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Manifolds with singularities accepting a metric of positive scalar curvature

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Abstract

We study the question of existence of a Riemannian metric of positive scalar curvature metric on manifolds with the Sullivan–Baas singularities. The manifolds we consider are *Spin* and simply connected. We prove an analogue of the Gromov–Lawson Conjecture for such manifolds in the case of particular type of singularities. We give an affirmative answer when such manifolds with singularities accept a metric of positive scalar curvature in terms of the index of the Dirac operator valued in the corresponding "K–theories with singularities". The key ideas are based on the construction due to Stolz, some stable homotopy theory, and the index theory for the Dirac operator applied to the manifolds with singularities. As a side-product we compute homotopy types of the corresponding classifying spectra.

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1 Introduction

1.1 Motivation It is well-known that the question of existence of positive scalar curvature metric is hard enough for regular manifolds. This question was studied extensively, and it is completely understood, see [9], [29], for simply connected manifolds and for manifolds with few particular fundamental groups, see [4], and also [23], [24] for a detailed discussion. At the same time, the central statement in this area, the Gromov–Lawson–Rosenberg Conjecture is known to be false for some particular manifolds, see [26]. To motivate our interest we first address a couple of naive questions. We shall consider manifolds with boundary, and we always assume that a metric on a manifold is product metric near its boundary. We use the abbreviation "psc" for "positive scalar curvature" throughout the paper.

Let (P, g_P) be a closed Riemannian manifold, where the metric g_P is not assumed to be of positive scalar curvature. Let X be a closed manifold, such that the product $X \times P$ is a boundary of a manifold Y.

Naive Question 1 *Does there exist a psc-metric g_X on X, so that the product metric $g_X \times g_P$ could be extended to a psc-metric g_Y on Y?*

Examples (1) Let $P = \langle k \rangle = \{k \text{ points}\}$, then a manifold Y with $\partial Y = X \times \langle k \rangle$ is called a $\mathbf{Z}/k$*–manifold.* When $k = 1$ (or $X = \partial Y$) the above question is essentially trivial. Say, if X and Y are simply connected *Spin* manifolds, and $\dim X = n - 1 \geq 5$, there is always a psc-metric g_X which could be extended to a psc-metric g_Y.

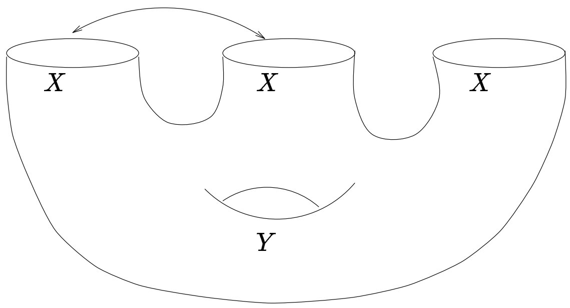


Figure 1: $\mathbf{Z}/k$–manifold

To see this one can delete a small open disk $D^n \subset Y$, and then push the standard metric on S^{n-1} through the cobordism $W = Y \setminus D^n$ to the manifold X using the surgery technique due to Gromov, Lawson [9] and Schoen, Yau [27].

(2) The case $P = \langle k \rangle$ with $k \geq 2$ is not as simple. For example, there are many simply connected *Spin* manifolds X of dimension $4k$ (for most k) which are not cobordant to zero, and, in the same time, two copies of X are. Let

$\partial Y = 2X$. It is not obvious that one can find a psc-metric g_X on X, so that the product metric $g_X \times \langle 2 \rangle$ extends to a psc-metric g_Y on Y.

(3) Let Σ^m (where $m = 8l+1$ or $8l+2$, and $l \geq 1$) be a homotopy sphere which does not admit a psc-metric, see [12]. We choose $k \geq 2$ disjoint discs $D_1^m, \dots, D_k^m \subset \Sigma^m$ and delete their interior. The resulting manifold Y^m has the boundary $S^{m-1} \times \langle k \rangle$. Clearly it is not possible to extend the standard metrics on the spheres $S^{m-1} \times \langle k \rangle$ to a psc-metric on the manifold Y since otherwise it would give a psc-metric on the original homotopy sphere Σ^m. However, it is not obvious that for *any choice of a psc-metric g on S^{m-1}* the metric $g \times \langle k \rangle$ could not be extended to a psc-metric on Y^m.

(4) Let P be again k points. Consider a Joyce manifold J^8 (*Spin*, simply connected, Ricci flat, with $\hat{A}(J^8) = 1$, and holonomy $Spin(7)$), see [16]. Delete k open disks $D_1^m, \dots, D_k^m \subset J^8$ to obtain a manifold M, with $\partial M = S^7 \times \langle k \rangle$. Let g_0 be the standard metric on S^7. Then clearly the metric $g_0 \times \langle k \rangle$ on the boundary $S^7 \times \langle k \rangle$ cannot be extended to a psc-metric on M since otherwise one would construct a psc-metric on J^8. However, there are so called "exotic" metrics on S^7 which are not in the same connective component as the standard metric. Nevertheless, as we shall see, there is no any psc-metric g' on S^7, so that the metric $g' \times \langle k \rangle$ could be extended to a psc-metric on M.

(5) Let $P = S^1$ with nontrivial *Spin* structure, so that $[P]$ is a generator of the cobordism group $\Omega_1^{Spin} = \mathbf{Z}/2$.

Let $d\theta^2$ be the standard metric on the circle. The analysis of the ring structure of Ω_*^{Spin} shows that there exist many examples of simply connected manifolds X which are not *Spin* cobordant to zero, however, the products $X \times P$ are, say $\partial Y = X \times P$.
Again, in general situation there is no obvious clue whether for some psc-metric g_X on X the product metric $g_X + d\theta^2$ on $X \times P$ could be extended to a psc-metric on Y or not.

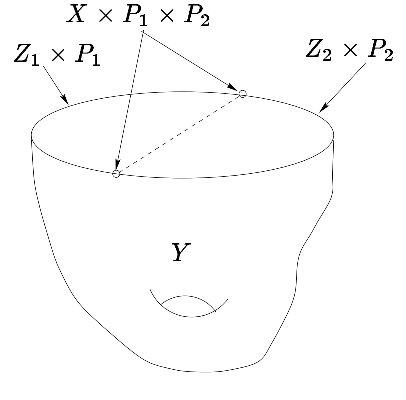


Figure 2

Now let (P_1, g_1), (P_2, g_2) be two closed Riemannian manifolds, again, the metrics g_1, g_2 are not assumed to be of positive scalar curvature. Let X be a closed manifold such that

- the product $X \times P_1$ is a boundary of a manifold Z_2,

- the product $X \times P_2$ is a boundary of a manifold Z_1,
- the manifold $Z = Z_1 \times P_1 \cup \epsilon Z_2 \times P_2$ is a boundary of a manifold Y (where ϵ is an appropriate sign if the manifolds are oriented), see Figure 2.

Naive Question 2 *Does there exist a psc-metric g_X on X, so that*

(a) *the product metric $g_X \times g_1$ on $X \times P_1$ could be extended to a psc-metric g_{Z_2} on Z_2,*

(b) *the product metric $g_X \times g_2$ on $X \times P_1$ could be extended to a psc-metric on g_{Z_1} Z_1,*

(c) *the metric $g_{Z_1} \times g_{P_1} \cup g_{Z_2} \times g_{P_2}$ on the manifold $Z = Z_1 \times P_1 \cup \epsilon Z_2 \times P_2$ could be extended to a psc-metric g_Y on Y?*

1.2 Manifolds with singularities Perhaps, one can recognize that the above naive questions are actually about the existence of a psc-metric on a manifold with the Baas–Sullivan singularities, see [28], [2]. In particular, a $\mathbf{Z}/k$–manifold M is a manifold with boundary ∂M diffeomorphic to the product $\beta M \times \langle k \rangle$. Then a metric g on M is a regular Riemannian metric on M such that it is product metric near the boundary, and its restriction on each two components $\beta M \times \{i\}$, $\beta M \times \{j\}$ are isometric via the above diffeomorphism. To get the singularity one has to identify the components $\beta M \times \{i\}$ with a single copy of βM. Similarly a Riemannian metric may be defined for the case of general singularities. We give details in Section 7.

Thus manifolds with the Baas–Sullivan singularities provide an adequate environment to reformulate the above naive question. Let $\Sigma = (P_1, \dots, P_q)$ be a collection of closed manifolds, and M be a Σ–manifold (or manifold with singularities of the type Σ), see [2], [19], [3] for definitions. For example, if $\Sigma = (P)$, where $P = \langle k \rangle$, a Σ–manifold M is $\mathbf{Z}/k$–manifold. Then the above questions lead to the following one:

Question *Under which conditions does a Σ–manifold M admit a psc-metric?*

Probably it is hard to claim anything useful for a manifold with arbitrary singularities. We restrict our attention to *Spin* simply connected manifolds and very particular singularities. Now we introduce necessary notation.

Let $\Omega_*^{Spin}(\cdot)$ be the *Spin*–cobordism theory, and $MSpin$ be the Thom spectrum classifying this theory. Let $\Omega_*^{Spin}(pt) = \Omega_*^{Spin}$ be the coefficient ring. Let $P_1 = \langle 2 \rangle = \{\text{two points}\}$, P_2 be a circle with a nontrivial *Spin* structure, so that $[P_2] = \eta \in \Omega_1^{Spin} \cong \mathbf{Z}/2$, and P_3, $[P_3] \in \Omega_8^{Spin}$, is a Bott manifold, ie, a simply-connected manifold such that $\widehat{A}(P_3) = 1$. There are different representatives of the Bott manifold P_3. Perhaps, the best choice is the Joyce manifold J^8, [16]. Let $\Sigma_1 = (P_1)$, $\Sigma_2 = (P_1, P_2)$, $\Sigma_3 = (P_1, P_2, P_3)$, and $\eta = (P_2)$. We denote by $\Omega_*^{Spin,\Sigma_i}(\cdot)$ the cobordism theory of *Spin*–manifolds with Σ_i–singularities, and by $MSpin^{\Sigma_i}$ the spectra classifying these theories, $i = 1, 2, 3$. We also study the theory $\Omega_*^{Spin,\eta}(\cdot)$, and the classifying spectrum for this theory is denoted as $MSpin^{\eta}$. We use notation Σ for the above singularities Σ_1, Σ_2, Σ_3 or η.

Let $KO_*(\cdot)$ be the periodic real K–theory, and KO be the classifying Ω–spectrum. The Atiyah–Bott–Shapiro homomorphism $\alpha\colon \Omega_*^{Spin} \longrightarrow KO_*$ induces the map of spectra

$$\alpha\colon MSpin \longrightarrow KO. \tag{1}$$

It turns out that for our choice of singularities Σ the spectrum $MSpin^{\Sigma}$ splits as a smash product $MSpin^{\Sigma} = MSpin \wedge X_{\Sigma}$ for some spectra X_{Σ} (see Theorems 3.1, 6.1). We would like to introduce the real K–theories $KO_*^{\Sigma}(\cdot)$ with the singularities Σ. We define the classifying spectrum for $KO_*^{\Sigma}(\cdot)$ by $KO^{\Sigma} = KO \wedge X_{\Sigma}$. The K–theories $KO_*^{\Sigma}(\cdot)$ may be identified with the well-known K–theories. Indeed,

$$KO_*^{\Sigma_1}(\cdot) = KO_*(\cdot; \mathbf{Z}/2), \qquad KO_*^{\eta}(\cdot) = K_*(\cdot), \qquad KO_*^{\Sigma_2}(\cdot) = K_*(\cdot; \mathbf{Z}/2),$$

see Corollary 5.4. The K–theory $KO_*^{\Sigma_3}(\cdot)$ is "trivial" since the classifying spectrum KO^{Σ_3} is contractible, see Corollary 6.4. Now the map α from (1) induces the map

$$\alpha^{\Sigma}\colon MSpin^{\Sigma} = MSpin \wedge X_{\Sigma} \xrightarrow{\alpha\wedge 1} KO \wedge X_{\Sigma} = KO^{\Sigma}$$

and the homomorphism of the coefficient rings

$$\alpha^{\Sigma}\colon \Omega_*^{Spin,\Sigma} \longrightarrow KO_*^{\Sigma}. \tag{2}$$

We define the integer $d(\Sigma)$ as follows:

$$d(\Sigma_1) = 6, \qquad d(\Sigma_2) = 8, \qquad d(\Sigma_3) = 17, \qquad d(\eta) = 7.$$

Recall that if M is a Σ–manifold, then (depending on the length of Σ), the manifolds $\beta_i M$, $\beta_{ij} M$, $\beta_{ijk} M$ (as Σ–manifolds) are defined in canonical way. In particular, for $\Sigma = \Sigma_1, \eta$, there is a manifold $\beta_i M$ such that $\partial M = \beta_i M \times P_i$, for $\Sigma = \Sigma_2$, there are Σ–manifolds $\beta_1 M$, $\beta_2 M$,$\beta_{12} M$, and for $\Sigma = \Sigma_3$ there are Σ–manifolds $\beta_i M$, $\beta_{ij} M$, $\beta_{ijk} M$. These manifolds may be empty. The manifolds $\beta_i M$, $\beta_{ij} M$ and $\beta_{ijk} M$ are called *Σ–strata of M*.

We say that a Σ–manifold M is *simply connected* if M itself is simply connected and all Σ–strata of M are simply connected manifolds.

1.3 Main geometric result The following theorem is the main geometric result of this paper.

Theorem 1.1 *Let M^n be a simply connected Spin Σ–manifold of dimension $n \geq d(\Sigma)$, so that all Σ–strata manifolds are nonempty manifolds. Then M admits a metric of positive scalar curvature if and only if $\alpha^{\Sigma}([M]) = 0$ in the group KO_n^{Σ}.*

We complete the proof of Theorem 1.1 only at the end of the paper. However, we would like to present here the overview of the main ingredients of the proof.

