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Calculus of clovers and finite type invariants of 3–manifolds

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Abstract

A clover is a framed trivalent graph with some additional structure, embedded in a 3–manifold. We define surgery on clovers, generalizing surgery on Y–graphs used earlier by the second author to define a new theory of finite-type invariants of 3–manifolds. We give a systematic exposition of a topological calculus of clovers and use it to deduce some important results about the corresponding theory of finite type invariants. In particular, we give a description of the weight systems in terms of uni-trivalent graphs modulo the AS and IHX relations, reminiscent of the similar results for links. We then compare several definitions of finite type invariants of homology spheres (based on surgery on Y–graphs, blinks, algebraically split links, and boundary links) and prove in a self-contained way their equivalence.

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1 Introduction

1.1 Finite type invariants

Polynomials play a fundamental role in mathematics. Here is one of the possible definitions of a polynomial using a discrete version of the n-th derivative. Let V be a vector space over a field k. For $(x_1, \dots, x_n) \in V^n$ and $\sigma \in \{0,1\}^n$, denote $x_\sigma = \sum_{i:\sigma_i=1} x_i$ and $|\sigma| = \sum_i \sigma_i$. A function $f\colon V \to k$ is a polynomial of degree less than n, iff $\sum_\sigma (-1)^{|\sigma|} f(x_\sigma) = 0$ for any $(x_1, \dots, x_n) \in V^n$. This definition has a significant advantage over the standard definition of a polynomial, since it can be easily considered in a more general setup. In particular, V and k may be just abelian groups.

Finite type invariants play the role of polynomial functions in topology. In this setting, however, the situation is somewhat more tricky, and involves a choice of an appropriate analog of both the zero element 0 and the addition operation $+$ in V. In the last decade, this general idea was successfully used in knot theory under the name of Vassiliev invariants. This paper is devoted exclusively to a study of a theory of finite type invariants of 3–manifolds, initiated by the second author.

The present paper was started in September 1998 in an attempt to present unpublished results by the second author. In June 1999, when this paper was still in its initial stage, the second author passed away in a tragic accident. As a result, many details and proofs were lost. We tried our best to reconstruct them and to present our results in his style and spirit.

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1.2 Finite type invariants of 3–manifolds

Several authors developed different theories of finite type invariants of integer homology spheres. All these approaches are based on the cut-and-paste technique along different types of handlebodies. Historically the first, the theory of T Ohtsuki [13] is based on surgery on algebraically split links and is by now relatively well understood. The first author [2] proposed then a theory based on surgery on boundary links, which remained relatively undeveloped. Subsequently, the first author and J Levine [3] considered theories based on surgery on blinks, and on gluings along surfaces using elements of the Torelli group. All these theories turn out to be equivalent (up to a 2–torsion), see Section 1.8.

The second author initiated in [5] a new theory of finite type invariants for arbitrary 3–manifolds, based on surgery along Y–graphs (see [4, 5]). The latter theory has some important advantages. Firstly, it allows a unified treatment of links, graphs and 3–manifolds (possibly with boundary or a Spin–structure). It is by now relatively well understood for rational homology spheres. Secondly, other theories mentioned above have a technical drawback: the corresponding classes of surgery links are not preserved under handle slides. Finally, this theory comes equipped with a powerful topological calculus (introduced in [4]), that is well-suited for a study of 3–manifolds and for explicit computations. This calculus is similar to Kirby's calculus of framed links, but instead of cutting, twisting and regluing solid tori, in the clover calculus one performs similar operations with solid handlebodies of higher genus. The role of framed links is played here by *clovers*, which are framed trivalent graphs with some additional structure.

The main goals of the paper are:

- To give an exposition of the calculus of clovers (see Sections 2–3), and to describe the graded spaces of the theory in terms of uni-trivalent graphs (see Sections 1.8 and 4.5).
- To show, as an application, in an elementary and self-contained way, that all the above mentioned theories of finite type invariants of integer homology spheres, properly indexed, coincide over $\mathbb{Z}[1/2]$ (see Section 1.8).

Independently and being uninformed about the results of the second author, K Habiro developed a theory of claspers [6] and used it to study finite type invariants of links. He also announced in [6, Section 8.4] that his theory can be extended to the study of finite type invariants of 3–manifolds and gave a brief outline of this extension. It is based on *allowable graph claspers,* which differ from the type of claspers considered in the rest of Habiro's paper. In particular, the basic claspers, which were Habiro's main tool in the theory for links, are not allowed in this setting. In general it is plausible that invariants of links can be extended to invariants of 3–manifolds. In the case in hand, however, this extension is rather nontrivial and involves new ideas presented here.

The initial data, the way to describe the basic objects, and the graphical calculus in the clasper and clover theories are somewhat different. However, while different in some aspects, these two theories essentially coincide, with surgery on allowable graph claspers being equivalent to surgery on clovers. This independent discovery of almost the same theory by two non-interacting researchers looks rather promising, ruling out an element of arbitrariness. Some results announced by K Habiro [6, Section 8.4] seem to overlap with those discussed in Sections 2–4 below. Unfortunately, it seems that presently there is no available manuscript with the exact statements and proofs of Habiro's results for 3–manifolds. We apologize in advance for any possible overlaps. It should be also mentioned that T Cochran and P Melvin [1] proposed an extension of Ohtsuki's theory to arbitrary 3–manifolds. Their theory, unlike the one considered in this paper, preserves triple cup-products, and the relation between these theories is more complicated.

1.3 Y–graphs

Throughout this paper by a manifold we mean a smooth oriented compact connected 3–manifold.

We recall some definitions from [5]. The graph $\Gamma \subset \mathbb{R}^3$ shown in Figure 1a is called a *standard Y–graph.* The edges of Γ are framed with a vector field normal to the plane of the picture. A framed graph G in a 3–manifold M is called a *Y–graph*, if it is the image of Γ under a smooth embedding $\phi_G\colon N \to M$ of a neighborhood N of Γ. The embedding ϕ_G can be recovered from G up to isotopy. Let L be the six component link in N shown in Figure 1b. All components of L are 0–framed. Surgery on M along the framed link $\phi_G(L)$ is called a *surgery along G*, or a *Y–surgery.* A Y–surgery can be realized by cutting a solid genus 3 handlebody $\phi_G(N)$ and regluing it in another way (see [10]). The resulting manifold is defined up to a diffeomorphism, which is the identity outside of a small neighborhood of the Y–graph. Denote it by M_G.

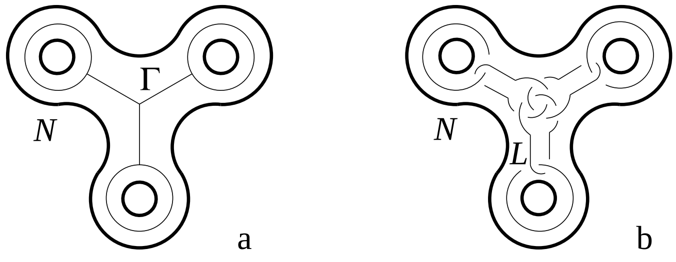


Figure 1: Y–graph and the corresponding surgery link

An equivalent surgery, under the name of *Borromean surgery* was considered by S Matveev. As shown in [10], two manifolds can be connected by a sequence of Borromean surgeries, if and only if they have the same homology and the linking pairing in $\operatorname{Tors} H_1$. In particular, M can be obtained from S^3 by Y–surgeries, iff M is an integer homology sphere ($\mathbb{Z}$HS in short).

A *Y–link* G in a manifold M is a collection of disjoint Y–graphs in M. By surgery on G we mean surgery on each Y–graph in G. Pushing out the Y–graphs of the following surgeries from the previously reglued handlebodies, we can present a sequence of Y–surgeries by surgery on a Y–link.

