(D, \partial D) = \beta_{n-i}(D)$.

7.4.3 If $(D, \partial D)$ is a GHD^{n-1}, then the homology and cohomology sequences of the pair $(D, \partial D)$ are isomorphic under Poincaré duality in the sense that the following diagram commutes up to sign,

$$\begin{tikzcd}
\arrow[r] & \mathfrak{h}_{i+1}(D,\partial D) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathfrak{h}_i(\partial D) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathfrak{h}_i(D) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathfrak{h}_i(D,\partial D) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & {} \\
\arrow[r] & \mathfrak{h}^{n-i-1}(D) \arrow[r] & \mathfrak{h}^{n-i-1}(\partial D) \arrow[r] & \mathfrak{h}^{n-i}(D,\partial D) \arrow[r] & \mathfrak{h}^{n-i}(D) \arrow[r] & {}
\end{tikzcd}$$

where the vertical isomorphisms are given by Poincaré duality (cf Theorem 1.1.5 of [9]). From this we deduce the following lemmas which we shall need in Section 9.

Lemma 7.4.4 *Suppose $(D, \partial D)$ is a GHD^{2k} and that $j_*\colon \mathfrak{h}_k(\partial D) \to \mathfrak{h}_k(D)$ is the map induced by the inclusion. Then*

$$\dim_{W_L}(\operatorname{Ker} j_*) = \dim_{W_L}(\overline{\operatorname{Im} j_*}) = \frac{1}{2}\beta_k(\partial D).$$

Proof By 7.4.3, the sequences

$$\mathfrak{h}_{k+1}(D, \partial D) \xrightarrow{\partial_*} \mathfrak{h}_k(\partial D) \xrightarrow{j_*} \mathfrak{h}_k(D)$$

and

$$\mathfrak{h}^k(D) \xrightarrow{j^*} \mathfrak{h}^k(\partial D) \xrightarrow{\partial^*} \mathfrak{h}^{k+1}(D, \partial D)$$

are isomorphic. In other words, under Poincaré duality, the connecting homomorphism ∂_* is isomorphic to j^*. Since j^* is the adjoint of j_*, 3.2.8 implies that

$$\beta_k(\partial D) = \dim_{W_L}(\operatorname{Ker} j_*) + \dim_{W_L}(\overline{\operatorname{Im} \partial_*}).$$

By the exact sequence of the pair, Lemma 7.2.1, $\operatorname{Ker} j_* = \overline{\operatorname{Im} \partial_*}$, so $\dim_{W_L}(\operatorname{Ker} j_*) = \frac{1}{2}\beta_k(\partial D)$. Then, by 3.2.7, we also have $\dim_{W_L}(\overline{\operatorname{Im} j_*}) = \frac{1}{2}\beta_k(\partial D)$, which proves the lemma. □

7.4.5 Suppose that $S = D_1 \cup D_2$ and $S_0 = D_1 \cap D_2$. Also suppose that S is a GHS^{n-1} and that (D_1, S_0) and (D_2, S_0) are GHD^{n-1}'s. By Lemma 7.2.3(2), $\mathfrak{h}^i(S, S_0) \cong \mathfrak{h}^i(D_1, S_0) \oplus \mathfrak{h}^i(D_2, S_0)$. Similarly to 7.4.3, the homology Mayer–Vietoris sequence of $S = D_1 \cup D_2$ is isomorphic, via Poincaré duality, to the exact sequence of the pair (S, S_0) in cohomology. In other words, the following diagram commutes up to sign,

$$\begin{array}{ccccccccc}
\to & \mathfrak{h}_{i+1}(S) & \to & \mathfrak{h}_i(S_0) & \to & \mathfrak{h}_i(D_1) \oplus \mathfrak{h}_i(D_2) & \to \\
& \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & \\
\to & \mathfrak{h}^{n-i-1}(S) & \to & \mathfrak{h}^{n-i-1}(S_0) & \to & \mathfrak{h}^{n-i}(D_1, S_0) \oplus \mathfrak{h}^{n-i}(D_2, S_0) & \to
\end{array}$$

where the first row is the Mayer–Vietoris sequence, the second is the exact sequence of the pair and the vertical isomorphisms are given by Poincaré duality. We record the special case of this where $n = 2k + 1$ and $i = k$ as the following lemma.

Lemma 7.4.6 *With hypotheses as in 7.4.5, suppose $n = 2k+1$. Then the map $i_*\colon \mathfrak{h}_k(S_0) \to \mathfrak{h}_k(S)$ induced by the inclusion is dual (under Poincaré duality) to the connecting homomorphism $\partial_*\colon \mathfrak{h}_{k+1}(S) \to \mathfrak{h}_k(S_0)$ in the Mayer–Vietoris sequence.*

Proof In this special case, the diagram in 7.4.5 becomes the following:

$$\begin{array}{ccccc}
\mathfrak{h}_{k+1}(S) & \xrightarrow{\partial_*} & \mathfrak{h}_k(S_0) & \longrightarrow & \mathfrak{h}_k(D_1) \oplus \mathfrak{h}_k(D_2) \\
\updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong \\
\mathfrak{h}^k(S) & \xrightarrow{i^*} & \mathfrak{h}^k(S_0) & \longrightarrow & \mathfrak{h}^{k+1}(D_1, S_0) \oplus \mathfrak{h}^{k+1}(D_2, S_0)
\end{array}$$

□

8 Variations on Singer's Conjecture

In this section, we will consider several conjectures, $\mathbf{I}(n)$, $\mathbf{II}(n)$, $\mathbf{III}(n)$, $\mathbf{IV}(n)$ and $\mathbf{V}(n)$, concerning the reduced ℓ^2–homology of Σ_L, where L is either a generalized homology sphere, usually denoted by S, or a generalized homology disk, denoted by D. Here the "n" refers to the dimension of Σ_L, where $L = S$ or D, so that $\dim L = n - 1$.

As usual all simplicial complexes are flag complexes and all subcomplexes are full.

8.1 Restatement of Singer's Conjecture

$\mathbf{I}(n)$ *If S is a GHS^{n-1}, then $\beta_i(S) = 0$ for all $i \neq \frac{n}{2}$.*

8.2 Singer's Conjecture for a disk

II(n) *Suppose* $(D,\partial D)$ *is a* GHD^{n-1}.

- *If* $n=2k$ *is even, then* $\beta_i(D)=\beta_i(D,\partial D)=0$ *for all* $i\neq k$.
- *If* $n=2k+1$ *is odd, then*
  (1) $\beta_i(D)=\beta_{i+1}(D,\partial D)=0$ *for all* $i\neq k$, *and*
  (2) $\beta_k(D)=\beta_{k+1}(D,\partial D)=\frac{1}{2}\beta_k(\partial D)$ *and the following sequence of the pair is weakly short exact,*

$$0\to\mathfrak{h}_{k+1}(D,\partial D)\to\mathfrak{h}_k(\partial D)\to\mathfrak{h}_k(D)\to 0.$$

8.2.1 Given a GHD, $(D,\partial D)$, let S denote the GHS formed by gluing on $C(\partial D)$ to D along ∂D. If v denotes the cone point, then $\partial D=S_v$ (the link of v) and $C(\partial D)=CS_v$. Conversely, given a GHS, call it S, and a vertex v, we obtain a GHD, $D=S-v$ with $\partial D=S_v$.

Next we consider some seemingly weaker statements in odd dimensions.

8.3 A weak form of the conjecture

III$(2k+1)$ *Suppose* (D,S_v) *is a* GHD^{2k} *and that* $S=D\cup CS_v$ *is as in 8.2.1. Then in the Mayer–Vietoris sequence, the map,*

$$j_*\oplus h_*\colon\ \mathfrak{h}_k(S_v)\to\mathfrak{h}_k(D)\oplus\mathfrak{h}_k(CS_v),$$

is a monomorphism.

8.3.1 (*Remark*) By Lemma 7.3.7, $h_*\colon\ \mathfrak{h}_k(S_v)\to\mathfrak{h}_k(CS_v)$ is surjective and the von Neumann dimension of its kernel is $\frac{1}{2}\beta_k(S_v)$. Similarly, by Lemma 7.4.4, $\dim_{W_S}(\operatorname{Ker} j_*)=\frac{1}{2}\beta_k(S_v)$. So, it is not unreasonable to expect that these subspaces intersect in general position: $\operatorname{Ker} j_*\cap\operatorname{Ker} h_*=0$, in other words, that **III**$(2k+1)$ is valid.

8.3.2 By Lemma 7.4.6, **III**$(2k+1)$ is equivalent to the following.

III$'(2k+1)$ *Suppose* (D,S_v) *is a* GHD^{2k} *and that* $S=D\cup CS_v$ *is as in 8.2.1. Then the map* $i_*\colon\ \mathfrak{h}_k(S_v)\to\mathfrak{h}_k(S)$, *induced by the inclusion, is the zero homomorphism.*

8.4 A weak form of the conjecture for a disk

IV$(2k+1)$ *Suppose* $(D, \partial D)$ *is a* GHD^{2k}*. Then* $\beta_{k+1}(D) = 0$.

8.5 The strong form of the conjecture The formulation of the last conjecture is key to our approach.

V(n) *Suppose* S *is a* GHS^{n-1} *and that* A *is any full subcomplex.*

- *If* $n = 2k$ *is even, then* $\beta_i(S, A) = 0$ *for all* $i > k$.
- *If* $n = 2k+1$ *is odd, then* $\beta_i(A) = 0$ *for all* $i > k$.

8.6 Joins It follows from the Künneth Formula, Lemma 7.2.4, that the above conjectures are compatible with the operation of taking joins. For example, let **J** stand for **I**, **III** or **V**. If S_1 and S_2 are GHS's of dimension $n_1 - 1$ and $n_2 - 1$ for which $\mathbf{J}(n_1)$ and $\mathbf{J}(n_2)$ hold, then $\mathbf{J}(n_1+n_2)$ holds for $S_1 * S_2$ (which is a $GHS^{n_1+n_2-1}$ by 4.3.7). Similarly, let **J** stand for **II** or **IV**. If S_1 is a GHS^{n_1-1} for which $\mathbf{I}(n_1)$ holds and D_2 is GHD^{n_2-1} for which $\mathbf{J}(n_2)$ holds, then $\mathbf{J}(n_1+n_2)$ holds for $S_1 * D_2$ (which is a $GHD^{n_1+n_2-1}$ by 4.3.7).

8.7 An m–gon Suppose S is an m–gon, $m \geq 4$. Then by 7.2.6, $\beta_0(S) = 0$ and by 7.2.7 (or by 7.3.1) $\beta_2(S) = 0$. So, $\mathbf{I}(2)$ holds. Similarly, $\beta_0(\mathbb{S}^0) = \beta_1(\mathbb{S}^0) = 0$, so $\mathbf{I}(1)$ holds.

8.8 Some implications Next we list some obvious implications among these conjectures.

8.8.1 Let **J** stand for **I**, **II**, **III**, **III**$'$, **IV**, or **V**. Then $\mathbf{J}(n) \implies \mathbf{J}(n-2)$.

Proof Suppose L^{n-3} is a GHS or GHD for which $\mathbf{J}(n-2)$ fails. Let S_1 be a 5–gon. Since $\beta_1(S_1) \neq 0$, the Künneth Formula 7.2.4 shows that $\mathbf{J}(n)$ also fails for $S_1 * L^{n-3}$. □

8.8.2 $\mathbf{II}(n) \implies \mathbf{I}(n-1)$.

Proof Let S be a GHS^{n-2}. By Lemma 7.3.7, if $\mathbf{II}(n)$ holds for (CS, S), then $\mathbf{I}(n-1)$ holds for S. □

8.8.3 $[\mathbf{I}(n-1) \text{ and } \mathbf{I}(n)] \implies \mathbf{II}(n)$.

Proof Suppose $\mathbf{I}(n-1)$ and $\mathbf{I}(n)$ hold, that $(D, \partial D)$ is a GHD^{n-1} and, as in 8.2.1, that $S = D \cup C(\partial D)$. If $n = 2k$, then since $\mathbf{I}(2k-1)$ holds for ∂D, $\beta_i(\partial D) = 0$ for all i. By $\mathbf{I}(2k)$, $\beta_i(S) = 0$ for $i \neq k$. The Mayer–Vietoris sequence then yields that $\beta_i(D) = 0$ for $i \neq k$ and that $\mathfrak{h}_k(D) \oplus \mathfrak{h}_k(C(\partial D)) \cong \mathfrak{h}_k(S)$. So, $\mathbf{II}(2k)$ holds for $(D, \partial D)$. If $n = 2k+1$, then by $\mathbf{I}(2k+1)$, $\beta_i(S) = 0$ for all i. Hence, the Mayer–Vietoris sequence yields, $\mathfrak{h}_i(\partial D) \cong \mathfrak{h}_i(D) \oplus \mathfrak{h}_i(C(\partial D))$. By $\mathbf{I}(2k)$, $\beta_k(\partial D) = 0$ for $i \neq k$. It then follows from 7.4.2 and Lemma 7.4.4 that $\mathbf{II}(2k+1)$ holds for $(D, \partial D)$. $\square$

8.8.4 $\mathbf{II}(2k) \implies \mathbf{I}(2k)$.

