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Hofer–Zehnder capacity and length minimizing Hamiltonian paths

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Abstract

We use the criteria of Lalonde and McDuff to show that a path that is generated by a generic autonomous Hamiltonian is length minimizing with respect to the Hofer norm among all homotopic paths provided that it induces no non-constant closed trajectories in M. This generalizes a result of Hofer for symplectomorphisms of Euclidean space. The proof for general M uses Liu–Tian's construction of S^1–invariant virtual moduli cycles. As a corollary, we find that any semifree action of S^1 on M gives rise to a nontrivial element in the fundamental group of the symplectomorphism group of M. We also establish a version of the area-capacity inequality for quasicylinders.

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1 Introduction

In this paper we provide a sufficient condition for a path ϕ_t, $0 \le t \le 1$ in the Hamiltonian group $\mathrm{Ham}(M)$ to be length minimizing with respect to the Hofer norm among homotopic paths with fixed endpoints. This extends the work done by Hofer [7], Bialy–Polterovich [2], Ustilovsky [31], and Lalonde–McDuff [10] on characterizing geodesics in $\mathrm{Ham}(M)$. We will work throughout on a closed symplectic manifold (M,ω), though our results extend without difficulty to the group $\mathrm{Ham}^c(M,\omega)$ of compactly supported Hamiltonian symplectomorphisms when M is noncompact and without boundary.

There are several more or less equivalent definitions of the Hofer norm. We will use Hofer's original definition. Namely, we define the length $L(H_t)$ of a time dependent Hamiltonian function $H_t\colon M \to \mathbf{R}$ for $0 \le t \le \tau$ to be

$$L(H_t) = \int_0^\tau \left(\max_{x\in M} H_t(x) - \min_{x\in M} H_t(x) \right) dt.$$

The length of the corresponding path $\phi_t^H, 0 \le t \le \tau$, in $\mathrm{Ham}(M)$ is also taken to be $L(H_t)$, and the Hofer norm $\|\phi\|$ of $\phi \in \mathrm{Ham}(M)$ is the infimum of the lengths of all of the paths from the identity to ϕ.[1] This norm does not change if we restrict attention to paths parametrized by $t \in [0,1]$, since this amounts to replacing $H_t, t \in [0,\tau]$, by $\tau H_{\tau t}, t \in [0,1]$. Hence, unless explicit mention is made to the contrary, all paths will be assumed to be so parametrized.

Although the Hofer norm is simply defined, it is difficult to calculate in general. One can separate this question into two: the first is to calculate the minimum of the lengths of paths between id and ϕ in some fixed homotopy class, and the other is to minimize over the set of all homotopy classes. We call paths that realise the first minimum *length minimizing in their homotopy class* (or simply *length minimizing*), and those realising the second *absolutely length minimizing.* It is hard to find absolutely length minimizing paths except in the very rare cases when $\pi_1(\mathrm{Ham}(M))$ is known. However the first problem is often more manageable. Also, in cases where there is a natural path from the identity to ϕ — for example if there is a path induced by a circle action such as a rotation — one can look for conditions under which this natural path is length minimizing.

A simple example of an absolutely length minimizing path is rotation of S^2 through π radians: see [10], II Lemma 1.7. The proof can be generalized to rotations of CP^2 and of the one-point blow up of CP^2: see Slimowitz [30].

[1] We fix signs by choosing ϕ_t^H to be tangent to the vector field X defined by $\omega(X,\cdot) = dH$.

Because the argument uses explicit embeddings of balls, it is too clumsy to work for general manifolds.

1.1 Statement of main results

In this note we concentrate on paths $\phi_t^H, t \in [0,1]$, that are generated by autonomous (ie time independent) Hamiltonian functions $H\colon M \to \mathbb{R}$. Our aim is to understand the set

$$\Lambda_H = \{\lambda : \text{the flow } \phi_t^{\lambda H}, t \in [0,1], \text{ of } \lambda H \text{ is length minimizing in its homotopy class}\}.$$

It is easy to see that this set is always a closed interval. It has nonempty interior by Proposition 1.14 in [10].[2] There are Hamiltonians H on manifolds with infinite fundamental group such that $\Lambda_H = [0,\infty)$, ie, the flow of λH is absolutely length minimizing for all $\lambda > 0$. Here the lower bound for the length is provided by the energy–capacity inequality on the universal cover: see [10] Lemma 5.7. When M is closed and simply connected, in all known examples (other than circle actions that have $\phi_1^{\lambda H} = id$ for some $\lambda > 0$) the distance between the identity and the symplectomorphism $\phi_1^{\lambda H}$ tends to infinity as $\lambda \to \infty$.[3] However, this path does not remain length minimizing for all λ. Thus in the simply connected case one expects Λ_H to be a compact interval $[0, \lambda_{\max}(H)]$ for all H.

The next result applies to all symplectic manifolds, and follows by an easy application of the curve shortening technique of [10] I Proposition 2.2.

Lemma 1.1 *Suppose that H is a Hamiltonian that assumes its maximum values on the set $X_{\max}$. Then, if there is a Hamiltonian symplectomorphism ϕ of M such that $\phi(X_{\max}) \cap X_{\max}$ is empty, $\lambda_{\max} < \infty$.*

[2]The papers [10] were written at a time when it was not yet understood how to define Gromov–Witten invariants for general symplectic manifolds M. Therefore, many of the results in part II have unnecessary restrictions. In particular, in Theorems 1.3 (i) and 1.4 and in Propositions 1.14 and 1.19 (i) one can remove the hypothesis that M has dimension ≤ 4 or is semi-monotone. The point is that these results rely on Proposition 4.1, and so use the fact that quasicylinders $Q = (M \times D^2, \Omega)$ have the nonsqueezing property. This is now known to hold for all M.

[3]Added Dec 01: In fact there are many other paths $\phi_1^{\lambda H}, \lambda \ge 0$, that remain a bounded distance from id. For example, if F has support in a ball B and $\psi(B) \cap B = \emptyset$, define $H = F - F\circ\psi$. Then $\phi_1^{\lambda H} = \phi_1^{\lambda F} \circ \psi \circ (\phi_1^{\lambda F})^{-1} \circ \psi^{-1}$ remains at a distance $2\|\psi\|$ from id.

In this case one can estimate $\lambda_{\max}$ by comparing the displacement energy of a neighborhood $\mathcal{N}$ of $X_{\max}$ with the growth of H on $\mathcal{N}$. For a discussion of related questions see Polterovich [25].

If H is generic and hence a Morse function, it follows from the above lemma that $\lambda_{\max}(H) < \infty$. However, one can get a sharper estimate for $\lambda_{\max}$ by looking at the linearized flow near a critical point p of λH. In suitable coordinates, this has the form $e^{-t\lambda JQ}$ where Q is the Hessian of H at p and J is the standard almost complex structure. We will say that p is *overtwisted* for λH if $A = -JQ$ has an imaginary eigenvalue $i\mu$ with $\mu > 2\pi$. This is equivalent to saying that the linearized flow of λH at p has a nonconstant periodic orbit of period < 1: see Section 3.2. Ustilovsky's analysis in [31] of the second variation equation for geodesics shows that the path $\phi_t^{\lambda H}, t \in [0,1]$, ceases to be length minimizing as soon as all the global maxima of λH are overtwisted. A similar result applies to minima, and also to certain degenerate H: see [10].

If p is an overtwisted local extremum of H, a celebrated result of Weinstein [32] implies that the nonlinear flow of λH near p also has nonconstant periodic orbits of period < 1. Hence it is natural to make the following conjecture.

Conjecture 1.2 The path $\phi_t^H, t \in [0,1]$, is length minimizing in its homotopy class whenever its flow has no nonconstant contractible periodic orbits of period < 1.

Hofer showed in [7] that this is true for compactly supported Hamiltonians on $\mathbb{R}^{2n}$ by using a variational argument that does not extend to arbitrary manifolds: see also Section 5.7 in [9]. It was also established in the cases when M has dimension two or is weakly exact in [10] Theorem 5.4. In this paper we extend the arguments in [10] to arbitrary manifolds. Unfortunately this does not quite allow us to prove the full conjecture. The problem is that there are functions H with no nonconstant periodic orbits but yet with overtwisted critical points, and, for technical reasons, our argument cannot cope with such points. However, it is well known that for generic H this problem does not occur; generic overtwisted critical points always give rise to 1–parameter families of contractible periodic orbits of period < 1. For the sake of completeness, we give a simple topological proof of this in Lemma 3.4 below and also describe Moser's example of an overtwisted Hamiltonian whose only periodic orbit is constant.

In view of this, it is useful to make the following definition.

Definition 1.3 A periodic orbit is called *fast* if its period is < 1. Given an (autonomous) Hamiltonian H we denote by $\mathcal{P}(H)$ the set of its fast contractible periodic orbits, and by $\mathcal{P}_{crit}(H)$ the set of fast periodic orbits of the linearized flows at its critical points. We will say that H is *slow* if the only elements in $\mathcal{P}(H)$ and $\mathcal{P}_{crit}(H)$ are constant paths.

Here is the main result of this paper.

Theorem 1.4 *Given a closed symplectic manifold* (M, ω), *let* ϕ_t^H, $0 \leq t \leq 1$, *be the path in* $\mathrm{Ham}(M)$ *generated by the autonomous Hamiltonian* $H\colon M \to \mathbf{R}$. *If* H *is slow, then this path is length minimizing among all homotopic paths between the identity and* ϕ_1^H.

Note that the path remains length minimizing in its homotopy class even if H has periodic orbits of period exactly equal to 1. To see this, first apply the theorem to $(1-\varepsilon)H$ for $\varepsilon > 0$ and then use the fact that the set Λ_H defined above is closed.