1.4 Key ideas and constructions of the proof There are two parts of Theorem 1.1 to prove. The first "if" part is almost "pure topological". The second "only if" part has more analytical flavor. We start with the topological ingredients.

The first key construction which allows to reduce the question on the existence of a psc-metric to a topological problem, is the Surgery Lemma. This fundamental observation originally is due to Gromov–Lawson [9] and Schoen–Yau [27]. We generalize the Surgery Lemma for simply connected *Spin* Σ–manifolds.

This generalization is almost straightforward, however we have to describe the surgery procedure for Σ–manifolds.
To explain the difference with the case of regular surgery, we consider the example when M is a $\mathbf{Z}/k$–manifold, ie, $\partial M = \beta M \times \langle k \rangle$. There are two types of surgeries here. The first one is to do surgery on the interior of M, and the second one is to do surgery on each manifold βM.

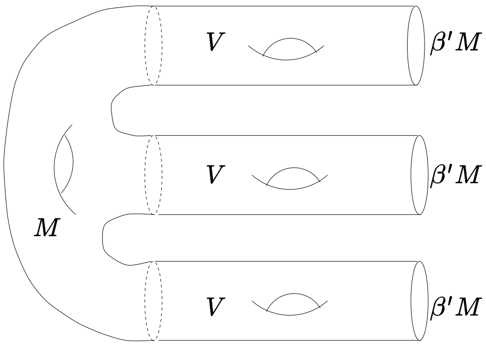


Figure 3: The manifold M'

We start with the second one. Let M be a $\mathbf{Z}/k$-manifold, with a psc-metric g_M. We have $\partial M = \beta M \times \langle k \rangle$, where $g_{\beta M}$ is a psc-metric. Let $S^p \times D^{n-p-1} \subset \beta M$, and V be a trace of the surgery along the sphere S^p, ie, $\partial V = -\beta M \cup \beta' M$. We assume that $n - p - 1 \geq 3$, so we can use the regular Surgery Lemma to push a psc-metric through the manifold V to obtain a psc-metric g_V which is a product near the boundary. Then we attach k copies of V to obtain a manifold $M' = M \cup_{\partial M} V \times \langle k \rangle$, see Figure 3. Clearly the metrics g_M and g_V match along a color of the common boundary, giving a psc-metric g' on M'.

The first type of surgery is standard. Let $S^\ell \times D^{n-\ell} \subset M$ be a sphere together with a tubular neighborhood inside the interior of the manifold M. Denote by M'' the result of surgery on M along the sphere S^ℓ. Notice that $\partial M'' = \partial M$. Then again the regular Surgery Lemma delivers a psc-metric on M''.

The case of two and more singularities requires a bit more care. We discuss the general Surgery procedure for Σ–manifolds in Section 7. The Bordism Theorem (Theorem 7.3) for simply connected Σ–manifolds reduces the existence question of a positive scalar curvature to finding a Σ–manifold within the cobordism class $[M]_\Sigma$ equipped with a psc-metric.

To solve this problem we use the ideas and results due to S Stolz [29], [30]. The magic phenomenon discovered by S Stolz is the following. Let us start with the quaternionic projective space $\mathbf{HP}^2$ equipped with the standard metric g_0 (of constant positive curvature). It is not difficult to see that the Lie group

$$G = PSp(3) = Sp(3)/\text{Center},$$

acts by isometries of the metric g_0 on $\mathbf{HP}^2$. Here Center $\cong \mathbf{Z}/2$ is the center of the group $Sp(3)$. Then given a smooth bundle $E \xrightarrow{p} B$ of compact *Spin*–manifolds, with a fiber $\mathbf{HP}^2$, and a structure group G, there is a straightforward construction of a psc-metric on the manifold E, the total space of this bundle. (A bundle with the above properties is called a *geometric* $\mathbf{HP}^2$*–bundle*.) The construction goes as follows. One picks an arbitrary metric g_B on a manifold B. Then locally, over an open set $U \subset B$, a metric on $p^{-1}(U) \cong U \times \mathbf{HP}^2$ is given as product metric $g_E|_{p^{-1}(U)} = g_B|_U \times g_0$. By scaling the metric g_0, one obtains that the scalar curvature of the metric $g_E|_{p^{-1}(U)}$ is positive. Since the structure group of the bundle acts by isometries of the metric g_0, one easily constructs a psc-metric g_E on E.

Perhaps, this general construction was known for ages. The amazing feature of geometric $\mathbf{HP}^2$–bundles is that their total spaces, the manifolds E, generate the kernel of the Atiyah–Bott–Shapiro transformation $\alpha\colon \Omega_n^{Spin} \longrightarrow KO_n$. In more detail, given an $\mathbf{HP}^2$–bundle $E^n \xrightarrow{p} B^{n-8}$, there is a classifying map $f\colon B^{n-8} \longrightarrow BG$ which defines a cobordism class $[(B,f)] \in \Omega_{n-8}^{Spin}(BG)$. The correspondence $[(B,f)] \mapsto [E] \in \Omega_n^{Spin}$ defines the transfer map

$$T\colon \Omega_{n-8}^{Spin}(BG) \longrightarrow \Omega_n^{Spin}.$$

Stolz proves [29] that $\text{Im}\, T = \text{Ker}\, \alpha$. Thus the manifolds E deliver representatives in each cobordism class of the kernel Ker α.

We adopt this construction for manifolds with singularities. First we notice that if a geometric $\mathbf{HP}^2$–bundle $E \xrightarrow{p} B$ is such that B is a Σ–manifold, then E is also a Σ–manifold. In particular we obtain the induced transfer map

$$T^{\Sigma}\colon \Omega_*^{Spin,\Sigma}(BG) \longrightarrow \Omega_{*+8}^{Spin,\Sigma}.$$

The key here is to prove that $\mathrm{Im}\; T^{\Sigma} = \mathrm{Ker}\; \alpha^{\Sigma}$. This requires complete information on the homotopy type of the spectra $MSpin^{\Sigma}$. Sections 3–6 are devoted to study of the spectra $MSpin^{\Sigma}$.

The second part, the proof of the "only if" statement, is geometric and analytic by its nature. We explain the main issues here for the case of $\mathbf{Z}/k$–manifolds. Recall that for a *Spin* manifold M the direct image $\alpha([M]) \in KO_n$ is nothing else but the *topological index* of M which coincides (via the Atiyah–Singer index theorem) with the *analytical index* $\mathrm{ind}(M) \in KO_n$ of the corresponding Dirac operator on M. Then the Lichnerowicz formula and its modern versions imply that the analytical index $\mathrm{ind}(M)$ vanishes if there is a psc-metric on M.

Thus if we would like to give a similar line of arguments for $\mathbf{Z}/k$–manifolds, we face the following issues. To begin with, we should have the Dirac operator to be well-defined on a *Spin* $\mathbf{Z}/k$–manifold. Then we have to define the $\mathbf{Z}/k$–version of the analytical index $\mathrm{ind}_{\mathbf{Z}/k}(M) \in KO_n^{\langle k\rangle}$ and to prove the vanishing result, ie, that $\mathrm{ind}_{\mathbf{Z}/k}(M) = 0$ provided that there is a psc-metric on M. Thirdly we must identify the analytical index $\mathrm{ind}_{\mathbf{Z}/k}(M)$ with the direct image $\alpha^{\langle k\rangle}([M]) \in KO_n^{\langle k\rangle}$, ie, to prove the $\mathbf{Z}/k$–mod version of the index theorem. These issues were already addressed, and, in the case of $Spin^c$–manifolds, resolved by Freed [5], [6], Freed & Melrose [7], Higson [11], Kaminker & Wojciechowski [14], and Zhang [34, 35]. Unfortunately, the above papers study mostly the case of $Spin^c$ $\mathbf{Z}/k$–manifolds (with the exception of [34, 35] where the mod 2 index is considered), and the general case of *Spin* $\mathbf{Z}/k$–manifolds is essentially left out in the cited work. The paper [22] by J. Rosenberg shows that the Dirac operator and its index are well-defined for $\mathbf{Z}/k$–manifolds and there the index vanishes if a *Spin* $\mathbf{Z}/k$–manifold has psc-metric. The case of general singularities Σ require more work. Here we use the results of [22] to prove that if a Σ–manifold M has a psc-metric, then $\alpha^{\Sigma}([M]) = 0$ in the group KO^{Σ}. In order to prove this fact we essentially use the specific homotopy features of the spectra $MSpin^{\Sigma}$.

The plan is the following. We give necessary definitions and constructions on manifolds with singularities in Section 2. The next four sections are devoted to homotopy-theoretical study of the spectra $MSpin^{\Sigma}$. We describe the homotopy type of the spectra $MSpin^{\Sigma_1}$, $MSpin^{\Sigma_2}$, and $MSpin^{\eta}$ in Section 3. We describe a product structure of these spectra in Section 4. In Section 5 we describe a splitting of the spectra $MSpin^{\Sigma}$ into indecomposable spectra. In

Section 6 we describe the homotopy type of the spectrum $MSpin^{\Sigma_3}$. We prove the Surgery Lemma for manifolds with singularities in Section 7. Section 8 is devoted to the proof of Theorem 1.1.

It is a pleasure to thank Hal Sadofsky for helpful discussions on the homotopy theory involved in this paper, and acknowledge my appreciation to Stephan Stolz for numerous discussions about the positive scalar curvature. The author also would like to thank the Department of Mathematics of the National University of Singapore for hospitality (this was Fall of 1999). The author is thankful to Jonathan Rosenberg for his interest to this work and useful discussions. Finally, the author thanks the referee for helpful suggestions.

2 Manifolds with singularities

Here we briefly recall basic definitions concerning manifolds with the Baas–Sullivan singularities. Let G be a stable Lie group. We will be interested in the case when $G = Spin$. Consider the category of smooth compact manifolds with a stable G–structure in their stable normal bundle.

2.1 General definition Let $\Sigma = (P_1, \dots, P_k)$, where $P_1, \dots, P_k$ are arbitrary closed manifolds (possibly empty). It is convenient to denote $P_0 = pt$. Let $I = \{i_1, \dots, i_q\} \subset \{0, 1, \dots, k\}$. We denote $P^I = P_{i_1} \times \dots \times P_{i_q}$.

Definition 2.1 We call a manifold M a Σ*–manifold* if there are given the following:

(i) a partition $\partial M = \partial_0 M \cup \partial_1 M \cup \dots \cup \partial_k M$ of its boundary ∂M such that the intersection $\partial_I M = \partial_{i_1} M \cap \dots \cap \partial_{i_q} M$ is a manifold for every collection $I = \{i_1, \dots, i_q\} \subset \{0, 1, \dots, k\}$, and its boundary is equal to

$$\partial\left(\partial_I M\right) = \bigcup_{j \notin I} \left(\partial_I M \cap \partial_j M\right);$$

(ii) compatible product structures (ie, diffeomorphisms preserving the stable G–structure)

$$\phi_I \colon \partial_I M \longrightarrow \beta_I M \times P^I.$$

Compatibility means that if $I \subset J$ and $\iota \colon \partial_J M \longrightarrow \partial_I M$ is the inclusion, then the map

$$\phi_I \circ \iota \circ \phi_J^{-1} \colon \beta_J M \times P^J \longrightarrow \beta_I M \times P^I$$

is identical on the direct factor P^I.

To get actual singularities we do the following. Two points x, y of a Σ–manifold M are *equivalent* if they belong to the same manifold $\partial_I M$ for some $I \subset \{0, 1, \ldots, k\}$ and $pr \circ \phi_I(x) = pr \circ \phi_J(y)$, where $pr\colon \beta_I M \times P^I \longrightarrow \beta_I M$ is the projection on the direct factor. The factor-space of M under this equivalence relation is called *the model of the* Σ*–manifold* M and is denoted by M_Σ. Actually it is convenient to deal with Σ–manifolds without considering their models. Indeed, we only have to make sure that all constructions are consistent with the projections $\pi\colon M \longrightarrow M_\Sigma$. The *boundary* δM *of a* Σ*–manifold* M is the manifold $\partial_0 M$. If $\delta M = \emptyset$, we call M a *closed* Σ*–manifold.* The boundary δM is also a Σ–manifold with the inherited decomposition $\partial_I(\delta M) = \partial_I M \cap \delta M$. The manifolds $\beta_I M$ also inherit a structure of a Σ–manifold:

$$\partial_j(\beta_I M) = \begin{cases} \emptyset & \text{if } j \in I, \\ \beta_{\{j\}\cup I} M \times P_j & \text{otherwise.} \end{cases} \tag{3}$$

Here we denote $\beta_I M = \beta_{i_1}\left(\beta_{i_2}\left(\cdots \beta_{i_q} M\right)\cdots\right)$ for $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$.

Let (X, Y) be a pair of spaces, and $f\colon (M, \delta M) \longrightarrow (X, Y)$ be a map. Then the pair (M, f) is a *singular* Σ*–manifold* of (X, Y) if the map f is such that for every index subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$ the map $f|_{\partial_I M}$ is decomposed as $f|_{\partial_I M} = f_I \circ pr \circ \phi_I$, where the map ϕ_I as above, $pr\colon \beta_I M \times P^I \longrightarrow \beta_I M$ is the projection on the direct factor, and $f_I\colon \beta_I M \longrightarrow X$ is a continuous map. The maps f_I should be compatible for different indices I in the obvious sense.

Remark 2.2 Let (M, f) be a *singular* Σ*–manifold*, then the map f factors through as $f = f_\Sigma \circ \pi$, where $\pi\colon M \longrightarrow M_\Sigma$ is the canonical projection, and $f_\Sigma\colon M_\Sigma \longrightarrow X$ is a continuous map. We also notice that singular Σ–manifolds may be identified with their Σ–models.