Proof Let S be a GHS^{2k-1} and v a vertex of S. Write $S = D \cup CS_v$, as in 8.2.1. Assume $\mathbf{II}(2k)$ holds. By 8.8.2, $\mathbf{I}(2k-1)$ holds for S_v, ie, $\beta_i(S_v) = 0$ for all i. From the sequence of the pair (S, S_v) we get: $\mathfrak{h}_i(S) \cong \mathfrak{h}_i(S, S_v) \cong \mathfrak{h}_i(D, S_v) \oplus \mathfrak{h}_i(CS_v, S_v)$. Since $\mathbf{II}(2k)$ holds, the last two terms are nonzero only in the middle dimension. Hence, $\mathbf{II}(2k) \implies \mathbf{I}(2k)$. $\square$

The next two implications, 8.8.5 and 8.8.6, are immediate.

8.8.5 $\mathbf{I}(2k+1) \implies \mathbf{III}(2k+1)$.

8.8.6 $\mathbf{II}(2k+1) \implies \mathbf{IV}(2k+1)$.

8.8.7 $\mathbf{V}(n) \implies \mathbf{I}(n)$.

Proof This follows from Poincaré duality. (If $n = 2k$, take $A = \emptyset$ to get $\beta_i(S) = 0$ for $i > k$. If $n = 2k+1$, take $A = S$, to get $\beta_i(S) = 0$ for $i > k$.) $\square$

8.8.8 $\mathbf{V}(n) \implies \mathbf{II}(n)$.

Proof We proceed as in 8.8.3. Given $(D, \partial D)$, set $S = D \cup C(\partial D)$, as before. Assume $\mathbf{V}(n)$ holds. If $n = 2k$, then $\beta_i(S) = 0$ for $i \neq k$ (by 8.8.7). For $i > k$, by $\mathbf{V}(2k)$, we have that

$$0 = \mathfrak{h}_i(S, \partial D) \cong \mathfrak{h}_i(D, \partial D) \oplus \mathfrak{h}_i(C(\partial D), \partial D).$$

So, for $i > k$, $\beta_i(D, \partial D) = 0$ and $\beta_{i-1}(\partial D) = 0$ (by Lemma 7.3.7(2)). By 7.4.1, the second equation implies that $\beta_i(\partial D) = 0$ for all i. It then follows from the exact sequence of the pair and 7.4.2, that $\mathbf{II}(2k)$ holds for $(D, \partial D)$. If $n = 2k+1$, then $\beta_i(S) = 0$ for all i (by 8.8.7). The sequence of the pair $(S, \partial D)$ then gives:

$$\mathfrak{h}_i(D, \partial D) \oplus \mathfrak{h}_i(C(\partial D), \partial D) \cong \mathfrak{h}_{i-1}(\partial D).$$

By $\mathbf{V}(2k+1)$, $\beta_j(\partial D) = 0$ for $j > k$; hence, by 7.4.1, it vanishes for $j \neq k$. Therefore, $\beta_i(D, \partial D) = 0$ for $i \neq k+1$ and by 7.4.2, $\beta_i(D) = 0$ for $i \neq k$. So, $\mathbf{II}(2k+1)$ holds for $(D, \partial D)$. □

Lemma 8.8.9 *Statement $\mathbf{V}(2k)$ implies that for any full subcomplex A of S (a GHS^{2k-1}), we have*

$$\beta_i(A) = 0 \quad \text{for all } i > k.$$

Proof Assume $\mathbf{V}(2k)$ holds. By 8.8.7, $\beta_i(S) = 0$ for $i \neq k$. Hence, in the exact sequence of the pair,

$$\mathfrak{h}_{i+1}(S, A) \to \mathfrak{h}_i(A) \to \mathfrak{h}_i(S),$$

the first and third terms vanish for all $i > k$. □

8.9 A conjecture for groups of finite type Recent work by Bestvina, Kapovich and Kleiner in [5] shows that if $A = P_3 * \cdots * P_3$ is an k–fold join of 3 points with itself, then W_A cannot act properly on a contractible $(2k-1)$–manifold. Their argument uses the well-known fact that A does not embed in $\mathbb{S}^{2k-2}$. We note that this well-known fact follows from Conjecture $\mathbf{V}(2k-1)$ since, by 7.2.4, $b_k^{(2)}(W_A) \neq 0$ which would contradict Conjecture $\mathbf{V}(2k-1)$ if A were a full subcomplex of some flag triangulation of $\mathbb{S}^{2k-2}$. (Here the ℓ^2–Betti numbers $b_i^{(2)}(W_A)$ are as defined in 3.3.7.) These remarks suggest the following generalization of Singer's Conjecture.

Conjecture 8.9.1 *Suppose that a discrete group G acts properly on a contractible n–manifold. Then*

$$b_i^{(2)}(G) = 0 \quad \text{for } i > \frac{n}{2}.$$

(In the case where G does not act cocompactly on its universal space $\underline{E}G$, define its ℓ^2–Betti numbers as in [12].)

9 Inductive Arguments

We describe a partially successful program for proving conjecture $\mathbf{V}(n)$. The idea is to use a double induction: first, induction on the dimension n and second, depending on the parity of n, induction either on the number of vertices of A or on the number of vertices in $S - A$.

9.1 Notation We set up some notation for the induction on the number of vertices. Suppose A and B are full subcomplexes of S, the vertex sets of which differ by only one element, say v. In other words, $B = A - v$, for some $v \in \mathcal{S}_0(A)$. Let A_v and S_v denote the link of v in A and S, respectively. Thus, $A = B \cup CA_v$ and $CA_v \cap B = A_v$. We note that S_v is a GHS of one less dimension than S and that A_v is a full subcomplex of S_v.

9.2 Induction on the number of vertices

Lemma 9.2.1 $\mathbf{V}(2k-1) \implies \mathbf{V}(2k)$.

Proof Suppose $\mathbf{V}(2k-1)$ holds. Let (S, A) be as in $\mathbf{V}(2k)$ and let $B = A - v$. Assume, by induction on the number of vertices in $S - A$, that $\mathbf{V}(2k)$ holds for (S, A). (The case $A = S$ being trivial.) We want to prove it also holds for (S, B), ie, that $\beta_i(S, B) = 0$ for $i > k$. Consider the exact sequence of the triple (S, A, B):

$$\to \mathfrak{h}_i(A, B) \to \mathfrak{h}_i(S, B) \to \mathfrak{h}_i(S, A) \to .$$

Suppose $i > k$. By inductive hypothesis, $\beta_i(S, A) = 0$. By excision, Lemma 7.2.2, $\beta_i(A, B) = \beta_i(CA_v, A_v)$. By Lemma 7.3.7(2), $\beta_i(CA_v, A_v) = \frac{1}{2}\beta_{i-1}(A_v)$. Since $\mathbf{V}(2k-1)$ holds for (S_v, A_v) and since $i - 1 > k - 1$, $\beta_{i-1}(A_v) = 0$. So, $0 = \beta_i(CA_v, A_v) = \beta_i(A, B)$. Consequently, $\beta_i(S, B) = 0$. □

Essentially the same argument proves the following lemma (which we will need in Section 11.4).

Lemma 9.2.2 *Assume that $\mathbf{V}(2k)$ holds. Suppose that a flag complex L is a polyhedral homology manifold of dimension $2k$ and that A is a full subcomplex. Then $\beta_i(L, A) = 0$ for $i > k + 1$.*

Proof We proceed as in the previous proof. If $B = A - v$, then $\beta_i(A, B) = \beta_i(CA_v, A_v) = \frac{1}{2}\beta_{i-1}(A_v)$. Since we are assuming $\mathbf{V}(2k)$ holds, Lemma 8.8.9 implies that $\beta_{i-1}(A_v) = 0$ for $i > k+1$. Hence, if we assume by induction that the lemma holds for (L, A), then it also holds for (L, B). $\square$

Lemma 9.2.3 $[\mathbf{V}(2k)$ *and* $\mathbf{III}(2k+1)] \implies \mathbf{V}(2k+1)$.

Proof Assume $\mathbf{V}(2k)$ and $\mathbf{III}(2k+1)$ hold. Let (S, A) be as in $\mathbf{V}(2k+1)$ and let $B = A - v$. Assume, by induction on the number of vertices in B, that $\mathbf{V}(2k+1)$ holds for B. (The case $B = \emptyset$ being trivial.) We want to prove that it also holds for A, ie, that $\beta_i(A) = 0$ for $i > k$.

First suppose that $i > k+1$. Consider the Mayer–Vietoris sequence for $A = B \cup CA_v$:

$$\mathfrak{h}_i(B) \oplus \mathfrak{h}_i(CA_v) \to \mathfrak{h}_i(A) \to \mathfrak{h}_{i-1}(A_v).$$

By $\mathbf{V}(2k)$ and Lemma 8.8.9, $\beta_{i-1}(A_v) = 0$ (since $i - 1 > k$) and hence, $\beta_i(CA_v) = 0$ (by Lemma 7.3.7(1)). By inductive hypothesis, $\beta_i(B) = 0$, and consequently, $\beta_i(A) = 0$.

For $i = k+1$, we compare the Mayer–Vietoris sequence of $A = B \cup CA_v$ with that of $S = D \cup CS_v$ (where $D = S - v$):

$$\begin{tikzcd}
 & & \mathfrak{h}_{k+1}(S_v, A_v) \arrow[d] & \\
0 \arrow[r] & \mathfrak{h}_{k+1}(A) \arrow[r] & \mathfrak{h}_k(A_v) \arrow[r, "j'_* \oplus h'_*"] \arrow[d, "f_*"'] & \mathfrak{h}_k(B) \oplus \mathfrak{h}_k(CA_v) \arrow[d] \\
 & & \mathfrak{h}_k(S_v) \arrow[r, "j_* \oplus h_*"] & \mathfrak{h}_k(D) \oplus \mathfrak{h}_k(CS_v)
\end{tikzcd}$$

By $\mathbf{V}(2k)$, $\beta_{k+1}(S_v, A_v) = 0$; hence, f_* is injective. By $\mathbf{III}(2k+1)$, $j_* \oplus h_*$ is injective. Hence, $j'_* \oplus h'_*$ is injective and therefore, $\beta_{k+1}(A) = 0$. $\square$

9.3 Induction on dimension

Our main result is the following.

Theorem 9.3.1 *Statement* $\mathbf{III}(2k-1)$ *implies that* $\mathbf{V}(n)$ *holds for all* $n \leq 2k$.

Proof By 8.2.1, $\mathbf{III}(2k-1)$ implies $\mathbf{III}(2l-1)$, for all $l \leq k$. Suppose, by induction on n, that $\mathbf{V}(n-1)$ holds for some $n \leq 2k$. If $n-1$ is odd, then by Lemma 9.2.1, $\mathbf{V}(n-1)$ implies $\mathbf{V}(n)$. If $n-1$ is even, then by Lemma 9.2.3, $\mathbf{V}(n-1)$ and $\mathbf{III}(n)$ imply $\mathbf{V}(n)$. $\square$

9.3.2 An n–dimensional simplicial complex L has *spherical links in codimensions* $\leq m$ if for each $\sigma \in \mathcal{S}(L)$ of dimension $n-i$, with $i \leq m$, its link $\mathrm{Link}(\sigma, L)$ is a GHS^{i-1}.

When $\dim L < m$, this condition means that L is a GHS (take $\sigma = \emptyset$). When $\dim L \geq m$, it means that the complement of its codimension–$(m+1)$ skeleton is a homology manifold. For example, for $m = 1$, it means that L is a pseudomanifold.

We note that the condition is inherited by links of vertices: if L has spherical links in codimensions $\leq m$, then so does L_v for any vertex v.