1.4 Colored Y–links and the filtration $\mathcal{F}_n^Y(M)$

An n*–coloring* of a Y–link G is its splitting into n disjoint subcollections G_1, G_2, ... , G_n of Y–graphs. The coloring is *simple*, if each subcollection G_i consists of a single Y–graph.

For an n–tuple $\sigma = \{\sigma_1, \dots, \sigma_n\} \in \{0,1\}^n$, let $|\sigma| = \sum_i \sigma_i$ be the number of ones in σ and put $G(\sigma) = \bigcup_{i:\sigma_i=1} G_i$. In an abelian group $\mathcal{F}$ generated by all 3–manifolds, put

$$[M,G] = \sum_\sigma (-1)^{|\sigma|} M_{G(\sigma)}$$

for a colored Y–link G in M.

Let $\mathcal{F}_n^Y(M)$ be the subgroup of $\mathcal{F}$, generated by $[M,G]$ for all Y–links G, colored in at least n colors. This defines a decreasing filtration $\mathcal{F}^Y(M) = \mathcal{F}_0^Y(M) \supset \mathcal{F}_1^Y(M) \supset \mathcal{F}_2^Y(M) \supset \dots$.

Given a Y–link G in M without a specified coloring, by $[M,G]$ we mean the above alternating sum for a simple coloring of G.

Lemma 1.1 *The subgroup $\mathcal{F}_n^Y(M)$ is generated by $[M,G]$ for all simply colored Y–links G with at least n components.*

Proof Let G be an n–colored Y–link with m components. We call $d = n - m$ the defect of the coloring. If $d = 0$, the coloring is simple. Suppose $d > 0$; then there are at least two components of the same color j. Therefore, the j–colored Y–sublink of G can be split into two disjoint non-empty parts G' and G''. It is easy to see that

$$[M,G] = [M, G \smallsetminus G'] + [M, G \smallsetminus G''] - [M, \widetilde{G}],$$

where $\widetilde{G}$ is obtained from G by recoloring all components of G'' in a new, $(n+1)$-th, color. The defect of the coloring for each of $G \smallsetminus G'$, $G \smallsetminus G''$ and $\widetilde{G}$ is less than d. The statement follows by the induction on d. □

Thus, without a loss of generality, one can assume that the colorings in the definition of $\mathcal{F}_n^Y(M)$ are simple.

1.5 The filtration $\mathcal{F}_n^Y$

Further on we will be mainly interested in M being a $\mathbb{Z}$HS. Denote $\mathcal{F}_n^Y(S^3)$ by $\mathcal{F}_n^Y$.

Lemma 1.2 *Let G be an n–colored Y–link in a $\mathbb{Z}$HS M. Then $[M,G] \in \mathcal{F}_n^Y$.*

Proof It suffices to prove, that there exist two Y–links G' and G'' in S^3, each colored in at least n colors, such that $[M,G] = [S^3,G''] - [S^3,G']$.

Since M is a $\mathbb{Z}$HS, it may be obtained from S^3 by Y–surgery on a Y–link G in S^3. Let G' be an n–colored Y–link in $S^3 \smallsetminus G$, such that its image under Y–surgery on G is isotopic to G. Let G'' be the $(n+1)$–colored Y–link, obtained from G' by an addition of G, with all components of G colored in a new color. Then $[S^3, G' \cup G] = [S^3_G, G] + [S^3, G']$ and hence $[S^3_G, G] = [S^3, G''] - [S^3, G']$. □

1.6 The filtrations $\mathcal{F}_n^{as}$ and $\mathcal{F}_n^b$

A link L in a $\mathbb{Z}$HS is *algebraically split* if all pairwise linking numbers of its components vanish. Such a link is *boundary* if all its components bound non-intersecting surfaces. A framing of L is *unimodular* if the self-linking of each component is ± 1. Surgery on (any sublink of) a unimodular algebraically split link gives again a $\mathbb{Z}$HS. Using these classes of links, T Ohtsuki [13] and the first author [2] introduced two different filtrations on a vector space generated by all $\mathbb{Z}$HS. Below we describe the corresponding filtrations on the free abelian group generated by all $\mathbb{Z}$HS.

For a framed link L in M, denote by M_L the result of surgery of M along L. In $\mathcal{F}$, let

$$[M,L] = \sum_{L' \subset L} (-1)^{|L'|} M_{L'},$$

where $|L'|$ is the number of components of a sublink L'. Let $\mathcal{F}_n^{as}$ (respectively $\mathcal{F}_n^b$) be the subgroup of $\mathcal{F}$, generated by $[M,L]$ for all algebraically split (respectively boundary) unimodular links L in $\mathbb{Z}$HS M with at least n components. As shown in [13] and [2], in the definitions of these filtrations it suffices to consider only $M = S^3$.

1.7 The filtration $\mathcal{F}_n^{bl}$

In [3], J Levine and the first author considered another filtration, based on a notion of a blink. A *blink* is a framed link B with a given splitting of the set of its components into pairs $(B_1^-, B_1^+), \dots, (B_n^-, B_n^+)$, such that:

- each pair (B_i^-, B_i^+) bounds an orientable surface Σ_i, so that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$;
- the surface Σ_i induces a preferred framing of B_i^- and B_i^+; this framing should differ by ± 1 from the given framing of $B_i^\pm$.

A *sub-blink* B' of B is obtained from B by a removal of some pairs (B_i^-, B_i^+). In $\mathcal{F}$, put

$$[M, B] = \sum_{B' \subset B} (-1)^{|B'|} M_{B'},$$

where $|B'|$ is the number of pairs in a sub-blink B'. Let $\mathcal{F}_n^{bl}(M)$ be the subgroup of $\mathcal{F}$, generated by $[M, L]$ for all blinks L in a manifold M with at least n pairs of components.

In the same paper [3] several other filtrations were introduced, using different subgroups of the mapping class group. Each of them was shown to be equivalent to one of the above filtrations.

1.8 The main results

We describe a topological calculus of surgery on Y–links, developed by the second author. Using this calculus, we describe the structure of the graded groups $\mathcal{F}_n^Y / \mathcal{F}_{n+1}^Y \otimes \mathbb{Z}[1/2]$, where $\mathbb{Z}[1/2] = \{n/2^m | n, m \in \mathbb{Z}\}$ is the group of binary rational numbers.

We also compare different filtrations:

- $\mathcal{F}_n^Y(M) = \mathcal{F}_n^{bl}(M)$;
- $\mathcal{F}_{2n}^Y \subset \mathcal{F}_{3n}^{as}$ and $\mathcal{F}_{2n}^Y \subset \mathcal{F}_n^b$;
- $\mathcal{F}_{2n-1}^Y \otimes \mathbb{Z}[1/2] = \mathcal{F}_{2n}^Y \otimes \mathbb{Z}[1/2]$;
- $\mathcal{F}_{2n}^Y \otimes \mathbb{Z}[1/2] = \mathcal{F}_{3n}^{as} \otimes \mathbb{Z}[1/2]$ and $\mathcal{F}_{2n}^Y \otimes \mathbb{Z}[1/2] = \mathcal{F}_n^b \otimes \mathbb{Z}[1/2]$.