This theorem applies in particular to semi-free Hamiltonian circle actions ϕ_t^H, $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Recall that these are actions in which the stabilizer subgroups of each point are either trivial or the full group. Thus in this case all nonfixed points lie on periodic orbits of period exactly 1. Moreover, because the flow ϕ_t^H on M is congugate to its linearization near the critical points, it is easy to see that none of these points are overtwisted.

Corollary 1.5 *Every semi-free symplectic* S^1 *action on a closed symplectic manifold* (M, ω) *represents a nontrivial element* γ *in* $\pi_1(\mathrm{Symp}(M, \omega))$. *Moreover, if the action is Hamiltonian, the corresponding loop has minimal length among all freely homotopic loops in* $\mathrm{Ham}(M, \omega)$.

Proof If the action is not Hamiltonian then the result is obvious (and the semi-free condition is not needed) since in this case the image of the loop under the flux homomorphism

$$\pi_1 \mathrm{Symp}(M, \omega) \to H^1(M, \mathbb{R})$$

is nonzero. For Hamiltonian loops, Theorem 1.4 implies that they are length minimizing paths from *id* to *id* in their homotopy class. Because the constant path to *id* is always shorter than the given loop the latter cannot be null homotopic. The last statement is an easy consequence of the conjugacy invariance of the norm. □

Somewhat surprisingly, there seems to be no elementary proof of the first statement in this corollary. It would be interesting to know if it remains true in the smooth category. In particular, do arbitrary smooth semi-free S^1 actions on M represent nontrivial elements in $\pi_1(\mathrm{Diff}(M))$ or even in $\pi_1(H(M))$, where $H(M)$ is the group of self-homotopy equivalences of M? This is true for nonHamiltonian symplectic loops, since the flux homomorphism extends to $\pi_1(H(M))$.[4]

Observe also that the semi-free condition is needed. Consider, for example, the S^1 action on CP^2 given by:

$$[z_0 : z_1 : z_2] \mapsto [e^{i\theta} z_0 : e^{-i\theta} z_1 : z_2].$$

This is null-homotopic, while points such as $[1 : 1 : 0]$ have $\mathbb{Z}/2\mathbb{Z}$ stabilizer. Clearly, a general Hamiltonian S^1 action remains length minimizing for time $1/k$ where k is the order of the largest isotropy group.

As a byproduct of the proof we also calculate a very slightly modified version of the Hofer–Zehnder capacity for cylinders $Z(a)$, where

$$Z(a) = (M \times D(a),\, \omega \times \sigma_a)$$

and $(D(a), \sigma_a)$ is a 2–disc with total area a. To explain this, we recall the definition[5] of the Hofer–Zehnder capacity c_{HZ}:

$$c_{HZ}(N, \omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}_{ad}(N, \omega)\}$$

where the set $\mathcal{H}_{ad}(N, \omega)$ of admissible Hamiltonians consists of all of the autonomous Hamiltonians on N such that

(a) For some compact set $K \subset N - \partial N$, $H|_{N-K} = \max(H)$ is constant;

(b) There is a nonempty open set U depending on H such that $H|_U = 0$;

(c) $0 \le H(x) \le \max(H)$ for all $x \in N$;

(d) All fast contractible periodic solutions of the Hamiltonian system $\dot{x} = X_H(x)$ on N are constant.

[4] Added in Dec 01: Claude LeBrun pointed out that the diagonal S^1 action on $\mathbb{C}^2$ given by multiplication by $e^{i\theta}$ induces a semifree action on S^4 that represents the trivial loop in $\pi_1(S0(5)) \subset \pi_1(\mathrm{Diff}(S^4))$. For further work on this subject see [19].

[5] Hofer originally considered Hamiltonian systems in $\mathbb{R}^{2n}$ and hence had no need to restrict to contractible periodic orbits in condition (d) below. In the definition of c_{HZ} given in [9], this condition is not imposed. We have inserted it here to make c_{HZ} as relevant to our problem as possible. This definition appears in Lu [14], who pointed out that the monotonicity axiom has to be suitably modified. It is called the π_1–sensitive Hofer–Zehnder capacity in Schwarz [27].

As explained above, our arguments are sensitive to the presence of overtwisted critical points. Hence we define the modified capacity c'_{HZ} as follows:

$$c'_{HZ}(N,\omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}'_{ad}(N,\omega)\}$$

where the set $\mathcal{H}'_{ad}(N,\omega)$ of admissible Hamiltonians consists of all autonomous Hamiltonians on N that satisfy conditions (a), (b), (c) above as well as the following version of (d):

(d′) H is slow.

These capacities are closely related. Clearly $c_{HZ} \le c'_{HZ}$. Our discussion above implies that the set $\mathcal{H}'_{ad}(N,\omega)$ has second category in $\mathcal{H}_{ad}(N,\omega)$: see Corollary 3.5. Furthermore the two capacities may agree: it is not hard to see that they both equal a on the 2–disc $(D(a),\sigma_a)$.[6] Since the capacity of the product $Z(a)$ is at least as large as that of $(D(a),\sigma_a)$, the difficult part of the next proposition is to find an upper bound for $c'_{HZ}(Z(a))$.

Proposition 1.6 *Let (M,ω) be any closed symplectic manifold. Then*

$$c'_{HZ}(M\times D(a),\, \omega\times\sigma_a) = a.$$

There are several ways in which one could try to generalize the main theorem. Siburg showed in [29] that the conjecture holds for flows generated by time dependent Hamiltonians on $\mathbb{R}^{2n}$ provided that these also have isolated and fixed extremal points. (The fixed extrema are needed to ensure that the path is a geodesic: see [2].) Although it seems very likely that Theorem 1.4 should hold on general M in the time dependent case, the method used here is not well adapted to tackle this problem. In fact, while our paper was being finished, Entov developed in [3] a rather different approach as part of a larger program that has some very interesting applications. It may well be that his method would be better in the time dependent case: see Remark 2.10.

It is also natural to wonder what happens when H does have nonconstant fast periodic orbits and/or overtwisted critical points. For example we might take an H that satisfies the conditions of the theorem and consider the flow of λH for $\lambda > 1$. It would seem plausible that if some critical point of index lying strictly between $0, 2n$ becomes overtwisted λH would remain length minimizing, at least for a while. One problem here is that a critical point that is just on the point of beoming overtwisted (ie, has eigenvalue $2\pi i$) is degenerate as far as Floer theory is concerned. The main step in our proof is to demonstrate that a particular moduli space of Floer trajectories is nonempty, which we do

[6] In fact, there are no known examples where they differ.

by a deformation argument. Thus we need to know that the relevant spaces of Floer trajectories are regular when λ varies from 0 to 1, and it is here that the overtwisted critical points would cause a problem: see Lemma 3.6. If degenerations occur, one must either carry through a detailed analysis of the degeneration or argue that this moduli space is nonempty for cohomological reasons. Since both approaches would take us rather far from the main theme of this paper, we will not pursue them further here.

1.2 Techniques of proof

The proofs of the above results employ the criteria for length minimizing paths developed in [10]. For the convenience of the reader, this is explained in Section 2 below. The idea is to compare the length of the path with the capacity of an associated region in $M \times \mathbb{R}^2$ that is roughly speaking a cylinder. In order to make the method work, it would suffice to know that the Hofer–Zehnder capacity c_{HZ} satisfies the *area–capacity inequality*

$$c_{HZ}(Z(a)) \le a,$$

on all cylinders. This is equivalent to saying that every Hamiltonian $H\colon Z(a) \to [0, c]$, that is identically zero on some open subset and equals its maximum value c on a neighborhood of the boundary $\partial Z(a)$, has fast periodic orbits as soon as $c > a$. In [8], Hofer and Viterbo prove this statement for weakly exact (M, ω), ie, when $\omega|_{\pi_2(M)} = 0$. Their argument was extended to all manifolds by Liu–Tian in [11]. As these authors point out, the "usual" theory of J–holomorphic curves is not much help even in the semi-positive case because one must use moduli spaces on which there is an action of S^1. Their paper establishes the needed technical basis — S^1–equivariant Gromov–Witten invariants and virtual moduli cycles — to prove Proposition 1.6 stated above. However, they do not consider arbitrary Hamiltonians but a special class that is relevant to the Weinstein conjecture, and their paper is organised in such a way that one cannot simply quote the needed results. This question is discussed further in Section 3.3.

In fact the above area–capacity inequality is more than is needed for the problem at hand, and it is convenient to consider another modification of c_{HZ} defined by maximizing over a restricted class of Hamiltonians that are compatible with the fibered structure of the cylinder. This makes the geometry of the problem more transparent and hence allows us to work with semi-positive M without using virtual moduli cycles at all.

Here is a version of our main technical result. It is somewhat simplified since we in fact need an analogous result to hold for quasicylinders, rather than just for cylinders: see Section 2. It will be convenient to think of the base disc $D(a)$ of $Z(a) = M \times D(a)$ as being a disc on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with center at ∞, and hence to call the central fiber $M_\infty = M \times \{\infty\}$.

Proposition 1.7 *Let $F\colon Z(a) \to [0,c]$, be a Hamiltonian function such that*

(i) *its only critical points occur in the sets M_∞ and $M \times \overline{U}_0$, where $\overline{U}_0$ is a connected neighborhood of the boundary $\partial D(a)$;*

(ii) *near the central fiber M_∞, $F = H_M + \beta(r)$ where H_M is a Morse function on M, and β is a function of the radial coordinate r that is $< \pi r^2$ near $r = 0$;*

(iii) *$F\colon Z(a) \to [0,c]$ is surjective, and is constant and equal to its maximum value on $M \times \overline{U}_0$.*

Then, if $c > a$, F is not slow, ie, it has either a nonconstant fast periodic orbit or an overtwisted critical point.