Theorem 9.3.3 *Assume that* $\mathbf{III}(2l+1)$ *is true. Let* L *be an* $(n-1)$–*dimensional flag complex,* $n \geq 2l+1$, *with spherical links in codimensions* $\leq 2l+1$. *Then for any full subcomplex* A *of* L,

$$\beta_{n-i}(A) = 0 \quad \text{for } i \leq l,$$

and if $n \neq 2l+1$, *then*

$$\beta_{n-i}(L, A) = 0 \quad \text{for } i \leq l.$$

Proof The proof is by induction on n, starting at $n = 2l+1$. If $n = 2l+1$, then L is a GHS^{2l} and the result follows from Theorem 9.3.1.

If $n > 2l+1$, then for any vertex $v \in A$, we have, by inductive hypothesis that $\beta_{(n-1)-i}(A_v) = 0$ for $i \leq l$. The proof of Lemma 9.2.1 then shows that $\beta_{n-i}(L, A) = 0$ and the first part of the proof of Lemma 9.2.3 shows that $\beta_{n-i}(A) = 0$ for $i \leq l$. □

Corollary 9.3.4 *Assume that* $\mathbf{III}(2l+1)$ *is true. Suppose that a flag complex* S *is a* GHS^{n-1}, $n \geq 2l+1$. *Then*

$$\beta_i(S) = \beta_{n-i}(S) = 0 \quad \text{for } i \leq l.$$

Proof This follows from the previous theorem (taking $L = A = S$) and Poincaré duality. □

Theorem 9.3.3 suggests the following generalization of Singer's Conjecture.

Conjecture 9.3.5 *Suppose that* X *is a contractible, geometric* G–*complex of dimension* n *and that* X *has spherical links in codimensions* $\leq 2l+1$, *where* $2l+1 \leq n$. *Then*

$$b^{(2)}_{n-i}(X; G) = 0 \quad \text{for } i \leq l.$$

10 The conjecture in dimension 3

10.1 Review of previous results In [32] Lott and Lück proved Singer's Conjecture for any closed, irreducible 3–manifold with infinite fundamental group for which Thurston's Geometrization Conjecture holds. In other words, for such a 3–manifold, the reduced ℓ^2–homology of its universal cover vanishes. As we shall see in 10.1.5, below, the Geometrization Conjecture holds for the 3–dimensional orbifolds which we are interested in. Hence, conjecture $\mathbf{I}(3)$, from Section 8.1, is true.

The calculation in [32] depends on the following two facts, stated as 10.1.1 and 10.1.2, below.

10.1.1 Suppose M is the compact 3–manifold formed by gluing together two compact 3–manifolds M_1 and M_2 along one or more boundary components which are incompressible tori. If the reduced ℓ^2–homology of their universal covers, $\widetilde{M_1}$ and $\widetilde{M_2}$, vanishes, then so does the reduced ℓ^2–homology of $\widetilde{M}$. (This follows the Mayer–Vietoris sequence, 2.4.4, and the vanishing of the reduced ℓ^2–homology of the universal cover of T^2.)

10.1.2 (*Theorem 5.14 of [32]*) Let M^3 be a compact 3–manifold with boundary such that its interior is homeomorphic to a complete hyperbolic manifold of finite volume. Then the reduced ℓ^2–homology of its universal cover vanishes.

We note that if M^3 is a compact 3–manifold formed by chopping off the cusps of a complete hyperbolic 3–manifold of finite volume, then each boundary component of M^3 is a 2–torus (or possibly a Klein bottle in the nonorientable case). Hence, the result of Lott–Lück follows from 10.1.1, 10.1.2 and a similar result for aspherical Seifert fiber spaces.

The result in 10.1.2 is, in turn, a consequence of the next two facts, stated as 10.1.3 and 10.1.4, below.

10.1.3 The reduced ℓ^2–homology of any odd-dimensional hyperbolic space, $\mathbb{H}^{2k+1}$, vanishes. (This is proved in [21].)

The next result is stated on page 226 of [28]. It is proved by Cheeger and Gromov in [11].

10.1.4 (*Bounded geometry*) Suppose X is a complete contractible Riemannian manifold with uniformly bounded geometry, ie, its sectional curvature is bounded and its injectivity radius is bounded away from 0. Let Γ be a discrete group of isometries of X with $\mathrm{Vol} X/\Gamma < \infty$. Then

$$b_k^{(2)}(\Gamma) = \dim_\Gamma(\mathcal{H}_k(X)).$$

(Here $b_k^{(2)}(\Gamma)$ is the ℓ^2–Betti number of Γ defined in 3.3.7.)

Thus, 10.1.2 follows from 10.1.4 and 10.1.3 in the case where $X = \mathbb{H}^3$.

10.1.5 (*Haken manifolds*) Thurston proved that the Geometrization Conjecture holds for Haken 3–manifolds. Suppose that S is a triangulation of the 2–sphere as a flag complex and that $M^3 = P_S$, the commutator cover of Σ_S/W_S considered in 6.4. Then M^3 is obviously Haken. Indeed, for any vertex s of S, the special subcomplex Σ_{S_s} is geodesically convex; hence, its image in M^3 is an incompressible surface. (See the argument in 14.1.6, below.) Therefore, the reduced ℓ^2–homology of Σ_S (the universal cover of M^3) vanishes.

In this special case, Thurston's Theorem is basically a consequence of Andreev's Theorem, which was proved several years earlier in [2],[3]. We explain Andreev's Theorem in Section 10.3, below. However, we first need to develop some material about triangulations of the 2–sphere.

10.2 Triangulations of $\mathbb{S}^2$

Let S be a triangulation of $\mathbb{S}^2$ as a flag complex.

10.2.1 The *valence* of a vertex s of S is the number of vertices in its link. In what follows we shall be concerned with the vertices of valence 4.

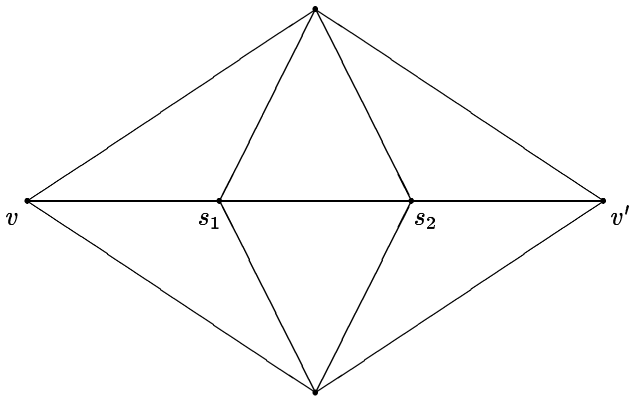
10.2.2 Let C be a circuit of length 4 in the 1–skeleton of S. Then C is an *empty* 4*–circuit* if (a) C is not the link of a vertex and (b) C is not the boundary of the union of two adjacent 2–simplices. Since S is a flag complex, it follows from (b) that any empty 4–circuit C is a full subcomplex.

Lemma 10.2.3 *Suppose that*

- (i) *S has no empty 4–circuits and*
- (ii) *S is not the suspension of a* 4– *or* 5*–gon.*

Then no two valence 4 *vertices of S are connected by an edge.*

Proof Suppose that s_1 and s_2 are valence 4 vertices which are connected by an edge. Then the star of that edge is the configuration pictured in the figure below.



The indicated vertices v and v' cannot coincide, since if they did S would contain an empty 3–circuit and hence, not be a flag complex. Similarly, the top and bottom vertices cannot be connected by an edge, since S would again contain empty 3–circuits. Let C be the boundary of the star in the figure. If C is the boundary of two adjacent 2–simplices, then S is the suspension of a 4–gon. If C is the link of a missing vertex, then S is the suspension of a 5–gon. Otherwise, C is an empty 4–circuit, contradicting (i). □

Lemma 10.2.4 *Let T be a set of valence 4 vertices of S, no two of which are connected by an edge. Then $\beta_i(S) = \beta_i(S - T)$ for all i.*

Proof By 7.3.6 $\mathfrak{h}_*(S_s)$ vanishes for any $s \in T$. Hence, it follows from the Mayer–Vietoris sequence, Lemma 7.2.3(1), that we can adjoin CS_s to $S - T$ without changing β_i. □

10.2.5 For $j = 1, 2$, suppose that S_j is a flag triangulation of $\mathbb{S}^2$ and that s_j is a vertex of valence 4 in S_j. Choose an identification of the link of s_1 with that of s_2. (They are both 4–gons.) Define a new triangulation $S_1 \square S_2$ of $\mathbb{S}^2$ by gluing together the 2–disks $S_1 - s_1$ and $S_2 - s_2$ along their boundaries.

10.2.6 Conversely, suppose C is an empty 4–circuit in S. Then C separates S into two 2–disks, D_1 and D_2. Let S_1 and S_2 denote the result of capping off D_1 and D_2, respectively (where "capping off" means adjoining a cone on the boundary). Then $S = S_1 \square S_2$.

The next lemma is a version of 10.1.1.

Lemma 10.2.7 $\beta_1(S_1 \Box S_2) = \beta_1(S_1) + \beta_1(S_2)$. *Thus, $\mathfrak{h}_*$ vanishes for $S_1 \Box S_2$ if and only if it vanishes for both S_1 and S_2.*

Proof This follows from the Mayer–Vietoris sequence as before. □

10.2.8 Suppose S satisfies the conditions of Lemma 10.2.3 and let T denote the set of valence 4 vertices of S. Consider a cellulation $[S-T]$ of $\mathbb{S}^2$ obtained by replacing stars of vertices of T by square cells. By Lemma 10.2.3, $[S-T]$ is a well-defined 2–complex homeomorphic to $\mathbb{S}^2$ with triangular and square faces. In fact, it is easy to see that, under the assumptions of Lemma 10.2.3, this complex is a *cell* complex in a strict sense that any nonempty intersections of two cells is a cell. It is a classical fact that any such complex is combinatorially dual to the boundary complex of a convex polytope, which we will denote $K_{[S-T]}$.

10.3 Andreev's Theorem

In [3] Andreev determined which convex polytopes could occur as fundamental chambers of classical reflection groups on $\mathbb{H}^3$. More precisely, given a convex polytope with assigned dihedral angles in $\left(0, \frac{\pi}{2}\right]$ on the edges, he gave necessary and sufficient conditions for it to be realized as a (possibly ideal) convex polytope in $\mathbb{H}^3$. A special case of his result is the following.

Theorem 10.3.1 (Andreev) *Suppose that S is a flag triangulation of $\mathbb{S}^2$ and that*

- (i) *S has no empty 4–circuits, and*
- (ii) *S is not the suspension of a 4– or 5–gon.*

Let T denote the set of valence 4 vertices of S and let $K_{[S-T]}$ be the dual of the cellulation $[S-T]$ of $\mathbb{S}^2$ obtained by replacing stars of vertices of T by square cells.

Then $K_{[S-T]}$ can be realized as an ideal, right-angled convex polytope in $\mathbb{H}^3$. (The ideal vertices correspond to the square faces of $[S-T]$, ie, to the vertices of valence 4 in S.) The resulting classical reflection group is the right-angled Coxeter group W_{S-T}.

Proof By 10.2.8, $K_{[S-T]}$ is combinatorially equivalent to the boundary complex of a convex polytope with vertices of valence 3 and 4 only. In Theorem 2 of [3], Andreev lists 6 conditions $\mathfrak{m}0$–$\mathfrak{m}5$ on assigned angles for such a polytope to be realized in $\mathbb{H}^3$.

The conditions $\mathfrak{m}0$ and $\mathfrak{m}1$ are immediate under our hypothesis, since all angles are $\frac{\pi}{2}$. The remaining conditions refer to certain configurations of faces of the polytope, and turn out to be vacuous in our case, since these configurations never appear under our hypothesis.

Indeed, since S is a flag triangulation, it follows that $[S-T]$ does not contain empty 3–circuits, and therefore, $K_{[S-T]}$ does not contain triangular prismatic elements and cannot be a triangular prism. Similarly, since S does not contain empty 4–circuits, every 4–circuit in $[S-T]$ is a boundary of either two adjacent triangles or of a square cell, and therefore, $K_{[S-T]}$ does not contain quadrangular prismatic elements. Thus, we have verified conditions $\mathfrak{m}2$, $\mathfrak{m}3$ and $\mathfrak{m}4$.

