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Positive scalar curvature, diffeomorphisms and the Seiberg–Witten invariants

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Abstract

We study the space of positive scalar curvature (psc) metrics on a 4–manifold, and give examples of simply connected manifolds for which it is disconnected. These examples imply that concordance of psc metrics does not imply isotopy of such metrics. This is demonstrated using a modification of the 1–parameter Seiberg–Witten invariants which we introduced in earlier work. The invariant shows that the diffeomorphism group of the underlying 4–manifold is disconnected. We also study the moduli space of positive scalar curvature metrics modulo diffeomorphism, and give examples to show that this space can be disconnected. The (non-orientable) 4–manifolds in this case are explicitly described, and the components in the moduli space are distinguished by a Pin^c eta invariant.

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1 Introduction

One of the striking initial applications of the Seiberg–Witten invariants was to give new obstructions to the existence of Riemannian metrics of positive scalar curvature on 4–manifolds. The vanishing of the Seiberg–Witten invariants of a manifold admitting such a metric may be viewed as a non-linear generalization of the classic conditions [18, 17, 26] derived from the Dirac operator. If a manifold Y has a metric of positive scalar curvature, it is natural to investigate the topology of the space $\mathrm{PSC}(Y)$ of all such metrics. Perhaps the simplest question which one can ask is whether $\mathrm{PSC}(Y)$ is connected; examples of manifolds for which it is disconnected were previously known in all dimensions greater than 4. This phenomenon is detected via the index theory of the Dirac operator, often in conjunction with the Atiyah–Patodi–Singer index theorem [2].

In the first part of this paper, we use a variation of the 1–parameter Seiberg–Witten invariant introduced in [27] to prove that on a simply-connected 4–manifold Y, $\mathrm{PSC}(Y)$ can be disconnected. Our examples cannot be detected by index theory alone, ie without the intervention of the Seiberg–Witten equations. These examples also yield a negative answer, in dimension 4, to the question of whether metrics of positive scalar curvature which are concordant are necessarily isotopic. Apparently (cf the discussion in [26, section 3 and section 6]) this is the first result of this sort in any dimension other than 2.

An *a priori* more difficult problem than showing that PSC is not necessarily connected is to find manifolds for which the "moduli space" $\mathrm{PSC}/\mathit{Diff}$ is disconnected. (The action of the diffeomorphism group on the space of metrics is by pull-back, and preserves the subset of positive scalar curvature metrics.) The metrics lying in different components of $\mathrm{PSC}(Y)$ constructed in the first part of the paper are obtained by pulling back a positive scalar curvature metric via one of the diffeomorphisms introduced in [27], and hence give no information about $\mathrm{PSC}/\mathit{Diff}$. Building on constructions of Gilkey [10] we give explicit examples of non-orientable 4–manifolds for which the moduli space is disconnected. These examples are detected, as in [10, 4], by an η–invariant associated to a Pin^c Dirac operator.

The Seiberg–Witten invariant for diffeomorphisms introduced in [27] is not very amenable to calculation, for reasons explained below. Hence, in the first section of this paper, we will give a modification of that construction which yields a more computable invariant. From this modified Seiberg–Witten invariant we will deduce the non-triviality of the isotopy class of the diffeomorphisms described in [27], the non-connectedness of PSC, and the fact that concordance does not imply isotopy. The modification has the disadvantage of not behaving very sensibly under composition of diffeomorphisms, but it suffices for the application to the topology of PSC.

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2 Seiberg–Witten invariants of diffeomorphisms

Let us briefly describe the approach taken in [27] to defining gauge-theoretic invariants of diffeomorphisms, and explain a modification of that approach which renders the Seiberg–Witten version of those invariants more usable in the current paper.

Notation A Spin^c structure on a manifold Z will be indicated by Γ, representing the Spin^c bundles $W^\pm$ and the Clifford multiplication $\Lambda^*(Z)\times W^\pm\to W^\mp$. The set of perturbations for the Seiberg–Witten equations will be denoted by:

$$\Pi=\{(g,\mu)\in\mathrm{Met}(Z)\times\Omega^2(Z)\mid *_g\,\mu=\mu\}$$

We will sometimes identify $\mathrm{Met}(Z)$ with its image $\{(g,0)|\, g\in\mathrm{Met}(Z)\}$, and so will write $g\in\Pi$ instead of $(g,0)\in\Pi$. A Riemannian metric g and a $U(1)$ connection A on $L=\det(W^+)$ determine the Dirac operator $D_A\colon\Gamma(W^+)\to\Gamma(W^-)$. For $(g,\mu)\in\Pi$, the perturbed Seiberg–Witten equations for A and $\varphi\in\Gamma(W^+)$ are written:

$$\begin{cases} D_A\varphi=0\\ F_A^+ + \imath\mu = q(\varphi)\end{cases}\tag{1}$$

The moduli space of solutions to the perturbed Seiberg–Witten equations (modulo gauge equivalence) associated to $h\in\Pi$ will be denoted $\mathcal{M}(\Gamma;h)$.

In principle, a Spin^c structure involves a choice of Riemannian metric on Z, but as explained in [27, section 2.2] a Spin^c structure using one metric gives rise canonically to a Spin^c structure for any metric.

Choose a homology orientation for Z, which is the data needed to orient the determinant line bundle associated to the Seiberg–Witten equations for every choice of Γ. Suppose that Γ is a Spin^c structure on Z, such that the moduli space has formal dimension equal to -1. Thus, for a generic $h_0 \in \Pi$, the moduli space $\mathcal{M}(\Gamma; h_0)$ will be empty. For a path $h_t \subset \Pi$, define the 1–parameter moduli space to be

$$\tilde{\mathcal{M}}(\Gamma; h) = \bigcup_{t \in [0,1]} \mathcal{M}(\Gamma; h_t).$$

If h is generic, then this space is 0–dimensional, and we can count its points with signs. It is shown in [27] that $\mathrm{SW}(\Gamma; h)$ only depends on the endpoints h_0 and h_1, and not on the choice of path.

Definition 2.1 For generic $h_0, h_1 \in \Pi$, and for a generic path $h\colon [0,1] \to \Pi$, we will denote by $\mathrm{SW}(\Gamma; h) = \mathrm{SW}(\Gamma; h_0, h_1)$ the algebraic count $\#\tilde{\mathcal{M}}(\Gamma; h)$.

The applications of Seiberg–Witten theory to the topology of PSC in this paper depend on the following simple observation.

Lemma 2.2 *Suppose $b_2^+(Y) \geq 2$, and that g_0, g_1 are generic Riemannian metrics in the same component of* $\mathrm{PSC}(Y)$*. Let $h_i = (g_i, \mu_i) \in \Pi$ be generic. Then $SW(\Gamma; h_0, h_1) = 0$ if the μ_i are sufficiently small.*

Suppose that $f\colon Z \to Z$ is a diffeomorphism satisfying

$$f^*(\Gamma) \cong \Gamma \text{ and } \alpha(f) = 1. \tag{2}$$

Here $\alpha(f) = \pm 1$ where the sign is that of the determinant of the map which f induces on $H^2_+(Z)$. Informally, $\alpha(f) = 1$ means that f preserves the orientation of the moduli space $\mathcal{M}(\Gamma)$. It is shown in [27] that the quantity $\mathrm{SW}(\Gamma; h_0, f^*(h_0))$ does not depend on the choice of h_0. In particular, if Y has a metric g_0 of positive scalar curvature, and a diffeomorphism satisfying (2) has $\mathrm{SW}(\Gamma; g_0, f^*(g_0)) \neq 0$, then g_0 and $f^*(g_0)$ are in different components of PSC. It also follows that f is not isotopic to the identity, although that is not our main concern in this paper.

In [27], certain diffeomorphisms were constructed that have nontrivial Donaldson invariants, but we were unable to evaluate their Seiberg–Witten invariants. The reason for this is that these diffeomorphisms are constructed as compositions of simpler diffeomorphisms, which do not satisfy (2). We will now describe how to modify Definition 2.1 so that the resulting invariants are defined and computable for a broader class of diffeomorphisms, including those discussed in [27].

Let $\mathrm{SPIN}^c(Y)$ denote the set of Spin^c structures on a manifold Y. For an orientation-preserving diffeomorphism $f\colon Y \to Y$, and a Spin^c structure Γ, let $\mathcal{O}(f,\Gamma)$ be the orbit of Γ under the natural action of f on SPIN^c via pullback.

Definition 2.3 Suppose that Γ is a Spin^c structure on Y such that the Seiberg–Witten moduli space $\mathcal{M}(\Gamma)$ has formal dimension -1. For an arbitrary generic point $h_0 \in \Pi$, define

$$\mathrm{SW}_{tot}(f,\Gamma) = \sum_{\Gamma' \in \ \mathcal{O}(f,\Gamma)} \mathrm{SW}(\Gamma'; h_0, f^* h_0). \tag{3}$$

If the orbit $\mathcal{O}(f,\Gamma)$ is finite (for instance if f preserves the Spin^c structure Γ) then the sum in Definition 2.3 makes evident sense. More generally, we have the following consequence of the basic analytical properties of the Seiberg–Witten equations.