The cobordism theory $\Omega_*^{G,\Sigma}(\cdot)$ of Σ–manifolds is defined in the standard way. In the case of interest, when $G = Spin$, we denote $MSpin^\Sigma$ a spectrum classifying the cobordism theory $\Omega_*^{Spin,\Sigma}(\cdot)$.

2.2 The case of two and three singularities We start with the case $\Sigma = (P_1, P_2)$. Then if M is a Σ–manifold, we have that the diffeomorphisms

$$\phi\colon \partial M \xrightarrow{\cong} \partial_1 M \cup \partial_2 M,$$

$$\phi_i\colon \partial_i M \xrightarrow{\cong} \beta_i M \times P_i, \quad i = 1, 2;$$

$$\phi_{12}\colon \partial_1 M \cap \partial_2 M \xrightarrow{\cong} \beta_{12} M \times P_1 \times P_2$$

are given. We always assume that the manifold $\beta_{12} M \times P_1 \times P_2$ is embedded into $\partial_1 M$ and $\partial_2 M$ together with a color:

$$\beta_{12}M \times P_1 \times P_2 \times I \subset \partial_1 M, \partial_2 M.$$

Thus we actually have the following decomposition of the boundary ∂M:

$$\partial M \cong \partial_1 M \cup (\beta_{12}M \times P_1 \times P_2 \times I) \cup \partial_2 M,$$

so the manifold $\beta_{12}M \times P_1 \times P_2$ is "fattened" inside ∂M. Also we assume that the boundary ∂M is embedded into M together with a color $\partial M \times I \subset M$, see Figure 4.

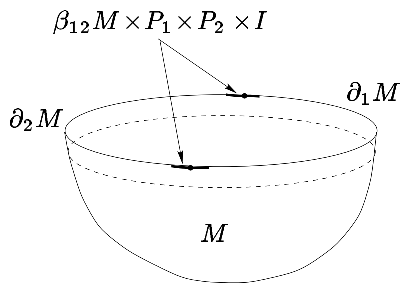


Figure 4

The case when $\Sigma = (P_1, P_2, P_3)$ is the most complicated one we are going to work with.

Let M be a closed Σ–manifold, then we are given the diffeomorphisms:

$$\phi \colon \partial M \xrightarrow{\cong} \partial_1 M \cup \partial_2 M \cup \partial_3 M,$$

$$\phi_i \colon \partial_i M \xrightarrow{\cong} \beta_i M \times P_i, \quad i = 1, 2, 3;$$

$$\phi_{ij} \colon \partial_i M \cap \partial_j M \xrightarrow{\cong} \beta_{ij} M \times P_i \times P_j,$$

$$\phi_{123} \colon \partial_1 M \cap \partial_2 M \cap \partial_3 M \xrightarrow{\cong} \beta_{123} M \times P_1 \times P_2 \times P_3$$

where $i, j = 1, 2, 3, \ \ i \neq j$, see Figure 5.

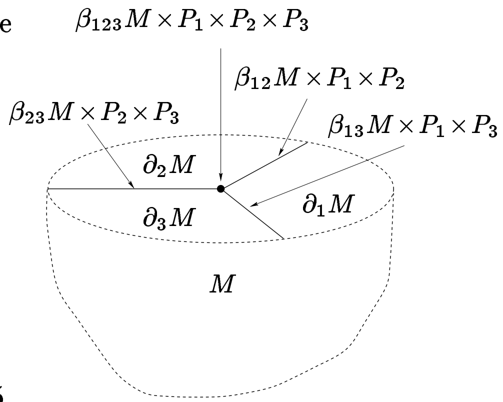


Figure 5

First, we assume here that the boundary ∂M is embedded into M together with a color $(0,1] \times \partial M$. The decomposition

$$\partial M \xrightarrow{\phi} \partial_1 M \cup \partial_2 M \cup \partial_3 M$$

gives also the "color" structure on ∂M.

We assume that the boundary $\partial(\partial_i M)$ is embedded into $\partial_i M$ together with the color $(0,1] \times \partial(\partial_i M)$.

Even more, we assume that the manifold $\beta_{123}M \times P_1 \times P_2 \times P_3$ is embedded into the boundary ∂M together with its normal tube:

$$\beta_{123}M \times P_1 \times P_2 \times P_3 \times D^2 \subset \partial M,$$

so that the colors of the manifolds

$$\beta_{ij}M \times P_i \times P_j \subset \partial_i M \cap \partial_j M$$

are compatible with this embedding, as is shown on Figure 6. As in the case of two singularities, the submanifolds

$$\beta_{ij}M \times P_i \times P_j \quad \text{and}$$

$$\beta_{123}M \times P_1 \times P_2 \times P_3$$

are "fattened" inside the boundary ∂M. Furthermore, we assume that there are not any corners in the above color decomposition.

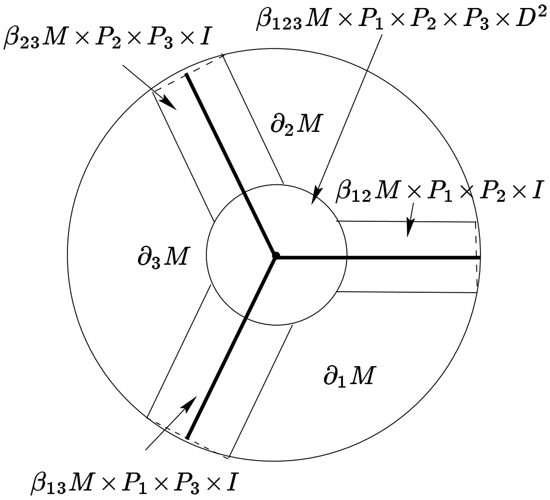


Figure 6

2.3 Bockstein–Sullivan exact sequence Let MG be the Thom spectrum classifying the cobordism theory $\Omega^G_*(\cdots)$. Let $\Sigma=(P)$, and $p=\dim P$. Then there is a stable map $S^p \xrightarrow{[P]} MG$ representing the element $[P]$. Then we have the composition

$$\cdot[P]\colon \Sigma^p MG = S^p\wedge MG \xrightarrow{[P]\wedge Id} MG\wedge MG \xrightarrow{\mu} MG$$

where μ is the map giving MG a structure of a ring spectrum. Then the cofiber, the spectrum MG^{Σ} of the map

$$\Sigma^p MG \xrightarrow{\cdot[P]} MG \xrightarrow{\pi} MG^{\Sigma} \tag{4}$$

is a classifying spectrum for the cobordism theory $\Omega^{G,\Sigma}_*$. The cofiber (4) induce the long exact Bockstein–Sullivan sequence

$$\cdots\to\Omega^G_{n-p}(X,A)\xrightarrow{\cdot[P]}\Omega^G(X,A)\xrightarrow{\pi}\Omega^{G,\Sigma}_{n-p}(X,A)\xrightarrow{\beta}\Omega^G_{n-p-1}(X,A)\to\cdots \tag{5}$$

for any CW–pair (X,A). Similarly, if $\Sigma_j=(P_1,\dots,P_j)$, $j=1,\dots,k$, then there is a cofiber

$$\Sigma^{p_j}MG^{\Sigma_{j-1}}\xrightarrow{\cdot[P_j]}MG^{\Sigma_{j-1}}\xrightarrow{\pi_j}MG^{\Sigma_j}$$

induce the exact Bockstein–Sullivan sequence

$$\cdots\xrightarrow{\beta_j}\Omega^{G,\Sigma_{j-1}}_{n-p_j}(X,A)\xrightarrow{\cdot[P_j]}\Omega^{G,\Sigma_{j-1}}_n(X,A)\xrightarrow{\pi_j}\Omega^{G,\Sigma_j}_n(X,A)\xrightarrow{\beta_j}\cdots \tag{6}$$

for any CW–pair (X,A). We shall use the Bockstein–Sullivan exact sequences (5), (6) throughout the paper.

3 The spectra $MSpin^{\Sigma_1}$, $MSpin^{\Sigma_2}$ and $MSpin^{\eta}$

Let $M(2)$ be the mod 2 Moore spectrum with the bottom cell in zero dimension, ie, $M(2) = \Sigma^{-1}\mathbf{RP}^2$. We consider also the spectrum $\Sigma^{-2}\mathbf{CP}^2$ and the spectrum $Y = M(2) \wedge \Sigma^{-2}\mathbf{CP}^2$ which was first studied by M Mahowald, [17]. Here is the result on the spectra $MSpin^{\Sigma_1}$, $MSpin^{\Sigma_2}$ and $MSpin^{\eta}$.

Theorem 3.1 *There are homotopy equivalences:*

(i) $MSpin^{\Sigma_1} \cong MSpin \wedge M(2)$,

(ii) $MSpin^{\eta} \cong MSpin \wedge \Sigma^{-2}\mathbf{CP}^2$,

(iii) $MSpin^{\Sigma_2} \cong MSpin \wedge Y$.

Proof Let $\iota\colon S^0 \longrightarrow MSpin$ be a unit map. The main reason why the above homotopy equivalences hold is that the elements $2, \eta \in \Omega_*^{Spin}$ are in the image of the homomorphism $\iota_*\colon S_*^0 \longrightarrow \Omega_*^{Spin}$. Indeed, consider first the spectrum $MSpin^{\eta}$. Let $S^1 \xrightarrow{\eta} S^0$ be a map representing $\eta \in \pi_1(S^0)$. We obtain the cofibration:

$$S^1 \xrightarrow{\eta} S^0 \xrightarrow{\pi} \Sigma^{-2}\mathbf{CP}^2 . \tag{7}$$

Then the composition $S^1 \xrightarrow{\eta} S^0 \xrightarrow{\iota} MSpin$ represents $\eta \in MSpin_1$. Let $\cdot\eta$ be the map

$$\cdot\eta\colon S^1 \wedge MSpin \xrightarrow{\iota\eta\wedge 1} MSpin \wedge MSpin \xrightarrow{\mu} MSpin,$$

where μ is a multiplication. Note that the diagram

$$\begin{array}{ccccc}
S^1 \wedge MSpin & \xrightarrow{\iota\eta\wedge 1} & MSpin \wedge MSpin & \xrightarrow{\mu} & MSpin \\
{\scriptstyle 1\wedge 1}\uparrow & & {\scriptstyle \iota\wedge 1}\uparrow & & {\scriptstyle 1}\uparrow \\
S^1 & \xrightarrow{\eta\wedge 1} & S^0 \wedge MSpin & \xrightarrow{\cong} & MSpin
\end{array}$$

commutes since the map $\iota\colon S^0 \longrightarrow MSpin$ represents a unit of the ring spectrum $MSpin$. We obtain a commutative diagram of cofibrations:

$$\begin{array}{ccccc}
S^1 \wedge MSpin & \xrightarrow{\cdot\eta} & MSpin & \xrightarrow{\pi_\eta} & MSpin^{\eta} \\
{\scriptstyle 1\wedge 1}\uparrow & & {\scriptstyle 1}\uparrow & & {\scriptstyle f_\eta}\uparrow \\
S^1 \wedge MSpin & \xrightarrow{\eta\wedge 1} & MSpin & \xrightarrow{\pi\wedge 1} & \Sigma^{-2}\mathbf{CP}^2 \wedge MSpin
\end{array} \tag{8}$$

where $f_\eta\colon MSpin^{\eta} \longrightarrow \Sigma^{-2}\mathbf{CP}^2 \wedge MSpin \cong MSpin \wedge \Sigma^{-2}\mathbf{CP}^2$ gives a homotopy equivalence by 5–lemma. The proof for the spectrum $MSpin^{\Sigma_1} = MSpin^{\langle 2\rangle}$ is similar.

Consider the spectrum $MSpin^{\Sigma_2}$. First we note that the bordism theory $\Omega_*^{Spin,\Sigma_2}(\cdot) = \Omega_*^{Spin,(P_1,P_2)}(\cdot)$ coincides with the theory $\Omega_*^{Spin,(P_2,P_1)}(\cdot)$, where the order of singularities is switched. In particular, the spectrum $MSpin^{\Sigma_2}$ is a cofiber in the following cofibration:

$$S^0 \wedge MSpin^{\eta} \xrightarrow{\cdot 2} MSpin^{\eta} \longrightarrow MSpin^{\Sigma_2}. \tag{9}$$

Here the map $\cdot 2$: $S^0 \wedge MSpin^{\eta} \longrightarrow MSpin^{\eta}$ is defined as follows. Let $S^0 \xrightarrow{2} S^0$ be a map of degree 2. Then the composition $S^0 \xrightarrow{2} S^0 \xrightarrow{\iota} MSpin$ represents $2 \in \Omega_0^{Spin}$. The spectrum $MSpin^{\eta}$ is a module (say, left) spectrum over $MSpin$, ie, there is a map $\mu'_L : MSpin \wedge MSpin^{\eta} \longrightarrow MSpin^{\eta}$ so that the diagram

$$\begin{tikzcd}
MSpin \wedge MSpin \arrow[r, "\mu"] \arrow[d, "1\wedge\pi_\eta"'] & MSpin \arrow[d, "\pi_\eta"'] \\
MSpin \wedge MSpin^{\eta} \arrow[r, "\mu'_L"] & MSpin^{\eta}
\end{tikzcd}$$

commutes. Then the map $\cdot 2$ is defined as composition:

$$S^0 \wedge MSpin^{\eta} \xrightarrow{2\iota\wedge 1} MSpin \wedge MSpin^{\eta} \xrightarrow{\mu'_L} MSpin^{\eta}.$$

Note that the diagram

$$\begin{tikzcd}
S^0 \wedge MSpin^{\eta} \arrow[r, "2\iota\wedge 1"] & MSpin \wedge MSpin^{\eta} \arrow[r, "\mu'_L"] & MSpin^{\eta} \\
S^0 \wedge MSpin^{\eta} \arrow[r, "2\wedge 1"] \arrow[u, "1\wedge 1"] & S^0 \wedge MSpin^{\eta} \arrow[r, "\cong"] \arrow[u, "\iota\wedge 1"] & MSpin^{\eta} \arrow[u, "1"]
\end{tikzcd}$$

commutes since $S^0 \xrightarrow{\iota} MSpin$ represents a unit, and $MSpin^{\eta}$ is a left module over the ring spectrum $MSpin$. We obtain the commutative diagram of cofibrations:

$$\begin{tikzcd}
S^0 \wedge MSpin^{\eta} \arrow[r, "\cdot 2"] & MSpin \arrow[r, "\pi_2"] & MSpin^{\Sigma_2} \\
S^0 \wedge MSpin^{\eta} \arrow[r, "2\wedge 1"] \arrow[u, "1\wedge 1"] & MSpin^{\eta} \arrow[r, "\pi\wedge 1"] \arrow[u, "1"] & M(2) \wedge MSpin^{\eta} \arrow[u, "f_2"]
\end{tikzcd} \tag{10}$$

The map f_2: $M(2) \wedge MSpin^{\eta} \longrightarrow MSpin^{\Sigma_2}$ gives a desired homotopy equivalence. Thus we have $MSpin^{\Sigma_2} \cong M(2) \wedge MSpin^{\eta} \cong MSpin^{\eta} \wedge M(2) = MSpin \wedge Y$. □

Remark 3.2 In the above proof, we did not use any specific properties of the spectrum $MSpin$ except that it is a ring spectrum. In fact, $MSpin$ may be replaced by any other classic Thom spectrum.