To verify condition $\mathfrak{m}5$ we note that if two faces of $K_{[S-T]}$ intersect at a vertex, but are not adjacent, then this vertex has to have valence 4. So this vertex corresponds to a square cell of $[S-T]$, and the two faces correspond to opposite corners of the square. The configuration in condition $\mathfrak{m}5$ has a third face, adjacent to both previous two, so the corresponding vertex in $[S-T]$ is connected to these corners. In S this square is subdivided by the diagonals and, since S does not contain empty 4–circuits, one of the remaining corners of the square in S must be connected to the vertex corresponding to the third face. This means that S contains a configuration pictured in Lemma 10.3.2, which according to that lemma is impossible. □

10.3.2 (*Remark*) Thurston gives a proof of Andreev's Theorem in [38]. Hypothesis (ii) does not occur in his statement of the result. The reason is that Thurston's statement is in terms of finding a collection of half-spaces in $\mathbb{H}^3$ with nonempty intersection such that their supporting planes intersect in the prescribed combinatorial pattern with the prescribed dihedral angles. When all the dihedral angles are strictly less than $\frac{\pi}{2}$, he shows that the intersection of half-spaces is a (possibly ideal) polytope. However, when some of the angles $= \frac{\pi}{2}$, the intersection can degenerate to a lower dimensional set. In the case of interest, all the angles are $\frac{\pi}{2}$. It is easy to see that when the intersection is a planar set, S must be the suspension of a 5–gon and similarly, when it is 0– or 1–dimensional, that S is the suspension of a 4–gon.

10.4 I(3) is true

Theorem 10.4.1 *Let S be a triangulation of the 2–sphere as a flag complex. Then*

$$\beta_i(S) = 0 \quad \text{for all } i.$$

Proof If S is the suspension of a 4– or 5–gon, then the theorem follows from Lemma 7.3.5. If S is not the suspension of a 4–gon or a 5–gon and if it has no empty 4–circuits, then by 10.1.3, 10.1.4 and 10.3.1, $\mathfrak{h}_i(S-T)$ vanishes for all i, where T denotes the set of valence 4 vertices. Hence, by Lemmas 10.2.3 and 10.2.4, $\mathfrak{h}_i(S)$ also vanishes.

In every other case, S has an empty 4–circuit which we can use to decompose S as, $S = S_1 \Box S_2$, as in 10.2.6. Since S_1 and S_2 each have fewer vertices than does S, this process must eventually terminate. So, the theorem follows from Lemma 10.2.7. □

11 Some consequences

11.1 V(3) and V(4) are true Since $\mathbf{I}(3)$ is true, Theorem 9.3.1 (together with 8.8.5) yields the following.

Theorem 11.1.1 *Statement $\mathbf{V}(n)$ (from 8.5) is true for $n \leq 4$.*

11.2 4–dimensional consequences Since $\mathbf{V}(4)$ implies $\mathbf{I}(4)$ (by 8.8.7), Singer's Conjecture holds for Σ_S, where S is any flag triangulation of a rational homology 3–sphere, ie, $\beta_1(S) = \beta_3(S) = 0$. By Atiyah's Formula, this implies that $\chi^{\mathrm{orb}}(\Sigma_S/W_S) = \beta_2(S) \geq 0$, and hence, by 6.3.4, that the Flag Complex Conjecture 0.2 is true in dimension 3. We restate this as follows.

Theorem 11.2.1 (The Flag Complex Conjecture in dimension 3) *Let S be any triangulation of a rational homology 3–sphere as a flag complex. Then*

$$\sum_{i=-1}^{3} \left(-\frac{1}{2}\right)^{i+1} f_i(S) \geq 0,$$

where $f_i(S)$ denotes the number of i–simplices in S and where $f_{-1} = 1$.

As explained in [10], this implies the following 4–dimensional result.

Theorem 11.2.2 *The Euler Characteristic Conjecture 0.1 holds for all nonpositively curved, piecewise Euclidean 4–manifolds which are cellulated by regular Euclidean cubes. In other words, for any such 4–manifold M^4,*

$$\chi(M^4) \geq 0.$$

In fact, one only need require M^4 to be a rational homology 4–manifold (rather than a 4–manifold).

11.3 Higher dimensional consequences From Theorem 9.3.3 and Corollary 9.3.4, we get the following.

Theorem 11.3.1 *Suppose L is an $(n-1)$–dimensional flag complex, $n \geq 3$, with spherical links in codimensions ≤ 3. Then for any full subcomplex A of L*

$$\beta_n(A) = \beta_{n-1}(A) = 0$$

and

$$\beta_n(L, A) = \beta_{n-1}(L, A) = 0.$$

Theorem 11.3.2 *Suppose S is a GHS^{n-1}, $n \geq 3$. Then*

$$\beta_1(S) = \beta_{n-1}(S) = 0.$$

11.4 3–dimensional consequences We restate $\mathbf{V}(3)$ as follows.

Theorem 11.4.1 *Let A be a finite flag complex of dimension ≤ 2. Suppose A is planar (ie, it can be embedded as a subcomplex of the 2–sphere). Then*

$$\beta_2(A) = 0.$$

Proof By Lemma 7.3.3, we may assume that A is connected. Suppose that A is piecewise linearly embedded in $\mathbb{S}^2$. By introducing a new vertex in the interior of each complementary region and then coning off the boundary of each region, we obtain a flag triangulation S of the 2–sphere with A embedded as a full subcomplex. By $\mathbf{V}(3)$, $\beta_2(A) = 0$. □

11.4.2 (*Example*) The contrapositive of Theorem 11.4.1 states that if $\beta_2(A) \neq 0$, then A is not planar. Kuratowski's graph $K_{3,3}$ is defined to be $P_3 * P_3$, the join of 3 points with itself. By Lemma 7.3.4, $\beta_2(K_{3,3}) = \frac{1}{4}$. So, as suggested in 8.9, we have a complicated proof of the classical fact that $K_{3,3}$ is not planar.

Statement $\mathbf{V}(3)$ implies the following generalization of $\mathbf{II}(3)$.

Proposition 11.4.3 *Suppose A is a flag triangulation of a 2–sphere with $g+1$ holes. Let $S_0, \dots, S_g$ be the boundary components of A and set*

$$\alpha = \frac{1}{2}(\beta_1(S_0) + \cdots + \beta_1(S_g)).$$

Then

$$\beta_i(A) = \begin{cases} \alpha & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

If, in addition, ∂A is a full subcomplex of A, then

$$\beta_i(A, \partial A) = \begin{cases} g + \alpha & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}$$

Proof As in Theorem 11.3.1, embed A in the flag triangulation S of $\mathbb{S}^2$ obtained by introducing new vertex s_i for each boundary component S_i. Since $\mathfrak{h}_i(S)$ vanishes,

$$\mathfrak{h}_i(A) \cong \mathfrak{h}_{i+1}(S, A) \cong \mathfrak{h}_{i+1}(C_{s_0}S_0 \cup \cdots \cup C_{s_g}S_g, S_0 \cup \cdots \cup S_g).$$

A simple calculation using Lemma 7.3.7 and 7.3.3 gives that this is nonzero only for $i+1 = 2$ and that

$$\beta_2(C_{s_0}S_0 \cup \cdots \cup C_{s_g}S_g, S_0 \cup \cdots \cup S_g) = \frac{1}{2}(\beta_1(S_0) + \cdots + \beta_1(S_g)).$$

The first formula follows.

To prove the second, consider the pair $(S, C_{s_0}S_0 \cup \cdots \cup C_{s_g}S_g)$. By excision, its homology is isomorphic to that of $(A, \partial A)$. Hence,

$$\beta_{i+1}(A, \partial A) = \beta_i(C_{s_0}S_0 \cup \cdots \cup C_{s_g}S_g)$$

and by Lemma 7.3.7 and 7.3.3, the second term is nonzero only for $i = 1$, in which case,

$$\beta_1(C_{s_0}S_0 \cup \cdots \cup C_{s_g}S_g) = g + \alpha.$$

□

11.5 Surfaces of higher genus Suppose L_g is a triangulation of a closed orientable surface of genus g as a flag complex. In [1], Akita points out that $\chi^{\mathrm{orb}}(\Sigma_{L_g}/W_{L_g}) = g$. This, together with the calculation in Proposition 11.4.3, makes the following generalization of $\mathbf{I}(3)$ a very plausible conjecture.

Conjecture 11.5.1 $\beta_i(L_g) = 0$ *for* $i \neq 2$ *and* $\beta_2(L_g) = g$.

Akita also proves in [1] that if A is a 1–dimensional flag complex (ie, if it is a simplicial graph without any circuits of length 3) and if A embeds in an orientable surface of genus g, then $\chi^{\mathrm{orb}}(\Sigma_A/W_A) \leq g$. If the above conjecture holds, then the following analog of Theorem 11.4.1 gives a stronger result.

Proposition 11.5.2 *Assume Conjecture 11.5.1. If a finite flag complex A can be embedded as a subcomplex of an orientable surface of genus g, then* $\beta_2(A) \leq g$.

Proof As in the proof of Theorem 11.4.1, we can assume that A is a full subcomplex of some flag triangulation L of the orientable surface of genus g. By Lemma 9.2.2, $\beta_3(L, A) = 0$; hence, the map $\mathfrak{h}_2(A) \to \mathfrak{h}_2(L)$ is injective. Since we are assuming $\beta_2(L) = g$, the result follows. □

11.5.3 (*Example*) We repeat an example from [1]. Let $K_{m,n}$ denote the join of m points and n points (a complete bipartite graph). By 7.3.6,

$$\beta_2(K_{m,n}) = (\frac{m}{2} - 1)(\frac{n}{2} - 1) = \frac{(m-2)(n-2)}{4}.$$

By [29, Theorem 4.5.3] the minimal genus of a surface in which $K_{m,n}$ embeds is the least integer $\geq$ this number.

12 Reflection type covers

In this section we use the fact that ℓ^2–Betti numbers are multiplicative with respect to finite coverings (cf 3.3.3). In particular, in 12.3, we use this to show that Conjecture $\mathbf{III}(2k+1)$ is implied by Conjecture $\mathbf{IV}(2k+1)$ (that $\beta_{k+1}(D) = 0$ whenever D is a GHD^{2k}).

12.1 Reflection subgroups Let L be a finite flag complex. To simplify notation we write W, Σ and K for W_L, Σ_L and K_L, respectively. In this subsection we will state some basic facts about W and Σ. Most of the proofs will be left as exercises for the reader. (They are all straightforward adaptations of standard arguments from the theory of classical reflection groups, for example, as explained in [6].)

12.1.1 (*Reflections, walls, half-spaces*) An element of W is a *reflection* if it is conjugate to a fundamental generator, ie, to an element of $\mathcal{S}_0(L)$. Given a reflection r, the fixed set of r on Σ is denoted by $\Sigma(r)$ and called the *wall* associated to r. Each wall separates Σ into two pieces, called the *half-spaces* bounded by the wall. To be more explicit, for each reflection r, let $P_r = \{w \in W \mid \ell(rw) > \ell(w)\}$ (where $\ell(\ )$ denotes word length) and let $H(r)$ denote the union of the chambers wK, with $w \in P_r$. Then $H(r)$ is the half-space bounded by $\Sigma(r)$ which contains the fundamental chamber K. The other half-space is $rH(r)$.

12.1.2 (*Convexity and half-spaces*) Each half-space is geodesically convex. (The proof uses the fact that there is a distance decreasing retraction $\Sigma \to H(r)$ called the "folding map".) If C is any convex union of chambers, then it is the intersection of the half-spaces which contain it.

12.1.3 (*Supporting walls*) Suppose C is a convex union of chambers. A wall $\Sigma(r)$ is a *supporting wall* of C if (a) C is contained in one of the half-spaces bounded by $\Sigma(r)$ and (b) the intersection $C \cap \Sigma(r)$ is nonempty and is not contained in any other wall. Let $\mathrm{Supp}(C)$ denote the set of reflections r such that $\Sigma(r)$ is a supporting wall of C.