Proposition 2.4 *For a generic path $h\colon [0,1] \to \Pi$, there are a finite number of* Spin^c *structures on Y for which $\mathcal{M}(Y;h)$ is non-empty.*

Proof Since $b^2_+ > 2$, no reducible solutions will arise in a generic path of metrics. In a neighborhood of a given path (g,μ) in Π, there is a uniform lower bound on the scalar curvature and the norm of μ. By a basic argument in Seiberg–Witten theory, this implies that there are a finite number of Spin^c structures for which the parameterized moduli space has dimension ≥ 0. But for generic paths, a moduli space with negative dimension will be empty. □

The main point we will need to verify concerning Definition 2.3 is that the quantity defined in (3) is independent of the choice of $h_0 \in \Pi$. In discussing the invariance properties of $\mathrm{SW}_{tot}(f,\Gamma)$ it is useful to rewrite it as a sum over a long path in Π. For $n \in \mathbf{Z}$, we write f_n for the n–fold composition of f, with the understanding that $f_0 = id$ and $f_{-n} = f_n^{-1}$. Correspondingly, given $h_0 \in \Pi$, we write $h_n = f_n^* h_0$. If $h\colon [0,1] \to \Pi$ is a path from h_0 to h_1, then there are obvious paths $f_{n-1}^* h$ between h_{n-1} and h_n, which fit together to give a continuous map $\mathbf{R} \to \Pi$. We distinguish two cases, according to the way in which f acts on the Spin^c structures.

(1) If $\mathcal{O}(f,\Gamma)$ is finite, then we can write

$$\mathrm{SW}_{tot}(f,\Gamma) = \sum_{n=1}^{N} \mathrm{SW}(f_{-n}^*\Gamma; h)$$

where N is the smallest positive integer with $f_N^*\Gamma \cong \Gamma$.

(2) If $\mathcal{O}(f,\Gamma)$ is infinite, then we can write

$$\mathrm{SW}_{tot}(f,\Gamma) = \sum_{n=-\infty}^{\infty} \mathrm{SW}(f_{-n}^{*}\Gamma; h)$$

keeping in mind that the sum has only finitely many non-zero terms, for a generic path h.

We will usually write $\sum_n \mathrm{SW}(f_{-n}^{*}\Gamma; h)$ so as to be able to to discuss the two cases simultaneously; the reason for using f_{-n}^{*} instead of f_n^* should become clear momentarily.

Lemma 2.5 *If $\alpha(f) = 1$, then $SW_{tot}(f,\Gamma)$ may be rewritten*

$$SW_{tot}(f,\Gamma) = \sum_n SW(\Gamma; f_n^* h) = \sum_n SW(\Gamma; f_n^* h_0, f_{n+1}^* h_0)$$

Proof The proof rests on the isomorphism of moduli spaces

$$f^*\colon\ \mathcal{M}(\Gamma; h) \to \mathcal{M}(f^*\Gamma; f^*h)$$

given by pulling back the spin bundle, spinors, and connection. This isomorphism preserves orientation precisely when $\alpha(f) = 1$. Note that $\alpha(f_n) = \alpha(f)^n$. Thus, if h is either a generic element of Π or a generic path in Π, we have a diffeomorphism

$$f_n^*\colon\ \mathcal{M}(f_{-n}^*\Gamma; h) \to \mathcal{M}(\Gamma; f_n^* h)$$

and the lemma follows by summing up over $\mathcal{O}(f,\Gamma)$. □

Theorem 2.6 *Suppose that $b_+^2(Y) > 2$ and that f is an orientation preserving diffeomorphism with $\alpha(f) = 1$. Let $h_0, k_0 \in \Pi$ be generic, and let $h, k\colon [0,1] \to \Pi$ be generic paths with $h_1 = f^* h_0$ and $k_1 = f^* k_0$. Then the sums defining $SW_{tot}(f,\Gamma)$ using h_0 and k_0 are equal:*

$$\sum_n SW(\Gamma; f_n^* h) = \sum_n SW(\Gamma; f_n^* k)$$

Proof of Theorem 2.6 As remarked before, each term in

$$\sum_n \mathrm{SW}(\Gamma; f_n^* h)$$

depends only on the endpoints h_n, h_{n+1} of the path. Start by choosing a generic path $K_{0,t} \subset \Pi$ from $h_0 = K_{0,0}$ to $k_0 = K_{0,1}$, and note that $K_{1,t} = f^* K_{0,t}$ is necessarily generic for $f^*\Gamma$. Now take generic paths h and k, and define

$K_{s,0} = h_s$ and $K_{s,1} = k_s$. Thus we have a well-defined loop in the contractible space Π, which may be filled in with a generic 2–parameter family $K_{s,t}$.

Consider the 2–parameter moduli space

$$\tilde{\tilde{\mathcal{M}}}(\Gamma; K) = \bigcup_{s,t} \mathcal{M}(\Gamma; K_{s,t})$$

which is a smooth compact 1–manifold with boundary. (For compactness, we need to ensure that there are no reducibles in a 2–parameter family, which is why we require that $b^2_+ > 2$.) Any boundary point lies in the interior of one of the sides of the rectangle indicated in Figure 1. Now form the union

$$\bigcup_n \mathcal{M}(\Gamma; f_n^* K)$$

where the union is taken over the same set of n as is the sum defining $\mathrm{SW}(f;\Gamma)$. Note that by construction the moduli space corresponding to the left side of the n^{th} square (ie, the 1–parameter moduli space corresponding to $f_n^* K_{0,t}$) matches up with the right side of the $(n-1)^{st}$ square (corresponding to $f_{n-1}^* K_{1,t}$). The argument diverges slightly according to whether $\mathcal{O}(f,\Gamma)$ is finite or infinite. In the former case, assume that the orbit has exactly N elements. Hence we have a diffeomorphism from the right hand boundary of the N^{th} square

$$f_{-N}^* \colon \mathcal{M}(\Gamma; \{f_N^* K_{0,t}\}) \to \mathcal{M}(f_{-N}^* \Gamma; \{K_{0,t}\}) \cong \mathcal{M}(\Gamma; \{K_{0,t}\}) \tag{4}$$

to the left hand boundary of the first square. Note that the last isomorphism, because it is defined up to gauge equivalence, is independent of the choice of isomorphism between Γ and $f_N^* \Gamma$. Moreover, by the assumption that $\alpha(f) = 1$, this isomorphism is orientation preserving. All of the other contributions from the side boundary components cancel, according to the observation in the last paragraph. So, counting with signs, the number of points on the top boundary of

$$\bigcup_n \mathcal{M}(\Gamma; f_n^* K)$$

is the same as the number of points on the bottom part of the boundary. It follows that the conclusion of theorem 2.6 holds in the case that $\mathcal{O}(f,\Gamma)$ is finite. When $\mathcal{O}(f,\Gamma)$ is infinite, we use the principle (Proposition 2.4) that for $|n| >$ some N_0, the moduli spaces $\mathcal{M}(\Gamma; f_n^* K)$ are all empty. Thus the union of all of the 2–parameter moduli spaces provides a compact cobordism between the union of 1–parameter moduli spaces corresponding to the initial point $h_0 \in \Pi$ and that corresponding to k_0. $\square$

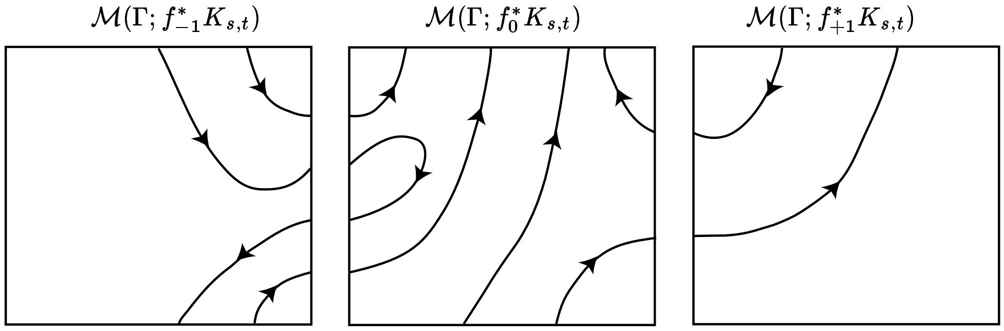

Figure 1

The proof is illustrated above in Figure 1, in the case that $\mathcal{O}(f,\Gamma)$ is infinite, and that $\mathcal{M}(\Gamma; f_n^* K) \neq \emptyset$ for $n = -1, 0, 1$. If $\mathcal{O}(f,\Gamma)$ is finite, say with N elements, there might be components of $\mathcal{M}(\Gamma; f_N^* K)$ going off the right-hand edge of the last box, but these would be matched up with the left-hand edge of the first box. In either case, the algebraic number of points on the top and bottom would be the same.

3 Basic properties

Theorem 3.1 *Suppose $b_2^+(Y) \geq 2$. If g is a generic Riemannian metric of positive scalar curvature, and there is a path in* $\mathrm{PSC}(Y)$ *from g_0 to $f^* g_0$, then $SW_{tot}(f,\Gamma) = 0$.*

Proof Since $b_2^+(Y) \geq 2$, there are no reducible solutions to the Seiberg–Witten equations for generic metrics or paths of metrics. Because $\mathrm{PSC}(Y)$ is open in the space of metrics, any path may be perturbed to be generic while staying in $\mathrm{PSC}(Y)$. Since $b_2^+(Y) \geq 2$, there are no reducible solutions to the Seiberg–Witten equations for generic metrics or paths of metrics. Now if $g \subset \mathrm{PSC}$ is a path from g_0 to $f^* g_0$, then all of the paths $f_k^*(g)$ lie in PSC. By the standard vanishing theorem, all of the moduli spaces

$$\tilde{\mathcal{M}}(\Gamma, f_k^*(g))$$

are empty, and so the invariant $\mathrm{SW}_{tot}(f,\Gamma)$ must vanish. □

The isotopy invariance of $\mathrm{SW}_{tot}(f,\Gamma)$ is a little more complicated to prove. It depends on the device of passing back and forth between the definition of SW_{tot} as a sum over Spin^c structures and its definition as a sum over paths in Π.