Later we prove that the homotopy equivalence

$$MSpin^{\Sigma_3} \sim MSpin \wedge \Sigma^{-2}\mathbf{CP}^2 \wedge V(1) \ ,$$

where $V(1)$ is the cofiber of the Adams map $A\colon \Sigma^8 M(2) \longrightarrow M(2)$. However, first we have to study the spectra $MSpin^{\Sigma_1}$, $MSpin^{\Sigma_2}$ and $MSpin^{\eta}$ in more detail.

4 Product structure

Recall that the spectrum $MSpin$ is a ring spectrum. Here we work with the category of spectra, and commutativity of diagrams mean commutativity up to homotopy. Let, as above, $\iota\colon S^0 \longrightarrow MSpin$ be the unit, and $\mu\colon MSpin \wedge MSpin \longrightarrow MSpin$ the map defining the product structure. Let $MSpin^{\Sigma}$ be one of the spectrum we considered above. The natural map $\pi\colon MSpin \longrightarrow MSpin^{\Sigma}$ turns the spectrum $MSpin^{\Sigma}$ into a left and a right module over the spectrum $MSpin$, ie, there are maps

$$\mu'_L\colon MSpin \wedge MSpin^{\Sigma} \longrightarrow MSpin^{\Sigma}, \quad \mu'_R\colon MSpin^{\Sigma} \wedge MSpin \longrightarrow MSpin^{\Sigma},$$

so that the diagrams

$$\begin{array}{ccc} MSpin \wedge MSpin & \xrightarrow{\mu} & MSpin \\ {\scriptstyle 1\wedge\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ MSpin \wedge MSpin^{\Sigma} & \xrightarrow{\mu'_L} & MSpin^{\Sigma} \end{array} \qquad \begin{array}{ccc} MSpin \wedge MSpin & \xrightarrow{\mu} & MSpin \\ {\scriptstyle \pi\wedge 1}\downarrow & & \downarrow{\scriptstyle \pi} \\ MSpin^{\Sigma} \wedge MSpin & \xrightarrow{\mu'_R} & MSpin^{\Sigma} \end{array}$$

commute. We say that the spectrum $MSpin^{\Sigma}$ has an *admissible ring structure*

$$\mu^{\Sigma}\colon MSpin^{\Sigma} \wedge MSpin^{\Sigma} \longrightarrow MSpin^{\Sigma}$$

if the map $S^0 \xrightarrow{\iota} MSpin \xrightarrow{\pi} MSpin^{\Sigma}$ is a unit, and the diagrams

$$\begin{array}{ccc} MSpin \wedge MSpin^{\Sigma} & \xrightarrow{\mu'_L} & MSpin^{\Sigma} \\ {\scriptstyle \pi\wedge 1}\downarrow & & \downarrow{\scriptstyle 1} \\ MSpin^{\Sigma} \wedge MSpin^{\Sigma} & \xrightarrow{\mu^{\Sigma}} & MSpin^{\Sigma} \end{array} \qquad \begin{array}{ccc} MSpin^{\Sigma} \wedge MSpin & \xrightarrow{\mu'_R} & MSpin^{\Sigma} \\ {\scriptstyle 1\wedge\pi}\downarrow & & \downarrow{\scriptstyle 1} \\ MSpin^{\Sigma} \wedge MSpin^{\Sigma} & \xrightarrow{\mu^{\Sigma}} & MSpin^{\Sigma} \end{array}$$

commute. The questions of existence, commutativity and associativity of an admissible product structure were thoroughly studied in [3], [19].

Theorem 4.1 (i) *The spectrum $MSpin^{\Sigma_1}$ does not admit an admissible product structure.*

(ii) *The spectra $MSpin^{\eta}$, $MSpin^{\Sigma_2}$ and $MSpin^{\Sigma_3}$ have admissible product structures μ^{η}, $\mu^{\Sigma_2}=\mu^{(2)}$, and $\mu^{\Sigma_3}=\mu^{(3)}$ respectively.*

(iii) *For any choice of an admissible product structure μ^{η}, it is commutative and associative. For any choice of admissible product structures $\mu^{(2)}$, and $\mu^{(3)}$, they are associative, but not commutative.*

Proof Recall that for each singularity manifold P_i there is an obstruction manifold P_i' with singularity. In the cases of interest, we have: $[P_1']_{\Sigma_1}=\eta\in\Omega_1^{Spin,\Sigma_1}$, which is non-trivial; and the obstruction $[P_2']\in\Omega_3^{Spin,\Sigma_2}=0$, and $[P_2']\in\Omega_3^{Spin,\eta}=0$. Thus [3, Lemma 2.2.1] implies that there is no admissible product structure in the cobordism theory $\Omega_*^{Spin,\Sigma_1}(\cdot)$, so the spectrum $MSpin^{\Sigma_1}$ does not admit an admissible product structure. The obstruction element $[P_3']_{\Sigma_3}\in\Omega_{17}^{Spin,\Sigma_3}$, and since $\dim P_3=8$ is even, the obstruction manifold P_3' is, in fact, a manifold without any singularities (see [19]), so the element $[P_3']_{\Sigma_3}$ is in the image $\mathrm{Im}\,(\Omega_{17}^{Spin}\longrightarrow\Omega_{17}^{Spin,\Sigma_3})$. However, the elements of Ω_{17}^{Spin} are divisible by η, so they are zero in the group $\Omega_{17}^{Spin,\eta}$, and, consequently, in $\Omega_{17}^{Spin,\Sigma_3}$.

The result of [3, Theorem 2.2.2] implies that the spectra $MSpin^{\eta}$, $MSpin^{\Sigma_2}$ and $MSpin^{\Sigma_3}$ have admissible product structures $\mu^{(2)}$ and μ^{η} respectively.

It is also well-known [33] that the element $v_1\in\Omega_2^{Spin,\Sigma_2}$ is an obstruction to the commutativity of the product structure $\mu^{(2)}$. An obstruction to the commutativity for the product structure μ^{η} lives in the group $\Omega_5^{Spin,\Sigma_2}=0$. The obstructions to associativity are 3–torsion elements, (see [3, Lemma 4.2.4]) so they all are zero. □

5 Homotopy structure of the spectra $MSpin^{\Sigma}$

First we recall the work of Anderson, Brown, and Peterson [1] on structure of the spectra $MSpin$, and of M Hopkins, M Hovey [13].

Let $KO_*(\cdot)$ be a periodic homological real K–theory, KO be a corresponding Ω–spectrum. Also let ko be the connected cover of KO, and $ko\langle 2\rangle$ denote the 2–connective cover of ko. It is convenient to identify the $2n$–fold connective covers of the spectrum KO. Indeed, the $4k$–fold connective cover of KO is $\Sigma^{4k}ko$ (when k is even), and the $(4k-2)$–fold connective cover is $\Sigma^{4k-2}ko\langle 2\rangle$. Let ku be a connected cover of the complex K–theory spectrum

K. Let $\mathbf{H}(\mathbf{Z}/2)$ denote the $\mathbf{Z}/2$–Eilenberg–MacLane spectrum. Recall that ko and ku are the ring spectra with the coefficient rings:

$$\begin{aligned} &ko_* \cong \mathbf{Z}[\eta,\omega,b]/(2\eta,\eta^3,\omega\eta,\omega^2-4b), \quad \deg\eta=1,\ \deg\omega=4, \deg b=8; \\ &ku_* \cong \mathbf{Z}[v], \quad \deg v = 2. \end{aligned} \tag{11}$$

Let $I=(i_1,\dots,i_r)$ be a partition (possibly empty) of $n=n(I)=\sum_{t=1}^r i_t$, $i_t>0$. Each partition I defines a map $\pi^I\colon MSpin \longrightarrow KO$ (which gives the KO–characteristic class, see [1]). If $I=\emptyset$ we denote $\pi^\emptyset$ by π^0, which coincides with the Atiyah–Bott–Shapiro orientation $\alpha\colon MSpin \longrightarrow KO$.

Remark 5.1 Let $\mathcal{P}$ be a set of all partitions, which is an abelian group. We can make the set $\mathbf{Z}[\mathcal{P}]$ of linear combinations into a ring, where multiplication of partitions is defined by set union, and then to into a Hopf algebra with the diagonal $\Delta(I)=\sum_{I_1+I_2=I} I_1\otimes I_2$.

Let $\mu\colon MSpin\wedge MSpin \longrightarrow MSpin$, $\mu'\colon KO\wedge KO\longrightarrow KO$ denote the ring spectra multiplications. The Cartan formula says that

$$\begin{array}{ccc} MSpin\wedge MSpin & \xrightarrow{\mu} & MSpin \\ {\scriptstyle \sum(\pi^{I_1}\wedge\pi^{I_2})}\Big\downarrow & & {\scriptstyle \pi^I}\Big\downarrow \\ KO\wedge KO & \xrightarrow{\mu'} & KO \end{array} \qquad \text{or} \quad \pi^{I_1}\mu = \sum_{I_1+I_2=I}\mu'(\pi^{I_1}\wedge\pi^{I_2}). \tag{12}$$

Theorem 5.2 [1]

(1) *Let $1\notin I$. Then if $n(I)$ is even, the map $\pi^I\colon MSpin\longrightarrow KO$ lifts to a map $\bar\pi^I\colon MSpin\longrightarrow \Sigma^{4n(I)}ko$. If $n(I)$ is odd, the map π^I lifts to a map $\bar\pi^I\colon MSpin\longrightarrow\Sigma^{4n(I)-4}ko\langle 2\rangle$.*

(2) *There exist a countable collection $z_k\in H^*(MSpin;\mathbf{Z}/2)$ such that the map*

$$\prod_{1\notin I}\bar\pi^I\times\prod_k z_k\colon MSpin\longrightarrow \prod_{\substack{1\notin I,\\ n(I)\text{ even}}}\Sigma^{4n(I)}ko\times\prod_{\substack{1\notin I,\\ n(I)\text{ odd}}}\Sigma^{4n(I)-4}ko\langle 2\rangle\times\prod_k\Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)$$

is a 2–local homotopy equivalence.

We use here the product symbol, however in the stable category of spectra the product and the coproduct, ie the wedge, are the same. We denote by ρ^I the left inverses of the maps $\bar\pi^I$ (when $1\notin I$). We denote also by b an element in

Ω_8^{Spin} which is the image of the Bott element under the map ρ^0. The following Lemma due to M Hovey and M Hopkins [13]. Since some fragments of its proof will be used later, we provide an argument which essentially repeats [13].

Lemma 5.3 [13, Lemma 1] *Let I be a partition. Then $\pi^I(b) = 0$ except for $\pi^0(b) = b$ and possibly $\pi^1(b) \in KO_8$ and $\pi^{1,1}(b) \in KO_8$. The elements $\pi^1(b)$, $\pi^{1,1}(b)$ are divisible by two in the group KO_8. Further, the image of the Bott element b is zero in MO_8.*

Proof In the case $1 \notin I$, $I \neq \emptyset$, the splitting shows that $\pi^I(b) = 0$. The map $\pi^I \colon MSpin \longrightarrow KO$ (for any partitions I) may be lifted to the $4n(I)$–connective cover of KO, as it is shown in [32]. Let $S^0 \longrightarrow ko$ be a unit map, and $\rho^0 \colon ko \longrightarrow MSpin$ be a left inverse of $\bar{\pi}^0$. The composition

$$S^0 \longrightarrow ko \xrightarrow{\rho^0} MSpin \xrightarrow{\pi^I} KO$$

is null-homotopic for $I \neq \emptyset$. Let $\eta \in MSpin_1 = \mathbf{Z}/2$ be a generator. It is well-known that the image of the map $S^0 \longrightarrow MSpin$ on positive dimensional homotopy groups is $\{b^n\eta, b^n\eta^2 \mid n \geq 0\}$. It implies that $\pi^I\rho^0(b^n\eta) = 0$ and $\pi^I\rho^0(b^n\eta^2) = 0$ for all partitions $I \neq \emptyset$. Since the unit map $S^0 \longrightarrow MSpin$ is a map of ring spectra, we have $\eta \cdot \pi^I\rho^0(b^n) = 0$, so the elements $\pi^I\rho^0(b^n)$ are even for all partitions $I \neq \emptyset$. In particular, $\pi^I(b)$ is even for all $I \neq \emptyset$.