12.1.4 (*The subgroup generated by* $\mathrm{Supp}(C)$) For each $r \in \mathrm{Supp}(C)$ denote $C \cap \Sigma(r)$ by C_r and call it the *mirror* of C associated to r. Let $G = \langle \mathrm{Supp}(C) \rangle$ be the subgroup of W generated by $\mathrm{Supp}(C)$. Next we want to give a standard argument which shows that G is a Coxeter group and that C is a fundamental domain for the G–action on Σ. Let $\hat{G}$ be the group defined by the following presentation: there is a generator $\hat{r}$ for each $r \in \mathrm{Supp}(C)$ and there are relations, $\hat{r}^2 = 1$, for each $r \in \mathrm{Supp}(C)$ and $(\hat{r}_1\hat{r}_2)^2 = 1$, whenever $C_{r_1} \cap C_{r_2} \neq \emptyset$. Thus, $\hat{G}$ is a right-angled Coxeter group. Let $\theta : \hat{G} \to G$ be the epimorphism defined by $\theta(\hat{r}) = r$. Let $\mathcal{U}(\hat{G}, C) = (\hat{G} \times C)/\sim$, where $\sim$ denotes the equivalence relation generated by $(\hat{g}, x) \sim (\hat{g}\hat{r}, x)$ whenever $x \in C_r$. Let $[\hat{g}, x]$ denote the image of $(\hat{g}, x)$ in $\mathcal{U}(\hat{G}, C)$. The group G acts naturally on $\mathcal{U}(\hat{G}, C)$. For each $x \in C$, let G_x (resp. $\hat{G}_x$) denote the subgroup of G (resp. $\hat{G}$) generated by the reflections across the mirrors of C which contain x and let U_x be an open neighborhood of x in C which intersects only those mirrors which contain x. Let $\mathcal{U}(\hat{G}_x, U_x)$ denote the image of $\hat{G}_x \times U_x$ in $\mathcal{U}(\hat{G}, C)$. Then G_xU_x is an open neighborhood of x in Σ, $\mathcal{U}(\hat{G}_x, U_x)$ is an open neighborhood of $[1, x]$ in $\mathcal{U}(\hat{G}, C)$ and both G_x and $\hat{G}_x$ are isomorphic to $(\mathbb{Z}_2)^m$ where m is the number of mirrors containing x. Let $f : \mathcal{U}(\hat{G}, C) \to \Sigma$ denotes the θ–equivariant map defined by $f([\hat{g}, x]) = \theta(\hat{g})x$. Using the fact that W is right-angled, it can be seen that U_x is a fundamental domain for the G_x–action on G_xU_x. It follows from this that f maps $\mathcal{U}(\hat{G}_x, U_x)$ homeomorphically onto G_xU_x and consequently, that f is a covering projection. Since Σ is simply connected, this implies that f is a homeomorphism and that θ is an isomorphism. Thus, G is a right-angled Coxeter group, C is a fundamental domain and $\mathrm{Supp}(C)$ is a fundamental set of generators.

The nerve of $\langle \mathrm{Supp}(C) \rangle$ (cf 5.1) is the flag complex $L(C)$ which can be defined as follows. The vertex set of $L(C)$ is $\mathrm{Supp}(C)$ and two distinct vertices r_1 and r_2 span an edge if and only if $(r_1r_2)^2 = 1$ (a flag complex is determined by its 1-skeleton). Thus, $\langle \mathrm{Supp}(C) \rangle \cong W_{L(C)}$.

12.1.5 Suppose W_A is a special subgroup of W. Then W_AK is a convex union of chambers. The corresponding subgroup $W_{L(W_AK)}$ can be identified with the kernel of the homomorphism $\varphi_A \colon W \to W_A$, defined by specifying its values on the generating set $\mathcal{S}_0(L)$ as follows:

$$\varphi_A(s) = \begin{cases} s & \text{if } s \in \mathcal{S}_0(A), \\ 1 & \text{if } s \notin \mathcal{S}_0(A). \end{cases}$$

We note that this kernel is of finite index in W if and only if A is a simplex.

12.1.6 (*Doubling along a vertex*) Suppose σ is a simplex of L. Denote the corresponding flag complex $L(W_\sigma K)$ by $d_\sigma L$. Thus, $W_{d_\sigma L}$ is a normal subgroup of index $2^{\dim\sigma+1}$ in W. The special case where σ is a vertex v will be denoted $d_v L$ and called the *double of* L *along* v.

12.1.7 (*Description of* $d_v L$) For each vertex s of $L-v$, we get two supporting walls of $W_v K$, namely, $\Sigma(s)$ and $v\Sigma(s) = \Sigma(vsv^{-1})$. When $s \in L_v$, $vsv^{-1} = s$ and these two walls coincide. Hence, $d_v L$ is formed by taking two copies of $L - v$ and gluing them together along the subcomplex L_v.

12.1.8 (*Iterated doubles*) Suppose s_1 and s_2 are two vertices of L which are not connected by an edge. For each positive integer N, let F_N be the set of the first N elements in the list $1, s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_2, \ldots$ (F_N is a subset of the infinite dihedral group generated by s_1 and s_2.) Then $F_N K$ is a convex union of chambers. The corresponding flag complex $d^N L$ is the N-fold *iterated double* of L along (s_1, s_2). By 12.1.4, $F_N K$ is a fundamental domain for the $W_{d^N L}$-action; hence, the subgroup $W_{d^N L}$ is of index N in W.

12.1.9 Suppose that $L = S$, a GHS^{n-1} and that C is a convex union of a finite number of chambers in Σ_S. Then C is contractible (since it is CAT(0)) and hence, a generalized homology n-disk. It follows that the flag complex $L(C)$ (which is "dual" to the boundary of C) is also a GHS^{n-1}.

12.2 Inequalities In this subsection we return to the situation of statement **III**$(2k+1)$ in 8.3: (D, S_v) is a GHD^{2k} and $S = D \cup CS_v$ is the generalized homology $2k$-sphere obtained by adjoining the cone on the boundary. (The cone point is v.) Set

$$\alpha_{k+1} = \dim_{W_S}(\overline{\mathrm{Im}}(i_*\colon \mathfrak{h}_{k+1}(D) \to \mathfrak{h}_{k+1}(S))).$$

By excision and Lemma 7.3.7 (1), $\beta_{k+1}(S,D) = \beta_{k+1}(CS_v, S_v) = \frac{1}{2}\beta_k(S_v)$. Hence, the sequence of the pair (S,D) gives the following inequality.

12.2.1 $\quad 0 \le \beta_{k+1}(S) - \alpha_{k+1} \le \frac{1}{2}\beta_k(S_v)$.

Next suppose $\beta_{k+1}(S_v) = 0$. (For example, this holds if $\mathbf{I}(2k)$ holds for the link S_v.) Then since $\beta_{k+2}(S, D) = \frac{1}{2}\beta_{k+1}(S_v) = 0$, the map $i_*\colon \mathfrak{h}_{k+1}(D) \to \mathfrak{h}_{k+1}(S)$ is injective and $\alpha_{k+1} = \beta_{k+1}(D)$. So, 12.2.1, can be rewritten as:

12.2.2 $\quad 0 \le \beta_{k+1}(S) - \beta_{k+1}(D) \le \frac{1}{2}\beta_k(S_v)$.

The next lemma shows that this inequality can be improved by a factor of 2.

Lemma 12.2.3 *Suppose, as above, that (D, S_v) is a GHD^{2k} and that $\beta_{k+1}(S_v) = 0$. Then*

$$\beta_{k+1}(S) - \beta_{k+1}(D) \le \frac{1}{4}\beta_k(S_v).$$

Proof By 12.1.9, the double of S, d_vS, is also a GHS^{2k}. By 12.1.7, d_vS is the union of two copies of D glued along S_v. So, we have a Mayer-Vietoris sequence,

$$0 \to \mathfrak{h}_{k+1}(D) \oplus \mathfrak{h}_{k+1}(D) \to \mathfrak{h}_{k+1}(d_vS) \xrightarrow{\partial} \mathfrak{h}_k(S_v) \to \mathfrak{h}_k(D) \oplus \mathfrak{h}_k(D).$$

By Lemma 7.4.4, the kernel of the map $\mathfrak{h}_k(S_v) \to \mathfrak{h}_k(D)$ into either factor has dimension $\frac{1}{2}\beta_k(S_v)$. Thus, the kernel of the map $\mathfrak{h}_k(S_v) \to \mathfrak{h}_k(D) \oplus \mathfrak{h}_k(D)$ has dimension $\le \frac{1}{2}\beta_k(S_v)$. Hence,

$$\beta_{k+1}(d_vS) \le 2\beta_{k+1}(D) + \frac{1}{2}\beta_k(S_v).$$

Substituting in $2\beta_{k+1}(S)$ for $\beta_{k+1}(d_vS)$ (by 12.1.6 and 3.3.3), we get the desired inequality. $\square$

12.3 $\mathbf{IV}(2k+1) \implies \mathbf{III}(2k+1)$ Suppose that $\mathbf{IV}(2k+1)$ is true (ie, that $\beta_{k+1}(D') = 0$ for any generalized homology $2k$-disk D'). Then the inequality 12.2.1 becomes,

12.3.1 $\quad \beta_{k+1}(S) \le \frac{1}{2}\beta_k(S_v)$.

As we shall see below, this inequality forces $\beta_{k+1}(S) = 0$. Since $\mathfrak{h}_{k+1}(S)$ is the previous term for the map in the Mayer-Vietoris sequence which is under consideration in $\mathbf{III}(2k+1)$, the next lemma shows that $\mathbf{IV}(2k+1)$ implies $\mathbf{III}(2k+1)$.

Lemma 12.3.2 *As in* $\mathbf{III}(2k+1)$, *let* (D, S_v) *be a* GHD^{2k} *and let* $S = D \cup CS_v$ *be the* GHS^{2k} *formed by adjoining a cone on the boundary. Assume* $\mathbf{IV}(2k+1)$ *is true. Then*

$$\beta_{k+1}(S) = 0.$$

Proof Case 1 Suppose $D - S_v$ is not a simplex. Then we can find vertices s_1, s_2 in $D - S_v$ which are not connected by an edge. Let $\widetilde{S}$ be the N–fold iterated double $d^N S$ along (s_1, s_2), as defined in 12.1.8. Then v has N preimages in $\widetilde{S}$ and the link of each is isomorphic to S_v. Choose one, say v_1, and set $\widetilde{D} = \widetilde{S} - v_1$. Since we are assuming $\mathbf{IV}(2k+1)$, we have, by 12.3.1, that $\beta_{k+1}(\widetilde{S}) \leq \frac{1}{2}\beta_k(S_{v_1})$. By 3.3.3, $\beta_{k+1}(\widetilde{S}) = N\beta_{k+1}(S)$. Hence,

$$\beta_{k+1}(S) \leq \frac{1}{2N}\beta_k(S_v).$$

Since this holds for any N, $\beta_{k+1}(S) = 0$.

Case 2 $D - S_v$ is a simplex σ. If $\dim \sigma = 0$, then S is a suspension and we are done by Lemma 7.3.5. If $\dim \sigma > 0$, then let $S' = d_\sigma S$ (defined in 12.1.6). By 3.3.3, $\beta_{k+1}(S') = m\beta_{k+1}(S)$ where $m = 2^{\dim \sigma + 1}$. Moreover, there are m preimages of v in S', no two of which are connected by an edge and such that the link of each is isomorphic to S_v. Choose one of these preimages, say v_1, and set $D' = S' - v_1$. Since D' contains $m-1$ preimages of v, $m - 1 \geq 2$, we can apply Case 1 to (D', S_v) to conclude that $0 = \beta_{k+1}(S') = m\beta_{k+1}(S)$. □

12.4 Atiyah's Conjecture In [4] Atiyah conjectured that ℓ^2–Betti numbers of any geometric G–complex X are rational numbers. A refinement of this states that if m denotes the least common multiple of the orders of the finite subgroups of G, then $mb_i^{(2)}(X, G)$ is an integer. An equivalent form (see [23]) of this conjecture is the following.

Conjecture 12.4.1 (Atiyah) *Suppose* $\phi\colon (\mathbb{Z}G)^p \to (\mathbb{Z}G)^q$ *is a homomorphism of free* $\mathbb{Z}G$*–modules and that* $\hat{\phi}\colon \ell^2(G)^p \to \ell^2(G)^q$ *is the induced map of Hilbert* G*–modules. As above, let* m *denote the least common multiple of the orders of the finite subgroups of* G, *and suppose that* m *is finite. Then*

$$m \dim_G(\operatorname{Ker} \hat{\phi}) \in \mathbb{N}.$$

12.4.2 The above conjecture implies that in the (weakly) exact sequence of any pair of geometric G–complexes, the von Neumann dimension of the kernel or image of any map is a nonnegative rational number with denominator dividing m.