Theorem 3.2 *Suppose that f and g are isotopic diffeomorphisms of Y for which $SW_{tot}(f,\Gamma)$ are defined. Then $SW_{tot}(f,\Gamma) = SW_{tot}(g,\Gamma)$.*

Proof In the proof we use the notation fg for the composition $f \circ g$ of functions, and $h \cdot k$ for the composition of paths (the path h followed by the path k.) We will use an isotopy between f and g to construct a diffeomorphism between the various 1–parameter moduli spaces, taking advantage of the fact that any with the correct endpoints may be used to calculate $\mathrm{SW}(\Gamma; g_n^* h_0, g_{n+1}^* h_0)$. First choose an isotopy $F_t\colon Y \to Y$ such that $F_0 = id$ and $F_1 = f_{-1}g$. Let $h_0 \in \Pi$ be generic, and choose a generic path h from h_0 to $f^* h_0$. By definition,

$$\mathrm{SW}_{tot}(f,\Gamma) = \sum_n \mathrm{SW}(\Gamma; h_n, h_{n+1})$$

where $h_n = f_n^* h_0$. We use the path $f_n^* h$ to calculate the n^{th} term in the sum. Let

$$F_{k,t} = f_{-k} F_t g_k, \ t \in [0,1]$$

which gives an isotopy of $f_{-k}g_k$ to $f_{-(k+1)}g_{k+1}$. We calculate $\mathrm{SW}_{tot}(g,\Gamma)$ using h_0 for the initial point. The path $F_{k,t}^* f_k^* h_t = g_k^* F_t^* h_t$ has endpoints

$$g_k^* F_0^* h_0 = g_k^* h_0$$

for $t = 0$ and

$$g_k^* F_1^* h_1 = g_k^* g^* f_{-1}^* f^* h_0 = g_{k+1}^* h_0$$

for $t = 1$. So the 1–parameter moduli spaces

$$\mathcal{M}(\Gamma; \{F_{k,t}^* f_k^* h_t\})$$

may be used to calculate $\mathrm{SW}_{tot}(g,\Gamma)$.

But $F_{k,t}$ is homotopic to the identity, and so acts as the identity on the set of Spin^c structures on Y. Hence there is an orientation preserving diffeomorphism

$$F_{k,t}^*\colon \mathcal{M}(\Gamma; F_{k,t}^* f_k^* h_t) \to \mathcal{M}((F_{k,t}^{-1})^*\Gamma; f_k^* h_t) \cong \mathcal{M}(\Gamma; f_k^* h_t)$$

which is the moduli space used to calculate $\mathrm{SW}_{tot}(f;\Gamma)$. (Note that the genericity of the path $F_{k,t}^* f_k^* h_t$ follows from this diffeomorphism.) Therefore $\mathrm{SW}_{tot}(f;\Gamma) = \mathrm{SW}_{tot}(g,\Gamma)$. □

One thing which would be useful to understand is the behavior of the invariant SW_{tot} under compositions of diffeomorphisms. By reversing the orientation of all paths, we have (trivially)

Lemma 3.3 $SW_{tot}(f^{-1},\Gamma) = -SW_{tot}(f,\Gamma)$.

The 1–parameter Donaldson invariants [27, 28] satisfy the rule

$$D(fg) = D(f) + D(g) \tag{5}$$

as does the Seiberg–Witten invariant in the special case that f and g preserve the Spin^c structure. Each of these statements is proved by concatenating the relevant paths of metrics or perturbations, and using the independence of the invariants from the initial point. So, for instance, for the Seiberg–Witten invariant, if h_0 is the initial point for calculating $\mathrm{SW}(f, \Gamma)$, then we use $f^* h_0$ as the initial point for calculating $\mathrm{SW}(g, \Gamma)$. However, when $f^*\Gamma \not\cong \Gamma$, there seems to be no choice of paths used for computing $\mathrm{SW}_{tot}(f, \Gamma)$, $\mathrm{SW}_{tot}(g, \Gamma)$, and $\mathrm{SW}(fg, \Gamma)$ for which the corresponding moduli spaces can reasonably be compared.

That the composition rule (if there is one) must be more complicated than (5) may be seen from the following special case, for compositions of a diffeomorphism with itself.

Theorem 3.4 *Let $f\colon Y \to Y$ be a diffeomorphism such that $SW_{tot}(f, \Gamma)$ is defined. Then $SW_{tot}(f_d, \Gamma)$ is defined and satisfies:*

(1) *If $\mathcal{O}(f, \Gamma)$ is infinite, then $SW_{tot}(f_d, \Gamma) = SW_{tot}(f, \Gamma)$ for all d.*

(2) *If $|\mathcal{O}(f, \Gamma)| = N < \infty$, then:*

$$SW_{tot}(f_d, \Gamma) = \frac{\mathrm{lcm}(d, N)}{N} SW_{tot}(f, \Gamma)$$

Proof Choose a generic $h_0 \in \Pi$ and path h with which to define $\mathrm{SW}_{tot}(f, \Gamma)$. If $|\mathcal{O}(f, \Gamma)| = \infty$, consider the concatenation of paths $H = h \cdot f^* h \cdot f_2^* h \cdots f_{d-1}^* h$, with endpoints h_0 and $f_d^* h_0$. The union of the moduli spaces $\mathcal{M}(\Gamma; f_{dk}^* H)$ corresponding to these paths may be used to define both $\mathrm{SW}_{tot}(f, \Gamma)$ and $\mathrm{SW}_{tot}(f_d, \Gamma)$, and so these invariants are equal.

The same argument works if the orbit of Γ is finite, except that each moduli space $\mathcal{M}(\Gamma; f_k^* h)$ is counted $\mathrm{lcm}(d, N)/N$ times when one takes the union of the moduli spaces

$$\mathcal{M}(\Gamma; f_{dk}^* H)$$

which defines $\mathrm{SW}_{tot}(f_d, \Gamma)$. □

We will show later in Corollary 5.4 that the invariant SW_{tot} can detect non trivial elements of $\pi_0(\mathit{Diff}_h)$, where Diff_h is the subgroup of diffeomorphisms which are homotopic to the identity. With a better understanding of the invariants of compositions of diffeomorphisms, it might be possible to recover the result of [28] that $\pi_0(\mathit{Diff}_h)$ is infinitely generated.

Corollary 3.5 *Let f be a diffeomorphism which is an isometry for some metric on Y. Then $SW_{tot}(f, \Gamma) = 0$ for any Γ.*

Proof If f were an isometry of a generic metric g, then this would be clear because one could choose a constant path from g to f^*g. Since g might not be generic, we make use instead of Theorems 3.4 and 3.2: the isometry group of Y is compact, so f_d is in the connected component of the identity for some d. But this implies that f_d is isotopic to the identity, and so $\mathrm{SW}_{tot}(f_d, \Gamma) = 0$. By Theorem 3.4, this implies that $\mathrm{SW}_{tot}(f, \Gamma) = 0$ as well. □

4 Calculation of some examples

In this section we show how to calculate SW_{tot} for the diffeomorphisms described in [27, 28]. These calculations will be applied to questions about the space PSC in the next section. The construction starts with oriented, simply connected manifolds X satisfying the following conditions.

(1) $b^2_+(X)$ is odd and at least 3.

(2) $\mathrm{SW}(\Gamma_X) \neq 0$ for some Spin^c structure Γ_X for which $\dim(\mathcal{M}(\Gamma_X)) = 0$.

The first condition could presumably be weakened to allow $b^2_+(X) = 1$; this could be accomplished by developing a version of SW_{tot} invariants involving a chamber structure.

Given X as above, let N denote the connected sum

$$\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2 \# \overline{\mathbf{CP}}^2$$

and let $Z = X \# N$. If X has a homology orientation, then Z gets a homology orientation using the natural isomorphism $H^2_+(Z) \cong H^2_+(X) \oplus H^2_+(N)$, with the added convention that the generator of $H^2_+(N)$ is dual to the (complex) embedded 2–sphere in $\mathbf{CP}^2$. In what follows, the homology orientations on X and Z will be related in this way, without further mention.

Let S, E_1, E_2 be the obvious embedded 2–spheres of self-intersection ± 1 in N, and let $\Sigma_\pm = S \pm E_1 + E_2$ be spheres of square -1. For $\Sigma = \Sigma_+$ or Σ_-, let ρ^Σ be the diffeomorphism of N which induces the map $\rho^\Sigma_*(x) = x + 2(x \cdot \Sigma)\Sigma$ on $H_2(N)$, where '$\cdot$' is the intersection product. (In cohomology, $(\rho^\Sigma)^*$ is given by the same formula, where Σ is replaced by $\sigma = PD(\Sigma)$ and $\cdot$ is replaced by $\cup$.) In particular, letting s, e_1, e_2 be the Poincaré duals of S, E_1, E_2, we have

$(\rho^\Sigma)^*(s) = s + 2(s \cup \sigma)\sigma = 3s + 2(e_2 \pm e_1)$. Hence ρ^Σ is orientation preserving on $H^2_+(N)$. Choose the orientation in which s is in the positive cone.

The diffeomorphism ρ^Σ glues together with the identity map of X to give a diffeomorphism f^Σ on the connected sum $Z = X\#N$. Define $f = f^{\Sigma_+} \circ f^{\Sigma_-}$, and note that $\alpha(f) = 1$. Consider the Spin^c structure Γ on Z whose restriction to X is Γ_X, and whose restriction Γ_N to N has $c_1 = s + e_1 + e_2$. It is easy to check that $\dim(\mathcal{M}(\Gamma_Z)) = -1$.