Let p_I be the Pontryagin class corresponding to a partition I. Anderson, Brown and Peterson show that the Chern character ch $(\pi^I(x) \otimes \mathbf{C}) = p_I(x) +$ (higher terms), for $x \in \Omega^*_{Spin}(X)$. It implies that $p_I(b)$ are even elements for all $I \neq \emptyset$. The Pontryagin classes p_2 and $p_{1,1} = p_1^2$ determine the oriented cobordism ring Ω_*^{SO} in dimension 8, so the Bott element goes to an even element in Ω_8^{SO} under the natural map $MSpin \longrightarrow MSO$. Thus the composition $MSpin \longrightarrow MSO \longrightarrow MO$ takes the Bott element b to zero. □

We define the K–theory spectra with singularities KO^{Σ_1}, KO^{η} and KO^{Σ_2}, as the cofibers:

$$\begin{aligned}
KO \wedge S^0 \xrightarrow{1\wedge 2} KO \wedge S^0 \xrightarrow{\pi} KO \wedge M(2) &= KO^{\Sigma_1}\\
KO \wedge S^1 \xrightarrow{1\wedge \eta} KO \wedge S^0 \xrightarrow{\pi} KO \wedge \Sigma^{-2}\mathbf{CP}^2 &= KO^{\eta}\\
KO^{\eta} \wedge S^0 \xrightarrow{1\wedge 2} KO^{\eta} \wedge S^0 \xrightarrow{\pi} KO^{\eta} \wedge M(2) &= KO^{\Sigma_2}
\end{aligned}$$

It is easy to derive (see, for example, [18]) the following statement.

Corollary 5.4 *The spectrum KO^{η} is homotopy equivalent (as a ring spectrum) to the spectrum K, classifying the complex K–theory, and the spectrum KO^{Σ_2} is homotopy equivalent (as a ring spectrum) to the spectrum $K(1)$ classifying the first Morava K–theory.*

We introduce also the notation:

$$ko^{\Sigma_1} = ko \wedge M(2), \quad ko\langle 2\rangle^{\Sigma_1} = ko\langle 2\rangle \wedge M(2), \mathbf{H}(\mathbf{Z}/2)^{\Sigma_1} = \mathbf{H}(\mathbf{Z}/2) \wedge M(2);$$

$$ko^{\eta} = ko \wedge \Sigma^{-2}\mathbf{CP}^2, ko\langle 2\rangle^{\eta} = ko\langle 2\rangle \wedge \Sigma^{-2}\mathbf{CP}^2, \ \mathbf{H}(\mathbf{Z}/2)^{\eta} = \mathbf{H}(\mathbf{Z}/2) \wedge \Sigma^{-2}\mathbf{CP}^2;$$

$$ko^{\Sigma_2} = ko^{\eta} \wedge M(2), \ ko\langle 2\rangle^{\Sigma_2} = ko\langle 2\rangle^{\eta} \wedge M(2), \mathbf{H}(\mathbf{Z}/2)^{\Sigma_2} = \mathbf{H}(\mathbf{Z}/2)^{\eta} \wedge M(2).$$

Let I be a partition as above. The KO–characteristic numbers

$$\pi^I \colon MSpin \longrightarrow KO$$

which are lifted to the connective cover $ko\langle 4n(I)\rangle$ give the characteristic numbers

$$\begin{aligned}
\pi^I_{\Sigma_1} &= \pi^I \wedge 1\colon MSpin^{\Sigma_1} = MSpin \wedge M(2) \longrightarrow KO \wedge M(2) &&= KO^{\Sigma_1},\\
\pi^I_{\eta} &= \pi^I \wedge 1\colon MSpin^{\eta} = MSpin \wedge \Sigma^{-2}\mathbf{CP}^2 \longrightarrow KO \wedge \Sigma^{-2}\mathbf{CP}^2 &&= KO^{\eta},\\
\pi^I_{\Sigma_2} &= \pi^I \wedge 1\colon MSpin^{\Sigma_1} = MSpin \wedge Y \longrightarrow KO \wedge Y &&= KO^{\Sigma_2},
\end{aligned}$$

together with the lifts to the corresponding connective covers:

$$\begin{aligned}
\bar\pi^I_{\Sigma_1} &= \bar\pi^I \wedge 1\colon \quad MSpin^{\Sigma_1} \longrightarrow ko\langle 4n(I)\rangle \wedge M(2) &&= ko\langle 4n(I)\rangle^{\Sigma_1}\\
\bar\pi^I_{\eta} &= \bar\pi^I \wedge 1\colon \quad MSpin^{\eta} \longrightarrow ko\langle 4n(I)\rangle \wedge \Sigma^{-2}\mathbf{CP}^2 &&= ko\langle 4n(I)\rangle^{\eta}\\
\bar\pi^I_{\Sigma_2} &= \bar\pi^I \wedge 1\colon \quad MSpin^{\Sigma_1} \longrightarrow ko\langle 4n(I)\rangle \wedge Y &&= ko\langle 4n(I)\rangle^{\Sigma_2}
\end{aligned}$$

Now we would like to identify the spectra ko^{Σ}, $ko\langle 4n(I)\rangle^{\Sigma}$ for $\Sigma = \Sigma_2$ or η for those partitions I, $1 \notin I$. It is enough to determine a homotopy type of the spectra ko^{Σ} and $ko\langle 2\rangle^{\Sigma}$.

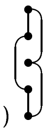
Let $\mathcal{A}(1)$ be a subalgebra of the Steenrod algebra $\mathcal{A}_2$ generated by $1, Sq^1, Sq^2$. The cohomology $H^*(ko)$ as a module over Steenrod algebra is $H^*(ko) \cong \mathcal{A}_2 \otimes_{\mathcal{A}(1)} \mathbf{Z}/2$. The Künneth homomorphism

$$H^*(ko \wedge X) \cong (\mathcal{A}_2 \otimes_{\mathcal{A}(1)} \mathbf{Z}/2) \otimes H^*(X) \cong \mathcal{A}_2 \otimes_{\mathcal{A}(1)} H^*(X)$$

and the ring change formula $\mathrm{Hom}_{\mathcal{A}_2}(\mathcal{A}_2 \otimes_{\mathcal{A}(1)} M, N) \cong \mathrm{Hom}_{\mathcal{A}(1)}(M, N)$ turn the ordinary mod 2 Adams spectral sequence into the one with the E_2–term

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}/2) \Longrightarrow ko_{t-s}(X).$$

Here we use regular conventions to draw the cell-diagrams for the spectra in question. Recall that

$H^*(ko) = \mathcal{A}_2 \otimes_{\mathcal{A}(1)} \bullet$ and $H^*(ko\langle 2\rangle) = \mathcal{A}_2 \otimes_{\mathcal{A}(1)}$  (the joker).

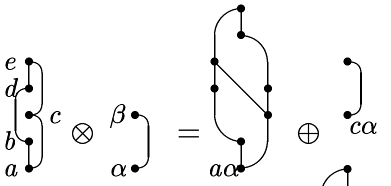
Let $k(1)$ be a connected cover of the first Morava k–theory spectrum $K(1)$ with the coefficient ring $k(1)_* \cong \mathbf{Z}/2[v_1]$. Here is the result for the spectra ko^{η}, ko^{Σ_2}:

Lemma 5.5 *There are the following homotopy equivalences*

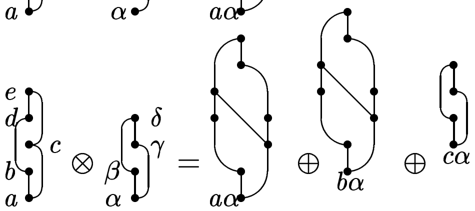
$$ko^{\eta} \cong ku, \qquad ko^{\Sigma_2} \cong k(1) \tag{13}$$

The following result one can prove by an easy computation:

Lemma 5.6 *There are isomorphisms of the following $\mathcal{A}(1)$–modules:*



(14)



(15)

Using the Adams spectral sequence for the spectra $ko\langle 2\rangle^{\eta}$ and $ko\langle 2\rangle^{\Sigma_2}$, one obtains the following result:

Lemma 5.7 *There are the following homotopy equivalences*

$$\begin{aligned} ko\langle 2\rangle^{\eta} &\cong \mathbf{H}(\mathbf{Z}/2) \vee \Sigma^2 ku, \\ ko\langle 2\rangle^{\Sigma_2} &\cong \mathbf{H}(\mathbf{Z}/2) \vee \Sigma\mathbf{H}(\mathbf{Z}/2) \vee \Sigma^2 k(1). \end{aligned} \tag{16}$$

It is convenient to denote:

$$\widehat{ko} = \prod_{\substack{1\notin I,\\ n(I)\neq 0,\ \text{even}}} \Sigma^{4n(I)} ko \times \prod_{\substack{1\notin I,\\ n(I)\ \text{odd}}} \Sigma^{4n(I)-4} ko\langle 2\rangle, \quad \text{and} \quad \mathbf{H}\widehat{(\mathbf{Z}/2)} = \prod_k \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2).$$

The spectra $\widehat{ko}^{\Sigma}$ are defined similarly for $\Sigma = \Sigma_1$, Σ_2, Σ_3 or η. Theorem 5.2 implies the following result:

Corollary 5.8 *There is the following homotopy equivalence of 2–local spectra:*

$$F^{\Sigma}\colon MSpin^{\Sigma} \longrightarrow ko^{\Sigma} \vee \widehat{ko}^{\Sigma} \vee \mathbf{H}(\widehat{\mathbf{Z}}/2)^{\Sigma}, \quad \textit{where } \Sigma = \Sigma_1,\ \Sigma_2,\ \textit{or } \eta.$$

Remark 5.9 The coefficient groups of the K–theories KO^{Σ} are well-known in homotopy theory. We give the table of the groups $KO_n^{\Sigma_1} = KO_n(pt, \mathbf{Z}/2)$ for convenience:

	0	1	2	3	4	5	6	7	8	
$KO_n^{\Sigma_1} = KO_n(pt, \mathbf{Z}/2)$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/4$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0	0	$\mathbf{Z}/2$	$\cdots$

We emphasize that $KO_{8k+2}(pt, \mathbf{Z}/2) \cong \mathbf{Z}/4$.

Remark 5.10 We notice that there is a natural transformation

$$r : \Omega_*^{Spin,\eta}(\cdot) \longrightarrow \Omega_*^{Spin^c}(\cdot).$$

Indeed, let M be an η–manifold, ie, $\partial M \cong \beta_2 M \times P_2$, where $P_2 = S^1$ with nontrivial *Spin* structure. Then P_2 is a boundary as a $Spin^c$–manifold, even more, $P_2 = \partial D^2$. Then the correspondence

$$(M, \partial M = \beta_2 M \times P_2) \mapsto (N = M \cup -\beta_2 M \times D^2)$$

determines the transformation r. In particular, r gives a map of classifying spectra: $r\colon MSpin^{\eta} \longrightarrow MSpin^c$. It is easy to see that there is a commutative diagram

$$\begin{array}{ccc} MSpin^{\eta} & \xrightarrow{\quad r \quad} & MSpin^c \\ \cong\downarrow & & \cong\downarrow \\ MSpin \wedge \Sigma^{-2}\mathbf{CP}^2 & \xrightarrow{Id\wedge\Sigma^{-2}j} & MSpin \wedge \Sigma^{-2}\mathbf{CP}^{\infty} \end{array}$$

where $j\colon \mathbf{CP}^2 \longrightarrow \mathbf{CP}^{\infty}$ is the standard embedding. There are simple geometric reasons which imply that the transformation r is not multiplicative. In fact, it is very similar to the transformation $\Omega_*^{SU,\eta}(\cdot) \longrightarrow \Omega_*^{U}(\cdot)$, where $\Omega_*^{SU,\eta}(\cdot)$ is the SU–cobordism theory with η–singularities. The cobordism theory $\Omega_*^{SU,\eta}(\cdot)$ may be easily identified with the Conner–Floyd theory $W(\mathbf{C}, 2)_*(\cdot)$, see [19].

6 The spectrum $MSpin^{\Sigma_3}$

Let $A\colon \Sigma^8 M(2) \longrightarrow M(2)$ be the Adams map. Let $V(1)$ be a cofiber:

$$\Sigma^8 M(2) \xrightarrow{A} M(2) \xrightarrow{p} V(1).$$

The objective of this section is to prove the following result.

Theorem 6.1 *There is a homotopy equivalence of spectra localized at 2:*

$$MSpin^{\Sigma_3} \cong MSpin \wedge \Sigma^{-2}\mathbf{CP}^2 \wedge V(1). \tag{17}$$

Proof Recall that the Adams map A induces a multiplication by the Bott element in KO_* and connected covers ko_* and $ko\langle 2\rangle_*$. Let, as above, $Y = \Sigma^{-2}\mathbf{CP}^2 \wedge M(2)$. We apply the Cartan formula (12)

$$\begin{tikzcd} MSpin \wedge MSpin \wedge Y \arrow[r, "\mu\wedge 1"] \arrow[d, "\sum(\pi^{I_1}\wedge\pi^{I_2})\wedge 1"'] & MSpin \wedge Y \arrow[d, "\pi^I"'] \\ KO \wedge KO \wedge Y \arrow[r, "\mu'\wedge 1"] & KO \wedge Y \end{tikzcd}$$

to obtain the formula:

$$\begin{tikzcd} MSpin \wedge MSpin^{\Sigma_2} \arrow[r, "\mu_{\Sigma_2}"] \arrow[d, "\sum(\pi^{I_1}\wedge\pi^{I_2}_{\Sigma_2})"'] & MSpin^{\Sigma_2} \arrow[d, "\pi^I_{\Sigma_2}"'] \\ KO \wedge KO^{\Sigma_2} \arrow[r, "\mu'_{\Sigma_2}"] & KO^{\Sigma_2} \end{tikzcd} \quad \text{or } \mu_{\Sigma_2}\pi^I_{\Sigma_2} = \sum_{I_1+I_2=I} \mu'_{\Sigma_2}(\pi^{I_1}\wedge\pi^{I_2}_{\Sigma_2}). \tag{18}$$

Let X be a space, $x \in \Omega^{Spin}_*(X)$, $b \in \Omega^{Spin}_8$ be the Bott element. Then $\mu_{\Sigma_2}(b,x) = b\cdot x \in \Omega^{Spin}_{*+8}(X)$.