This paper is organized in the following way. The second section describes the criteria for length minimizing paths developed by Lalonde and McDuff in [10] and explains the role of Hofer–Zehnder capacities. The third gives the proofs of the area–capacity inequality and of Proposition 1.7. We discuss in detail some technicalities about the intersections of bubbles and Floer trajectories, that are omitted from standard references such as [5].

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2 Criteria for length minimizing paths

We briefly describe the Lalonde–McDuff criterion for finding paths that are length minimizing in their homotopy class. In [10], they first derive a geometric way of detecting that $L(H_t) \leq L(K_t)$ for two Hamiltonians H_t and K_t on M.

Then, they determine sufficient conditions involving symplectic capacities for this geometric requirement to be satisfied.

For technical reasons it is convenient to restrict to Hamiltonians H_t that are identically 0 for t near $0, 1$. This restriction does not cause any problems: it is easy to see that every time independent Hamiltonian H may be replaced by one of the form $\beta(t)H$ that satisfies the above condition and has the same length and time 1–map as before.

2.1 Estimating Hofer length via quasicylinders

To begin, we must make a few definitions and set some notation. Suppose we have H_t, a time dependent Hamiltonian function on the closed symplectic manifold (M^{2n}, ω). We may assume[7] that for each t,

$$\min_{x\in M} H_t(x) = 0.$$

We denote the graph Γ_H of H_t by

$$\Gamma_H = \{(x, H_t(x), t)\} \subset M \times \mathbb{R} \times [0,1].$$

Now, given some small $\nu > 0$ choose a function $\ell(t)$: $[0,1] \to [-2\nu, 0]$ such that $\int_0^1 -\ell(t)dt = \nu$. A *thickening of the region under* Γ_H is

$$R_H^-(\nu) = \{(x,s,t) \,|\, \ell(t) \leq s \leq H_t(x)\} \subset M \times [\ell(t), \infty) \times [0,1].$$

Since $H_t \equiv 0$ for t near $0,1$ we may arrange that R_H^- is a manifold with corners along $s = 0, t = 0, 1$ by choosing the function $\ell(t)$ so that its graph is tangent to the lines $t = 0, t = 1$.

Similarly, we can define $R_H^+(\nu)$ to be a slight thickening of the region above Γ_H:

$$R_H^+(\nu) = \{(x,s,t) \,|\, H_t(x) \leq s \leq \mu_H(t)\} \subset M \times \mathbb{R} \times [0,1]$$

where $\mu_H(t)$ is chosen so that

$$\mu_H(t) \geq \max_{(} t) = \max_{x\in M} H_t(x), \qquad \int_0^1 (\mu_H(t) - \max_t)dt = \nu.$$

We define

$$R_H(2\nu) \;=\; R_H^-(\nu) \cup R_H^+(\nu) \;\subset M \times \mathbb{R} \times [0,1].$$

[7] There is a slight technical problem here when the function $t \mapsto \min(t) = \min_{x\in M} H_t(x)$ is not smooth. In this case, we replace H_t by $H_t + m(t)$ where $m(t)$ is a smooth function that is everywhere $\leq \min(t)$ and is such that $\min(t) - m(t)$ has arbitrarily small integral. This slightly changes the areas of the regions $R_H^{\pm}$. However, this can be absorbed into the ν fudge factor: we only need to measure lengths exactly for time independent H.

We equip $R_H^-(\nu)$, $R_H^+(\nu)$, and $R_H(2\nu)$ with the product symplectic form $\Omega = \omega \times \sigma$ where $\sigma = ds \wedge dt$. In particular, for any Hamiltonian H_t, $(R_H(\nu), \Omega)$ is symplectomorphic to the product $(M \times D(a), \Omega)$ where $D(a)$ denotes the 2–disc D^2 with area $a = L(H) + 2\nu$.

Now, suppose H_t and K_t are two Hamiltonians on M such that $\phi_1^H = \phi_1^K$ and the path ϕ_t^H for $0 \le t \le 1$ is homotopic (with fixed endpoints) to the path ϕ_t^K in $\mathrm{Ham}(M)$. There is a map $g\colon \Gamma_K$ to Γ_H defined by

$$g(x, s, t) = (\phi_t^H \circ (\phi_t^K)^{-1}(x), s - K(x) + H(\phi_t^H \circ (\phi_t^K)^{-1}(x)), t).$$

This map g extends to a symplectomorphism of $R_K^+(\nu)$, and we define

$$(R_{H,K}(2\nu), \Omega) = R_H^-(\nu) \cup_g R_K^+(\nu).$$

We assume that the functions ℓ and μ_H are chosen so that $R_{H,K}(2\nu)$ is a smooth manifold with boundary. The contractibility of the loop $\phi_t^H \circ (\phi_t^K)^{-1}$ in $\mathrm{Ham}(M, \omega)$ implies that $(R_{H,K}(2\nu), \Omega)$ is diffeomorphic to a product $(M \times D, \Omega)$ by a diffeomorphism that is the identity near the boundary and is symplectic on each fiber. However Ω may not be a product, and so we make the following definition.

Definition 2.1 Let (M, ω) be a closed symplectic manifold and D a set diffeomorphic to a disc in $(\mathbf{R}^2, \sigma)$ where $\sigma = ds \wedge dt$. Then, the manifold $Q = (M \times D, \Omega)$ endowed with the symplectic form Ω is called a *quasicylinder* if

(i) Ω restricts to ω on each fibre $M \times \{pt\}$;

(ii) Ω is the product $\omega \times \sigma$ near the boundary $M \times \partial D$.

If $\Omega = \omega \times \sigma$ everywhere, not just near the boundary, Q is called a *split* quasicylinder. The *area* of any quasicylinder $(M \times D, \Omega)$ is defined to be the number A such that

$$\mathrm{vol}\ (M \times D, \Omega) = A \cdot\ \mathrm{vol}\ (M, \omega).$$

Thus if $(M \times D(a), \Omega)$ is split, its area is simply a.

Since $(R_{H,K}(2\nu), \Omega)$ has trivial monodromy round its boundary, it is not hard to see that it is a quasicylinder: see [10] IISection 2.1. However, it may not be split.

The key to the analysis is the following lemma taken from [10] II, Lemma 2.1, whose proof we include for the convenience of the reader. It shows that if the areas of both quasicylinders $(R_{H,K}(2\nu), \Omega)$ and $R_{K,H}(2\nu), \Omega)$ are greater than or equal to $L(H_t)$ for all ν, then $L(H_t) \le L(K_t)$.

Lemma 2.2 *Suppose that $L(K_t) < L(H_t) = A$. Then, for sufficiently small $\nu > 0$, at least one of the quasicylinders $(R_{H,K}(2\nu), \Omega)$ and $(R_{K,H}(2\nu), \Omega)$ has area $< A$.*

Proof Choose $\nu > 0$ so that

$$L(K_t) + 4\nu < L(H_t).$$

Evidently,

$$\begin{aligned}\mathrm{vol}(R_{H,K}(2\nu)) + \mathrm{vol}(R_{K,H}(2\nu)) &= \mathrm{vol}(R_H(2\nu)) + \mathrm{vol}(R_K(2\nu)) \\ &= (\mathrm{vol}M) \cdot (L(H_t) + L(K_t) + 4\nu) \\ &< 2(\mathrm{vol}M) \cdot L(H_t)\end{aligned}$$

where $R_H(2\nu) = R_H^-(\nu) \cup R_H^+(\nu)$. □

To proceed, one needs some way of finding lower bounds for the area of a quasicylinder. The arguments in [10] use symplectic capacities, which are functions from the set of symplectic manifolds to $\mathbb{R} \cup \{\infty\}$ satisfying certain properties; in particular, they are invariant under symplectomorphisms.

Suppose we have chosen a particular capacity c and symplectic manifold (M, ω). We say the *area–capacity inequality* holds for c on M if

$$c(M \times D, \Omega) \leq \text{ area of } (M \times D, \Omega)$$

holds for all quasicylinders $(M \times D, \Omega)$. It is useful to make the following definition.

Definition 2.3 The *capacity* $c(H_t)$ of a Hamiltonian function H_t is defined as

$$c(H_t) = \min\{\inf_{\nu>0} c(R_H^-(\nu)), \inf_{\nu>0} c(R_H^+(\nu))\}.$$

Now, take a manifold M and a capacity c such that the area–capacity inequality holds for c on M, and suppose that we have a Hamiltonian H_t: $M \to \mathbb{R}$ for which

$$c(H_t) \geq L(H_t).$$

Then, for any Hamiltonian K_t generating a flow ϕ_t^K which is homotopic with fixed end points to ϕ_t^H (and thus has $\phi_1^K = \phi_1^H$), we can embed $R_H^-(\nu)$ into $R_{H,K}(2\nu)$ and $R_H^+(\nu)$ into $R_{K,H}(2\nu)$. Thus, we know

$$\begin{aligned}L(H_t) \ &\leq \ c(H_t) \ \leq \ c(R_H^-(\nu)) \ \leq \ c(R_{H,K}(2\nu)) \\ L(H_t) \ &\leq \ c(H_t) \ \leq c(R_H^+(\nu)) \ \leq \ c(R_{K,H}(2\nu)),\end{aligned}$$

with the last inequality in both lines holding by the monotonicity property of capacities. Since the area–capacity inequality holds, we know that the areas of both quasicylinders $R_{H,K}(2\nu)$ and $R_{K,H}(2\nu)$ must be greater than or equal to their capacities and hence greater than or equal to $L(H_t)$. Therefore, by Lemma 2.2, $L(K_t) \geq L(H_t)$. This proves the following result (Proposition 2.2 from [10], Part II.)