Let F be a free abelian group on a set S of generators, equipped with a decreasing filtration $F = F_0 \supset F_1 \supset \dots$. Given an abelian group A, a function $S \to A$ is called a *finite type invariant* of degree at most d, if its extension to a homomorphism $F \to A$ vanishes on F_{d+1}. Each filtration $\mathcal{F}_n^{as}$, $\mathcal{F}_n^b$, $\mathcal{F}_n^{bl}$

and $\mathcal{F}_n^Y$ defines a notion of a finite type invariant on the set of all $\mathbb{Z}$HS. The above comparison of filtrations implies the following theorem announced in [5, Theorem 1]. If 2 is invertible in A, these definitions are equivalent, with the following relation of degrees:

$$2d^{as} = 6d^b = 3d^{bl} = 3d^Y.$$

In [3], weaker results $\mathcal{F}_n^b \otimes \mathbb{Q} \subset \mathcal{F}_{2n}^{bl} \otimes \mathbb{Q} = \mathcal{F}_{3n}^{as} \otimes \mathbb{Q}$ were obtained by using different methods that involved the study of the mapping class group and several of its subgroups (building on the results of D Johnson [7, 8]).

2 Surgery on clovers

In this section we introduce a generalization of Y–links, which we call clovers. They turn out to be quite useful from a technical point of view, and are closely related to uni-trivalent graphs appearing in the study of the graded quotients $\mathcal{F}_n^Y/\mathcal{F}_{n+1}^Y$. Similar objects were called *allowable graph claspers*[1] by K Habiro [6]. We start by recalling some basic facts about surgery (see [9]).

2.1 Surgery on links

We will call framed links L and L' in a manifold M *surgery equivalent*, and denote $L \sim L'$, if M_L is diffeomorphic to $M_{L'}$. Recall (see [9]), that links L and L' in S^3 are surgery equivalent, iff one can pass from L to L' by a sequence of *Kirby moves* K_1 and K_2:

K_1: Add to L a small ± 1 framed unknot, unlinked with the other components of L.

K_2: Add L_0 to L_1 by replacing L_1 with $L_1 \#_b \widetilde{L_0}$, where $\#_b$ is a band connected sum and $\widetilde{L_0}$ is a push-off of L_0 along the framing.

It is convenient to introduce three additional moves K_3–K_5, which can be expressed via $K_1^{\pm 1}$ and K_2 (see [9] for K_4, K_5, and [11] for K_3):

K_3: Let L_0 be a closed, 0–framed component of L bounding an embedded disk D, which intersects $L \smallsetminus L_0$ in exactly two points, belonging to different components L_1 and L_2 of L. Replace $L_0 \cup L_1 \cup L_2$ by $L_1 \#_b L_2$, where the band b intersects D along the middle line of b.

[1]See `http://www.dictionary.com` for some unexpected meanings of the word clasper.

K_4: Delete a ±1–framed unknot, at the expense of the full left- or right-hand twist on the strings linked with it.

K_5: Let L_0 be a closed, 0–framed component of L which bounds an embedded disk D. Suppose that $D \cap (L \smallsetminus L_0)$ consists of exactly one point lying on some component $L_1 \subset L$.
Delete L_0 and L_1 from L.

The moves K_2–K_5 are shown in Figure 2. Here, and throughout the paper, we

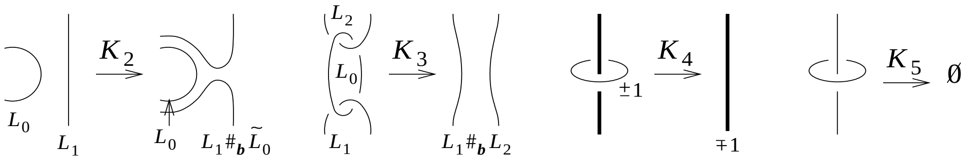


Figure 2: Kirby moves

think about a surgery presentation of M and use thick solid lines to depict an arbitrary union of surgery components comprising M (possibly together with some edges of embedded graphs). Also, we use a usual convention that all depicted links, or graphs, coincide outside a ball shown in the picture. In all figures the framings are assumed to be orthogonal to the plane of the picture, unless explicitly indicated otherwise. Some examples of the moves K_2, K_4 and K_5 are given in Figure 3. By a repeated application of K_3 we obtain the

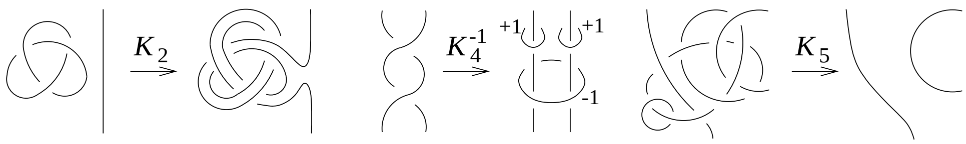
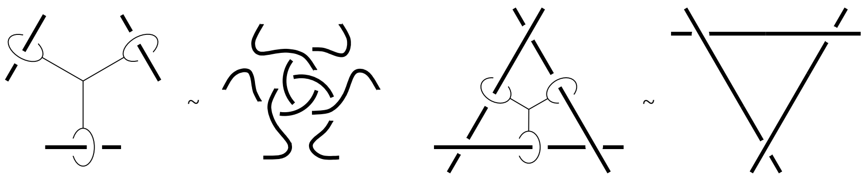


Figure 3: Examples of Kirby moves

following relation (which explains the term *Borromean surgery*, used in [10] for Y–surgery in a manifold M, and the term Δ*–move*, used in [12]):

Lemma 2.1



2.2 Clovers

Let G be a framed trivalent graph, smoothly embedded into a manifold M. We will call G a *clover*, if:

- Each point of G has a neighborhood, diffeomorphic to a neighborhood of some point of a standard Y–graph (including the framing), by an orientation-preserving diffeomorphism.
- Different edges meeting in a vertex have different tangent lines.

If both endpoints of an edge coincide, the two corresponding tangent lines may coincide. In this case the edge is called a *leaf*, and the incident vertex is called an *external vertex*. A vertex not incident to a leaf is *internal*. We call an edge *internal*, if both its endpoints are internal vertices. A *degree* of G is the number of internal vertices. For example, a Y–graph is a degree 1 clover. It has one internal vertex, three external vertices and three leaves, see Figure 1a.

Clovers of degree 0 play an important role in an analogous theory of finite type invariants of links. However, we will exclude them from the theory of finite type invariants of 3–manifolds. Therefore all throughout the paper we will always assume that each connected component of a clover is of degree at least one.

In figures we will always assume that the framing of G is orthogonal to the plane of the picture.

2.3 Surgery on clovers

Let G be a clover in M. We construct a framed link $L(G)$ in a small neighborhood of G. In Figure 4 we show the structure of $L(G)$ near an edge, a leaf, and an internal vertex of G. The framings of the fragments of G and $L(G)$ appearing in Figure 4 are assumed to be orthogonal to the plane of the picture. The construction of $L(G)$ is illustrated in Figure 5. The surgery of M along

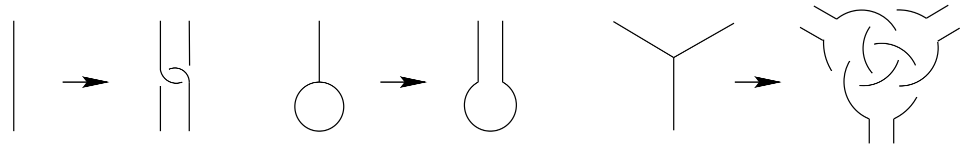


Figure 4: Construction of the surgery link from a clover

$L(G)$ is called a *surgery along* G, and the manifold $M_{L(G)}$ is denoted by M_G.