Theorem 4.1 *The invariant $SW_{tot}(f, \Gamma_Z)$ is defined and equals $SW(\Gamma_X)$.*

As in [27], this will be proved by a combination of gluing and wall-crossing arguments. As a preliminary to these arguments, we set up the notation for wall-crossing. Consider the hyperbolic space:

$$\mathcal{H} = \{x^2 - y^2 - z^2 = 1,\ x > 0\} \subset H^2(N)$$

For a metric g^N, let $\omega(g^N) \in \mathcal{H}$ be the unique self-dual form satisfying

$$\int_N \omega \wedge \omega = 1 \ \text{ and} \int_N \omega \wedge s > 0. \tag{6}$$

The Seiberg–Witten equations on N admit a reducible solution if and only if $\omega(g^N)$ lies on the wall $\mathcal{W}$ in $\mathcal{H}$ defined by

$$\int_N \omega \wedge (s + e_1 + e_2) + 2\pi \int_N \omega \wedge \mu = 0. \tag{7}$$

The wall is transversally oriented by the convention that the local intersection number of a path ω_t meeting $\mathcal{W}$ at $t = 0$ is given by the sign of $f'(0)$, where

$$f(t) = \int_N \omega_t \wedge (s + e_1 + e_2 + 2\pi\mu) \tag{8}$$

at $t = 0$.

The Seiberg–Witten equations may be defined in the context of cylindrical-end manifolds [20, 30, 22], giving rise to moduli spaces of finite-energy solutions. We refer the reader to the book [23] for details; the full strength of the theory is not needed in our case because the ends of all the manifolds we consider have positive scalar curvature. Consider the manifold Y_0 gotten by removing an open ball from a 4–manifold Y, and the cylindrical-end manifold $\hat{Y} = Y_0 \cup S^3 \times [0, \infty)$. Fix a standard round metric on S^3, and define a cylindrical-end metric $\hat{g}$ on $\hat{Y}$ to be one whose restriction to the end is a product of this round metric with the usual metric on $[T, \infty)$ for some $T > 0$. For a pair $\hat{h} = (\hat{g}, \hat{\mu})$, where $\hat{\mu}$ is a self-dual form of compact support, one defines [23, section 4] the moduli space $\mathcal{M}(\Gamma^Y, \hat{h})$ consisting of finite-energy solutions to the Seiberg–Witten equations. Likewise, for a path $\hat{h}_t$ which is constant in the end, one has a finite-energy parameterized moduli space. In what follows, objects associated to a cylindrical-end manifold will be indicated by a ˆ on top.

Because $b^2_+(\hat{N}) = 1$, the cylindrical-end manifold $\hat{N}$ has wall-crossing behavior analogous to that on N itself. Note the cohomology isomorphism $H^2(\hat{N}) \cong H^2(N)$, and also note that classes in $H^2(\hat{N})$ are uniquely represented by L^2 harmonic forms [2]. A cylindrical-end metric $\hat{g}$ determines a unique L^2 self-dual harmonic form $\omega(\hat{g}^N) \in \mathcal{H}$ satisfying (6). (To ensure that the integrals in equations (6), (7) and (8) make sense, assume that the generators e_1, e_2, and s are represented by 2–forms with compact support.) The wall $\hat{\mathcal{W}}$ is defined precisely as above, and the transverse orientation of $\hat{\mathcal{W}}$ is given by the sign of the derivative of the function

$$\hat{f}(t) = \int_{\hat{N}} \omega_t \wedge (s + e_1 + e_2 + 2\pi\hat{\mu}) \tag{9}$$

as in the compact case. One readily verifies that there are reducible finite–energy solutions to the Seiberg–Witten equations on $\hat{N}$ precisely when $\omega(\hat{g}^N) \in \hat{\mathcal{W}}$.

Beyond the wall-crossing picture, the main ingredient in the calculation of SW_{tot} is a special case of the gluing principle for solutions to the 1–parameter Seiberg–Witten equations. Cutting and pasting of monopoles (solutions to the usual Seiberg–Witten equations) is discussed comprehensively in section 4.5 of [23]. The analytical details of gluing in the 1–parameter case are almost identical.

Write the connected sum $Z = X \# N$ as $X_0 \cup N_0$ where the union is along the common S^3 boundary. For any $r > 0$, we can view Z as diffeomorphic to $X_0 \cup S^3 \times [-r, r] \cup N_0$. We consider the perturbed Seiberg–Witten equations, where the metric on Z is required to be glued together from a product metric on the length $2r$ cylinder $S^3 \times [-r, r]$ and metrics $g^X \in \Pi(X_0)$ and $g^N \in \Pi(N_0)$. Similarly, we assume that the g–self-dual form used in perturbing the Seiberg–Witten equations vanishes on the cylinder. Such a pair (g, μ) will be written $h = h^X \#_r h^N$, and the set of such pairs will be denoted $\Pi_r(Z)$. Note that $h \in \Pi_r(Z)$ determines cylindrical end metrics on $\hat{N}$ and $\hat{X}$, with self-dual 2–forms of compact support. Also, $h \in \Pi_r(Z)$ determines metrics on the closed manifolds N and X, by cutting along $S^3 \times \{0\}$ and gluing in a standard round 4–ball.

An important remark is that if r is sufficiently large, then the functions f and $\hat{f}$ associated to metrics on N determined by $h \in \Pi_r(Z)$ are close in $\mathcal{C}^1$. This follows ([5]) from the exponential decay (cf [2]) of harmonic forms on $\hat{N}$. To take advantage of this, consider a path $h_t \subset \Pi_r(Z)$ such that $\omega(g_t)$ crosses the wall $\mathcal{W}$ transversally at $t=0$. Then the path $\omega(\hat{g}_t)$ crosses the wall $\hat{\mathcal{W}}$ transversally at $t=0$ and with the same sign.

The gluing procedure detailed in [23, section 4.5] relates monopoles on Z to those on X and N in two stages. The first is to relate solutions on Z (for metrics in the space $\Pi_r(Z)$ where r is sufficiently large) to finite energy solutions on $\hat{N}$ and $\hat{X}$. A second application of the gluing procedure relates these, in turn, to solutions on X and N, where the metrics on these manifolds are as described at the end of the preceding paragraph.

The following basic local calculation leads quickly to the proof of theorem 4.1. Note that it evaluates the invariant $\mathrm{SW}(\Gamma; h)$ for the special case of $h \subset \Pi_r(Z)$. Using the invariance properties of SW_{tot}, this will suffice for our applications.

Proposition 4.2 *There exists an r_0 such that for all $r > r_0$, the following statements hold. Let h be a path in $\Pi_r(Z)$ such that $h_t = h^X \#_r h_t^N$ for all t, where $h^X \in \Pi(X)$ is generic and $h_t^N \subset \Pi(N)$ is a generic path.*

(1) *If $\omega(h_t^N)$ is disjoint from $\mathcal{W}$, then $SW(\Gamma; h) = 0$.*

(2) *Suppose that $\omega(h_t^N)$ crosses $\mathcal{W}$ at $t=0$ transversally, and with positive orientation. Then $SW(\Gamma; h) = SW(\Gamma_X)$.*

Proof To prove these statements, we describe the 1–parameter Seiberg–Witten moduli space on Z in terms of the moduli spaces on $\hat{N}$ and $\hat{X}$. The idea is basically the same as the 'fundamental lemma' in [6]; details of a very similar argument (without the extra parameter) are given in the proof of Theorem 4.5.19 of [23]. As a first step, note that the path $h_t \in \Pi_r(Z)$ induces a path $\hat{h}_t^N$ on $\hat{N}$ and a constant path $\hat{h}^X$ on $\hat{X}$. The assumption that h^X is generic implies that $\hat{h}^X$ is generic. Moreover, the moduli space $\mathcal{M}(\Gamma^X, h^X)$ is diffeomorphic to the cylindrical end moduli space associated to $\hat{h}^X$.

To establish the first part of the proposition, note that if $\omega(h_t^N)$ is disjoint from $\mathcal{W}$, then the same will be true for $\omega(\hat{h}_t^N)$. Since the finite-energy Seiberg–Witten moduli space on $\hat{N}$ has formal dimension -2 and contains no reducibles, the parametrized moduli space $\mathcal{M}(\Gamma^{\hat{N}}, \hat{h}_t^N)$ will be empty. Because the gluing map is a diffeomorphism, the parameterized moduli space on Z will also be empty.

The second part is a little more complicated; we need to work out the Kuranishi model for the 1–parameter gluing problem for the Seiberg–Witten equation for a path of perturbations for which the equations on N admit a reducible solution

$(A_0^N, 0)$ at $t = 0$. As above, this path gives rise to a path of perturbations on $\hat{N}$. Using the relation between wall-crossing on N and $\hat{N}$, there will be exactly one small δ for which the Seiberg–Witten equations with perturbation $\hat{h}_\delta^N$ admit a reducible solution $(\hat{A}_\delta, 0)$. To simplify notation, we will assume $\delta = 0$.