Proposition 2.4 *Let M be any symplectic manifold and $H_{t\in[0,1]}$ a Hamiltonian generating an isotopy ϕ_t^H from the identity to $\phi = \phi_1^H$. Suppose there exists a capacity c such that the following two conditions hold:*

(i) *$c(H_t) \geq L(H_t)$ and*

(ii) *for all Hamiltonian isotopies ϕ_t^K homotopic rel endpoints to ϕ_t^H , $t \in [0,1]$, the area–capacity inequality holds (with respect to the given capacity c) for the quasicylinders $R_{H,K}(2\nu)$ and $R_{K,H}(2\nu)$.*

Then, the path $\{\phi_t^H\}_{t\in[0,1]}$ minimizes length among all homotopic Hamiltonian paths from id to ϕ.

Hence, to show that H_t generates a length minimizing path $\{\phi_t^H\}_{t\in[0,1]}$, we need only produce a capacity c that satisfies the above conditions (i) and (ii). Various results were obtained in [10] by using the Gromov capacity c_G and the Hofer–Zehnder capacity c_{HZ}. It seems to be best to use c_{HZ}, since condition (i) holds for it almost by definition whenever H has no nontrivial fast periodic orbits, while (i) is very restrictive for c_G. On the other hand, the existence of Gromov–Witten invariants on general symplectic manifolds allows one to show easily that condition (ii) holds for c_G, while the proof of (ii) for c_{HZ} is more subtle. Liu–Tian consider a very closely related question in [11], and using their methods one can prove that (ii) holds for the very slightly modified version c'_{HZ} of c_{HZ} on any manifold: see Section 3.3.

In view of the complexity of the constructions in [11], we present in the next section a different modification of the Hofer–Zehnder capacity for which one can prove condition (ii) without too much difficulty in the semi-positive case. This capacity c_f is defined for fibered spaces such as quasicylinders, satisfies (i) whenever H is slow and also satisfies (ii) for any closed M. It depends on some extra structure that we need to choose and so is not defined for all symplectic manifolds. Note that the only properties of the capacity c that we used above are that it is defined for sets such as $R_H^{\pm}(\nu)$ and that it has the monotonity property

$$c(R_H^-(\nu)) \leq c(R_{H,K}(2\nu)), \qquad c(R_H^+(\nu)) \leq c(R_{K,H}(2\nu)).$$

2.2 The Hofer–Zehnder capacity for fibered spaces

We first explain what is meant by a fibered symplectic manifold.

Definition 2.5 We will say that the symplectic manifold (Q,Ω) is *fibered* with fiber (M,ω) if there is a submersion $\pi\colon Q\to D^2$ such that Ω restricts to a nondegenerate form on each fiber $M_b=\pi^{-1}(b)$, where (M_b,ω_b) is symplectomorphic to (M,ω) for one and hence all b. In this case, because D^2 is contractible one can use Moser's theorem to choose an identification s_Q of Q with $M\times D^2$ so that $\omega_b=\omega$ for all b. s_Q is said to *normalize* Q if in addition there is a small closed disc $\overline{U}_\infty$ in D^2 with center ∞ so that Ω restricts to $\omega\times\sigma$ on $M\times U_\infty$, where σ is the area form $ds\wedge dt$ as before. A symplectic embedding $\psi\colon Q\to Q'$ is said to be *normalized* if it takes the central fiber M_∞ in Q to that in Q' and if

$$\psi=(s_{Q'})^{-1}\circ s_Q$$

on some neighborhood of M_∞ that need not be the whole of $\pi^{-1}\overline{U}_\infty$.

Using the symplectic neighborhood theorem it is easy to see that every fibered space can be normalized near any fiber. Further, every quasicylinder (Q,Ω) is fibered, though in general the identification $Q\to M\times D^2$ that occurs in the definition of a quasicylinder is a normalization only near fibers that are sufficiently close to the boundary. It is also not hard to see that the spaces $(R_H^{\pm}(\nu),\Omega)$ can be fibered with fibers $\pi^{-1}(b)$ of the form $\{(x,s_b(x),t_b) : x\in M\}$: the restriction of Ω to such sets equals ω since t_b is fixed. We will assume that the fibers lying in the part of $R_H^-(\nu)$ with $s<0$ are flat, ie, also have fixed s–coordinate $s_b(x)=s_b$. This normalizes $R_H^-(\nu)$ near some fiber M_0 with $s<0$. Similarly, the fibration of $R_H^+(\nu)$ is chosen to have flat fibers $s=const$ near its upper boundary $s=\mu_H(t)$. This means that spaces such as $R_{H,K}(\nu)$ have two possible normalizations, one at a fiber where $s<0$ and the other near its upper boundary. However, it is not hard to see that there is a fiberwise symplectomorphism taking one to the other so that they are equivalent.

Definition 2.6 Given a normalized fibered space Q, we define the set $\mathcal{H}_{f,ad}(Q)$ of admissible Hamiltonians to be the set of all functions $F\colon Q\to[0,\infty)$ such that:

(i) in some neighborhood $M\times\overline{U}_\infty$ of the central fiber M_∞, $F=H_M+\beta(r)$ where H_M is a Morse function on M, and β is a function of the radial coordinate r of the disc that is $<\pi r^2$;

(ii) $F\geq 0$ everywhere and is constant and equal to its maximum on a product neighborhood $M\times\overline{U}_0$ of the boundary;

(iii) the only critical points of F occur on M_∞ and in $M \times \overline{U}_0$;

(iv) F is slow.

Definition 2.7 We define the Hofer–Zehnder capacity of a normalized fibered space Q by
$$c_f(Q) = \sup\{\max(F) \,|\, F \in \mathcal{H}_{f,ad}(Q)\}$$

Clearly, this capacity c_f has the appropriate monotonicity property, ie, $c_f(Q) \le c_f(Q')$ whenever there is a normalized symplectic embedding $Q \to Q'$. In particular,
$$c_f(R_H^-(\nu)) \le c_f(R_{H,K}(2\nu)), \qquad c_f(R_H^+(\nu)) \le c_f(R_{K,H}(2\nu)).$$
The following proposition, which is proved in Section 3, shows that c_f also satisfies condition (ii) in Proposition 2.4.

Proposition 2.8 *For any normalized quasicylinder (Q,Ω) of area A,*
$$c_f(Q) \le A.$$

We next check condition (i).

Lemma 2.9 *If $H\colon M \to \mathbb{R}$ is slow, then $c_f(H) \ge L(H)$.*

Proof This is essentially [10] II, Proposition 3.1. We will prove that $c_f(R_H^-(\nu)) \ge L(H)$. The case of $R_H^+(\nu)$ is similar: indeed $R_H^+(\nu)$ is symplectomorphic to $R_{m-H}^-(\nu)$, where $m = \max H$.

By assumption, H has minimum value 0. Let m be its maximum, and consider the set
$$S_{H,\nu} = \{(x,\rho,\tau) \in M \times D(m+\nu/2) \mid 0 \le \rho \le H(x) + \nu/2\},$$
where (ρ,τ) are the action-angle coordinates on the disc given in terms of polar coordinates (r,θ) by
$$\rho = \pi r^2, \quad \tau = \frac{\theta}{2\pi}.$$
This space $S_{H,\nu}$ is essentially the same as $R_H^-(\nu)$. Indeed, it is not hard to check that there is a symplectic embedding $S_{H,\nu} \to R_H^-(\nu)$ of the form $(x,\rho,\tau) \mapsto (x,\phi(\rho,\tau))$ for some area preserving map $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$. Moreover, $S_{H,\nu}$ is fibered with central fiber at $(\rho,\tau) = (0,0)$, and we may choose this embedding so that

it respects suitable normalizations of both spaces. Hence it suffices to show that for all $\varepsilon > 0$

$$c_f(S_{H,\nu}) \geq L(H) - \varepsilon.$$

To see this, first consider the function $F = m - H(x) + \rho$. This is constant and equal to $m + \nu/2$ on $\partial S_{H,\nu}$, and its flow is given by

$$\phi_F^t \colon (x, \rho, \tau) \mapsto (\phi_H^t(x), \rho, \tau + t).$$

Since H is slow and the critical points of H give rise to periodic orbits for F with period precisely 1, F is also slow. Now smooth out F to $F_\varepsilon : S_{H,\nu} \to \mathbb{R}$, where

$$F_\varepsilon(x, \rho, \tau) = \begin{cases} (1-\varepsilon)\,(m - H(x) + \alpha_\nu(\rho))\,, & \text{if } \ \rho < \nu/4, \\ (1-\varepsilon)F(x, \rho, \tau) & \text{if } \ \nu/4 \leq \rho \leq H(x) + \nu/4, \\ (1-\varepsilon)\,(m - \alpha_\nu(H(x) + \nu/4 - \rho))\,, & \text{if } \ H(x) + \nu/4 \leq \rho \leq \\ & \qquad\qquad H(x) + \nu/2. \end{cases}$$

Here $\varepsilon > 0$, and $\alpha_\nu(\lambda)$ is a increasing smooth surjection $\lambda \colon [0, \nu] \to [0, \nu]$ that is $\leq \lambda^2$ near 0 and equals λ when $\lambda \geq \nu/6$. Since the flow of $(1-\varepsilon)F$ goes slower than that of F when $\varepsilon > 0$, $(1-\varepsilon)F$ is slow. Now the bump function $\alpha_\nu(\rho)$ must have derivative slightly > 1 somewhere. Hence when we turn it on the flow in the τ–direction goes slightly faster. However, for each given ε we can clearly choose α_ν so that the product $(1-\varepsilon)\alpha_\nu(\rho)$ is slow. A similar remark applies to the smoothing at $\partial S_{H,\nu}$. Hence F_ε is slow and has maximum value $m - \varepsilon = L(H) - \varepsilon$.