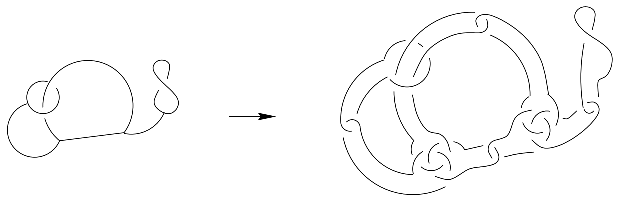


Figure 5: A clover and the corresponding link

We call two clovers G and G' in M *surgery equivalent*, and denote $G \sim G'$, if M_G is diffeomorphic to $M_{G'}$.

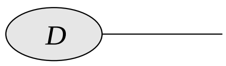
Denote by $G_1, \dots, G_n$ the connected components of G. For an n–tuple $\sigma = \{\sigma_1, \dots, \sigma_n\} \in \{0,1\}^n$ we put

$$[M,G] = \sum_{\sigma} (-1)^{|\sigma|} M_{G(\sigma)},$$

where $G(\sigma) = \bigcup_{i:\sigma_i=1} G_i$ and $|\sigma| = \sum_i \sigma_i$. If the degree of each G_i is 1, we recover the definition of $[M,G]$ for Y–links.

Suppose that a leaf or a cycle of edges of G in M bounds an embedded surface Σ in M. Then the intersection number of its push-off along the framing with Σ does not depend on the choice of Σ and (by a slight abuse of terminology) will be also called the framing of this leaf or cycle. A leaf is *trivial*, if it is 0–framed and bounds a disc D in M whose interior does not intersect G.

Lemma 2.2 *Let $G \subset M$ be a connected clover, which contains a trivial leaf l. Then surgery on G preserves a neighborhood of $G \cup D$, where D is the disc bounding l. In particular, $M_G = M$ and $[M,G] = 0$:*



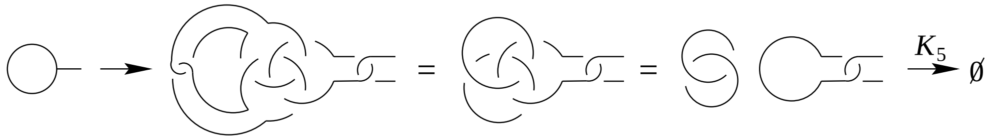
$\sim \quad \emptyset$

Proof Follows from the construction of $L(G)$ by an application of K_5. □

An edge of a clover is a *trivial loop*, if both its endpoints coincide, it is 0–framed, and bounds a disc D whose interior does not intersect G.

Lemma 2.3 *Let $G \subset M$ be a connected clover, which contains a trivial looped edge e. Then surgery on G preserves a neighborhood of $G \cup D$, where D is the disc bounding e. In particular, $M_G = M$ and $[M,G] = 0$.*

Proof Follows from the construction of $L(G)$ by an application of K_5:

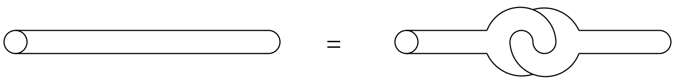


□

Theorem 2.4 *Let G be a clover in M, and let G' be obtained from G by cutting an internal edge of G and inserting there two small Hopf-linked leaves:*

Then $G' \sim G$ and $[M, G'] = \pm[M, G]$, where the sign is negative if the edge is splitting G, and positive otherwise.

Proof The equality $G \sim G'$ follows from the construction of $L(G)$ by an application of K_3. Topologically, this corresponds to cutting the corresponding solid handlebody introducing a pair of complimentary handles as shown below:



Let G_0 be the component of G containing this edge, and G_1, G_2 be the components of G' replacing G_0. Note that each subclover of G' which contains exactly one of the components G_1 and G_2, has a trivial leaf. Thus by the definition of $[\cdot, \cdot]$ and Lemma 2.2 $[M, G'] = [M, G]$ if G' has the same number of components as G, and $[M, G'] = -[M, G]$ otherwise. □

Corollary 2.5 *Let G be a degree n clover. Then $[M, G] \in \mathcal{F}_n^Y(M)$.*

3 Topological calculus of clovers

Below we describe some important moves Y_1–Y_4 on Y–links, which are shown to preserve the surgery equivalence classes. One can apply these moves also to clovers, in view of Theorem 2.4.

Moves Y_1 and Y_2 (similar to the Kirby moves K_1, K_2) are shown in Figure 6:

Y_1: Add to G a connected clover G', which has a 0–framed leaf l bounding an embedded disc whose interior does not intersect $G \cup G'$.

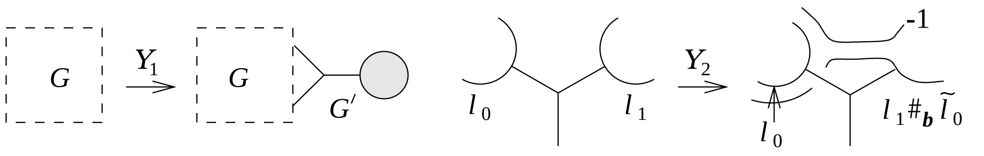


Figure 6: Blow-up and leaf slide

Y_2: Add a leaf l_0 to another leaf l_1 of the same Y–graph along a band b by replacing l_1 with $l_1 \#_b \widetilde{l_0}$, where $\widetilde{l_0}$ is a push-off of l_0 along the framing; then change the framing by -1.

Let l be a leaf of a Y–graph G_0 and e be the adjacent edge. Let K be a knot. The move Y_3 is sliding an edge e along K, see Figure 7:

Y_3: Denote by G_1 Y–graph obtained from G_0 by adding K to e along a band b. Construct a Y–graph G_2 as shown in Figure 7: one of its leaves is a push-off of K (and the adjacent edge goes along b), another is a push-off of l, and the third leaf is a 0–framed unknot linked once with l. Replace G_0 by $G_1 \cup G_2$.

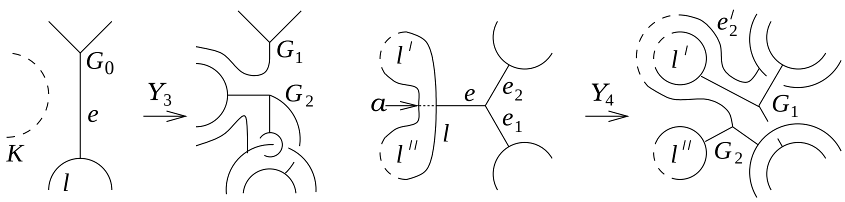


Figure 7: Edge slide and leaf-cutting

Let l be a leaf of a Y–graph G_0, e, e_1 and e_2 be its edges, with e adjacent to l. An arc a, starting at $e \cap l$ and ending on l, cuts l into two parts l' and l''. The move Y_4 is cutting the leaf l along a, see Figure 7:

Y_4: Let G_1 be an Y–graph obtained from G_0 by replacing the leaf l by $l' \cup a$. Let G_2 be obtained from G_0 by replacing the leaf l by $l'' \cup a$, adding l' to e_2 along the band e, and taking a push-off copy e_2', as shown in Figure 7. Replace G_0 by $G_1 \cup G_2$.