Suppose that (A^X, φ^X) is a smooth irreducible solution to the Seiberg–Witten equations on X. As remarked above, it determines a unique finite-energy solution on $\hat{X}$. For small t, say in an interval $(-\epsilon, \epsilon)$, we have the constant configuration $(\hat{A}_0, 0)$ on $\hat{N}$. Note that this will be a solution to the Seiberg–Witten equations with perturbation $\hat{h}_t^N$ only when $t = 0$. Consider the 1–parameter family of configurations on Z gotten by gluing (A^X, φ^X) to $(\hat{A}_0, 0)$, and the problem of deforming this family to give elements in the 1–parameter moduli space on Z parameterized by $h_t = h^X \#_r h_t^N$. Adapting [23, Theorem 4.5.19] to this context, the portion of the parameterized moduli space close to the glued-up path near $t = 0$ is described (for r sufficiently large) in terms of the Kuranishi model. In other words, there is an S^1–equivariant map

$$\psi\colon\ (-\epsilon, \epsilon) \times \ker(D_{\hat{A}_0^N}) \times S^1 \to \operatorname{coker}(D_{\hat{A}_0^N}) \times H^2_+(\hat{N})$$

such that the parameterized moduli space on Z is diffeomorphic to $\psi^{-1}(0)/S^1$. The assumption that the path $\hat{h}_t$ is generic means that $\operatorname{coker}(D_{\hat{A}_0^N})$ vanishes. From the Atiyah–Patodi–Singer index theorem, the index of $D_{\hat{A}_0^N}$ is zero, so that

$$\ker(D_{\hat{A}_0^N}) \cong \operatorname{coker}(D_{\hat{A}_0^N}) \cong \{0\}.$$

As in [16], the derivative $\partial\psi/\partial t(0, 0, \theta)$ is given by $\hat{f}'(0)$ where $\hat{f}$ is defined in equation (9). Hence $\psi^{-1}(0)/S^1$ consists of exactly one point; the orientation of this point is the same as the orientation of the corresponding point in $\mathcal{M}(\Gamma^X, h^X)$. Each point of the moduli space $\mathcal{M}(\Gamma^X, h^X)$ contributes therefore ± 1 to $\mathrm{SW}(\Gamma, h)$. By part (1), the rest of the path (where $\omega(\hat{h}_t)$ misses $\hat{\mathcal{W}}$) doesn't contribute at all, and the result follows. □

Proposition 4.2 may be viewed as stating that $\mathrm{SW}(\Gamma; h)$, for a path which is constant on X, is given by the intersection number of the path $\omega(h_t^N)$ with $\mathcal{W}$. Note that this in turn only depends on the endpoints (as an ordered pair) of the path.

Proof of Theorem 4.1 Let $h_0 = h^X \# h_0^N \in \Pi$, where h^X and h_0^N are generic, and choose a generic path $h^N \in \Pi$ from h_0^N to $\rho^*(h_0^N)$. This glues up to give a path h which we can use to compute $\mathrm{SW}_{tot}(f, \Gamma)$. According to Proposition 4.2, we need to know the intersection number of the wall $\mathcal{W}$ with the path $(\rho^{\Sigma_+}\circ\rho^{\Sigma_-})_k^* h_N$ for each k. Now $(\rho^{\Sigma_+}\circ\rho^{\Sigma_-})^*$, viewed as a transformation of the hyperbolic space $\mathcal{H}$, is a parabolic element, with fixed point $\imath$ in the unit disc model of $\mathcal{H}$. In this model, $\mathcal{W}$ is the geodesic meeting the boundary of the unit disc orthogonally at $\imath$ and 1. Hence for any generic starting point h_0, there is a unique n for which $f_n^* h_0$ is on the left side of $\mathcal{W}$ and $f_{n+1}^* h_0$ is on the right side.

It follows that $\mathrm{SW}(\Gamma, f_k^* h) = 0$ for $k \neq n$, and that $\mathrm{SW}(\Gamma, f_n^* h) = \mathrm{SW}(\Gamma_X)$. Hence $\mathrm{SW}_{tot}(f,\Gamma) = \mathrm{SW}(\Gamma_X)$. □

5 Isotopy and concordance of positive scalar curvature metrics

The combination of Theorems 3.1 and 3.4 give rise to a method of detecting different components of PSC.

Theorem 5.1 *Suppose that Y is a manifold with a* Spin^c *structure Γ and diffeomorphism $f\colon Y \to Y$ such that $SW_{tot}(f,\Gamma) \neq 0$.*

(1) *If Y admits a metric of positive scalar curvature, then* $\mathrm{PSC}(Y)$ *has infinitely many components. In fact, if $g_0 \in$* PSC, *then the metrics $f_n^* g_0$ for $n \neq 0$ are all in different path components of* $\mathrm{PSC}(Y)$.

(2) *If f' is another diffeomorphism such that $SW_{tot}(f',\Gamma) \neq SW_{tot}(f,\Gamma)$, then for some k, $f_k^* g_0$ and $(f'_k)^* g_0$ are in different path components of* $\mathrm{PSC}(Y)$.

Proof By Theorem 3.4, the invariants $\mathrm{SW}_{tot}(f_d,\Gamma)$ are all non–zero, as long as $d \neq 0$. Suppose that $f_l^* g_0$ and $f_k^* g_0$ are in the same path component of PSC. Hence they are joined by a path $h\colon [0,1] \to \mathrm{PSC}$. As in the proof of Theorem 3.1, use the translates of h by f_{l-k} to show that $\mathrm{SW}_{tot}(f_{l-k},\Gamma) = 0$. But this implies that $l = k$.

For the second part, suppose to the contrary that $f_k^* g_0$ and $(f'_k)^* g_0$ are connected by a path of positive scalar curvature metrics for all k. Then by the usual cobordism argument we would have

$$\mathrm{SW}(\Gamma; f_k^* g_0, f_{k+1}^* g_0) \neq \mathrm{SW}(\Gamma; (f'_k)^* g_0, (f'_{k+1})^* g_0).$$

(The assumption that the metrics are connected in PSC provides a 2–parameter family in which the 1–parameter moduli spaces corresponding to the 'sides' are empty.) This would imply that the corresponding SW_{tot} invariants are equal, contradicting our assumption. □

Corollary 5.2 *There exist simply-connected 4–manifolds Y for which* $\mathrm{PSC}(Y)$ *is non-empty and has infinitely many components. More precisely, for any $n \geq 2$ and $k > 10n$, the connected sum*

$$Z = \#_{2n}\mathbf{CP}^2 \#_k \overline{\mathbf{CP}}^2$$

has infinitely many components in $\mathrm{PSC}(Z)$.

Proof Let E denote the elliptic surface with $b_+^2 = 2n-1$, blown up at $k - 10n$ points, and let $Z = E\#N$ as in Theorem 4.1. According to [21, 19, 12], Z decomposes as a connected sum as in the statement of the corollary. Therefore ([13, 29]) Z admits a metric g_0 of positive scalar curvature. By Theorem 4.1, Z supports a diffeomorphism with $\mathrm{SW}_{tot}(f, \Gamma) = \mathrm{SW}_E(\Gamma) \neq 0$ (for an appropriate choice of Γ) and so by Theorem 5.1, $\mathrm{PSC}(Z)$ has infinitely many components. □

As mentioned in the introduction, all of the known obstructions to isotopy are in fact obstructions to the weaker relation of *concordance*. By definition, a concordance between $g_0, g_1 \in \mathrm{PSC}(X)$ is a positive scalar curvature metric on $X \times I$, which in a collar neighborhood of $X \times \{i\}$ is a product $g_i + dt^2$. A path in $\mathrm{PSC}(X)$, suitably modified [13, 8] gives a concordance, but the converse is an important open question in all dimensions other than 2 where it is known to hold. Although the metrics constructed in Corollary 5.2 are not isotopic, it is not at all clear whether they are concordant. We present two ways of finding concordant but not isotopic metrics of positive scalar curvature, corresponding to the two parts of Theorem 5.1. The manifolds in question are the same as in Corollary 5.2, ie, connected sums of projective space (with both orientations). The simpler version, corresponding to the second part of that theorem, uses the principle that concordant (also known as *pseudoisotopic*) diffeomorphisms give rise to concordant metrics.

Theorem 5.3 *There are concordant, but not isotopic, metrics of positive scalar curvature on simply-connected 4–manifolds.*

Proof For any $n \geq 2$, we start with two simply-connected 4–manifolds X, X' satisfying

($1'$) $b_+^2(X) = 2n-1$.

($2'$) $X' \simeq X$, inducing a correspondence between $\mathrm{Spin}^c(X'$ and $\mathrm{Spin}^c(X)$.

($3'$) For corresponding Spin^c structures $\Gamma_{X'}$ and Γ_X we have that $\mathrm{SW}(\Gamma_{X'}) \neq \mathrm{SW}(\Gamma_X)$.

(4′) $X'\#\mathbf{CP}^2$ and $X\#\mathbf{CP}^2$ decompose as a connected sum of $\mathbf{CP}^2$'s and $\overline{\mathbf{CP}}^2$'s.

There are plenty of sources of such manifolds. One could, for example take X to be a (blown-up) elliptic surface, and build the other X' to be the corresponding decomposable manifold. An infinite family $X^{(i)}$ of such manifolds arises via the Fintushel–Stern [7] knot surgery construction on X. The decomposition (4′) of the stabilized manifold $X^{(i)}\#\mathbf{CP}^2$ follows readily from the analogous fact for X.

Given X and X', we get, as above, diffeomorphisms f, f' of a single manifold Z with the property that for an appropriate Spin^c structure on Z, the invariants $\mathrm{SW}_{tot}(f,\Gamma)$ and $\mathrm{SW}_{tot}(f',\Gamma)$ are distinct. This Spin^c structure has the property that $\mathcal{O}(f,\Gamma)$ is infinite for all i. It is explained in [27] and [28] how to arrange the identification of the stabilized manifolds so that f and f' are homotopic to one another.