If H were a Morse function, F_ε would be admissible, ie, belong to $\mathcal{H}_{f,ad}(S_{H,\nu})$, and the proof would be complete. Hence the last step is to alter F_ε near the central fiber by replacing H with a function that is independent of ρ for ρ near 0 and restricts to a Morse function H_M on M_∞. This is easy to do without introducing any nonconstant fast periodic orbits since we just need to change H in directions along which its second derivative is small. See, for example, Lemma 12.27 in [17] that shows that H is slow whenever its second derivative is sufficiently small. $\square$

Proof of Theorem 1.4

This follows by the preceding lemma and by the remarks at the end of Section 2.1.

Remark 2.10 Suppose that H_t is a time dependent Hamiltonian. The space R_H^- is again essentially the same as $S_{H,\nu}$ where this is defined to be the set of points (x, ρ, τ) with $0 \le \rho \le H_\tau(x)$, and we can define the (time independent) Hamiltonian F near its boundary ∂S to be (a smoothing of) $m - H_\tau(x) + \rho$ as before. The problem is that this function is not well defined on the central fiber M_∞ since τ is not a coordinate there, and there seems to be no satisfactory way of understanding when one can make such an extension. In particular, it seems one would need the restriction of F to M_∞ to have the same norm as H_t and yet be slow. Entov in [3] connects the Hamiltonian H to the geometry of a fibered space via the choice of suitable connection rather than by the construction of the Hamiltonian F. The condition on the connection is local while our condition on F (that it should be slow) is global. Hence his approach seems better adapted to this problem.

3 The area–capacity inequality

We begin by sketching the proof of this inequality for semi-positive M using the set up in Hofer–Viterbo [8]. Section 3.2 contains more technical details, and Section 3.3 discusses the case of general M.

3.1 Outline of the proof

For simplicity, we will assume for now that M is semi-positive, ie, that one of the following conditions holds:

(a) the restriction to $\pi_2(M)$ of the first Chern class $c_1(M)$ of M is positively proportional to $[\omega]$ – the monotone case; or

(b) the minimal Chern number N of M is $> n-2$, where $2n = \dim M$.

In this case the Gromov–Witten invariants on M can be defined naively, ie, bubbles can be avoided, simply by choosing a generic J on M: see [18]. It is not necessary to use the virtual moduli cycle. Notice that usually one asks that $N > n-3$ in (b). Strengthening this requirement allows us to say that no element of a generic 2–parameter family of almost complex structures on M admits a holomorphic curve of negative Chern number.

We will assume in what follows that (Q, Ω) is a quasicylinder and that F is an admissible Hamiltonian in the sense of Definition 2.6. In particular, this means that for all $\lambda \le 1$ the only 1–periodic orbits of the flow of λF on M_∞ are constant and occur at the critical points p_k of F. Thus every Floer trajectory for λF on M_∞ converges to these critical points. Our aim is to show:

Proposition 3.1 *If F is an admissible Hamiltonian on the quasicylinder (Q,Ω) and if M is semi-positive then $\|F\| \le \operatorname{area} Q$.*

Because (Q,Ω) is a product near its boundary ∂Q we can identify this to a single fiber M_0 and so replace Q by $(V = M \times S^2, \Omega)$ where Ω restricts to ω on each fiber.

Definition 3.2 An Ω–tame almost complex structure J on V will be said to be *normalized* if each fiber is J–holomorphic and if in addition it is a product near both M_0 and M_∞.

Thus each such J defines a 2–parameter family of ω–tame almost complex structures on M, and by our assumptions on M we can assume that there are no J–holomorphic spheres that have Chern number < 0 and lie in a fiber of V. Since the existence of such curves is what necessitates the introduction of virtual moduli cycles, we will be able to count curves in V (and hence define appropriate Gromov–Witten invariants) provided that we are in a situation where the only bubbles that appear lie in its fibers.

The idea of the proof is to assume that $\|F\| > \operatorname{area} Q$ and to find a contradiction. Let $A = [pt \times S^2] \in H_2(V)$. It is shown in [10] that there is a family of noncohomologous symplectic forms Ω_s on V starting with $\Omega_0 = \Omega$ such that Ω_1 is a product. Hence the fibered space (V,Ω) is deformation equivalent to a product, which implies that $Gr(A) = 1$, where the Gromov invariant $Gr(A)$ counts the number of J–holomorphic A–spheres in V going through some fixed point p in V for sufficiently generic J. We will choose p to be some minimum $p_\infty \in M_\infty$ of F, and will fix the parametrizations u of the spheres by requiring that

$$u(0) \in M_0, \quad u(1) \in M_1, \quad u(\infty) = p_\infty \in M_\infty,$$

where M_1 is some fiber distinct from M_0, M_∞. The arguments given in Section 3.2 below show that one can calculate $Gr(A)$ using generic normalized J. Hence, for such J the number of these curves will sum up to 1 when counted with the appropriate signs. (In fact, in this semi-positive case, one can use mod 2 invariants and so ignore the sign.)

We now "turn on" the perturbation corresponding to the Hamiltonian flow of λF for increasing $\lambda \ge 0$.[8] The resulting trajectories u have domain $\mathbb{C}$ and in

[8] One must be very careful with signs here since there are many different conventions in use. We have chosen to use the upward gradient flow of F (even though it is more usual to use the downward flow) because this fits in with our set-up. Since F takes its maximum on M_0 we need to consider trajectories going from this maximum to a minimum: see Lemma 3.3 below.

terms of the coordinates (s,t) of $(-\infty,\infty)\times S^1$ satisfy the following equation for some λ:

$$\begin{aligned} \partial_s u + J(u)\partial_t(u) &= \lambda\,(\operatorname{grad} F)\circ u, && (1)\\ \lim_{s\to-\infty} u(s,t)\in M_0, &\qquad \lim_{s\to\infty} u(s,t) = p_\infty, && (2)\end{aligned}$$

where $\operatorname{grad} F$ is the gradient of F with respect to the metric defined by Ω and J. Because $dF = 0$ near M_0 the map u is J–holomorphic for $s << 0$ and so, by the removable singularity theorem, does extend to a holomorphic map $\mathbb{C}\to V$. Thus u is a generalized Floer trajectory of the kind considered in [8, 21], and we will call it a λ–trajectory. Because its limit at ∞ is a point, it also extends to a continuous map $S^2\to V$ that represents the class A. It is shown in [8] that the algebraic number of solutions to this equation is still 1 for small λ.

Given F and a normalized J, let $\mathcal{C}=\mathcal{C}_A$ be the moduli space consisting of all pairs (u,λ) where $\lambda\in[0,1]$ and u: $\mathbb{R}\times S^1\to V$ satisfies equations (1), (2) as well as the following normalization condition:

$(*)$ $u(0,0)\in M_1$ where M_1 is a fiber of Q distinct from M_0, M_∞.

Note that $\Omega(A)$ is precisely the area of Q. The crucial ingredient that ties the solutions of the above equation to the area–capacity inequality is the fact that the size $\|F\|$ of F gives an upper bound for λ.

Lemma 3.3 *If $(u,\lambda)\in\mathcal{C}_A$ then $\lambda\|F\| < \Omega(A) =$ area Q.*

Proof A standard calculation shows that the action functional

$$a(s) = \int_{(-\infty,s]\times S^1} u^*\Omega + \int_0^1 \lambda H(u(s,t))dt$$

is a strictly increasing function of s. Since $F(p_\infty) = 0$ and $F|_{M_0} = \|F\|$ by construction, the action $a(s)$ satisfies

$$\lim_{s\to-\infty} a(s) = \lambda\|F\|,\quad \lim_{s\to\infty} a(s) = \Omega(A).$$

Hence $\lambda\|F\| < \Omega(A)$ as claimed. □

Note that if p_∞ is a nonovertwisted critical point of F of Morse index k, then the formal dimension of $\mathcal{C}$ is $1+k$ (see for example [21]) and so equals 1 with the current choice of p_∞. Because A is not a multiple class, it follows from the standard theory that for any M we can regularize the moduli space $\mathcal{C}$ by choosing a generic normalized J: see Section 3.2. Hence for such a choice $\mathcal{C}$ is a manifold of dimension 1 lying over $[0,1]$ via the projection

$$\mathrm{pr}\colon \mathcal{C} \to [0,1], \qquad (u,\lambda) \mapsto \lambda.$$

Because λ is restricted to the interval $[0,1]$, $\mathcal{C}$ could have boundary over $\lambda = 0,1$. As mentioned above, 0 is a regular value for pr for generic J, and the algebraic number of points in $\mathrm{pr}^{-1}(0)$ is 1. On the other hand, we know from Lemma 3.3 above that, if $\|F\| \geq \operatorname{area} Q$, the set $\mathrm{pr}^{-1}(\lambda)$ is empty for $\lambda = 1$. The only way to reconcile these statements is for $\mathcal{C}$ to be noncompact.

Noncompactness of $\mathcal{C}$

Noncompactness in a moduli space of J–holomorphic Floer trajectories is caused either by the bubbling off of J–holomorphic spheres or by the splitting of Floer trajectories. Now bubbling is a codimension 2 phenomenon, and so, provided that we can make everything regular by choosing a suitably generic J, it will not occur along the 1–dimensional space $\mathcal{C}$. It is easy to see that all bubbles have to lie in some fiber. Hence, by our choice of normalization for J, we can avoid all bubbles. (There are some extra details here that are discussed in Section 3.2 below.)