A convenient way to draw the moves Y_1–Y_4 is by depicting them in a standard handlebody, which then can be embedded into a manifold in an arbitrary way. Below is such an interpretation of the moves Y_3, Y_4 as moves in a standard handlebody of genus 4:

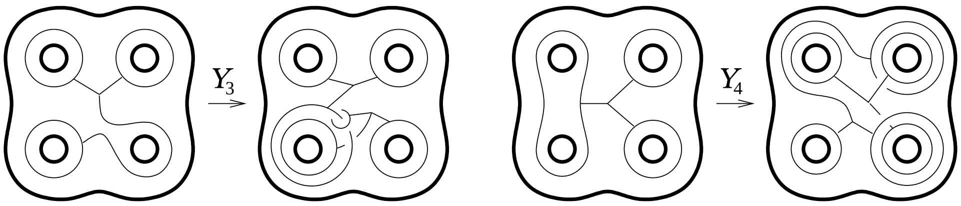


Figure 8: Y_3 and Y_4 as moves in a handlebody

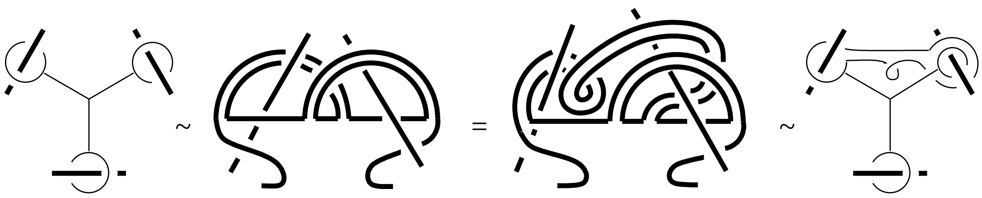
For example, when the handlebody is embedded so that one of the handles links another handle as in Figure 9a, the corresponding move Y_4 is shown in Figure 9b.

Figure 9: An embedding and the induced Y_4 move

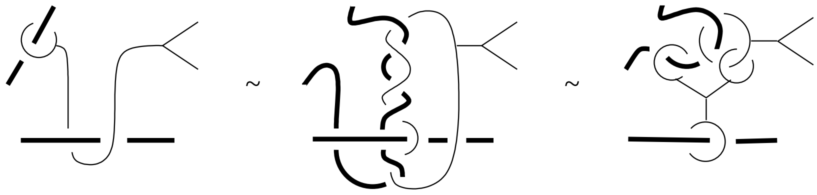
Theorem 3.1 *The moves Y_1–Y_4 preserve the classes of surgery equivalence of Y–links in a manifold M.*

Proof It suffices to prove the surgery equivalence of Y–links obtained by the moves Y_1–Y_4 in the standard handlebody. Instead of drawing the handlebodies we will draw thick lines passing through the handles (encoding a set of surgery and clover components), similarly to Lemma 2.1.

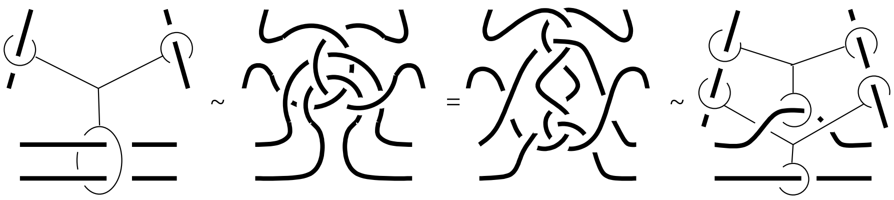
By Lemma 2.2, Y_1 is surgery equivalence. To verify Y_2, depict the Borromean linking with one component passing on the boundary of an embedded surface with two handles, and then twist one of the handles along the other:



To verify Y_3, use an isotopy and Lemma 2.1:



The verification of Y_4 is similar:



□

The following theorem was announced in [5] in case of a Y–graph G.

Theorem 3.2 *For any clover G in a manifold M, there exists a clover G^{-1} in a neighborhood of G, such that the result of surgery on both clovers is the original manifold: $M_{G\cup G^{-1}} = M$. The construction of G^{-1} for a Y–graph G is shown in Figure 10a. Another presentation of G^{-1} by a 2–component Y–link $G_1 \cup G_2$ is shown in Figure 10b.*

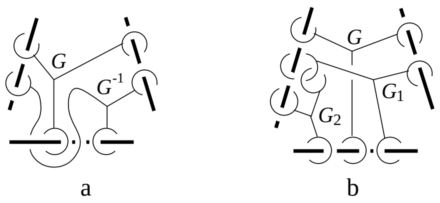
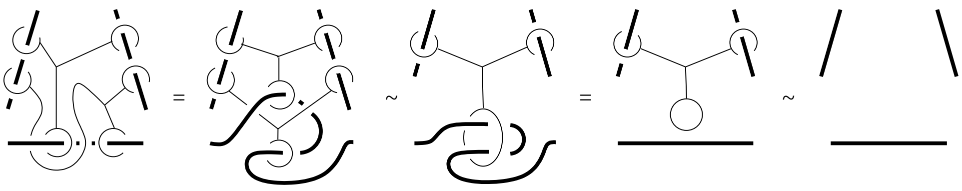


Figure 10: Two presentations of the inverse of a Y–graph

Proof Let us first verify the statement for a Y–graph. By an isotopy and a subsequent application of Y_4 and Y_1, we get



The general case now follows by Theorem 2.4. Finally, $G^{-1} \sim G_1 \cup G_2$ by Y_3. □

4 The structure of the graded spaces $\mathcal{G}_n$

This section is devoted to the study of the graded spaces

$$\mathcal{G}_n(M) = \mathcal{F}_n^Y(M)/\mathcal{F}_{n+1}^Y(M).$$

Denote $[M,G] \underset{n}{=} [M,G']$ iff $[M,G] - [M,G'] \in \mathcal{F}_{n+1}^Y(M)$.

4.1 Graded versions of Y_2–Y_4

Using 2.4 and Lemma 2.2, we obtain the following immediate corollaries of Theorem 3.1.

Corollary 4.1 *Let G be a clover of degree n in a manifold M, and let G' be obtained from G by sliding one of its leaves along an adjacent leaf by Y_2. Then $[M,G'] = [M,G]$, and hence $[M,G'] \underset{n}{=} [M,G]$.*

Let G_0 be a Y–graph in M, and let $G_1 \cup G_2$ be obtained from G_0 by an application of Y_3, see Figure 7. Note that $M_{G_2} = M$ by Y_1, thus

$$[M,G_0] - [M,G_1] = M_{G_1 \cup G_2} - M_{G_1} = [M,G_1 \cup G_2] \in \mathcal{F}_2^Y(M).$$

Hence, using Theorem 2.4 we obtain:

Corollary 4.2 *Let G be a clover of degree n in a manifold M, and let $K \subset M$ be a knot. Let G' be obtained from G by sliding an edge of G along K. Then $[M,G'] \underset{n}{=} [M,G]$.*

In Lemma 4.10 we will strengthen this result by computing the difference $[M,G'] - [M,G]$ in $\mathcal{G}_{n+1}(M)$.

Let G_0 be a Y–graph in M, and let $G_1 \cup G_2$ be obtained from G_0 by an application of Y_4, see Figure 7. By the previous Lemma, the value of $[M,G_2]$ in $\mathcal{G}_1(M)$ does not change when we slide its edge e_2' along a knot. Thus we can pull e_2' off l', making it back into e_2. Hence, using again Theorem 2.4 we obtain:

Corollary 4.3 *Let G be a clover of degree n in a manifold M and l be a leaf of G. An arc a starting in the external vertex incident to l and ending in other point of l, splits l into two arcs l' and l''. Denote by G' and G'' the clovers obtained from G by replacing the leaf l with $l' \cup a$ and $l'' \cup a$ respectively, see Figure 11. Then $[M,G] \underset{n}{=} [M,G'] + [M,G'']$.*

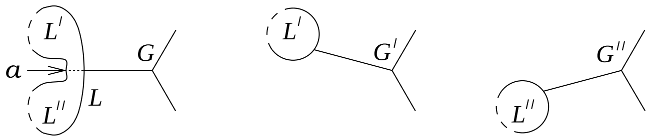


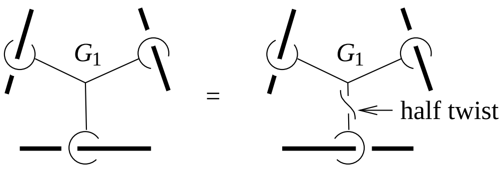
Figure 11: Splitting a leaf

4.2 The dependence on framings

Theorem 3.2 allows us to deduce the dependence of $[M,G]$ on the framings of edges.