Use diffeomorphisms f, f' to pull back a positive scalar curvature metric g_0 on Z to get metrics g, g'. A result of Kreck [14] implies that f is concordant (or pseudoisotopic) to f'. In other words, there is a diffeomorphism F of the 5–manifold $Z \times I$, which is the $f \times id$ in a collar of $Z \times \{0\}$ and $f' \times id$ in a collar of $Z \times \{1\}$. Pulling back the product metric $g_0 + dt^2$ via F gives the required concordance between g and g'. But by the second part of Theorem 5.1, these metrics are not isotopic. □

The nontriviality of the diffeomorphism $f' f_{-1}$ was the main result of [27]. Although it is not our main concern in this paper, we note that the SW_{tot} invariants also recover this result.

Corollary 5.4 *The diffeomorphisms $f'_k f_{-k}$ are homotopic to the identity, but are not isotopic to the identity.*

Proof By construction, f'_k is homotopic to f_k. If $f'_k f_{-k}$ were isotopic to the identity, then f'_k would be isotopic to f_k. But, using Theorem 3.4, this would imply that

$$\mathrm{SW}_{tot}(f,\Gamma) = \mathrm{SW}_{tot}(f_k,\Gamma) = \mathrm{SW}_{tot}(f'_k,\Gamma) = \mathrm{SW}_{tot}(f',\Gamma)$$

which is a contradiction. □

Using some additional topological ingredients, we can prove a stronger result highlighting the difference between concordance and isotopy of PSC metrics.

Theorem 5.5 *Let Z be a connected sum $\#_{2n}\mathbf{CP}^2\#_k\overline{\mathbf{CP}}^2$ as in theorem 5.2. Then any concordance class of* PSC *metrics on Z contains infinitely many isotopy classes.*

Proof The main topological ingredient is the computation of the group Δ_n of bordisms of diffeomorphisms. Recall that Δ_n consists of pairs (X, f) where f is a diffeomorphism of X, modulo cobordisms over which the diffeomorphism extends. (All manifolds are oriented, and cobordisms and diffeomorphisms are to respect the orientations.) The group was computed by Kreck [15] and Quinn [25]; we will recall Kreck's computation of Δ_4 shortly. To make use of this computation, we need the following observation.

Claim Let A be a non-spin, simply-connected 4–manifold, with a metric g_0 of positive scalar curvature. Suppose that f is a diffeomorphism which is bordant to id_A. Then f^*g_0 is concordant to g_0.

Proof of Claim By definition, there is a manifold B^5 with two boundary components $\partial_0 B \cong A \cong \partial_1 B$, and a diffeomorphism $F\colon B \to B$ with

$$F|\partial_0 B = id_A \quad \text{and} \quad F|\partial_1 B = f.$$

According to [15, Remark 11.5] we can (and will) assume that B is simply connected. Now there is a cobordism W^6, relative to the boundary of B, to the product $A \times I$. Again, by preliminary surgery on circles if necessary, we can assume that W is simply connected. By surgery on 2–spheres in W, we can also assume that

$$\pi_2(W, B) = \pi_2(W, A \times I) = 0.$$

(It is at this point that we used the assumption that A is not spin, in order to be able to choose a basis of 2–spheres with trivial normal bundle.) As a consequence, we can find a handle decomposition for W, relative to the boundary, with only 3–handles. In other words, B is obtained from $A \times I$ by surgery on 2–spheres, and *vice-versa.*

Now $A \times I$ has the product metric $g_0 + dt^2$, which can be pushed across the cobordism W to give a metric of positive scalar curvature on B which is a product near the boundaries. Use the diffeomorphism F to pull this metric back to give a new positive scalar curvature metric on B which is $g_0 + dt^2$ near $\partial_0 B$ and $f^*g_0 + dt^2$ near $\partial_1 B$. Push this new metric *back* across W to give a positive scalar curvature metric on $A \times I$; since the metric is not changed near the boundary components, this is a concordance. (Cf [9] for an argument of this sort.) $\square$ Claim

To make use of the claim, we need to show that the diffeomorphism f on $Z = Z\#N \cong \#_{2n}\mathbf{CP}^2\#_k\overline{\mathbf{CP}}^2$ constructed in Theorem 4.1 is bordant to the identity. By the basic theorem of [15], this is determined by the action of f_* on $H_2(Z)$, as follows. Let q denote the intersection form on $H_2(Z)$. Then f is bordant to id_Z if (and only if) there is a 1/2–dimensional subspace

$$V \subset H_2(Z) \oplus H_2(Z)$$

which is $f_* \oplus id$ invariant and on which the form $q \oplus -q$ vanishes. (V is called a metabolizer for the isometric structure $(H_2(Z) \oplus H_2(Z), f_* \oplus id, q \oplus -q)$.) This will clearly hold if there is a metabolizer for the action of $f_* \oplus id$ on the summand $H_2(N) \oplus H_2(N)$ of $H_2(Z) \oplus H_2(Z)$. Now a straightforward calculation shows that on $H_2(N)$, with respect to the basis $\{S, E_1, E_2\}$ the action of f_* is given by the matrix:

$$\begin{pmatrix} 9 & 4 & -8 \\ 4 & 1 & -4 \\ 8 & 4 & -7 \end{pmatrix}$$

The reader can check that with respect to the basis given above, the vectors $(1,0,1,0,0,0), (0,1,0,0,1,0)$, and $(1,0,1,1,0,1)$ span a metabolizer in $H_2(N) \oplus H_2(N)$. It follows that f is bordant to the identity.

To conclude the proof, let g_0 be a representative of any concordance class in $\mathrm{PSC}(Z)$. (Because of the decomposition of Z as a connected sum, this space is non-empty.) Then all of the metrics $f_k^* g_0$ are concordant, by the claim and the preceding calculation. On the other hand, these metrics are mutually non-isotopic, because of Theorem 5.1. □

6 The moduli space of positive scalar curvature metrics

As mentioned in the introduction, one can also study the space of metrics modulo the diffeomorphism group. From this point of view, the different path components of PSC detected in the previous sections are really the same because they are related by a diffeomorphism.

Definition 6.1 Let X be a manifold for which $\mathrm{PSC}(X)$ is nonempty. Then we define the moduli space $\mathcal{M}^+(X)$ to be $\mathrm{PSC}(X)/\mathit{Diff}(X)$, where the action of $\mathit{Diff}(X)$ is by pullback.

The main goal of this section is to show that $\mathcal{M}^+(X)$ can have some nontrivial topology.

Theorem 6.2 *For any N, there is a smooth 4 manifold X such that $\mathcal{M}^+(X)$ has at least N components.*

The manifolds we construct to prove Theorem 6.2 are non-orientable. It would be of some interest to find orientable, or even simply connected 4–manifolds with $\mathcal{M}^+$ disconnected; these would have no known analogues among higher (even) dimensional manifolds. At the end of the paper, we will suggest some potential examples.

6.1 Construction of examples

The phenomenon underlying Theorem 6.2 is that the same C^∞ manifold can be constructed in a number of ways. Each construction gives rise to a metric of positive scalar curvature; eventually we will show that these metrics live in different components of $\mathcal{M}^+$. The basic ingredient is a non-orientable manifold described in [10]. Start with the 3–dimensional lens space $L(p,q)$, which carries a metric of positive curvature as a quotient of S^3. Assuming, without loss of generality, that q is odd, there is a double covering

$$L(p,q) \to L(2p,q)$$

with covering involution $\tau\colon L(p,q) \to L(p,q)$. Let

$$M(p,q) = \left(S^1 \times L(p,q)\right)/((z,x) \sim (\bar{z}, \tau(x))).$$

This is a non-orientable manifold, because complex conjugation on the S^1 factor reverses orientation. The covering translation is an isometry of the product metric on $S^1 \times L(p,q)$, and so descends to a metric of positive scalar curvature on $M(p,q)$. A second useful description of $M(p,q)$ results from the projection of $S^1 \times L(p,q)$ onto the second factor. The descends to an S^1 fibration of $M(p,q)$ over $L(2p,q)$. From the fibration over S^1, the fundamental group of $M(p,q)$ is given as an extension

$$\{1\} \to \mathbf{Z}/p \times \mathbf{Z} \to \pi_1(M(p,q)) \to \mathbf{Z}/2 \to \{1\}$$

whereas the second description yields the exact sequence:

$$\{1\} \to \mathbf{Z} \to \pi_1(M(p,q)) \to \mathbf{Z}/(2p) \to \{1\} \tag{10}$$

Proposition 6.3 *If $M(p,q) \cong M(p',q')$, then $L(p',q') \cong L(p,q)$.*

This is straightforward; if the manifolds are diffeomorphic, then their orientable double covers are diffeomorphic. But it is a standard fact that $S^1 \times L(p,q)$ determines the lens space up to diffeomorphism.

In view of this proposition, it is somewhat surprising that most of the information about L (except for its fundamental group) is lost when one does a single surgery on M.

Let $\gamma \subset M$ be a fiber over a point of $L(2p,q)$; by construction it is an orientation preserving curve. Note that we can arrange that γ meets every lens space fibre in the fibration $M(p,q) \to S^1$ transversally in one point. Choose a trivialization of the normal bundle of γ, and do surgery on it, to obtain a manifold $X(p,q)$. The analogous oriented construction (ie, surgery on $S^1 \times \Sigma^3$) is called the spin of Σ, so we will call $X(p,q)$ the *flip-spun* lens space. From the exact sequence (10), we see that $\pi_1(X) \cong \mathbf{Z}/(2p)$. Moreover, the evident cobordism $W(p,q)$ between $M(p,q)$ and $X(p,q)$ has fundamental group $\mathbf{Z}/(2p)$. The inclusion of $X(p,q)$ into this cobordism induces an isomorphism on π_1, while the inclusion of $M(p,q)$ induces the projection in the sequence (10).