12.4.3 If W_L is a right-angled Coxeter group, then $m = 2^{\dim L+1}$.

12.4.4 Taken together with Atiyah's Conjecture, Lemma 12.2.3 provides some convincing evidence for the truth of $\mathbf{III}(2k+1)$. Let S, S_v and D be as above and assume $\mathbf{I}(2k)$ holds for S_v. Then the largest possible denominator for $\beta_k(S_v)$ $(= (-1)^k\chi^{(2)}(S_v))$ is 2^{2k}. If Conjecture 12.4.1 holds for W_S, then the largest possible denominator for $\beta_{k+1}(S) - \beta_{k+1}(D)$ is 2^{2k+1}. So, if $\beta_k(S_v)$ has the smallest possible nonzero value, namely $(1/2)^{2k}$, and if Conjecture 12.4.1 is true, then Lemma 12.2.3 implies that $\beta_{k+1}(S) = \beta_{k+1}(D)$. This implies that in the Mayer–Vietoris sequence for $S = D \cup CS_v$, the map $\mathfrak{h}_{k+1}(D) \oplus \mathfrak{h}_{k+1}(CS_v) \to \mathfrak{h}_{k+1}(S)$ is surjective. and hence, that the map $\mathfrak{h}_k(S_v) \to \mathfrak{h}_k(D) \oplus \mathfrak{h}_k(CS_v)$ is injective, ie, that $\mathbf{III}(2k+1)$ holds for the pair (S, S_v).

13 Inclusions of walls

Let S be a flag triangulation of a generalized homology sphere of dimension $2k$. Suppose that s is a vertex of S and that S_s denote its link in S. By 8.3.2 and Theorem 9.3.1, our conjecture has been reduced to $\mathbf{III}'(2k+1)$, which asserts that the map $i_*\colon \mathfrak{h}_k(S_s) \to \mathfrak{h}_k(S)$, induced by inclusion, is zero.

In this section we shall make a series of observations about this problem. Our eventual point is made in 13.3: Conjecture $\mathbf{III}'(2k+1)$ is essentially equivalent to a certain estimate on the rate of growth of the norms of k–dimensional homology classes in the hypersurface Σ_{S_s} as they are pushed onto an "equidistant hypersurface".

To simplify notation set $\Sigma = \Sigma_S$ and $W = W_S$. We recall (from 7.1) that $\mathfrak{h}_k(S_s)$ stands for $\mathcal{H}_k(W\Sigma_{S_s})$, where Σ_{S_s} is the special subcomplex corresponding to S_s. We also note that Σ_{CS_s} can be identified with $\Sigma_{S_s} \times [-1, 1]$, where the wall $\Sigma(s)$ (defined in 12.1.1) corresponds to $\Sigma_{S_s} \times 0$. In particular, Σ_{S_s} is W_{S_s}–equivariantly homeomorphic to the wall $\Sigma(s)$.

13.1 Reduction to a single wall Since $W\Sigma_{S_s}$ is the disjoint union of copies of $\Sigma(s)$, one for each coset of W_{S_s} in W, the Hilbert space $\mathcal{H}_k(W\Sigma_{S_s})$ is an orthogonal sum of copies of $\mathcal{H}_k(\Sigma(s))$. Hence, to prove that $i_*\colon \mathcal{H}_k(W\Sigma_{S_s}) \to \mathcal{H}_k(\Sigma)$ is the zero map, it is necessary and sufficient to show that its restriction to one summand, $\mathcal{H}_k(\Sigma(s))$, is zero.

13.2 The map into unreduced homology The map $i_*\colon \mathcal{H}_k(\Sigma(s)) \to \mathcal{H}_k(\Sigma)$ factors as a composition $p \circ \widehat{i_*}$, where the map $\widehat{i_*}\colon \mathcal{H}_k(\Sigma(s)) \to H_k^{(2)}(\Sigma)$ is induced by the inclusion of the harmonic k–cycles into the ℓ^2–cycles on Σ and where the map $p\colon H_k^{(2)}(\Sigma) \to \mathcal{H}_k(\Sigma)$ is projection onto the harmonic cycles. (Recall, from 2.3.3, that $H_k^{(2)}(\ )$ denotes unreduced ℓ^2–homology.)

Lemma 13.2.1 *If $i_*\colon \mathcal{H}_k(\Sigma(s)) \to \mathcal{H}_k(\Sigma)$ is the zero map, then the map $\widehat{i_*}\colon \mathcal{H}_k(\Sigma(s)) \to H_k^{(2)}(\Sigma)$ is injective.*

Proof Suppose that x is a harmonic k–cycle in $\mathcal{H}_k(\Sigma(s))$ such that $\widehat{i_*}(x) = 0$ in $H_k^{(2)}(\Sigma)$. In other words, $x = d(y)$ for some $(k+1)$–chain y in $C_{k+1}(\Sigma)$. (We identify x with its image under the inclusion of chains $C_k(\Sigma(s)) \hookrightarrow C_k(\Sigma)$.) The wall $\Sigma(s)$ divides Σ into two half-spaces; let us call them Σ_+ and Σ_-. We first claim that we can find a $(k+1)$–chain $y' \in C_{k+1}(\Sigma)$ so that $x = d(y')$ and so that y' is supported on only one half-subspace, say Σ_+. To see this, first write $y = y_+ + y_-$, where y_+ (respectively, y_-) is supported on Σ_+ (respectively, Σ_-) and therefore, $d(y_+)$ and $d(y_-)$ are both supported on $\Sigma(s)$. Then set $y' = y_+ + sy_-$. Since s fixes $\Sigma(s)$,

$$d(y') = d(y_+) + sd(y_-) = d(y_+ + y_-) = d(y) = x.$$

Set $z = y' - sy'$. Then

$$d(z) = d(y') - sd(y') = 0,$$

so z is a $(k+1)$–cycle in $C_{k+1}(\Sigma)$. Let $\overline{z}$ denote its image in reduced ℓ^2–homology $\mathcal{H}_{k+1}(\Sigma)$.

Consider the Mayer–Vietoris sequence of $\Sigma = \Sigma_+ \cup \Sigma_-$ in unreduced ℓ^2–homology. Let $\partial\colon H_{k+1}^{(2)}(\Sigma) \to H_k^{(2)}(\Sigma(s))$ be the connecting homomorphism and let $\partial_*\colon \mathcal{H}_{k+1}(\Sigma) \to \mathcal{H}_k(\Sigma(s))$ be the induced map of quotients. It follows from the definition of ∂, that $\partial([z]) = [x]$ in unreduced homology and therefore, $\partial_*(\overline{z}) = x$, since x is harmonic. On the other hand, just as in 7.4.5, the map $i_*\colon \mathcal{H}_k(\Sigma(s)) \to \mathcal{H}_k(\Sigma)$ is isomorphic, under Poincaré duality to $\partial^*\colon \mathcal{H}_k(\Sigma(s)) \to \mathcal{H}_{k+1}(\Sigma)$. Hence, if i_* is the zero map, then so is ∂_*. Therefore, our hypothesis implies that $x = \partial_*(\overline{z}) = 0$ and consequently, that $\widehat{i_*}$ is injective. □

13.2.2 An alternative proof of this lemma can be constructed as follows. If $i_*\colon \mathcal{H}_k(\Sigma(s)) \to \mathcal{H}_k(\Sigma)$ is zero, then, by Theorem 9.3.1, $\mathcal{H}_{k+1}(\Sigma) = 0$. As before, suppose $\widehat{i_*}(x) = 0$ and define z as before. If $x \neq 0$, then we can find a $u \in \mathcal{H}_k(\Sigma(s))$ with nonzero intersection number with x. It is geometrically clear that the intersection number of u and $\overline{z}$ in Σ is the same. Hence, $u \neq 0$, contradicting $\mathcal{H}_k(\Sigma(s)) = 0$.

13.3 Equidistance hypersurfaces There is an infinite sequence of disjoint W_{S_s}–stable hypersurfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_n, \dots$ in Σ and W_{S_s}–equivariant homotopy equivalences $p_n\colon \Sigma_n \to \Sigma(s)$.

To define these, let X_n denote the union of all cells in the half-space Σ_+ of combinatorial distance $\le n$ from $\Sigma(s)$. (This definition is intentionally vague; there are several possible definitions of "combinatorial distance" and at least two possible cell structures on Σ — one is the cellulation by cubes and the other is the dual cellulation by chambers.) The boundary of X_n has two components, one is $\Sigma(s)$, the other is denoted by Σ_n.

There is a W_{S_s}–equivariant retraction of X_n onto $\Sigma(s)$. Its restriction to Σ_n is p_n. Since X_n is W_{S_s}–cocompact and W_{S_s}–homotopy equivalent to $\Sigma(s)$ we have that $i_* \circ p_{n*} = i_{n*}$ where $i_n\colon \Sigma_n \to \Sigma$ denotes the inclusion.

13.3.1 Given a cycle $x \in C_k(\Sigma(s))$ we can find a cycle $x_n \in C_k(\Sigma_n)$ with $p_{n*}(x_n) = x$ in reduced homology. So, x_n will be homologous to x in Σ. Therefore, any linear combination $a_1x_1 + \cdots + a_nx_n$ with $a_1 + \cdots + a_n = 1$ is also homologous to x in Σ. Let y_n denote a linear combination of the above form which has the minimal norm. Since the cycles x_i are supported on disjoint sets, they are mutually orthogonal in $C_k(\Sigma)$. Then an easy inductive argument shows that the norm of y_n is given by

$$\frac{1}{\|y_n\|^2} = \frac{1}{\|x_1\|^2} + \cdots + \frac{1}{\|x_n\|^2}.$$

Hence, if the series $\sum_{n=1}^{\infty} \frac{1}{\|x_n\|^2}$ is divergent, then $\lim_{n\to\infty}\|y_n\| = 0$ and therefore, $i_*(x) = 0$ (since there would be a sequence of cycles representing $i_*(x)$, with norms going to 0). For example, this argument works if $\|x_n\|^2$ grows sublinearly in n.

13.3.2 The cell structure on Σ_n can be obtained from that of Σ_{n-1} by a subdivision process which can be described by a regular procedure which depends only on the initial data. It follows that x_{n-1} can be pushed to Σ_n by a process which replaces each k–cell in x_{n-1} by a k–chain in Σ_n and that the maximum norm of this chain can be bounded above by a constant D which is independent of n. This gives $\|x_n\| \le D\|x_{n-1}\|$ and hence, that $\|x_n\| \le D^n\|x\|$, an estimate that is much worse than what we want. On the other hand, there are many possible choices for the "pushing procedure" of associating a k–chain to each k–cell. Roughly, the hope is that one can show that there are at least D such choices of disjoint k–chains. We could then choose x_n to be the average of D disjoint pushes of x_{n-1}, obtaining $\|x_n\| \le \|x\|$, the desired result.

13.4 Equidistance hypersurfaces in hyperbolic space In the case of hyperbolic $(2k+1)$–space $\mathbb{H}^{2k+1}$, the above argument can be made precise. Let $\mathbb{H}^{2k}$ be a totally geodesic hyperplane in $\mathbb{H}^{2k+1}$. We claim that the map $\mathcal{H}_k(\mathbb{H}^{2k}) \to \mathcal{H}_k(\mathbb{H}^{2k+1})$, induced by inclusion, is the zero map. This is of course true, since $\mathcal{H}_*(\mathbb{H}^{2k+1}) = 0$ by [21], but our proof below does not depend on that, and, in fact, can be used to give an alternative proof of Singer's Conjecture for hyperbolic space. Our argument uses L^2–de Rham cohomology theory and is dual to the argument in 13.3.1–13.3.2. We will show that the map $\mathcal{H}_k(\mathbb{H}^{2k+1}) \to \mathcal{H}_k(\mathbb{H}^{2k})$, induced by restriction of forms, is the zero map.

Let N_t be the hypersurface in $\mathbb{H}^{2k+1}$ consisting of the points of (oriented) distance t from $\mathbb{H}^{2k}$. Let $p_t\colon N_t \to \mathbb{H}^{2k}$ be the projection which takes a point in N_t to the closest point in $\mathbb{H}^{2k}$. Then p_t is a homothety. Let $\phi_t\colon \mathbb{H}^{2k} \to N_t$ be its inverse. Also, let $i\colon \mathbb{H}^{2k} \to \mathbb{H}^{2k+1}$ and $i_t\colon N_t \to \mathbb{H}^{2k+1}$ be the inclusions. Thus, i and $i_t \circ \phi_t$ are properly homotopic.