Lemma 6.2 *The KO^{Σ_2}–characteristic numbers $\pi^I_{\Sigma_2}\colon MSpin^{\Sigma_2} \longrightarrow KO^{\Sigma_2}$ commutes with a multiplication by the Bott element, ie, $\pi^I_{\Sigma_2}(b\cdot x) = b\cdot\pi^I_{\Sigma_2}(x)$.*

Proof The Cartan formula (18) and Lemma 5.3 gives:

$$\pi^I_{\Sigma_2}(b\cdot x) = \sum_{I_1+I_2=I}(\pi^{I_1}(b)\pi^{I_2}_{\Sigma_2}(x) = b\cdot\pi^I_{\Sigma_2}(x) + \pi^{(1)}(b)y + \pi^{(1,1)}(b)z =$$
$$b\cdot\pi^I_{\Sigma_2}(x) + (2c)\cdot y + (2d)\cdot z = b\cdot\pi^I_{\Sigma_2}(x) + c\cdot(2y) + d\cdot(2z) = b\cdot\pi^I_{\Sigma_2}(x).$$

Here $\pi^{(1)}(b) = 2c$, $y,z \in KO^{\Sigma_2}_*(X)$, and $\pi^{(1,1)}(b) = 2d$ by Lemma 5.3. We note that $2y = 0$ and $2z = 0$ since the cobordism theory $\Omega^{Spin,\Sigma_2}_*(\cdot)$ has an admissible product structure by Theorem 4.1. $\square$

Let I be a partition, and $1 \notin I$. The map $\pi^I_{\Sigma_2}\colon MSpin_{\Sigma_2} \longrightarrow KO_{\Sigma_2}$ lifts to connective cover: $\bar\pi^I_{\Sigma_2}\colon MSpin^{\Sigma_2} \longrightarrow ko\langle 4n(I)\rangle^{\Sigma_2}$. Let $S^8 \xrightarrow{b} MSpin$ be a map representing the Bott element $b \in \Omega^{Spin}_8$. We denote by $\cdot b$ the composition

$$\Sigma^8 MSpin = S^8 \wedge MSpin^{\Sigma_2} \xrightarrow{b\wedge 1} MSpin \wedge MSpin^{\Sigma_2} \xrightarrow{\mu''} MSpin^{\Sigma_2}.$$

Note that the diagram

$$\begin{array}{ccc} \Sigma^8 MSpin^{\Sigma_2} & \xrightarrow{\quad\cdot b\quad} & MSpin^{\Sigma_2} \\ \cong\downarrow & & \cong\downarrow \\ \Sigma^8 MSpin \wedge M(2) \wedge \Sigma^{-2}\mathbf{CP}^2 & \xrightarrow{\cdot b\wedge 1\wedge 1} & MSpin \wedge M(2) \wedge \Sigma^{-2}\mathbf{CP}^2 \end{array} \tag{19}$$

commutes since $MSpin^{\Sigma_2}$ is a module over $MSpin$.

Lemma 6.3 *Let I be a partition, so that $1 \notin I$. The following diagrams commute:*

$$\begin{array}{ccc} \Sigma^8 MSpin^{\Sigma_2} & \xrightarrow{\quad\cdot b\quad} & MSpin^{\Sigma_2} \\ \downarrow \Sigma^8\pi^I_{\Sigma_2} & & \downarrow \pi^I_{\Sigma_2} \\ \Sigma^8 ko\langle n(I)\rangle^{\Sigma_2} & & ko\langle n(I)\rangle^{\Sigma_2} \\ \downarrow\cong & & \downarrow\cong \\ ko\langle n(I)\rangle \wedge \Sigma^8 M(2) \wedge \Sigma^{-2}\mathbf{CP}^2 & \xrightarrow{1\wedge A\wedge 1} & ko\langle n(I)\rangle \wedge M(2) \wedge \Sigma^{-2}\mathbf{CP}^2 \end{array}$$

$$\begin{array}{ccc} \Sigma^8 MSpin^{\Sigma_2} & \xrightarrow{\quad\cdot b\quad} & MSpin^{\Sigma_2} \\ \downarrow \Sigma^8 z_k^{\Sigma_2} & & \downarrow \Sigma^8 z_k^{\Sigma_2} \\ \Sigma^{8+\deg z_k}\mathbf{H}(\mathbf{Z}/2) \wedge M(2) \wedge \Sigma^{-2}\mathbf{CP}^2 & & \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)\wedge M(2)\wedge\Sigma^{-2}\mathbf{CP}^2 \\ \downarrow\cong & & \downarrow\cong \\ \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)\wedge\Sigma^8 M(2)\wedge\Sigma^{-2}\mathbf{CP}^2 & \xrightarrow{1\wedge A\wedge 1} & \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)\wedge M(2)\wedge\Sigma^{-2}\mathbf{CP}^2 \end{array} \tag{20}$$

Proof A commutativity of the first diagram follows from Lemma 6.2 and the diagram (19). Recall that a projection of the Bott element into the homotopy group of $\Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)$ is zero. Let X be a finite spectrum. The map

$$1\wedge A\wedge 1\wedge 1\colon \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)\wedge\Sigma^8 M(2)\wedge\Sigma^{-2}\mathbf{CP}^2\wedge X \longrightarrow \Sigma^{\deg z_k}\mathbf{H}(\mathbf{Z}/2)\wedge M(2)\wedge\Sigma^{-2}\mathbf{CP}^2\wedge X$$

in homotopy coincides with the homomorphism in mod 2 homology groups

$$\Sigma^{\deg z_k} H_*(\Sigma^8 M(2)\wedge\Sigma^{-2}\mathbf{CP}^2\wedge X) \xrightarrow{A_*\otimes 1\otimes 1} \Sigma^{\deg z_k} H_*(M(2)\wedge\Sigma^{-2}\mathbf{CP}^2\wedge X)$$

and is trivial for any space X since A has the Adams filtration 4. It implies that $1\wedge A\wedge 1$ is a trivial map. A commutativity of (20) now follows. □

To complete the proof of Theorem 6.1 we notice that Lemmas 6.3 and 6.2 give the commutative diagram

$$\begin{array}{ccccc} \Sigma^8 MSpin^{\Sigma_2} & \xrightarrow{\cdot b} & MSpin^{\Sigma_2} & \xrightarrow{\pi_3} & MSpin^{\Sigma_3} \\ \downarrow \Sigma^8 F^{\Sigma_2} & & \downarrow F^{\Sigma_2} & & \downarrow F^{\Sigma_3} \\ \Sigma^8 MSpin^\eta\wedge\Sigma^8 M(2) & \xrightarrow{1\wedge A} & MSpin^\eta\wedge M(2) & \xrightarrow{p} & MSpin^\eta\wedge V(1) \end{array}$$

where the map F^{Σ_3} exists since the both rows are cofibrations. The five-lemma implies that F^{Σ_3} is a homotopy equivalence. □

Corollary 6.4 *The spectrum* $KO^{\Sigma_3} = KO \wedge \Sigma^{-2}\mathbf{CP}^2 \wedge V(1)$ *is a contractible spectrum.*

Remark 6.5 The connective spectrum ko^{Σ_3} is of some interest. It is certainly not contractible, and it is very easy to see that

$$ko^{\Sigma_3}_j = \begin{cases} \mathbf{Z}/2 & \text{if } j = 0, 2, 4, 6, \\ 0 & \text{otherwise,} \end{cases}$$

and the Postnikov tower of ko^{Σ_3} has the operation Q_1 as its k–invariants.

The technique we used above may be applied to prove the following result:

Corollary 6.6 *There is such admissible product structure* $\mu^{(2)}$ *of the spectrum* $MSpin^{\Sigma_2}$*, so that the map* $\bar{\pi}^0_{\Sigma_2}\colon MSpin^{\Sigma_2} \longrightarrow ko^{\Sigma_2} = k(1)$ *is a ring spectra map, moreover, there is an inverse ring spectra map* $\rho^0_{\Sigma_2}\colon ko^{\Sigma_2} \longrightarrow MSpin^{\Sigma_2}$*. In other words,* ko^{Σ_2} *splits off of the spectrum* $MSpin^{\Sigma_2}$ *as a ring spectrum.*

7 Surgery Lemma for Σ–manifolds

7.1 A Riemannian metric on a Σ–manifold Here we describe what do we mean by a Riemannian metric on manifold with singularities. We consider the case when a manifold has of at most three singularities, $\Sigma_3 = (P_1, P_2, P_3)$. We denote $\Sigma_1 = (P_1)$, $\Sigma_2 = (P_1, P_2)$. We assume that there are given Riemannian metrics g_{P_i} on the manifolds P_i, $i = 1, 2, 3$. As we mentioned earlier, the metrics g_{P_i} are not assumed to be psc-metrics.

If M is a Σ_3–manifold, we assume that it is given a decomposition of the boundary ∂M:

$$\begin{aligned} \partial M = {} & (\beta_1 M \times P_1 \cup \beta_2 M \times P_2 \cup \beta_2 M \times P_2) \cup \ \beta_{123} M \times P_1 \times P_3 \times P_2 \times D^2 \\ \cup \ & (\beta_{12} M \times P_1 \times P_2 \times I_{12} \cup \beta_{23} M \times P_2 \times P_3 \times I_{23} \cup \beta_{13} M \times P_1 \times P_3 \times I_{13}) \end{aligned}$$

glued together as it is shown on Figure 7 (a). We start with a Riemannian metric g_{123} on the manifold $\beta_{123} M$. We assume that the manifold

$$\beta_{123} M \times P_1 \times P_2 \times P_3 \times D^2$$

has product metric $g_{123} \times g_{P_1} \times g_{P_2} \times g_{P_3} \times g_0$, where g_0 is the standard flat metric on the disk D^2.

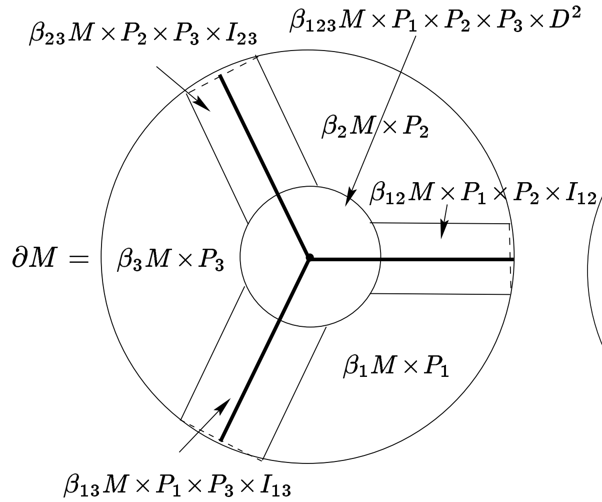
Besides, we assume that the manifold $\beta_{123}M \times P_1 \times P_2 \times P_3$, being common boundary of the manifolds

$$\beta_{12}M \times P_1 \times P_2, \quad \beta_{13}M \times P_1 \times P_3, \quad \text{and} \quad \beta_{23}M \times P_2 \times P_3,$$

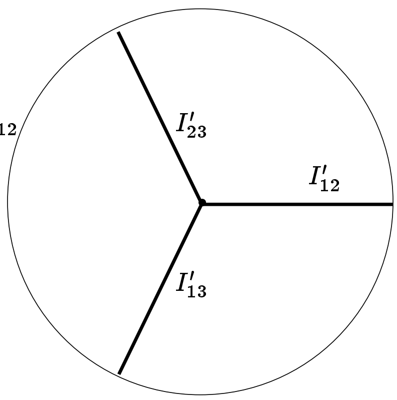
is embedded together with the colors (see Figure 7 (a)):

$$\beta_{123}M \times P_1 \times P_2 \times P_3 \times I'_{12} \subset \beta_{12}M \times P_1 \times P_2,$$
$$\beta_{123}M \times P_1 \times P_2 \times P_3 \times I'_{13} \subset \beta_{13}M \times P_1 \times P_3,$$
$$\beta_{123}M \times P_1 \times P_2 \times P_3 \times I'_{23} \subset \beta_{23}M \times P_2 \times P_3.$$

Here I'_{ij} are the intervals embedded into the flat disk D^2 as it is shown on Figure 7 (b).



(a) The decomposition of ∂M



(b) Flat disk D^2

Figure 7

Let g_{ij} be metrics on the manifolds $\beta_{ij}M$. We assume that the product metric

$$g_{ij} \times g_{P_i} \times g_{P_j} \quad \text{on the manifold} \quad \beta_{ij}M \times P_i \times P_j$$

coincides with the product metric on the color $\beta_{123}M \times P_1 \times P_2 \times P_3 \times I'_{ij}$ near its boundary. Finally if g_i is a metric in $\beta_i M$ $(i = 1,2,3)$, then we assume that the product metric $g_i \times g_{P_i}$ on $\beta_i M \times P_i$ coincides with the above product metrics on the manifold $\beta_{ij}M \times g_{P_i} \times g_{P_j} \times I_{ij}$. Furthermore, the product metric $g_i \times g_{P_i}$ on $\beta_i M \times P_i$ restricted on the manifold

$$(\beta_{123}M \times P_1 \times P_2 \times P_3 \times D^2) \cap (\beta_i M \times P_i)$$

coincides with the product metric $g_{123}\times g_{P_1}\times g_{P_2}\times g_{P_3}\times g_0$. Finally the metric g on the manifold M is assumed to be product metric near the boundary ∂M. Let M, as above be a Σ–manifold with the same singularities $\Sigma=(P_1,P_2,P_3)$. We say that a *metric g on M is of positive scalar curvature*, if, besides the above conditions, the metrics g on M, g_i on $\beta_i M$, g_{ij} on $\beta_{ij}M$, and g_{123} on $\beta_{123}M$ have positive scalar curvature functions.