Floer splitting is harder to deal with since it occurs in codimension 1: a generic 1–parameter family of Floer trajectories can degenerate into a pair of such trajectories. For example, the trajectories in $\mathcal{C}$ could converge to the concatenation of a λ–trajectory $u\colon \mathbb{C} \to V$ in class $A-B$ that converges to some critical point p_k on M_∞ of index k together with a Floer λ–trajectory in M_∞ from p_k to p_∞ in class $B \in H_2(M)$. We will see in Lemma 3.7 below that these are the only degenerations that happen generically. Observe also that these degenerations do not occur in the situation treated by Hofer–Viterbo because of their topological assumptions on M.

To analyse this situation further, denote by

$$\mathcal{C}_{A-B}(p_k)$$

the space of all pairs (u,λ), where $u\colon \mathbb{C} \to V$ is a solution to equations (1), (2) with p_∞ replaced by p_k, that is normalised by condition $(*)$ and represents the class $A-B$. Similarly, denote by

$$\mathcal{F} = \mathcal{F}_B(p_k)$$

the space of all pairs (v,λ) where $v\colon \mathbb{R} \times S^1 \to M_\infty$ is a Floer trajectory for λF from p_k to p_∞ in class B. Note that the classes B that occur here are constrained by the inequality $\omega(B) < \omega(A)$. Moreover, since our assumption is that $\|F\| > \text{area}\, Q$, we can slightly perturb F within the class of admissible Hamiltonians to make H_M slow and generic in the sense of Lemma 3.6. That lemma then says that we can choose J so that all the relevant moduli spaces of simple trajectories are regular, ie, have dimension equal to their formal dimension. Thus $\mathcal{C}_{A-B}(p_k)$ will have dimension $-2c_1(B) + k + 1$, where $k = \text{index}\, p_k$. Further if $B \neq 0$ is a simple (ie nonmultiple) class, then $\mathcal{F}$ has dimension $2c_1(B) - k + 1$. Because F and J_M are independent of the time coordinate t and because the trajectories in $\mathcal{F}$ limit on fixed points rather than nonconstant periodic orbits, there is a 2–dimensional reparametrization group acting on the trajectories in $\mathcal{F}$. Thus we need $2c_1(B) - k + 1 \geq 2$ for $\mathcal{F}$ to be nonempty, while we need $-2c_1(B) + k + 1 \geq 0$ for $\mathcal{C}_{A-B}(p_k)$ to be nonempty. Therefore, if these spaces are both nonempty, $\mathcal{F}$ has dimension 2 and $\mathcal{C}_{A-B}(p_k)$ has dimension 0. Hence these spaces both consist of discrete sets of points, which, for generic J, will project to disjoint sets in the λ–parameter space. Thus this kind of degeneration does not occur for generic J.

The crucial point in this argument is that the elements in $\mathcal{F}$ have an S^1 symmetry. This presents a problem, since in general one cannot regularize Floer moduli spaces containing multiply covered trajectories unless one allows either the Hamiltonian F or the almost complex structure J to depend on t: see [5]. The usual way to deal with this is to assume that M is monotone: see Floer [4]. However, we now show that in our special situation this assumption is unnecessary.

First observe that we must also avoid the case when the trajectory itself is independent of t, since then the S^1 action becomes vacuous. But this could only happen if $B = 0$ and our choice of p_∞ implies both that $k \geq 0$ and that $B \neq 0$. (Because the action $a(s)$ is strictly increasing and $F(p_k) \geq F(p_\infty)$ we must have $\omega(B) > 0$.) The above argument shows that we need $2c_1(B) - k + 1 \geq 2$ and hence $c_1(B) > 0$ for $\mathcal{F}$ to be nonempty when B is simple and J is generic. Moreover, if there is a multiply covered trajectory in class $\ell B, \ell > 1$, from p_k to p_∞ then it covers an underlying simple trajectory in class B between these points. Therefore we must have $c_1(B) > 0$ and $2c_1(B) - k + 1 \geq 2$ in this case too. But then the formal dimension $-2\ell c_1(B) + k + 1$ of $\mathcal{C}_{A-\ell B}(p_k)$ is always negative. But, because $A - \ell B$ is not a multiple class, this moduli space consists of simple trajectories. Therefore our assumptions imply that it is regular and hence empty for generic J.

It follows (modulo a few details discussed in Section 3.2 below) that there are no degenerations of the trajectories in $\mathcal{C}$ for $\lambda \in [0, 1]$. But we saw earlier that if $\|F\| \geq \operatorname{area} Q$ these trajectories must degenerate, ie, $\mathcal{C}$ cannot be compact. Therefore $\|F\| < \operatorname{area} Q$.

We have used the fact that none of the critical points of F are overtwisted twice in the above argument. First, it implies that the contribution of each critical point p_k to the dimension of $\mathcal{C}_A$ is just its Morse index k and so is ≥ 0. Second, we need the space of λ–trajectories to p to be regular for *each* $\lambda \in [0,1]$ which is impossible if the linearized flow at p has a periodic orbit of period λ.

3.2 More details

We first discuss the behavior of the flow near overtwisted critical points, and then give more details of the transversality arguments needed to understand the compactification of $\mathcal{C}$.

Overtwisted critical points

Since this question is local, we consider Hamiltonians $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ with a nondegenerate critical point at 0. We denote the Hessian by Q so that the linearized flow at 0 is e^{At} where $A = -J_0 Q$. The eigenvalues of A occur in real or imaginary pairs $\pm\lambda, \pm i\lambda$, $\lambda \in \mathbb{R}$, or in quadruplets $\pm\mu, \pm\overline{\mu}, \mu \in \mathbb{C} - (\mathbb{R} \cup i\mathbb{R})$. Correspondingly, $\mathbb{R}^{2n}$ decomposes as a symplectically orthogonal sum of eigenspaces, one for each pair or quadruplet. We will be concerned with the partial decomposition

$$\mathbb{R}^{2n} = E \oplus \sum_{j=1}^{k} E_j$$

where the purely imaginary eigenvalues of A are $\pm\lambda_1, \dots, \pm\lambda_k$ and $E_j \otimes \mathbb{C}$ is the sum of the eigenspaces for the pair $\pm i\lambda_j$, and $E \times \mathbb{C}$ is the sum of the others. Observe that each E_j contains a subspace of dimension at least 2 that is filled out by periodic orbits of e^{At} of period $2\pi/\lambda_j$. Indeed, for each eigenvector $v \in \mathbb{C}^{2n}$ in $E_j \otimes \mathbb{C}$ the intersection of E_j with the subspace $\mathbb{C}v \oplus \mathbb{C}\overline{v}$ consists entirely of such periodic orbits. Hence, if A has imaginary eigenvectors the linearized flow always has nonconstant periodic orbits.

However this is not necessarily true for the nonlinear flow ϕ_t^H. Moser considers the following example in [20]:[9]

$$H(z_1, z_2) = \frac{1}{2}(|z_1|^2 - |z_2|^2) + (|z_1|^2 + |z_2|^2)\Re(z_1 z_2).$$

[9]He uses complex variables. Observe that if $z_k = x_k + iy_k$ the Hamiltonian flow with our sign conventions can be written as $\dot{z}_k = -2i(\partial H/\partial \overline{z}_k)$.

Clearly, the eigenvalues of A_H are $\pm i$. However, it is easy to check that the time derivative of the function $\Im(z_1 z_2)$ is strictly negative whenever $(z_1, z_2) \neq (0,0)$. Hence there are no nonconstant periodic orbits.

The problem here is that the two eigenvalues are equal. More generally, similar phenomena can occur if any pair $i\lambda, i\lambda'$ of eigenvalues are resonant, ie, if the ratio λ'/λ is integral. The next result is well known, and is proved in the real analytic case in Siegel–Moser [28] Section 16.

Lemma 3.4 *Suppose in the above situation that $i\lambda$ is an imaginary eigenvalue of A of multiplicity 1 that is nonresonant in the sense that the ratio λ'/λ is nonintegral for all other imaginary eigenvalues $i\lambda'$ of A. Then the flow ϕ_t^H of H has a periodic orbit of period close to $2\pi/\lambda$ on every energy surface close to zero.*

Proof The linearized flow around $\{0\}$ is e^{At} where $A = -J_0 Q$. As above $\mathbb{R}^{2n}$ decomposes as a symplectically othogonal sum $E_0 \oplus E_\lambda$, where E_λ is a 2–dimensional space filled by periodic orbits of period $2\pi/\lambda$ and the restriction of A to E_0 has no eigenvalues of the form $ik\lambda, k \in \mathbb{Z}$. Consider the level set

$$S_1 = \{x \in \mathbb{R}^{2n} : H_Q(x) = 1\}$$

of the quadratic part H_Q of H. By construction, it intersects E_λ in a periodic orbit γ for e^{tA} of period $T = 2\pi/\lambda$. The first return map ϕ_γ of this orbit can be identified with the restriction e^{TA_0} of e^{TA} to E_0. Hence our assumptions on the eigenvalues of A imply that its only fixed point is at the origin. Thus its Gauss map

$$g\colon S^{2n-3} \to S^{2n-3}, \quad v \mapsto \frac{\phi_\gamma(v) - v}{\|\phi_\gamma(v) - v\|}$$

is well defined. Observe that g has degree 1. In fact it is injective. For, otherwise there would be vectors v, w lying on different rays in E_0 such that $\phi_\gamma(v) - v = \phi_\gamma(w) - w$. Since ϕ_γ is linear, this would imply that it has 1 as an eigenvalue, contrary to hypothesis.