By studying the detailed properties of torus actions on these manifolds, P Pao [24] showed that the spin of the lens space $L(p,q)$ is (non-equivariantly) diffeomorphic to the spin of $L(p,q')$. We show that the same holds true for the flip-spun lens spaces. The proof is an adaptation of an argument for the orientable case; the 4–dimensional version was explained to me by S Akbulut, and the 5–dimensional version was known to I Hambleton and M Kreck.

Theorem 6.4 *The manifolds $X(p,q)$ and $X(p,q')$ are diffeomorphic for any choice of q and q', and are independent of the choice of framing of the surgery circle.*

As a consequence of the theorem, we will refer to all of these manifolds as $X(p)$. Note that the diffeomorphism described in the proof below respects of the generator of $\pi_1(X(p,q))$ corresponding to the core of the 1–handle. So there is also a preferred generator of $\pi_1(X(p))$.

Proof There are two framings for the surgery on γ, which differ by twisting by the non-trivial element in $\pi_1(\mathrm{SO}(3))$. But since $L(p,q)$ has a circle action with a circle's worth of fixed points, this rotation can be undone via an isotopy of $M(p,q)$ with support in a neighborhood of a copy of $L(p,q)$ of $M(p,q)$. Thus the two surgeries give the same manifold.

The key observation for showing that $X(p,q)$ does not depend on q is that it has a very simple structure as a double of a manifold with boundary. Here is a more precise description: First note that splitting the S^1 fibers into pieces with $\mathrm{Re}(z) \geq 0$ or $\mathrm{Re}(z) \leq 0$ exhibits $M(p,q)$ as the double of the non-orientable $\mathbf{I}$–bundle $L(2p,q)\tilde{\times}\mathbf{I}$. L_0 be $L(2p,q)$ minus an open disc, and consider the restriction $L_0\tilde{\times}\mathbf{I}$ of this $\mathbf{I}$–bundle. Then $X(p,q)$ is the double of $L_0\tilde{\times}\mathbf{I}$.

To see this, note that the curve γ along which the surgery is done is the union of two $\mathbf{I}$ fibres whose endpoints are identified when $L(2p,q)\tilde{\times}\mathbf{I}$ is doubled. $M(p,q)$ minus a neighborhood of γ is clearly gotten by doubling $L_0\tilde{\times}\mathbf{I}$ along the part of its boundary lying in the boundary of $L(2p,q)\tilde{\times}\mathbf{I}$. When doing the surgery, one glues in a copy of D^2 for each circle in the boundary of a neighborhood of γ. This has the effect of the doubling along the rest of the boundary of $L_0\tilde{\times}\mathbf{I}$.

The rest of the argument is handlebody theory, and can be expressed in either 4 or 5–dimensional terms. Both versions start with the observation that L_0 has a handlebody decomposition with a 0–handle, a 1–handle, and a single 2–handle. $X(p,q)$, being the double of $L_0\tilde{\times}\mathbf{I}$, is the boundary of the 5–manifold $L_0\tilde{\times}\mathbf{I}\times\mathbf{I}$, which has a handlebody decomposition with handles which are thickenings of those of L_0 and therefore have the same indices. But the isotopy class of the attaching map of the 2–handle, being an embedding of a circle in the 4–manifold $S^1\tilde{\times}S^3$, depends only on its homotopy class and its framing. By considering the 4–dimensional picture, the framing is also seen to be independent of q. But this homotopy class is $2p$ times a generator, and is therefore independent of q. By considering the 4–dimensional picture, the framing is also seen to be independent of q. It follows that $X(p,q) \cong X(p,q')$ for any q, q'.

The 4–dimensional version is a little more subtle; the fact that $X(p,q)$ is a double implies that it has a (4–dimensional) handle decomposition with a (non-orientable) 1–handle, two 2–handles, a 3–handle, and a 4–handle. As usual, we can ignore the handles of index > 2. The 0 and 1–handles, plus the first 2–handle, give a handlebody picture of $L_0\tilde{\times}\mathbf{I}$ as in [12, Figure 4.39]. If one draws this first 2–handle as in Figure 2 below, then it has framing $2pq$, because the framing is the same as the normal framing of the $(2p,q)$ torus knot which is the attaching region in the usual Heegaard splitting of $L(2p,q)$. The 1–handle is of course non-orientable; this means that in the diagram, a curve going into the front of one of the attaching spheres of the 1–handle comes out the back of the other one. The second 2–handle is attached, with framing 0, along a meridian to the first 2–handle. By sliding the first 2–handle over the second, one can change crossings; however this is not quite enough to change the diagram for $X(p,q)$ into the one for $X(p,q')$.

The reason for this is that in the diagram for $X(p,q)$ the 2–handle is drawn as a tangle of p arcs going from one attaching sphere of the 1–handle to the other, and changing crossings does not change the endpoints of this tangle. Associated to the tangle is a permutation of the $2p$ points where the 2–handle crosses the attaching sphere of the 1–handle. For differing q, q', these may not be the same permutation. However they are both $2p$–cycles and are therefore conjugate. Choose a permutation conjugating one to the other, and a $2p$–string braid σ inducing this permutation on the endpoints of the strings. Without changing $X(p,q)$, we can alter the 2–handle by inserting σ on one end and $\bar{\sigma}$ on the other end, where $\bar{\sigma}$ is σ^{-1} with all of the crossings reversed. (Note that σ^{-1} becomes $\bar{\sigma}$ when pushed over the 1—handle.) Now we can use the second 2—handle to change crossings until the diagrams for the two manifolds are identical. Since $2pq$ and $2pq'$ are both even, we can arrange by further handle slides to make the framings the same. □

The diagram below shows the proof for the diffeomorphism $X(4,1) \cong X(4,3)$. The top picture is a handle decomposition for $X(4,1)$. The second is equivalent to the first (push the braid at the left over the 1–handle and cancel with the braid on the right). The result (exercise!) is the diagram for $X(4,3)$, except that the framing is 8 rather than 24. But, as in the proof, this can be remedied by sliding 8 times over the second 2–handle. Because 2 is a small value for p, one does not have to change any crossings.

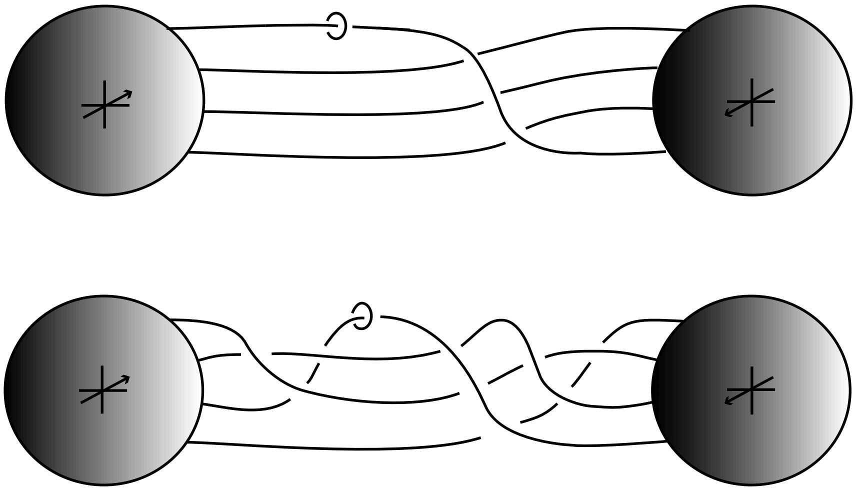

Figure 2

Corresponding to each of its descriptions as surgery on $M(p,q)$, the manifold $X(p)$ has a metric of positive scalar curvature. As described above, the manifold $M(p,q)$ has a positive scalar curvature metric $g_{p,q}$ which is locally a product of a metric on S^1 with the positive curvature metric on $L(p,q)$ which it inherits from S^3. According to the surgery construction of Gromov-Lawson/Schoen-Yau [13, 29], if the normal disc bundle to γ has sufficiently small radius, then $X(p)$ acquires a Riemannian metric $g_{p,q}$ of positive scalar curvature. (Technically speaking, this metric is only well-defined up to certain choices; none of these will affect what follows.) Note that $g_{p,q}$ depends, at least in principle, on q and not just on p. Moreover [8], the cobordism $W(p,q)$ between $M(p,q)$ and $X(p)$ has a metric of positive scalar curvature which is a product along both boundary components.

6.2 η–invariants for twisted $\mathrm{Pin}^{\pm}$ structures

We briefly review the invariant we will use to detect the different components of $\mathcal{M}^+$. Let Y be a Riemannian manifold with a $\mathrm{Pin}^{\pm}$ structure. A unitary representation α of $\pi_1(Y)$ gives rise to a twisted Dirac D operator on Y. Extending the work of Atiyah–Patodi–Singer [2] to this context, Gilkey [10] proved that the η function associated to D had an analytic continuation which was regular at the origin, and defined the invariant

$$\eta(Y,g,\alpha) = \frac{1}{2}(\eta(0,D) + \dim\ker(D)).$$

(We have omitted any notation specifying the $\mathrm{Pin}^{\pm}$ structure.)