Lemma 4.4 *Let G be a clover of degree n in a manifold M. Let G' be obtained from G by twisting the framing of an edge by a half twist. Then $[M,G'] \underset{n}{=} -[M,G]$.*

Proof Let G and $G_1 \cup G_2 \sim G^{-1}$ be the Y-graphs depicted in Figure 10b. Note that G_1 looks exactly like G, except for the way its lower leaf links the thick line. Turning this leaf to the same position changes the framing of the adjacent edge by a half twist:



By Theorem 3.2, $M_{G \cup G_1 \cup G_2} = M$; also, $M_{G \cup G_2} = M_G$ and $M_{G_2} = M$ by Lemma 2.2. Thus

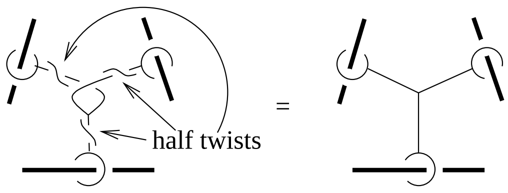
$$\begin{aligned} [M,G] + [M,G_1] &= 2M - M_G - M_{G_1} \\ &= [M, G \cup G_1] + [M, G_1 \cup G_2] - [M, G \cup G_1 \cup G_2] \underset{1}{=} 0 \end{aligned}$$

and the lemma follows.

Alternatively, one can show that $[M,G]+[M,G^{-1}] \underset{1}{=} 0$ for a Y–graph G^{-1} depicted in Figure 10a and pull its edge off the lower thick line by 4.2 to obtain once again G_1. □

Corollary 4.5 *Let G be a clover of degree n in M. Adding a kink, ie, a full twist to the framing of an edge preserves an n–equivalence class of $[M,G]$.*

Now, note that the following two Y–graphs are isotopic:



Thus, applying Lemma 4.4 and Theorem 2.4, we obtain:

Corollary 4.6 *Let G and G' be clovers of degree n in a manifold M, which coincide everywhere except for a fragment shown in Figure 12. Then $[M,G'] \underset{n}{=} -[M,G]$.*

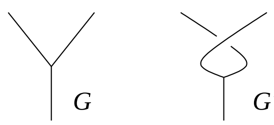


Figure 12: The AS relation

4.3 Simplifying the leaves

Denote $\mathcal{G}_n(S^3)$ by $\mathcal{G}_n$. We want to show that the space $\mathcal{G}_n \otimes \mathbb{Z}[1/2]$ is generated by clovers with only internal vertices, ie, without leaves.

A leaf of a clover G is *special*, if it is ±1–framed and bounds an embedded disc, whose interior does not intersect G. A leaf l of a clover G is *simple*, if it either bounds an embedded i–framed disc D whose interior intersects G in at most one point, or is special.

For $M=S^3$, the graded space $\mathcal{G}_n$ is generated by clovers of degree n all leaves of which are simple. Indeed, suppose we are given an arbitrary clover of degree

n in S^3. We may split each of its non-trivial leaves into small pieces and apply Corollary 4.3 to present G in $\mathcal{G}_n$ as a linear combination of clovers with simple leaves.

Suppose that all leaves of clover G in a manifold M are simple. If two simple leaves are linked with each other (see Figure 13a), we replace them by a new edge by Theorem 2.4. Suppose that at least one simple leaf l still remains after this procedure. We will show below that in this case $[M, G] \in \mathcal{F}^Y_{n+1}(M) \otimes \mathbb{Z}[1/2]$.

We should consider three cases, see Figure 13b–d. Firstly, it may be that l is trivial. Secondly, it may be that l is 0–framed, and the disc D intersects an edge of G. Thirdly, it may be that l is special. In the first case, $[M, G] = 0$ by

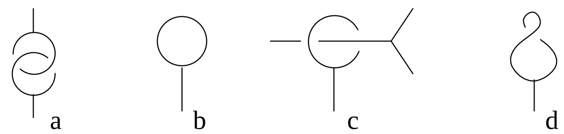


Figure 13: Four types of simple leaves

Lemma 2.2. The second case can be reduced to the first case after unlinking the edge from the corresponding leaf by Corollary 4.2 (sliding the edge along a small unknot linked once with the leaf). Thus we readily obtain:

Lemma 4.7 *Let G be a clover of degree n in a manifold M. Suppose that G contains a 0–framed leaf bounding an embedded disc, whose interior intersects G in exactly one point, belonging to an edge. Then* $[M, G] \underset{n}{=} 0$.

In the third case we encounter 2–torsion.

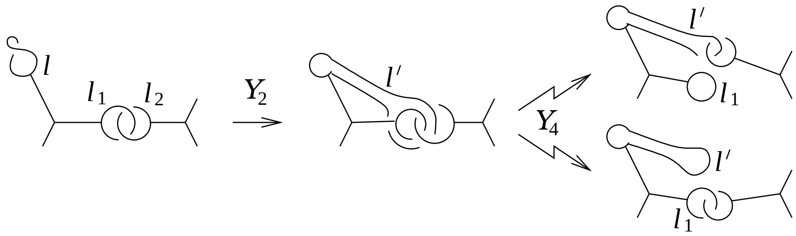
Lemma 4.8 *Let G be a clover of degree n in a manifold M, which contains a special leaf. Then* $2[M, G] \underset{n}{=} 0$.

Proof Rotating the special leaf we can change the framing of the adjacent edge while preserving the isotopy class of G. Thus by Lemma 4.4 $[M, G] \underset{n}{=} -[M, G]$ and the lemma follows. □

Over the integers, we have the following inclusion:

Lemma 4.9 *Let G and l be as in Lemma 4.8. Suppose that the connected component of G which contains l is of degree at least two. Then* $[M, G] \underset{n}{=} 0$.

Proof Cutting the neighboring internal edge we obtain two new leaves l_1 and l_2. Without a loss of generality suppose that the framing of l is $+1$. Then slide l along l_1 by Y_2 as shown below:



Notice that Y_2 changes the framing of a leaf by -1, so the new leaf l' is 0–framed. Splitting l_2 as shown above by Corollary 4.3, we obtain two clovers each of which contains a trivial leaf (either l' or l_1), and so can be removed by Y_1. □

4.4 The IHX relation

Let $G = \Gamma \cup G_0$ be an n–component Y–link which contains a Y–graph G_0, and let K be a knot in M. Choose a band connecting an edge of G_0 to K. Sliding this edge of G_0 along K by Y_3, we obtain a Y–graph G_1, as shown in Figure 7. Denote $G' = \Gamma \cup G_1$. By Corollary 4.2, $[M,G] - [M,G'] \in \mathcal{F}^Y_{n+1}(M)$. There is a simple expression for this difference modulo $\mathcal{F}^Y_{n+2}(M)$. It is easier to visualize the picture in a neighborhood N of $G_0 \cup b \cup K$, which is a genus 4 handlebody embedded into M.