Now consider the functions $x \mapsto \varepsilon^{-2} H(\varepsilon x)$. Since they converge to H_Q as ε decreases to 0, for each fixed sufficiently small ε the orbits that start near γ remain near γ for $t \in [0,T]$. Hence the first return map given by following these orbits round γ is a perturbation ϕ_γ^ε of ϕ_γ. Hence its Gauss map is also defined and has degree 1 for small ε. But this means that the Gauss map cannot extend over the interior of S^{2n-3}; in other words, ϕ_γ^ε must have a fixed point. This corresponds to a closed periodic orbit of $\varepsilon^{-2} H(\varepsilon x)$ that is close to γ and has period T_ε close to T. Since $\varepsilon^{-2} H(\varepsilon x)$ is conjugate to H, this implies that H also has a periodic orbit of period T_ε. □

Corollary 3.5 *If a generic H has an overtwisted critical point, ie, if its Hessian has imaginary eigenvalue $i\lambda$ with $\lambda > 2\pi$, then its flow has a nonconstant periodic orbit of period < 1.*

Proof The hypotheses of the above lemma are satisfied for generic H. □

Lemma 3.6 *Suppose that the Hamiltonian H on M is slow. Then H has arbitrarily small perturbations H' such that for generic J the moduli spaces of simple Floer trajectories for $\lambda H', \lambda \in [0,1]$, in classes $B \in H_2(M)$ are all regular.*

Proof If necessary, we first replace H by cH for some c close to 1 so that neither H nor its linearized flows have nonconstant periodic orbits of period ≤ 1. Then slightly perturb H so that it is also a Morse function. Finally, note that by [5] Remark 7.3 we may perturb H to H' so that for all $\lambda \in [0,1]$ the critical points of $\lambda H'$ satisfy the nondegeneracy conditions of [5] Lemma 7.2 with respect to a generic set of J and for all λ. Thus simple (ie nonmultiply covered) Floer trajectories all have regular injective points in the sense of [5] Section 7. The result now follows by [5] Theorem 7.4. □

As always, it is not enough to know that trajectory spaces are regular. One also needs to show that their closures have the right dimension. This will follow from Lemma 3.8 below.

Structure of the stable maps in the closure of $\mathcal{C}$

Next let us check that the degenerations of the elements in $\mathcal{C}$ really are compatible with the fibration. By the standard compactness theorem, these degenerations consist of a finite number of Floer λ–trajectories $u_i\colon \mathbb{R} \times S^1 \to V$, $i = \ell, \dots, k$ that are laid end to end together with some bubbles $v_j\colon S^2 \to V$. Here, the u_i are labelled in order, so that

$$\lim_{s\to\infty} u_i \quad = \quad \lim_{s\to-\infty} u_{i+1}, \quad \ell < i < k.$$

Since the only critical points are either near M_0 or on M_∞ there has to be at least one trajectory going between these manifolds. Pick one of them and call it u_1. (We will see that in fact there is only one such trajectory.) Because F is slow, the u_i converge to critical points of F at each end and so represent some homology classes in V. In the proof of the next result it is convenient to allow ourselves to decrease the component $\beta(r)$ of F that is perpendicular to the fiber at M_∞. Since we assumed $\beta < \pi r^2$ for small r, we can reduce β to εr^2 on $r < \delta/2$ for any ε without introducing any nonconstant fast periodic orbits.

Lemma 3.7 *Let (u_i, v_j) be a limit of elements of $\mathcal{C}$ as described above. If ε is sufficiently small, each bubble v_j is contained in some fiber, and the $u_i, i \neq 1$, are Floer λ–trajectories in M_∞. Moreover, $\ell = 1$ and the homology class represented by u_1 has the form $A - B$, for some $B \in H_2(M)$ with $0 \leq \omega(B) < \omega(A)$.*

Proof Suppose that $(u^\alpha, \lambda^\alpha)$ is a sequence of elements of $\mathcal{C}$ that converges weakly to a limit of the above type, where $u^\alpha: \mathbb{C} \to V$. Fix α and consider the composite map

$$\overline{u}^\alpha = \pi \circ u^\alpha: \mathbb{C} \to V \to S^2.$$

Since J is a product near M_0 this map is holomorphic over the inverse image of the neighborhood $\overline{U}_0$ of $0 \in S^2$. Hence, because it has degree 1, the projection from the image of u^α to the base is injective over $\overline{U}_0$.

Let z_j be the set of points in $\mathbb{C}$ at which $|du^\alpha(z)| \to \infty$. Then the restriction of u^α to compact pieces of $\mathbb{C} - \cup z_j$ converges to a map whose projection to the base is holomorphic and nonconstant over $\overline{U}_0$. Thus this limit is the trajectory u_1. Since its intersection with the fiber class is 1, it must represent some class of the form $A - B$, with $B \in H_2(M)$.

Now consider the bubbles. These are always J–holomorphic and so their projections to the base are holomorphic near M_0. Further, because the fibers are J–holomorphic they intesect each fiber positively. Hence each bubble either is entirely contained in a single fiber or represents a class $kA + B$ with $k > 0$. But in the latter case they must intersect each fiber of $M \times \overline{U}_0$ which is impossible because the projection from the image of u^α to the base is injective over $\overline{U}_0$ and, as noted above, these points converge to the component u_1.

Finally, consider the Floer trajectories. Suppose there was a trajectory that came before u_1 and so had endpoint on M_0. The previous argument applies to show that it is entirely contained in M_0 and therefore satisfies the unperturbed Cauchy–Riemann equation and should be considered as a bubble. In particular there is only one Floer trajectory that meets both M_0 and M_∞ namely u_1. Hence the other Floer trajectories begin and end at points in M_∞, and we claim that for sufficiently small ε they are completely contained in M_∞.

To see this, note that if ε were 0, then F would depend only on the fiber coordinates in the neighborhood $r < \delta/2$ of M_∞. Thus the Floer trajectories would project to holomorphic trajectories in the base and positivity of intersections with the fiber would imply as before that the trajectories are entirely contained in M_∞. Therefore, because we are only interested in trajectories lying in a finite set of homology classes and with a finite set of possible endpoints, standard compactness arguments imply that for sufficiently small ε all trajectories must be contained in the neighborhood $M_\infty \times \{r < \delta/2\}$ of M_∞. Thus these trajectories would project to nullhomologous Floer trajectories in S^2 for the function εr^2 that begin and end at the point $r = 0$. But these do not exist because the action functional could not increase strictly along such a trajectory.

It remains to prove the statement about the class $A-B$ represented by u_1. Let B_i, B_j be the classes represented by the other u_i and the bubbles v_j. Clearly each $\omega(B_j) > 0$. Further each $\omega(B_i) > 0$ because a strictly increases along each trajectory and p_∞ is a minimum of F: see Lemma 3.3. Similarly, $\omega(A-B) > 0$ since $u_1(0)$ lies at a maximum of F. Since $\omega(B)$ is the sum of the $\omega(B_i), \omega(B_j)$, the result follows. □

Transversality of intersections of bubbles with trajectories

First observe that by the previous lemma the only classes $B \in H_2(M)$ that occur as a component u_i or v_j of a limiting trajectory in the closure of $\mathcal{C}$ have $\omega(B) < \omega(A) = \operatorname{area} Q$. Hence only a finite number of classes can occur. As already noted, standard theory tells us that we can regularize the moduli spaces of vertical bubbles in V and make all their intersections transverse by choosing generic normalized J on V. Thus all spaces of bubble trees (or cusp-curves) can be assumed to be of the right dimension.

Similarly, as we noted in Lemma 3.6, spaces of nonmultiply covered Floer trajectories in M_∞ as well as the moduli spaces $\mathcal{C}_{B,p_k}$ can be regularized by a time independent J by [5]. Thus there is a subset $\mathcal{J}_{reg}$ of second category in the space of all normalized almost complex structures on Q such that all spaces of bubble trees and of simple trajectories are regular.

In order to make the "usual" theory of J–holomorphic curves work we must also ensure that these moduli spaces intersect transversally. The basic arguments that establish this for spheres are given in [18] and the case of Floer trajectories is discussed in [5]. However, the standard proof that spaces of bubbles can be assumed to intersect transversally uses the fact that if two distinct simple bubbles $\operatorname{im} u$ and $\operatorname{im} v$ intersect at some point $x = u(z) = v(w)$ then there is a small annulus α around z whose image by u does not intersect $\operatorname{im} v$: see [18] Propositions 6.3.3 and 2.3.2. This holds because otherwise the two curves are infinitely tangent at x and so must coincide. This argument breaks down for bubbles and Floer trajectories since they satisfy different equations. Since this detail seems to have been ignored in standard references such as [5], we deal with it now.

For simplicity, we will suppose that there is just one bubble and so will consider the intersection of the space of unparametrized bubbles in class B with the moduli space $\mathcal{C}_{B'} = \mathcal{C}_{B',p_\infty}$. It suffices to consider the intersection of the corresponding parametrized curves. Hence let $\mathcal{X}$ be the space of all maps

$$u\colon (S^2, 0, \infty) \to (Q, M_0, p_\infty)$$

in the class $A - B'$, let $\mathcal{Y}$ be the space of all maps $v\colon S^2 \to Q$ representing the class B, and consider the space $\mathcal{U}$ of all tuples

$$(u, v, \lambda, z, J) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \times S^2 \times \mathcal{J}$$

satisfying the following conditions:

(i) u is a Floer λ–trajectory with respect to J;

(ii) the bubble v is J holomorphic.

We want to show that when J lies in a subset $\mathcal{J}_{reg}$ of second category in $\mathcal{J}$ the space

$$\{(u, v, z) : (u, v, \lambda, z, J) \in \mathcal{U}, u(z) = v(0)\}$$

is a manifold of the correct dimension. This follows in the usual way from the next lemma.

Lemma 3.8 *The evaluation map*

$$ev\colon \mathcal{U} \to Q \times Q : (u, v, \lambda, z, J) \mapsto (u(z), v(0))$$

is transverse to the diagonal.