7.2 Surgery theorem in the case of manifolds without singularities
Here we briefly review key results on the connection between positive scalar curvature metric and surgery for manifolds without singularities. The first basic result is due to Gromov–Lawson [9, Theorem A] and to Schoen–Yau [27]. A detailed "textbook" proof may be found in [25, Theorem 3.1].

Theorem 7.1 (Gromov–Lawson [9], Schoen–Yau [27]) *Let M be a closed manifold, not necessarily connected, with a Riemannian metric of positive scalar curvature, and let M' is obtained from M by a surgery of codimension ≥ 3. Then M' also admits a metric of positive scalar curvature.*

To get started with Σ–manifolds we need an "improved version" of Theorem 7.1 which is due to Gajer [8].

Theorem 7.2 (Gajer [8]) *Let M be a closed manifold, not necessarily connected, with a Riemannian metric g of positive scalar curvature, and let M' is obtained from M by a surgery of codimension ≥ 3. Then M' also admits a metric g' of positive scalar curvature. Furthermore, let W be the trace of this surgery (ie, a cobordism W with $\partial W = M\sqcup -M'$). Then there is a positive scalar curvature metric $\bar{g}$ on W, so that $\bar{g}=g+dt^2$ near M and $\bar{g}=g'+dt^2$ near M'.*

In order to use the above Surgery Theorems, one has to specify certain structure of manifolds under consideration. This structure (known as γ–structure) is determined by the fundamental group $\pi_1(M)$, and the Stiefel–Whitney classes $w_1(M)$, and $w_2(M)$. Indeed, it is well-known that the fundamental group π is crucially important for the existence question. Then there is clear difference when a manifold M is oriented or not (which depends on $w_1(M)$). On the other hand, a presence of the *Spin*–structure (which means that $w_2(M)=0$) gives a way to use the Dirac operator on M to control the scalar curvature via the vanishing formulas. Stolz puts together those invariants to define a γ–structure, see [31]. In the case we are interested in, all manifolds are simply-connected and *Spin*, thus we will state only a relevant Bordism Theorem (see, say, [25, Theorem 4.2] for a general result).

Theorem 7.3 *Let M be a simply connected Spin manifold,* $\dim M \geq 5$. *Then M admits a metric of positive scalar curvature if and only if there is some simply-connected Spin–manifold M' of positive scalar curvature in the same Spin–bordism class.*

7.3 Surgery theorem in the case of manifolds with singularities Let M be a Σ–manifold with $\Sigma = (P_i)$, (P_i, P_j) or (P_i, P_j, P_k). Here P_i are arbitrary closed manifolds. Let $\dim M = n$, and $\dim P_i = p_i$, $i = 1, 2, 3$. Then we denote $\dim \beta_i M = n_i = n - p_i - 1$, $\dim \beta_{ij} M = n_{ij} = n - p_i - p_j - 2$, and $\dim \beta_{123} M = n_{123} = n - p_1 - p_2 - p_3 - 3$. The manifolds $\beta_i M$, $\beta_{ij} M$ and $\beta_{ijk} M$ are called Σ–strata of M.

We say that a Σ–manifold M is *simply connected* if M itself is simply connected and all Σ–strata of M are simply connected manifolds.

Theorem 7.4 *Let M be a simply connected Spin Σ–manifold,* $\dim M = n$, *so that all Σ–strata manifolds are nonempty, and satisfying the following conditions:*

(1) *if $\Sigma = (P_i)$, then $n - p_i \geq 6$;*

(2) *if $\Sigma = (P_i, P_j)$, then $n - p_i - p_j \geq 7$;*

(3) *if $\Sigma = (P_i, P_j, P_k)$, then $n - p_i - p_j - p_k \geq 8$.*

Then M admits a positive scalar curvature if and only if there is some simply-connected $Spin^{\Sigma}$–manifold M' of positive scalar curvature in the same $Spin^{\Sigma}$–bordism class.

Remark 7.5 The role of the manifolds M and M' are not symmetric here. For instance, it is important that M has all Σ–strata manifolds nonempty, however, the manifold M' may have empty singularities.

Proof (1) Let W be a $Spin^{\Sigma}$–cobordism between M and M'. Then $\beta_i W$ is a *Spin*–cobordism between $\beta_i M$ and $\beta_i M'$. By condition, $\beta_i M'$ is simply connected, and $\dim \beta_i M' = \dim M \geq 5$. We notice that there is a sequence of surgeries on the manifold $\beta_i W$ (relative to the boundary $\partial \beta_i M'$) so that the resulting manifold is 2–connected (see an argument given in [9, Proof of Theorem A]). Let V be a trace of this surgery. Then its boundary is decomposed as

$$\partial V = \beta_i W \cup (\beta_i M \times I) \cup (\beta_i M' \times I) \cup L_i.$$

We glue together the manifolds W and $-V\times P_i$:

$$W' := W\cup_{\beta_i W\times P_i} -V\times P_i.$$

Then the boundary of WW' (as a $Spin^{\Sigma}$–manifold) is

$$\delta W' = (M\cup(\beta_i M\times I\times P_i))\sqcup(M'\cup(\beta_i M'\times I\times P_i))\cong M\sqcup M',$$
$$\text{and}\quad \beta_i W' = L_i,\quad \text{with}\quad \partial L_i = \beta_i M\sqcup\beta_i M'.$$

Now we use Theorem 7.2 to "push" a positive scalar curvature metric from $\beta_i M'$ through L_i to $\beta_i M$ keeping it a product metric near the boundary. At this point a psc-metric g_i on L_i may be such that the product metric $g_i\times g_{P_i}$ is not of positive scalar curvature. We find $\epsilon>0$ so that the product metric $\epsilon g_i\times g_{P_i}$ has positive scalar curvature, and then we attach one more cylinder $L_i\times P_i\times[0,a]$ with the metric

$$g_i(t) := \frac{a-t}{a}g_i\times g_{P_i}+\frac{t}{a}\epsilon g_i\times g_{P_i}+dt^2.$$

We use metric $g_i(t)$ to fit together the metric already constructed on W'' with the metric on $L_i\times P_i\times[0,a]$. In particular, there is $a>0$ so that the restriction of $g_i(t)$ on $\beta_i M'\times P_i\times[0,a]$ has positive scalar curvature (since an isotopy of positive scalar curvature metrics implies concordance). By small perturbation, we can change $g_i(t)$, so that it has positive scalar curvature and it is a product near the boundary. Then we do surgeries on the interior of W' to make it 2–connected. Let W'' be the resulting manifold. In particular, $\beta_i W''=\beta_i W' = L_i$. Finally we use "push" a positive scalar curvature metric from M' to M through W'' keeping it a product metric near the singular stratum $\beta_i\beta_i W' = L_i$.

(2) Let M be a simply connected $Spin$ Σ–manifold, with $\Sigma=(P_i,P_j)$, and $n-p_i-p_j\geq 7$. By condition, the singular stratum $\beta_{ij}M\neq\emptyset$. Let W be a $Spin^{\Sigma}$–cobordism between M and M'. In particular, we have $\partial\beta_{ij}W = \beta_{ij}M\sqcup\beta_{ij}M'$. Recall that $\beta_{ij}W\times P_i\times P_j$ is embedded to the union

$$(\beta_i W\times P_i)\cup(\beta_j W\times P_j)$$

together with the colors

$$\beta_{ij}W\times P_i\times P_j\times[-\epsilon,\epsilon]$$

By conditions, the manifolds $\beta_{ij}M$ $\beta_{ij}M'$ are simply connected, and $\dim\beta_{ij}M = \dim\beta_{ij}M'\geq 5$. As above, there is a surgery on $\beta_{ij}W$ (relative to the boundary $\partial\beta_{ij}W=\beta_{ij}M\sqcup\beta_{ij}M'$) so that a resulting manifold is 2–connected. Let V_{ij} be the trace of this surgery:

$$\partial V = \beta_{ij}W\cup(\beta_{ij}M\times I)\cup(\beta_{ij}M'\times I)\cup L_{ij}.$$

We glue together the manifolds

$$W \quad \text{and} \quad -V\times[-\epsilon,\epsilon]\times P_i\times P_j$$

to obtain a manifold W', where we identify

$$\beta_{ij}W\times P_i\times P_j\times[-\epsilon,\epsilon]\subset(\beta_iW\times P_i)\cup(\beta_jW\times P_j) \quad \text{and}$$
$$-\beta_{ij}W\times P_i\times P_j\times[-\epsilon,\epsilon]\subset -\partial V\times[-\epsilon,\epsilon]\times P_i\times P_j,$$

see Figure 8.

The resulting manifold W' (after smoothing corners and extending metric according with the Surgery Theorem construction) is such that $\beta_{ij}W'=L_{ij}$ is 2–connected cobordism between $\beta_{ij}M$ and $\beta_{ij}M'$. Thus we can "push" a positive scalar curvature metric from $\beta_{ij}M'$ to $\beta_{ij}M$ through the cobordism $\beta_{ij}W'$. Thus we obtain a psc-metric g_{ij} on $\beta_{ij}M'$ which is a product near boundary. In general, the product metric $g_{ij}\times g_{P_i}\times g_{P_j}$ on $\beta_{ij}W\times P_i\times P_j$ is not of positive scalar curvature. Then we have to attach one more cylinder

$$\beta_{ij}W'\times[-\epsilon,\epsilon]\times I\times P_i\times P_j$$

to "scale" the metric $g_{ij}\times g_{P_i}\times g_{P_j}$ to a positive scalar curvature metric $\epsilon_{ij}g_{ij}\times g_{P_i}\times g_{P_j}$ through an appropriate homotopy.

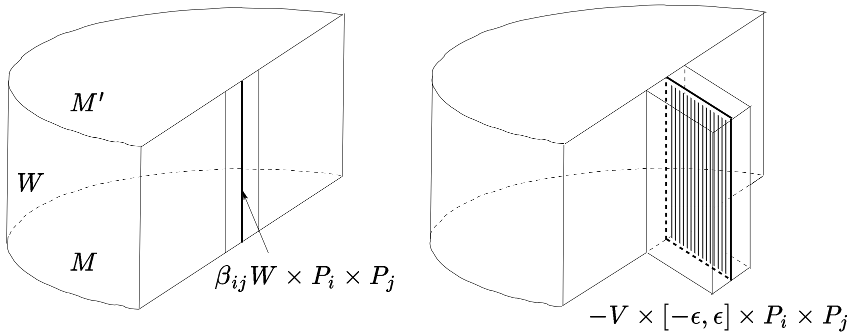


Figure 8

Then we consider the manifolds β_iW' and β_jW'. Again, we perform surgeries on the interior of β_iW', β_jW' to get 2–connected manifolds L_i and L_j. Let V_i, V_j be the traces of these surgeries:

$$\partial V_i=\beta_iW'\cup\beta_{ij}W'\times P_j\cup L_i,\quad \partial V_j=\beta_iW'\cup\beta_{ij}W'\times P_i\cup L_j.$$

Now we attach the manifolds $-V_i\times P_i$ and $-V_j\times V_j$ to W' by identifying

$$\beta_iW'\times P_i\subset W' \qquad \text{and} \quad \beta_jW'\times P_j\subset W' \qquad \text{with}$$
$$-\beta_iW'\times P_i\subset-\partial V_i, \qquad \text{and} \quad -\beta_jW'\times P_j\subset-\partial V_j$$

respectively. Let W'' be the resulting manifold (after an appropriate smoothing and extending a metric), see Figure 9. Notice that W'' is still a $Spin^{\Sigma}$–cobordism between M and M'.

This procedure combined with an appropriate metric homotopy gives W'' together with a metric g'' on W'', so that it is a product metric near the boundary, its restriction on M' has positive scalar curvature, and its restriction on the manifolds

$$\beta_i W'' \times P_i, \quad \beta_j W'' \times P_j, \quad \beta_i W'' \times [-\epsilon, \epsilon] \times P_i \times P_j$$

are psc-metrics

$$g_i \times g_{P_i}, \quad g_j \times g_{P_j} \quad g_{ij} \times g_{P_i} \times g_{P_j} + dt^2$$

respectively (for some psc-metrics g_i, g_j, g_{ij}). It remains to perform surgeries on the interior of W'' to get a 2–connected manifold, and finally push a psc-metric from M' to M relative to the boundary.