Lemma 4.10 *Let G_0, G_1 and G_H be the clovers of Figure 14 in a handlebody V embedded into M. Let Γ be a degree $n-1$ clover in the complement of N. Put $G = \Gamma \cup G_0$, $G' = \Gamma \cup G_1$ and $G'' = \Gamma \cup G_H$. Then $[M,G] - [M,G'] \underset{n+1}{=} [M,G'']$.*

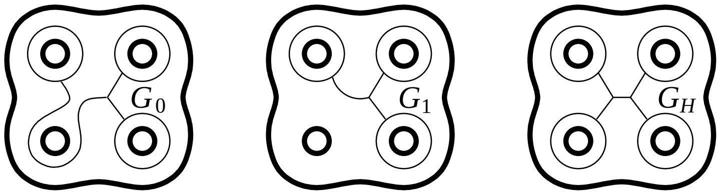
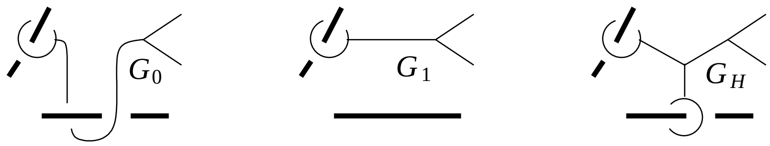
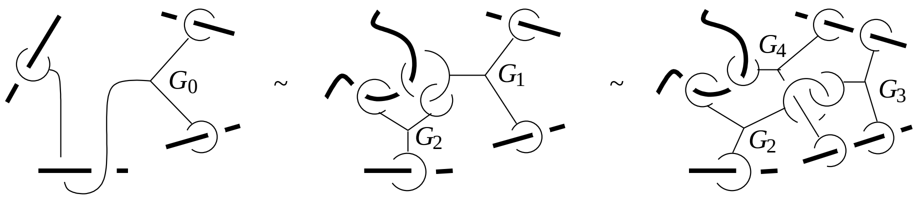


Figure 14: Sliding an edge and computing the difference

Here is another graphical expression for the above graphs:



Proof We use Y_3 to pass from G_0 to G_1 and G_2 with $G_0 \sim G_1 \cup G_2$ as in the proof of Theorem 3.1. Then we split the leaf l of G_1 by Corollary 4.3 introducing new Y–graphs G_1' and G_1'' with $[M, \Gamma \cup G_1 \cup G_2] \underset{n+1}{=} [M, \Gamma \cup G_1' \cup G_2] + [M, \Gamma \cup G_1'' \cup G_2]$:



But $[M, \Gamma \cup G_1'' \cup G_2] = -[M, G'']$ by Theorem 2.4, and $[M, \Gamma \cup G_1' \cup G_2] = 0$ by Lemma 2.2. Hence $[M, \Gamma \cup G_1 \cup G_2] \underset{n+1}{=} -[M, G'']$. On the other hand,

$$[M, \Gamma \cup G_1 \cup G_2] = \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|}(-M_{\Gamma' \cup G_1} + M_{\Gamma' \cup G_1 \cup G_2}) = [M, G'] - [M, G]$$

The comparison of two above expressions for $[M, \Gamma \cup G_1 \cup G_2]$ proves the theorem. □

Theorem 4.11 *Let G_I, G_H and G_X be clovers of degree n in a manifold M, which coincide everywhere except for a fragment shown in Figure 15. Then $[M, G_I] + [M, G_X] - [M, G_H] \underset{n}{=} 0$.*

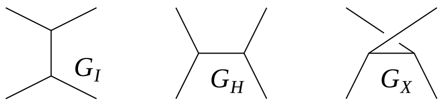


Figure 15: The IHX relation

Proof It suffices to prove the statement for $n = 2$; the general case then follows by Theorem 2.4 and Lemma 2.2. Consider a standard Y–graph G in a handlebody N of genus 3 and attach to N an additional handle h. We are to slide all three edges of Y along h in a genus 4 handlebody $N \cup h$ as indicated

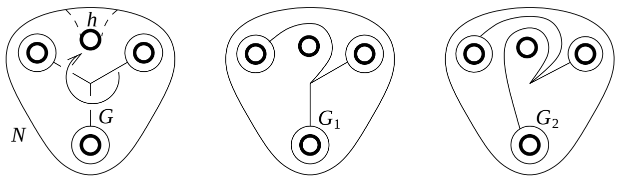


Figure 16: Sliding the edges along a handle

in Figure 16. Each time we will use Lemma 4.10 to compute the corresponding change of $[M,G]$. Sliding the first edge of G, we obtain a new Y–graph G_1 with $[M,G_1]-[M,G] \underset{2}{=} [M,G_I]$. Sliding the next edge of G (or rather of G_1), we obtain a new Y–graph G_2 with $[M,G_2]-[M,G_1] \underset{2}{=} [M,G'_X]$. After sliding the third edge we return back to G and get $[M,G]-[M,G_2] \underset{2}{=} [M,G'_H]$. Here a degree 2 clovers G_I, G'_X, and G'_H are shown in Figure 17. Summing up these

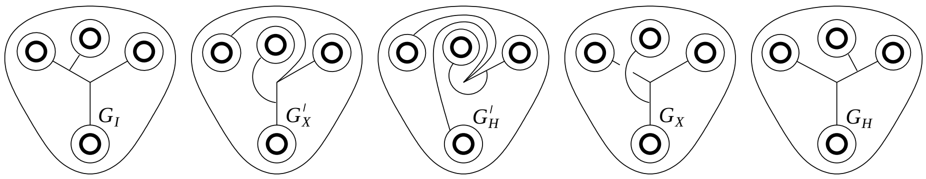


Figure 17: Computing the difference

three equalities we get

$$[M,G_I]+[M,G'_X]+[M,G'_H] \underset{2}{=} 0. \tag{1}$$

But G'_X differs from G_X only by an edge slide, and G'_H differs from G_H by an edge slide and a cyclic ordering of edges in a vertex, see Figure 17. Hence $[M,G'_X] \underset{2}{=} [M,G_X]$ by Corollary 4.2, and $[M,G'_H] \underset{2}{=} -[M,G_H]$ by Corollaries 4.2 and 4.6. A substitution of two these expressions into (1) proves the theorem. □

Remark 4.12 There is a topological version of the IHX relation, which may be deduced similarly using the Y_3 and Y_4 moves to slide each edge of a Y–graph through a handle in a handlebody of genus 4. We leave the details to an interested reader.

4.5 Trivalent graphs and $\mathcal{G}_n$

Consider an abelian group $\widetilde{\mathcal{A}}_k$ freely generated by abstract (not necessarily connected) trivalent graphs with $2k$ vertices and without looped edges, equipped with a cyclic ordering of the incident edges in each vertex. Denote by $\mathcal{A}_k$ the quotient of $\widetilde{\mathcal{A}}_k$ by the following AS and IHX relations:

AS: Let G' be obtained from G by reversing the cyclic ordering of edges in some vertex, see Figure 12. Then $G' = -G$.

IHX: Let G_I, G_H and G_X coincide everywhere except for a fragment shown in Figure 15. Then $G_I = G_H - G_X$.

Denote by Y a Y–graph in M with three special leaves. For each graph $G \in \widetilde{\mathcal{A}}_k$ pick an arbitrary embedding of G into M. Equip it with a framing so that the framing along each cycle of edges is integer, and take its disjoint union with m copies of Y. The resulting framed graph in M may be considered as a clover of degree $n = 2k + m$. Denote it by $\phi(G)$. Put $\widetilde{\psi}_n(G) = [M, \phi(G)]$ and extend $\widetilde{\psi}_n$ to $\widetilde{\psi}_n\colon\ \oplus_{2k \leq n} \widetilde{\mathcal{A}}_k \to \mathcal{F}$ by linearity. Note that $\widetilde{\psi}_n(\widetilde{\mathcal{A}}) \subset \mathcal{F}^Y_n$ by Corollary 2.5.