Proof If $z = 0$ then u is J–holomorphic near z and the argument of [18] Propositions 6.3.3 works. The case $z = \infty$ is somewhat special since the moduli space of u–trajectories does not have a tangent space at this point. However, this does not matter since $u(z)$ is fixed for all J because it is the endpoint of the Floer trajectory. Instead we look at the space of v–bubbles and can appeal to Theorem 6.1.1 of [18] that says that the map from the space of all pairs (v, J) in $\mathcal{U}$ to Q given by evaluation

$$ev_2\colon (v, J) \mapsto v(0)$$

is surjective.

When $z \neq 0, \infty$, we can identify the domain of u with $\mathbb{C}$ and by reparametrization fix $z = 1$. The domain of the linearization D_u of the defining equation for the Floer trajectory equation at u is then the space $W^{1,p}(u^*TQ)$ which is defined to be the closure with respect to the $(1, p)$–Sobolev norm of the space of compactly supported C^∞–sections of u^*TQ that are tangent to the fiber at $z = 0$: see [5] Section 5. Thus we may replace $\mathcal{U}$ by the space $\mathcal{U}'$ of tuples (u, v, λ, J). The tangent space of $\mathcal{U}'$ at (u, v, λ, J) consists of elements (ξ_1, ξ_2, r, Y) with $\xi_1 \in W^{1,p}(u^*TQ)$, $\xi_2 \in W^{1,p}(v^*TQ)$ and such that

$$D_u(\xi_1) + \frac{1}{2}Y(u) \circ du \circ i = rg_F, \qquad (*)$$
$$D_v(\xi_2) + \frac{1}{2}Y(v) \circ dv \circ i = 0 \qquad (**).$$

(Here g_F is the appropriate term coming from the variation in λF.) Moreover the derivative $d(ev)$ of the evaluation map is given by

$$d(ev)(\xi_1, \xi_2, Y) = (\xi_1(1), \xi_2(0)) \in T_{(x,x)}(Q \times Q).$$

We know by Theorem 6.1.1 in [18] that the map $(\xi_2, Y) \to \xi_2(0) \in T_xQ$ is surjective. Hence given $a \in T_xQ$ there is (ξ_2^a, Y^a) that satisfy (**) with $\xi_2^a(0) = a$. Note that we cannot assume that the support of Y^a is disjoint from the image of u though we can make it in an arbitrarily small neighborhood of the intersection point $v(0)$. Thus the element $\nu = \frac{1}{2}Y^a \circ du \circ i$ may well be nonzero. Clearly, it will suffice to find (ξ_1, Y) so that

$$\xi_1(1) = 0, \quad L(\xi_1, Y) = -\nu, \quad Y = 0 \text{ in the support of } Y^a$$

where

$$L(\xi_1, Y) = D_u(\xi_1) + \frac{1}{2}Y(u) \circ du \circ i.$$

The usual proof of transversality (as in [18] Proposition 3.4.1 or [5] Theorem 7.4) shows that the operator L is surjective if ξ_1 ranges freely in $W^{1,p}(u^*TQ)$ and Y is constrained to have support near any injective point of u. In particular, the condition that $\xi_1(0)$ be tangent to the fiber can be fulfilled by adding a suitable vector tangent to the group of Möbius transformations of S^2 that fix ∞ and 1. Since the image of v lies in a fiber distinct from M_0 and u is injective near there we can easily arrange that the support of Y is disjoint from that of Y^a. Thus the only problem is the question of how to deal with the condition $\xi_1(1) = 0$.

To do this, we must consider more closely the proof that L is surjective. The argument goes as follows. Since

$$D_u \colon W^{1,p}(u^*TQ) \to L^p(\Omega^{0,1}u^*TQ)$$

is Fredholm, the image of L is closed and it suffices to show that it is dense. If not, there is η in the dual space $L^q((\Omega^{0,1}u^*TQ)^*)$ that vanishes on $\operatorname{im} L$. In the standard case this implies that η is a weak solution of the adjoint equation

$D_u^*\eta = 0$ since it vanishes on all the elements $D_u\xi_1$. Hence, by elliptic regularity, it is a strong solution of this equation. It also must vanish in some open set because it pairs to zero with all the elements $L(0, Y)$. Hence $\eta = 0$ as required.

In our case ξ_1 is not an arbitrary element of $W^{1,p}(u^*TQ)$ but rather is in the image of the map

$$W^{1,p}(u^*TQ \otimes E) \xrightarrow{\phi} W^{1,p}(u^*TQ)$$

where E is a holomorphic bundle over S^2 with Chern class -1 and ϕ tensors the sections of $u^*TQ \otimes E$ by a holomorphic section s of the dual bundle E^* that vanishes at 1. Since s is holomorphic there is a commutative diagram

$$\begin{array}{ccc} W^{1,p}(u^*TQ \otimes E) & \xrightarrow{D_u^E} & L^p(\Omega^{0,1}u^*TQ \otimes E) \\ \otimes s \downarrow & & \otimes s \downarrow \\ W^{1,p}(u^*TQ) & \xrightarrow{D_u} & L^p(\Omega^{0,1}u^*TQ). \end{array}$$

It follows that the image $\eta^E = \phi^*(\eta) = \eta \otimes s$ of η in $L^q((\Omega^{0,1}u^*TQ \otimes E)^*)$ is a weak solution of the adjoint equation $(D_u^E)^*\eta^E = 0$. The standard argument applies to show that $\eta^E = \eta \otimes s$ is zero. Hence the L^q–section η also vanishes. □

3.3 The case of general M

To construct the virtual moduli cycle as in [12] for curves in some manifold (V, ω) one looks at the configuration space $\mathcal{B}$ of all pointed stable maps in some class A that are nearly holomorphic. Roughly speaking, $\mathcal{B}$ is an orbifold that supports a orbibundle $\mathcal{L}$ whose fiber L_u at the map $u\colon \Sigma \to V$ is the Sobolev space of $L^{k,p}$–smooth sections of the bundle $\Omega^{0,1}(\Sigma, u^*(TV))$ of $(0,1)$–forms on the nodal Riemann surface Σ. For each J, the delbar operator $\overline{\partial}_J$ defines a section of $\mathcal{L}$ whose zero set is the set $\overline{\mathcal{M}}_J$ of J–holomorphic stable maps. If the derivative

$$D_u\colon L^{k+1,p}(\Sigma, u^*(TV)) \to L_u$$

of this map is surjective for all $(\Sigma, u) \in \overline{\mathcal{M}}_J$, this zero set is an orbifold of the right dimension and its fundamental cycle can be used to define Gromov–Witten invariants. Although $\overline{\mathcal{M}}_J$ is always compact with respect to the weak topology of $\mathcal{B}$, it might well be that for all J' near J this derivative is badly behaved, so that $\overline{\mathcal{M}}_{J'}$ has components of too large dimension. What one does to remedy the situation is define, over some orbifold neighborhood $\mathcal{W}$ of $\overline{\mathcal{M}}_J$ in $\mathcal{B}$, a finite-dimensional subspace R of the set of sections of $\mathcal{L}$ such that the map

$$D_u \oplus \iota_u\colon L^{k+1,p}(\Sigma, u^*(TV)) \oplus R \to L_u$$

is surjective for all (Σ, u) in some smaller neighborhood $\mathcal{W}_R$ of $\overline{\mathcal{M}}_J$, where ι_u denotes evaluation at u. This implies that for a generic small element $\nu \in R$ the set of solutions of the perturbed Cauchy–Riemann equation

$$\partial_J(u) + \iota_u(\nu) = 0$$

has the right dimension and supports a fundamental cycle. This is often called the virtual moduli cycle or regularized moduli space $\overline{\mathcal{M}}^\nu$.

This is the briefest outline of Liu–Tian's method. Many more details can be found in [12, 13, 15]. The main point is the construction of R. The idea is to find a suitable perturbation space R_i over each subset U_i of an open cover of $\overline{\mathcal{M}}_J$ and then to patch these together.

In our situation we start with an action of S^1 by reparametrization on the space of J-holomorphic Floer trajectories in $V = M$ between two points p and q and want to construct the regularization $\overline{\mathcal{M}}^\nu$ so that it also supports an S^1–action. To do this one must first extend the original action to the neighborhood $\mathcal{W}$. This extension will not simply be an action of S^1: if a trajectory splits into two, or more generally k, pieces there will be an S^1 action on each part, and one has to make everything equivariant with respect to this. In particular, one must choose the initial covering $\{U_i\}$ so that each set U_i is invariant under this generalized action.

It is shown in [13] that these methods allow one to carry through the arguments in Section 3.1. Hence Proposition 2.8 holds for general M.

Once we have this powerful method there is no need to cling to all the special conditions that we put on F that adapted it to the fibration on $M \times S^2$. For the argument to make sense, we need F to be constant and equal to its absolute maximum (resp. minimum) in a neighborhood of one fiber and to assume its absolute minimum (resp. maximum) at some point that plays the role of p_∞. The other important condition is that F be slow. Thus F is admissible in that it belongs to the set $\mathcal{H}'_{ad}(M \times S^2)$ defined in Section 1. Using the methods of Liu–Tian to regularize the closure of the trajectory space $\mathcal{C}$ in $V = M \times S^2$ for these more general functions F, we obtain the following result.

Proposition 3.9 *Given any closed symplectic manifold (M, ω) and any quasi-cylinder $(Q = M \times D, \Omega)$ the capacity c'_{HZ} satisfies the area-capacity inequality*

$$c'_{HZ}(Q, \Omega) \leq \text{area}\,(Q, \Omega).$$

Proposition 1.6 clearly follows.

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