Let ω be a closed L^2-k–form on $\mathbb{H}^{2k+1}$. We claim that the restriction $i^*(\omega)$ of ω to $\mathbb{H}^{2k}$ represents the zero class in reduced L^2–cohomology. Indeed, suppose $[i^*(\omega)] \neq 0$. Then $\|i^*(\omega)\| \geq \|[i^*(\omega)]\| \geq 0$, where $\|[i^*(\omega)]\|$ denotes the norm of the harmonic representative of the cohomology class $[i^*(\omega)]$. Since ϕ_t is a conformal diffeomorphism, it follows that it preserves norms of middle-dimensional forms: $\|\phi_t^*(i_t^*(\omega)\| = \|i_t^*(\omega)\|$. Since i and $i_t \circ \phi_t$ are properly homotopic, $[\phi_t^*(i_t^*(\omega)] = [i^*(\omega)]$, so it follows that $\|i_t^*(\omega)\| \geq \|[i^*(\omega)]\|$. Now, since $i_t^*(\omega)$ is just a restriction of ω, we have a pointwise inequality $\|\omega\|_x \geq \|i_t^*(\omega)\|_x$. Therefore, using Fubini's Theorem, we obtain

$$\|\omega\|^2 = \int_{\mathbb{H}^{2k+1}} \|\omega\|_x^2 \, dV = \int_{\mathbb{R}} \int_{N_t} \|\omega\|_x^2 \, dA \, dt \geq$$
$$\int_{\mathbb{R}} \int_{N_t} \|i_t^*(\omega)\|_x^2 \, dA \, dt = \int_{\mathbb{R}} \|i_t^*(\omega)\|^2 \, dt \geq \int_{\mathbb{R}} \|[i^*(\omega)]\|^2 \, dt = \infty.$$

This contradicts our assumption that ω is L_2–form and completes the proof.

14 Virtual fibrations over the circle

In this section we discuss some ideas for another possible attack on the conjecture that $\mathbf{I}(2k+1)$ is true.

14.1 Another conjecture As in 1.1 suppose that X is a simply connected geometric G–complex. The orbihedron X/G *virtually fibers over* $\mathbb{S}^1$ if there is a subgroup Γ of finite index in G such that Γ acts freely on X and such that X/Γ fibers over $\mathbb{S}^1$.

Theorem 14.1.1 (Lück [33]) *If a finite complex fibers over $\mathbb{S}^1$, then the reduced ℓ^2–homology of its universal cover vanishes in all dimensions.*

14.1.2 It follows that if X is simply connected and if X/G virtually fibers over $\mathbb{S}^1$, then $\mathcal{H}_i(X)=0$ for all i.

14.1.3 There is an obvious obstruction for X/G to virtually fiber over $\mathbb{S}^1$: its orbihedral Euler characteristic must vanish. We note, however, that if X/G is a closed odd-dimensional orbifold, then this obstruction always vanishes.

In the late 1970's Thurston asked if the following conjecture were true.

Conjecture 14.1.4 (Thurston) *Let M^3 be a closed irreducible 3–manifold (or developable orbifold) with infinite fundamental group. Then M^3 virtually fibers over $\mathbb{S}^1$.*

From now on we suppose that S is a smooth triangulation of $\mathbb{S}^{n-1}$ as a flag complex. As usual, to simplify notation, we set $W = W_S$, $K = K_S$ and $\Sigma = \Sigma_S$. Then Σ/W can be given the structure of a smooth n–dimensional orbifold. In the following conjecture we shall also assume that n is odd.

Conjecture 14.1.5 *Suppose S is a smooth triangulation of $\mathbb{S}^{2k}$ as a flag complex. Then Σ/W virtually fibers over $\mathbb{S}^1$.*

14.1.6 By Lück's Theorem, this conjecture implies $\mathbf{I}(2k+1)$ (at least in the case where S is a sphere rather than just a generalized homology sphere).

There are reasons for believing that, in odd dimensions, the orbifolds Σ/W should be viewed as being analogous to 3–manifolds. One such reason is the following. Suppose that M^n is a manifold covering of Σ/W corresponding to a normal torsion-free subgroup Γ of finite index in W. Let $\Sigma(r)$ be a wall of Σ and let N^{n-1} denote its image in M^n. Since $\Sigma(r)$ is the fixed point set of an isometric reflection on Σ it is a geodesically convex subset. A well-known argument of [34] then shows that N^{n-1} is a totally geodesic hypersurface in M^n. (This argument goes as follows. Let H denote the centralizer of r in Γ. Suppose that for some $\gamma \in \Gamma$ there is a point x in $\Sigma(r) \cap \gamma\Sigma(r)$. The element $r\gamma r\gamma^{-1}$ fixes x. Since Γ is normal, $r\gamma r\gamma^{-1} \in \Gamma$ and since Γ is torsion-free, $r\gamma r\gamma^{-1} = 1$. Consequently, $\gamma \in H$. Therefore, $\Sigma(r)/H = N$ and $\Sigma(r) \to N$

is a covering projection.) In particular, N is aspherical, $\pi_1(N) = H$ and the induced homomorphism $\pi_1(N) \to \pi_1(M)$ is injective. The fact that M has many such "incompressible hypersurfaces" N means that M is a higher-dimensional analog of a Haken 3–manifold.

We will discuss below two ideas for attacking Conjecture 14.1.5. The first idea is to find a nowhere-zero, closed 1–form on M. It is discussed in subsections 14.2 and 14.3. The second idea is to find an incompressible hypersurface F in M to serve as the fiber of a fibration over $\mathbb{S}^1$. It is described in subsections 14.4 to 14.7.

14.2 Nowhere-zero closed 1–forms It is well-known that a smooth closed manifold M^n fibers over $\mathbb{S}^1$ if and only if it admits a nowhere-zero closed 1–form. Indeed, if $p\colon M^n \to \mathbb{S}^1$ is a smooth submersion, then $p^*(d\theta)$ is such a 1–form. Conversely, suppose ω is a nowhere-zero closed 1–form on M^n. After adding a closed 1–form (of small pointwise norm) to ω we may assume that, in addition to being nowhere-zero, ω has rational periods. In other words, we may assume that its cohomology class $[\omega]$ actually lies in $H^1(M^n;\mathbb{Q})$. Then after replacing ω by a suitable multiple, we may assume that it has integral periods, ie, that $[\omega]$ lies in the image of $H^1(M^n;\mathbb{Z})$ in $H^1(M^n;\mathbb{R})$. The cohomology class $[\omega]$ then defines a homomorphism $\phi_{[\omega]}\colon \pi_1(M^n) \to \mathbb{Z}$. Finally, after choosing a basepoint, integration of ω along paths yields a submersion $p_\omega\colon M^n \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$, with $p_\omega^*(d\theta) = \omega$.

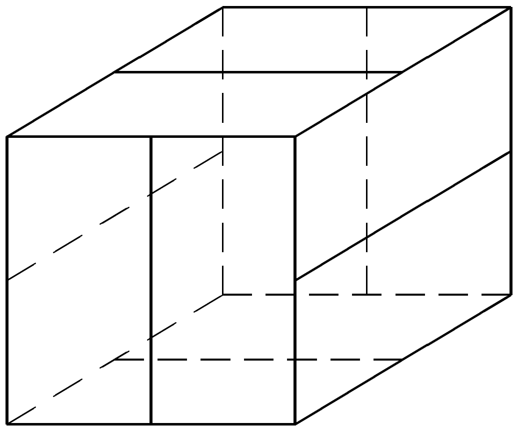
14.2.1 (*Remark*) In [25] Farber gives a direct argument for showing that if M^n admits a nowhere-zero closed 1–form, then the reduced ℓ^2–homology of its universal cover vanishes. This gives another proof of Lück's Theorem in the smooth case.

14.2.2 (*Example*) Suppose S is the boundary of an n–dimensional octahedron. Then K is an n–cube, $W = (D_\infty)^n$ and $\Sigma = \mathbb{R}^n$. The commutator cover of Σ/W is an n–torus T^n, which, of course, fibers over $\mathbb{S}^1$. (Any nonzero element of $H^1(T^n;\mathbb{R})$ can be represented by a linear 1–form, which is never zero.) So, in this case, Σ/W virtually fibers over $\mathbb{S}^1$.

14.3 Double branched covers Suppose $\pi\colon M_1^n \to M_2^n$ is a smooth branched covering, branched along a codimension–2 submanifold $B^{n-2} \subset M_2^n$. At any point $x \notin \pi^{-1}(B)$, the differential $d\pi_x\colon T_x(M_1^n) \to T_x(M_2^n)$ is an isomorphism. On the other hand, for $x \in \pi^{-1}(B)$, $d\pi_x$ maps $T_x(\pi^{-1}(B))$ isomorphically onto $T_x(B)$ and it maps a complementary 2–plane to 0.

Now suppose that M_2^n admits a nowhere-zero closed 1–form ω such that the restriction of ω to the tangent bundle of B is also nowhere-zero. Then $\pi^*(\omega)$ is a nowhere-zero closed 1–form on M_1^n. Another way to say essentially the same thing is the following. If $p\colon M_2^n \to \mathbb{S}^1$ is a smooth fibration and if the branch set B is never tangent to the fibers, then $p \circ \pi\colon M_1^n \to \mathbb{S}^1$ is also a smooth fibration.

14.3.1 (*An example of Thurston*) The following example of Thurston is explained in [36]. Let S and S' be the triangulations of $\mathbb{S}^2$ as a boundary of an octahedron and an icosahedron, respectively. Then K_S is a cube and $K_{S'}$ is a dodecahedron. Drawing the dodecahedron as below, shows that there is a map of orbifolds from $K_{S'}$ to K_S.



Take an 8–fold cover of K_S to get the 3–torus T^3. The induced covering of $K_{S'}$ is still topologically a torus, but orbifold singularities remain along six circles (which are the inverse images of the edges which were introduced in the subdivision of the cube above). These circles are parallel to the coordinate circles T^3. Let $\pi\colon M^3 \to T^3$ be the 2–fold branched covering of T^3, branched along these circles. (By Andreev's Theorem M^3 can be given the structure of the hyperbolic 3–manifold.) Now choose a linear fibration $p\colon T^3 \to \mathbb{S}^1$ such that the coordinate circles are transverse to the fibers. Then $p \circ \pi\colon M^3 \to \mathbb{S}^1$ is a fibration. In other words, $\Sigma_{S'}/W_{S'}$ virtually fibers over $\mathbb{S}^1$.

14.3.2 In the above example we started by subdividing each codimension-one face of of a cube. The process of dividing a codimension-one face of K in two is dual to the "edge subdivision process" of Section 5.3 in [10], applied to an appropriate edge of S. So, the method of the above example shows that we can construct many further examples of right-angled Coxeter orbifolds (in any dimension n) which virtually fiber over $\mathbb{S}^1$ by using the following three steps.

(1) Start with a triangulation S so that some finite cover M^n of Σ/W fibers over $\mathbb{S}^1$ (or admits a nowhere-zero closed 1–form ω).

(2) Subdivide an edge of S to obtain a new triangulation S' and a double branched cover $\pi\colon M' \to M$.

(3) If necessary, perturb the 1–form ω on M so that it is transverse to the branch set. If this is possible, then $\pi^*(\omega)$ will be a nowhere-zero closed 1–form on M'.

14.4 Finding a fiber Suppose F^{n-1} is a hypersurface in M^n. Let $\widehat{M}(F)$ denote the result of cutting open M along F. If $\widehat{M}(F)$ is homeomorphic to $F\times[0,1]$, then M fibers over $\mathbb{S}^1$ with fiber F. (Proof: M is obtained by gluing together two ends of $F\times[0,1]$, ie, it is the mapping torus of a homeomorphism.)

There are some obvious necessary conditions on F for it to be a fiber. For example, its Betti numbers must be fairly large. Indeed, by the Wang sequence, we have the inequality: $b_i(F) + b_{i-1}(F) \geq b_i(M)$. Furthermore, if M is aspherical, then F must be also aspherical and the induced homomorphism $\pi_1(F) \to \pi_1(M)$ must be an injection onto a normal subgroup.

Suppose that $p\colon M \to \mathbb{S}^1$ is a fibration with fiber F and with fibration 1–form $\omega = p^*(d\theta)$. Assume that M is oriented. Give F the induced orientation and let $[F]$ denote the image of its fundamental class in $H_{n-1}(M^n;\mathbb{R})$. Then $[F]$ is Poincaré dual to $[\omega]$.

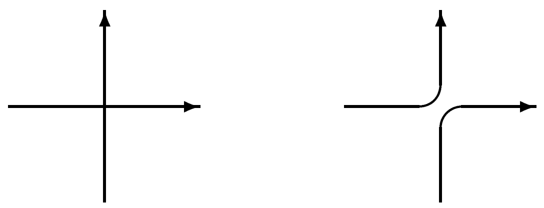
If ω is any nowhere-zero 1–form, then $\operatorname{Ker}\omega$ is an oriented $(n-1)$–dimensional subbundle of the tangent bundle of M^n. Let $e_\omega \in H^{n-1}(M^n;\mathbb{Z})$ denote the Euler class of this bundle. When ω is the 1–form of a fibration, we clearly have that $e_\omega([F]) = \chi(F)$.