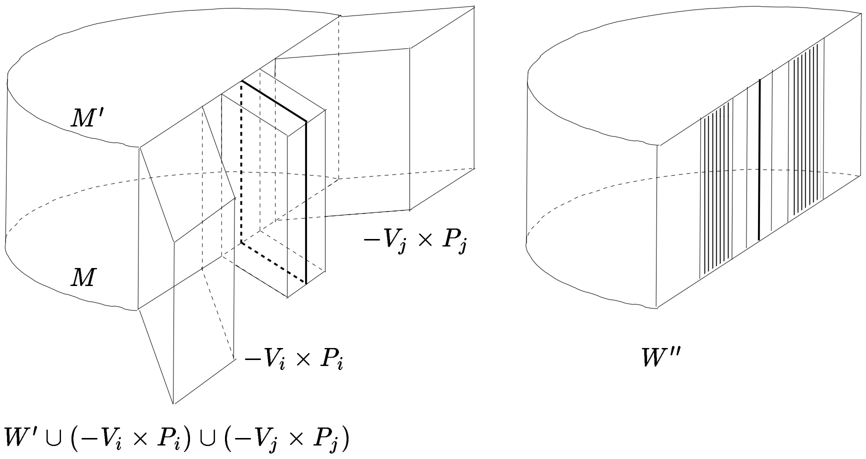


Figure 9

A proof of (3) is similar. □

8 Proof of Theorem 1.1

First we recall the main construction from [29]. Let $G = PSp(3) = Sp(3)/(\mathbf{Z}/2)$, where $\mathbf{Z}/2$ is the center of $Sp(3)$. Let g_0 be the standard metric on $\mathbf{HP}^2$. Recall that the group G acts on $\mathbf{HP}^2$ by isometries of the metric g_0. Let $E \longrightarrow B$ be a geometric $\mathbf{HP}^2$–bundle, ie, $E \longrightarrow B$ is a bundle with a fiber $\mathbf{HP}^2$ and structure group G. Each geometric $\mathbf{HP}^2$–bundle $E \longrightarrow B$ is given by a map

$f\colon B \longrightarrow BG$ by taking first the associated principal G–bundle, and then by "inserting" $\mathbf{HP}^2$ as a fiber employing the action of G. Assume that B is a *Spin* manifold. Then the correspondence $(B,f) \mapsto E$ gives the homomorphism $T\colon \Omega^{Spin}_{n-8}(BG) \longrightarrow \Omega^{Spin}_n$. Let X be a finite CW–complex. The homomorphism T actually gives the transformation

$$T\colon \Omega^{Spin}_{n-8}(X \wedge BG_+) \longrightarrow \Omega^{Spin}_n(X),$$

which may be interpreted as the transfer, and induces the map at the level of the classifying spectra

$$T\colon MSpin \wedge \Sigma^8 BG_+ \longrightarrow MSpin, \tag{21}$$

see details in [29]. Consider the composition

$$\Omega^{Spin}_{n-8}(X \wedge BG_+) \xrightarrow{\ T\ } \Omega^{Spin}_n(X) \xrightarrow{\ \alpha\ } ko_n(X) \tag{22}$$

Here is the result due to S Stolz, [30]:

Theorem 8.1 *Let X be a CW–complex. Then there is an isomorphism at the 2–local category:*

$$ko_n(X) \cong \Omega^{Spin}_n(X)/\operatorname{Im} T$$

Let $\Sigma = \Sigma_1$ or Σ_2, or Σ_3 or η, and X_Σ be the corresponding spectrum, so that $MSpin^\Sigma \cong MSpin \wedge X_\Sigma$. The map $T\colon MSpin \wedge \Sigma^8 BG_+ \longrightarrow MSpin$ induces the map

$$T^\Sigma\colon MSpin \wedge \Sigma^8 BG_+ \wedge X_\Sigma \longrightarrow MSpin \wedge X_\Sigma.$$

Consider the composition

$$MSpin \wedge \Sigma^8 BG_+ \wedge X_\Sigma \xrightarrow{\ T^\Sigma\ } MSpin \wedge X_\Sigma \xrightarrow{\ \alpha^\Sigma\ } ko \wedge X_\Sigma.$$

We use Theorems 3.1 and 6.1 to derive the following conclusion from Theorem 8.1.

Corollary 8.2 *Let $\Sigma = \Sigma_1$ or Σ_2, or Σ_3 or η. Then there is an isomorphism at the 2–local category:*

$$ko^\Sigma_n \cong MSpin^\Sigma_n/\operatorname{Im} T^\Sigma.$$

We remind here that the homomorphism $ko^\Sigma_n \to KO^\Sigma_n$ is a monomorphism for $n \geq 0$.

Corollary 8.2 describes the situation in 2–local category. Now we consider what is happening when we invert 2. Consider first the case when $\Sigma = \Sigma_1$. Then we have a cofibration:

$$MSpin[\tfrac{1}{2}] \xrightarrow{\ \cdot 2\ } MSpin[\tfrac{1}{2}] \longrightarrow MSpin^{\Sigma_1}[\tfrac{1}{2}]$$

Clearly the map $\cdot 2\colon\ MSpin[\frac{1}{2}] \to MSpin[\frac{1}{2}]$ is a homotopy equivalence. Thus $MSpin^{\Sigma_1}[\frac{1}{2}] \cong pt$. The case $\Sigma = (\eta)$ is more interesting. Here we have the cofibration:

$$S^1 \wedge MSpin[\tfrac{1}{2}] \xrightarrow{\cdot\eta} MSpin[\tfrac{1}{2}] \xrightarrow{\pi^\eta} MSpin^{(\eta)}[\tfrac{1}{2}].$$

Notice that $\eta = 0$ in p–local homotopy $\Omega_1^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$. Thus we have a short exact sequence:

$$0 \to \Omega_n^{Spin} \otimes \mathbf{Z}[\tfrac{1}{2}] \xrightarrow{\pi_*^\eta} \Omega_n^{Spin,\eta} \otimes \mathbf{Z}[\tfrac{1}{2}] \xrightarrow{\beta} \Omega_{n-2}^{Spin} \otimes \mathbf{Z}[\tfrac{1}{2}] \to 0.$$

This sequence has very simple geometric interpretation. Let $w \in \Omega_2^{Spin,\eta} \cong \mathbf{Z}$ be an element represented by an (η)–manifold W, so that $\beta W = 2$. Let $t = \frac{w}{2} \in \Omega_2^{Spin,\eta} \otimes \mathbf{Z}[\frac{1}{2}]$.

Now let $c \in \Omega_n^{Spin,\eta} \otimes \mathbf{Z}[\frac{1}{2}]$, and $\beta c = b$. Let $a = c - tb$, then $c = a + tb$ for $a \in \Omega_n^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$, $b \in \Omega_{n-2}^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$ for any element $c \in \Omega_n^{Spin,\eta} \otimes \mathbf{Z}[\frac{1}{2}]$. Furthermore, this decomposition is unique once we choose an element t. Recall that $\Omega_*^{Spin} \otimes \mathbf{Z}[\frac{1}{2}] \cong \Omega_*^{SO} \otimes \mathbf{Z}[\frac{1}{2}]$ is a polynomial algebra $\mathbf{Z}[\frac{1}{2}][x_1, x_2, \dots, x_j, \dots]$ with $\deg x_j = 4j$. Thus we obtain

$$\Omega_n^{Spin,(\eta)} \otimes \mathbf{Z}[\tfrac{1}{2}] \cong \begin{cases} \Omega_n^{Spin} \otimes \mathbf{Z}[\frac{1}{2}] & \text{if } n = 4k \\ \Omega_{n-2}^{Spin} \otimes \mathbf{Z}[\frac{1}{2}] & \text{if } n = 4k+2 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Furthermore, as it is shown in [15, Proposition 4.2] there are generators $x_j = [M^{4j}]$ of the polynomial algebra $\Omega_*^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$, so that the manifolds M^{4j} are total spaces of geometric $\mathbf{HP}^2$–bundles (for all $j \geq 2$). In particular, it means that the groups $\Omega_{4j}^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$ are in the ideal $\operatorname{Im} T \subset \Omega_*^{Spin} \otimes \mathbf{Z}[\frac{1}{2}]$. Now the formula (23) shows that the groups $\Omega_n^{Spin,(\eta)} \otimes \mathbf{Z}[\frac{1}{2}]$ are in the ideal $\operatorname{Im} T^\eta$. We obtain the isomorphism in integral homotopy groups: $ko_n^\Sigma \cong \Omega_n^{Spin,\eta} / \operatorname{Im} T^\eta$.

The cases $\Sigma = \Sigma_2$ or $\Sigma = \Sigma_3$ are similar to the case $\Sigma = \Sigma_1$: here we have that $MSpin^{\Sigma_i}[\frac{1}{2}] \cong pt$ for $i = 2, 3$.

Thus in all cases we conclude that any element $x \in \operatorname{Im} T^\Sigma$ may be represented by a simply connected Σ–manifold admitting a psc-metric. Here the restriction that $\dim x \geq d(\Sigma)$ is essential. Thus we conclude that if a simply-connected *Spin* Σ manifold M with $\dim M \geq d(\Sigma)$ is such that $[M] \in \operatorname{Ker} \alpha^\Sigma$, then M admits a psc-metric.

Now we prove the necessity.

Let M be a simply connected $Spin$–manifold of dimension $\dim M \geq d(\Sigma)$. What we really must show is that if there is a psc-metric on M, then $\alpha^{\Sigma}([M]) = 0$ on the group KO^{Σ}_{*}.

The case $\Sigma_1 = (P_1) = \langle 2\rangle$ is done in [22], where it is shown that $\alpha^{\Sigma}([M]) \in KO^{\langle 2\rangle}_{*}$ coincides with the index of the Dirac operator on M, and that the index $\alpha^{\Sigma}([M])$ vanishes if M has a psc-metric.

The next case to consider is when $\Sigma = \eta = (P_2)$. Let M be a closed Σ–manifold, ie, $\partial M = \beta_2 M \times P_2$, where P_2 is a circle with the nontrivial $Spin$ structure. Let g be a psc-metric on M. In particular, we have a psc-metric $g_{\beta_2 M}$. Then, as we noticed earlier, the circle P_2 is zero-cobordant as $Spin^c$–manifold. More precisely, we choose a disk D^2 with $\partial D^2 = P_2$, and construct the manifold

$$\overline{M} = M \cup \left(-\beta_2 M \times D^2\right)$$

where we identify $\partial M \beta_2 M \times P_2$ with $\partial(-\beta_2 M \times D^2)$. There is a canonical map

$$h\colon \overline{M} \longrightarrow \mathbf{CP}^1$$

which sends $M \subset \overline{M} = M \cup \left(-\beta_2 M \times D^2\right)$ to the point, and $\beta_2 M \times D^2$ to $\mathbf{CP}^1 = S^2$ by the composition

$$\beta_2 M \times D^2 \longrightarrow D^2 \longrightarrow D^2/S^1 = \mathbf{CP}^1.$$

The map h composed with the inclusion $\mathbf{CP}^1 \subset \mathbf{CP}^\infty$ gives the map $\bar{h}\colon \overline{M} \longrightarrow \mathbf{CP}^\infty$, and, consequently, a linear complex bundle $\xi \longrightarrow \overline{M}$ which is trivialized over M. The $Spin$–structure on M together with the linear bundle $\xi \longrightarrow M$ determines a $Spin^c$–structure on $\overline{M}$. To choose a metric g_0 on the disk, We identify D^2 with the standard hemisphere S^2_+ with a small color attached to the circle S^1, so that the metric $g_0|_{S^1}$ is the standard flat metric $d\theta^2$. Then we have the product metric $g_{\beta_2 M} \times g_0$ on $\beta_2 M \times D^2$. Together with the metric g on M, it gives a psc-metric $\overline{g}$ on $\overline{M}$. We choose a $U(1)$–connection on the linear bundle $\xi \longrightarrow M$, and let F be its curvature form. We notice that since ξ is trivialized over $M \subset \overline{M}$, the form F is supported only on the submanifold $-\beta_2 M \times D^2 \subset \overline{M}$. Moreover, we have defined the bundle $\xi \longrightarrow \overline{M}$ as a pull-back from the tautological complex linear bundle over $\mathbf{CP}^1$. Thus locally we can choose a basis $e_1, e_2, \dots e_n$ of the Clifford algebra, so that $F(e_1, e_2) \neq 0$, and $F(e_i, e_j) = 0$ for all other indices i, j. Notice also that the scalar curvature function $R_{\overline{g}} = R_{g_{\beta_2 M}} + R_{g_0}$. Let D be the Dirac operator on the canonical bundle $S(\overline{M})$ of Clifford modules over $\overline{M}$. We have the BLW–formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}\left(R_{\overline{g}} + R_{g_0}\right) + \frac{1}{2}F(e_1, e_2) \cdot e_1 \cdot e_2. \tag{24}$$

Now we scale the metric $g_{\beta_2 M}$ to the metric $\epsilon^2 \cdot g_{\beta_2 M}$ with the scalar curvature $R_{\epsilon^2 \cdot g_{\beta_2 M}} = \epsilon^{-2} \cdot R_{g_{\beta_2 M}}$. Clearly this scaling does not effect the connection form since the scaling is in the "perpendicular direction". Let $\epsilon > 0$ be such that the term

$$\frac{1}{4}\left(\epsilon^{-2} \cdot R_{g_{\beta_2 M}} + R_{g_0}\right)$$

will dominates the connection term $\frac{1}{2}F(e_1, e_2) \cdot e_1 \cdot e_2$. Then we attach the cylinder $\beta_2 M \times [0,a] \times D^2$ (for some $a > 0$) with the metric $g_{\beta_2 M}(t) \times g_0$, where

$$g_{\beta_2 M}(t) = \frac{a-t}{a} g_{\beta_2 M} + \frac{t}{a}\epsilon^2_{\beta_2 M} + dt^2,$$

so that the metric $g_{\beta_2 M}(t) \times g_0$ has positive scalar curvature, and is a product metric near the boundary. Thus with that choice of metric, the right-hand side in (24) becomes positive, which implies that the Dirac operator D is invertible, and hence $\operatorname{ind}(D) \in K_*$ vanishes. This completes the case of η–singularity.

Remark 8.3 Here the author would like to thank S Stolz for explaining this matter.

The case $\Sigma_2 = (P_1, P_2)$ is just a combination of the above argument and the BLW–formula for *Spin*^{c} $\mathbf{Z}/k$–manifolds given by Freed [5].

The last case, when $\Sigma = \Sigma_3 = (P_1, P_2, P_3)$ there is nothing to prove since KO^{Σ_3} is a contractible spectrum, and thus any Σ_3–manifold has a psc-metric. Indeed, we have that

$$\Omega_n^{Spin,\Sigma_3} \cong \operatorname{Im} T, \quad \text{if} \quad n \geq 17.$$

This completes the proof of Theorem 1.1.

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