When Y is even-dimensional, $\eta(Y,g,\alpha)$ modulo $\mathbf{Z}$ is independent of the metric, and thus is an $\mathbf{R}/\mathbf{Z}$–valued invariant of the underlying smooth manifold (even up to an appropriate notion of bordism). On the other hand, as a function of the metric, it is well-defined in $\mathbf{R}$. From the Atiyah–Patodi–Singer [2] theorem, plus the Lichnerowicz formula, one deduces a stronger invariance property.

Lemma 6.5 ([4], Lemma 2.1) *Let W be a $\mathrm{Pin}^{\pm}$ manifold with boundary $Y \coprod Y'$, and suppose that there is a unitary representation of $\pi_1(W)$ restricting to representations α, α' on the boundary. If W has a metric of positive scalar curvature which is a product near its boundary, then*

$$\eta(Y,g,\alpha) = \eta(Y',g',\alpha').$$

In particular, metrics of positive scalar curvature which are in the same path component of $\mathrm{PSC}(Y)$ have the same η–invariant, for every unitary representation of $\pi_1(Y)$.

6.3 Proof of Theorem 6.2

We will actually prove the following more precise version of Theorem 6.2.

Theorem 6.6 *Suppose that p is odd. If the metrics $g_{p,q}$ and $g_{p,q'}$ on $X(p)$ lie in the same component of $\mathcal{M}^+(X(p))$, then $q' \equiv (q)^{\pm 1} \pmod{2p}$.*

By taking p sufficiently large, so that there are many residue classes in $\mathbf{Z}^*_{2p}$, we get arbitrarily many components in $\mathcal{M}^+(X(p))$ as stated in Theorem 6.2. Note that the converse of Theorem 6.6 is trivially true.

Proof of Theorem 6.6 According to Lemma 6.5, if $g_{p,q}$ and $g_{p,q'}$ are in the same component of $\mathcal{M}^+(X(p))$, the corresponding η–invariants must coincide. We will show that this implies the stated equality of q' and $q^{\pm 1}$. First we must compute the η–invariants.

Claim For at least one choice of framing, the cobordism W is a Pin^+ cobordism from $M(p,q)$ to $X(p)$.

Proof of claim The obstruction to the extending a Pin^+ structure over the cobordism is a relative w_2, lying in $H^2(W,M;\mathbf{Z}/2)$. This is the same as the obstruction to extending the restriction of the Pin^+ structure to a neighborhood N of the attaching curve γ over the 2–handle. There are two Pin^+ structures on N, which differ by twisting by the non-trivial element of $\pi_1 \mathrm{SO}(3)$; one of them extends over the 2–handle. If, with respect to one framing, the Pin^+ structure does not extend, then it will if the handle is attached by the other framing, which also differs by twisting by the non-trivial element of $\pi_1 \mathrm{SO}(3)$. □

Fix a Pin^+ structure on $M(p,q)$, and choose the framing of the 2–handle so that W gives a Pin^+ cobordism to $X(p)$. Let α_s, $s = 0,\dots,2p-1$ be the $U(1)$ representations of $\pi_1(X(p)$. By extending over the cobordism $W(p,q)$, these give rise to $U(1)$ representations of $\pi_1(M(p,q))$, for which we will use the same name. Each of these representations gives rise to a twisted Pin^+ structure on $W(p,q)$. According to Lemma 6.5:

$$\eta(X(p), g_{p,q}, \alpha_s) = \eta(M(p,q), g_{p,q}, \alpha_s) \tag{11}$$

A Pin^+ structure twisted by a $U(1)$ representation is in a natural way a $\mathrm{Pin}^{\mathrm{c}}$ structure, and so the η invariants in (11) may be calculated as in [10, Theorem 5.3]:

$$\eta(X(p), g_{p,q}, \alpha_s) = \rho_{\alpha_s}(L(2p,q)) - \rho_{\alpha_{s+p}}(L(2p,q)) \tag{12}$$

where the reduced η–invariant $\rho_{\alpha_s}(L(2p,q))$ (in the original notation of [3], but not of [10]) is given by:

$$\frac{1}{2p}\sum_{\substack{\lambda^{2p}=1\\ \lambda\neq 1}}\frac{(\lambda^s-1)(\lambda^q)}{(\lambda^q-1)(\lambda-1)}$$

Note that with the given metrics, these formulas are valid over $\mathbf{R}$, not just reduced modulo $\mathbf{Z}$. This follows, for example, from [11] where $\rho_{\alpha_s}(L(2p,q))$ is computed via a spectral decomposition for the relevant Dirac operators in terms of spherical harmonics. Thus:

$$\eta(X(p),g_{p,q},\alpha_s)=\frac{1}{2p}\sum_{\substack{\lambda^{2p}=1\\ \lambda\neq 1}}\frac{(\lambda^s-\lambda^{s+p})(\lambda^q)}{(\lambda^q-1)(\lambda-1)} \tag{13}$$

Dividing the $2p^{th}$ roots of unity into those for which $\lambda^p=\pm 1$, this may be rewritten as:

$$\eta(X(p),g_{p,q},\alpha_s)=\frac{1}{p}\sum_{\lambda^p=-1}\frac{(\lambda^s)(\lambda^q)}{(\lambda^q-1)(\lambda-1)}$$

When p is odd, we can substitute $-\lambda$ for λ to get

$$\eta(X(p),g_{p,q},\alpha_s)=\frac{1}{p}\sum_{\lambda^p=1}\frac{(-1)^{s+1}(\lambda^s)(\lambda^q)}{(\lambda^q+1)(\lambda+1)}$$

where we have also used the fact that q is odd.

Suppose now that $g_{p,q}$ and $g_{p,q'}$ are connected in $\mathcal{M}^+(X(p))$. Taking into account that a diffeomorphism might permute the Pin$^{\mathrm{c}}$ structures on $X(p)$, this means that for some $a\in\mathbf{Z}/(2p)^*$, we must have

$$\eta(X(p),g_{p,q},\alpha_s)=\eta(X(p),g_{p,q'},\alpha_{as})\quad \forall s=0,\dots,2p-1. \tag{14}$$

Under the assumption that p is odd, we will show that equation (14) implies that either $q\equiv q' \pmod{2p}$ with $a=1$, or $qq'\equiv 1 \pmod{2p}$ with $a=-q'$. Since p is odd, and we have assumed that q,q' are odd as well, it suffices to show that one of these congruences holds mod p.

Let $\omega=e^{2\pi i/p}$ be a primitive p^{th} root of unity, and note that for any $j=$

$1, \ldots, p-1$ we have

$$\begin{aligned}
\sum_{s=0}^{p-1}(-1)^{s+1}\eta(X(p), g_{p,q}, \alpha_s)\omega^{-js} &= \frac{1}{p}\sum_{s=0}^{p-1}\sum_{k=0}^{p-1}\frac{\omega^{(k-j)s}\omega^{kq}}{(\omega^{kq}+1)(\omega^k+1)} \\
&= \frac{1}{p}\sum_{k=0}^{p-1}\frac{\omega^{kq}}{(\omega^{kq}+1)(\omega^k+1)}\sum_{s=0}^{p-1}\omega^{(k-j)s} \\
&= \frac{\omega^{jq}}{(\omega^{jq}+1)(\omega^j+1)} = \frac{1}{(\omega^{-jq}+1)(\omega^j+1)} \\
&= \frac{(\omega^{-2jq}-1)(\omega^{2j}-1)}{(\omega^{-jq}-1)(\omega^j-1)}
\end{aligned}$$

Suppose that the metrics $g_{p,q}$ and $g_{p,q'}$ are connected in $\mathcal{M}^+(X)$, so that the η–invariants correspond as in equation (14). By repeating these same calculations for $\eta(X(p), g_{p,q'}, a\alpha_s)$, we get that

$$\frac{(\omega^{-2jq}-1)(\omega^{2j}-1)}{(\omega^{-jq}-1)(\omega^j-1)} = \frac{(\omega^{-2bjq'}-1)(\omega^{2bj}-1)}{(\omega^{-bjq'}-1)(\omega^{bj}-1)}$$

where $ab \equiv 1 \pmod{p}$.

We now use Franz's independence lemma [1] which states that there are no non-trivial multiplicative relations amongst the algebraic numbers $\omega^k - 1$, ie, the terms on either side of this equality must match up in pairs. Eliminating some trivial possibilities we find that either $b = -q$ (implying $qq' \equiv 1 \pmod{p}$) or $b = 1$ which implies $q \equiv q' \pmod{p}$. Theorem 6.6 follows. □

7 Concluding Remarks

The arguments in the previous section mostly extend to twisted products of higher-dimensional lens spaces; and so one might be able to use our construction to get explicit higher-dimensional manifolds with $\mathcal{M}^+$ disconnected. The only missing step is the diffeomorphism between the different flip-spun lens spaces. The handlebody arguments do not work in higher dimensions, so one would have to resort to surgery theory.

As mentioned earlier, the spin of $L(p,q)$ ($S(L)$, given by surgery on $S^1 \times L$) is diffeomorphic to the spin of $L(p,q')$. Hence $S(L)$ supports a variety of metrics $g_{p,q}$ of positive scalar curvature, and it is natural to speculate that these are in different components of $\mathcal{M}^+$. Unfortunately, there does not seem to be a non-trivial η–invariant for even-dimensional orientable manifolds which will detect these components. In principle, the Seiberg–Witten equations could give rise to obstructions to homotopy of these metrics in PSC or $\mathcal{M}^+$, because the dimension of the Seiberg–Witten moduli space (for any Spin^c structure on $S(L)$) is -1. The definition of 1–parameter invariants, following the scheme in the first part of this paper, becomes complicated by the presence of a reducible solution. We hope to resolve these issues in future work.

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