14.5 Cutting and pasting hypersurfaces In this subsection we shall discuss a procedure for amalgamating oriented hypersurfaces in an arbitrary closed oriented n–manifold M^n.

14.5.1 (*Switching sheets*) Suppose N_1^{n-1} and N_2^{n-1} are oriented hypersurfaces in M^n intersecting transversely. As in [37] we can associate to this situation a new oriented hypersurface N_{12} in M with the following two properties:

(1) $[N_{12}] = [N_1] + [N_2]$ and
(2) $\chi(N_{12}) = \chi(N_1) + \chi(N_2)$.

The procedure can be described as follows. A neighborhood of $N_1 \cap N_2$ in $N_1 \cup N_2$ is homeomorphic to the product of $N_1 \cap N_2$ with the cone over 4 points. Replace this neighborhood by two copies of $(N_1 \cap N_2) \times [-1,1]$ by gluing each side of $N_1 \cap N_2$ in N_1 to the appropriate side of $N_1 \cap N_2$ in N_2, as indicated in the picture below.



$N_1 \cup N_2$ $\qquad$ N_{12}

After a small perturbation we may assume that the result is still embedded in M.

We note that in this procedure each point of $N_1 \cap N_2$ is replaced by two points in N_{12}.

Since N_{12} is just a small perturbation of the $(n-1)$–cycle $N_1 + N_2$, property 1 holds. If n is even, then property 2 is automatic (since the Euler characteristic of an odd-dimensional manifold vanishes). If n is odd, then the codimension–2 submanifold $N_1 \cap N_2$ has Euler characteristic 0. Hence, $\chi(N_1 \cup N_2) = \chi(N_1) + \chi(N_2) = \chi(N_{12})$ so property 2 holds.

14.5.2 (*Iterating this procedure*) Next suppose $N_1, \ldots, N_m$ are oriented hypersurfaces in general position in M. We can assume that N_3 and N_{12} are in general position. Define N_{123} to be the result of applying the switching sheets procedure to N_{12} and N_3. Continuing in this fashion, we eventually arrive to an embedded hypersurface $N_{12\ldots m}$. As before,

(1) $[N_{12\ldots m}] = \sum [N_i]$ and
(2) $\chi(N_{12\ldots m}) = \sum \chi(N_i)$.

We note that in this procedure each point of $N_1 \cup \cdots \cup N_m$ which lies in a j–fold intersection is blown up into j points in $N_{12\ldots m}$.

In the next subsection we shall give another description of this procedure which is independent of the ordering of the N_i. To give this description it suffices to consider the local model.

14.6 The local model For each i, $1 \leq i \leq n$, let P_i denote the coordinate hyperplane in $\mathbb{R}^n$ defined by $x_i = 0$. Orient P_i by requiring its unit normal vector e_i to be positively oriented.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be any function from $\{1, \ldots, n\}$ to $\{-1, 0, +1\}$. Let $z(\lambda)$ denote the number of zeroes in $(\lambda_1, \ldots, \lambda_n)$ and $n(\lambda)$ the number of (-1)'s.

The *quadrant* Q_λ corresponding to λ is the subset of $\mathbb{R}^n$ defined by

$$Q_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \;|\lambda_i x_i \geq 0 \text{ if } \lambda_i \neq 0;\; x_j = 0 \text{ if } \lambda_j = 0\}.$$

Clearly, Q_λ is isomorphic to the cone on a simplex of dimension $n - z(\lambda) - 1$. It is a manifold with corners of codimension $z(\lambda)$ in $\mathbb{R}^n$.

Each hyperplane P_i is divided into 2^{n-1} $(n-1)$–dimensional quadrants. Thus, in total there are $n2^{n-1}$ $(n-1)$–dimensional quadrants. We shall now reassemble these into n different sheets. For $l = 0, \ldots, n-1$, let $\mathcal{E}(l)$ denote the set of functions λ with $z(\lambda) = 1$ and $n(\lambda) = l$. Define

14.6.1 $$P(l) = \bigcup_{\lambda \in \mathcal{E}(l)} Q_\lambda.$$

As we shall see below, $P(l)$ is a piecewise differentiable submanifold of $\mathbb{R}^n$ which is homeomorphic to $\mathbb{R}^{n-1}$. Moreover, $P_{12\ldots n}$ can be identified with the disjoint union of the $P(l)$.

The whole arrangement of the Q_λ is isomorphic to the cone over the triangulation O^{n-1} of $\mathbb{S}^{n-1}$ as the boundary of the standard n–dimensional octahedron. The vertex set of O^{n-1} is $\{\pm e_i\}_{1 \leq i \leq n}$ and the simplex corresponding to λ is the spherical $(n - z(\lambda) - 1)$–simplex spanned by $\{\lambda_j e_j\}_{\lambda_j \neq 0}$.

Lemma 14.6.2 *Let O^{n-1} be the boundary of the n–dimensional octahedron. For $l = 0, \ldots, n-1$, let $B(l)$ denote the union of the $(n-1)$–simplices in O^{n-1} with $n(\lambda) \leq l$. Then $B(l)$ is a topological ball.*

Proof $B(0)$ is an $n-1$–simplex and $B(l)$ collapses onto $B(l-1)$. □

14.6.3 Clearly, $\partial B(l)$ is the union of the $(n-2)$–simplices in O^{n-1} with $n(\lambda) = l$. Hence, $P(l)$ (defined in 14.6.1) is homeomorphic to the cone on $\partial B(l)$. By the above lemma, $\partial B(l)$ is a triangulation of $\mathbb{S}^{n-2}$. Thus, $P(l)$ is homeomorphic to $\mathbb{R}^{n-1}$.

14.6.4 The $P(l)$ are not disjointly embedded (since $\bigcup P(l) = \bigcup P_i$). To remedy this we alter the embedding of $P(l)$ in $\mathbb{R}^n$ by a small isotopy as follows. Choose a decreasing sequence of real numbers $\mu_0 > \mu_1 > \cdots > \mu_{n-1}$. Let e be the vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$. Finally let $P'(l)$ be the subset of $\mathbb{R}^n$ defined by

$$P'(l) = P(l) + \mu_l e.$$

The $P'(l)$ are now disjointly embedded. This gives the desired local description of $P_{12\ldots n}$: it is the union of the $P'(l)$.

14.7 Potential fibers We return to our consideration of the orbifold Σ/W. Let Γ be a normal, torsion-free subgroup of finite index in W and set $M^n = \Sigma/\Gamma$. It follows from the assumptions that Γ is normal and torsion-free that, given any wall of Σ, its image in M^n is an embedded hypersurface N^{n-1}. We call such an N a *standard hypersurface* in M. A standard hypersurface is totally geodesic in the nonpositively curved cubical structure on M.

By passing to a deeper subgroup of finite index if necessary, we may assume that M is orientable and that each standard hypersurface is orientable.

If W does not split as product with an infinite dihedral group, then a standard hypersurface can never be the fiber of a fibration over $\mathbb{S}^1$. However, the cutting and pasting procedure of 14.5 applied to various collections of oriented standard hypersurfaces $\{N_1, \ldots, N_m\}$ yields a good source of candidates for fibers.

14.7.1 (*Fundamental domains for the commutator subgroup*) In this subsection Γ is the commutator subgroup of W and M^n is the commutator cover of Σ/W (see 6.4). The quotient group W/Γ is $(\mathbb{Z}_2)^{\mathcal{S}_0(S)}$ where $\mathcal{S}_0(S)$ denotes the vertex set of S. Order elements of $\mathcal{S}_0(S)$: s_1, s_2, $\ldots$, s_p. Next we shall inductively define an increasing sequence, $D(0) \subset \cdots \subset D(p)$, such that each $D(i)$ is a convex union of chambers and such that for all $j > i$, $\Sigma(s_j)$ is a supporting wall of $D(i)$. (See 12.1.)

Put $D(0) = K$. Assuming $D(i)$ has been defined for some $i < p$, define $D(i{+}1)$ to be the double of $D(i)$ along $\Sigma(s_{i+1})$. Set $D = D(p)$. We claim that D has the following properties:

(1) D is a convex union of chambers.

(2) D is a fundamental domain for the Γ–action on Σ.

(3) The codimension-one faces of D are identified in pairs by elements of Γ.

(4) The image of the boundary ∂D of D in M is a union of standard hypersurfaces $\{N_1, \ldots, N_m\}$.

Property 1 is immediate. To see property 2 first observe that any chamber in D has the form $s_{i_t} \cdots s_{i_1} K$ where $(i_1, \dots, i_t)$ is an increasing sequence of integers (possibly the empty sequence) in $[1,p]$. Since the group elements corresponding to such sequence map bijectively onto the quotient group $W/\Gamma = (\mathbb{Z}_2)^p$, we see that D is a fundamental domain.

Similarly, any supporting wall of D can be written in the form $s_{i_t} \cdots s_{i_1} \Sigma(s_j)$, where $(i_1, \dots, i_t)$ is a nonempty increasing sequence of integers in $[1,p]$ and $j \geq i_t$, and where $s_{i_t} \cdots s_{i_1}$ does not commute with s_j. In particular, consider the supporting walls $w\Sigma(s_j)$ and $s_j w \Sigma(s_j)$ where $w = s_{i_t} \cdots s_{i_1}$ and $j > i_t$. The element $s_j w s_j w^{-1}$ takes the first wall to the second and this element lies in the commutator subgroup Γ. Property 3 follows. The image of ∂D in M is the same as the image of the union of the supporting walls of D in M. Hence, property 4 holds.

14.7.2 Once again, Γ is an arbitrary normal, torsion-free subgroup of finite index in W such that M and the standard hypersurfaces are orientable. We further suppose that Γ has a fundamental domain satisfying properties 1 through 4 in 14.7.1. (By 14.7.1 such Γ exist.) Since D is convex, it is a disk. Let $N_1, \dots, N_m$ be the standard hypersurfaces coming from the supporting walls of D. We shall now describe an attempt to fiber M over $\mathbb{S}^1$ which sometimes works.

Let R be a regular neighborhood of $\bigcup N_i$ in M. Since $M - R$ can be identified with the complement of a collared neighborhood of ∂D in D, it is a disk. Thus, M is formed by attaching a n–disk to $\bigcup N_i$. Since M is orientable, the attaching map $\partial D \to \bigcup N_i$ is trivial on homology. Hence, for $j < n$, $H_j(M) \cong H_j(\bigcup N_i)$. Using Mayer–Vietoris sequences it is easy to see that we have an injection from $\bigoplus H_{n-1}(N_i)$ $(\cong \mathbb{Z}^m)$ into $H_{n-1}(\bigcup N_i)$. (In favorable circumstances this injection is an isomorphism.)

Now choose an orientation for each N_i and a positively oriented normal vector field v_i on N_i. Pulling this back to D we obtain a normal vector on each of its codimension-one faces. If two faces are identified by an element of Γ, then the vectors point in opposite directions. Call a codimension-one face *positive* (respectively, *negative*) if the normal vector is outward pointing (respectively, inward pointing). Let D_+ (respectively D_-) denote the union of the positive (respectively, negative) codimension-one faces. Thus, each choice of the orientations of the N_i leads to a partition of ∂D into positive and negative regions.

Proposition 14.7.3 *With hypothesis as above, suppose it is possible to choose orientations for N_i such that the positive region D_+ is a disk (of codimension 0) in ∂D. Then $N_{12\dots m}$ is the fiber of a fibration of M over $\mathbb{S}^1$.*

Proof Let $F = N_{12\dots m}$ and let $\widetilde{F}$ denote the inverse image of F in Σ. Then D_+ (or D_-) can be regarded as a fundamental domain for the Γ–action on $\widetilde{F}$. Let $\widehat{M}(F)$ denote the result of cutting M open along F. Take a component of the complement of $\widetilde{F}$ in Σ and let $\widetilde{M}$ be its closure. Then $\widetilde{M}$ is a covering space of $\widehat{M}(F)$ with group of covering transformations Γ'. Furthermore, D is a fundamental domain for the Γ'–action on $\widetilde{M}$. The only points of D that are identified under Γ' lie on the common boundary of D_+ and D_-. It follows that $\widehat{M}(F)$ is a quotient space of D by an equivalence relation $\sim$. Since by hypothesis, D is homeomorphic to $D_+ \times [0,1]$, it follows that the quotient space is homeomorphic to $(D_+/\sim) \times [0,1]$, where $(D_+/\sim) = F$. The proposition follows. □

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