Theorem 4.13 *The map $\widetilde{\psi}_n\colon\ \oplus_{2k \leq n} \widetilde{\mathcal{A}}_k \to \mathcal{F}^Y_n$ induces a quotient map $\psi_n\colon\ \oplus_{2k \leq n} \mathcal{A}_k \to \mathcal{G}_n$, which is surjective and does not depend on the choice of ϕ. The image of $\oplus \mathcal{A}_{2k<n}$ is a 2–torsion.*

Proof The map $\widetilde{\psi}_n$ factors through AS and IHX relations by Corollary 4.6 and Theorem 4.11. The independence on the choice of $\phi(G)$ follows from Corollary 4.2 and Lemma 4.4. The surjectivity follows from the results of Section 4.3 and Lemma 2.3. The torsion result follows from Lemma 4.8. □

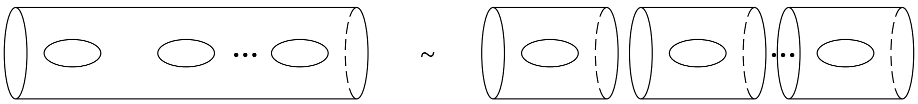
Note that $\oplus_{2k \leq 2n-1} \mathcal{A}_k = \oplus_{2k < 2n-1} \mathcal{A}_k$, hence $\mathcal{G}_{2n-1}$ consists of a 2–torsion and we obtain:

Corollary 4.14 *For any $n \in \mathbb{N}$, $\mathcal{F}^Y_{2n-1} \otimes \mathbb{Z}[1/2] = \mathcal{F}^Y_{2n} \otimes \mathbb{Z}[1/2]$.*

5 The equivalence of the $\mathcal{F}^{bl}$, $\mathcal{F}^{as}$ and $\mathcal{F}^Y$ filtrations

5.1 Equivalence of $\mathcal{F}^{bl}$ and $\mathcal{F}^Y$

We can present surgery on an arbitrary blink of genus g as surgery on g blinks of genus one, slicing the surface into pieces of genus one as shown below (see also [3]):



Also, as noticed already in [10], surgery on a blink of genus one can be presented as Y–surgery (and vice versa). Indeed, depicting the Borromean linking with one component passing on the boundary of an embedded surface with two bands, and then using K_3^{-1}, we obtain:

Lemma 5.1

Therefore the following theorem holds:

Theorem 5.2 *For each integer n we have $\mathcal{F}^{bl}_n(M) = \mathcal{F}^Y_n(M)$.*

5.2 Comparison of $\mathcal{F}^{as}$ and $\mathcal{F}^Y$: plan of the proof

The rest of the section is arranged in the following way.

We will first present any Y–link of degree d by a trivial d–component Y–link in S^3, together with a trivial unimodular link O which links T in a special way. We shall then prove an inclusion $\mathcal{F}^Y_{2n} \subset \mathcal{F}^{as}_{3n}$ by an easy counting argument.

The opposite inclusion is similar in spirit. Now we present any n–component algebraically split link by a trivial link O, together with a trivial Y–link T linking O in a special way. We then consider an appropriately modified version of some results of Section 4. Building on these results, we prove the inclusion $\mathcal{F}^{as}_{3n} \subset \mathcal{F}^Y_{2n} \otimes \mathbb{Z}[1/2]$ using a similar counting argument.

5.3 Undoing Y–graphs

A link O in $M \smallsetminus G$ *laces a clover* G, if O is trivial unimodular, and each of the (pairwise disjoint) discs bounding its components intersects G in at most two points, which belong to the leaves of G. A Y–link in M is *trivial*, if it consists of Y–graphs standardly embedded in n disjoint balls.

Lemma 5.3 *Let T be a trivial n–component Y–link in S^3. For any n–component Y–link G in S^3, there exists a link O in S^3 which laces T, such that $[S^3, G] = [S^3_O, T]$.*

Proof Any Y–link in S^3, in particular G, can be made into a trivial Y–link by framing twists and crossing changes. Moreover, it suffices to use crossing changes which involve only the leaves of Y–graphs. Indeed, instead of a crossing change which involves an edge of a Y–graph, one can do two subsequent crossing changes with the neighboring leaf of this graph (by sliding first the other branch towards the leaf). Using K_4 we can realize each of these framing and crossing changes by surgery on a trivial unimodular surgery link, as illustrated in Figure 3. The resulting collection of these surgery components comprises O. □

For a Y–link G in a manifold M and a link $L \subset M \smallsetminus G$ denote by $[M, G, L]$ the *double alternating sum*:

$$[M, L, G] = [[M, L], G] = [[M, G], L] = \sum_{G' \subset G} \sum_{L' \subset L} (-1)^{|G'|+|L'|} M_{G' \cup L'}$$

Corollary 5.4 *The space $\mathcal{F}^Y_n$ is generated by all $[S^3, O, T]$, where T is a trivial Y–link in S^3 of degree at least n, and the link O laces T.*

Proof By Lemma 5.3, $\mathcal{F}^Y_n$ is generated by all $[S^3_O, T]$, with O and T as above. It remains to notice that $S^3_O = \sum_{O' \subset O} (-1)^{|O'|} [S^3, O']$ and that any sublink O' of O also laces T. □

Theorem 5.5 *For each integer n we have $\mathcal{F}^Y_{2n} \subset \mathcal{F}^{as}_{3n}$.*

Proof Let T be a trivial Y–link in S^3 of degree at least $2n$ and O be an arbitrary link lacing T. In view of Corollary 5.4, it suffices to prove that $[S^3, O, T]$ belongs to $\mathcal{F}^{as}_{3n}$. Suppose that some leaf of T is not linked with O, ie, bounds a disc which does not intersect O; then $[S^3, O, T] = 0 \in \mathcal{F}^{as}_{3n}$ by Lemma 2.2. Otherwise, all (ie, at least $6n$) leaves of T are linked with O. But each component of O is linked with at most two leaves of T; hence the number of components of O is at least $3n$. Therefore $[S^3, O, T] \in \mathcal{F}^{as}_{3n}$. □

5.4 Undoing an AS–link

Let L be a framed link in S^3. We call L an *AS–link*, if it is algebraically split and unimodular. In Section 5.3 above we presented any Y–link by a trivial Y–link T, together with a trivial unimodular link O lacing it. In this section we shall do the opposite: we present any AS–link by a trivial unimodular link O, together with a trivial Y–link linking it in a special way, which we will also call "lacing".

A clover G in $M \smallsetminus L$ *laces a link* L, if each leaf of G either is trivial and links L once (ie, bounds a disc which intersects L in one point), or is unlinked with L (ie, bounds a surface which does not intersects L).[2] A pair (O, G) consisting of a trivial link O in M and a Y–link G lacing O is called a *lacing pair*. A lacing pair (O, G) is trivial, if G is trivial. Surgery on a trivial lacing pair in S^3 was called a Borromean surgery in [10] and a Δ–move in [12]. It was shown in [10, 12], that one can pass from a link L in S^3 to any other link with the same linking matrix by surgery on a trivial Y–link lacing L. Applying this result to AS–links, we deduce:

Lemma 5.6 *Let O be a trivial unimodular n–component link in a S^3. For any AS–link L in S^3, there exists a trivial lacing pair (O, G), such that $[S^3, L] = [S^3_T, O]$.*

Corollary 5.7 *The space $\mathcal{F}^{as}_n$ is generated by all $[S^3, O, T]$, where O is a trivial unimodular link in S^3 with at least n components and a trivial Y–link G laces O.*

Proof By Lemma 5.6, $\mathcal{F}^{as}_n$ is generated by all $[S^3_T, O]$, with T and O as above. It remains to notice that $S^3_T = \sum_{T' \subset T} (-1)^{|T'|}[S^3, T']$ and that any Y–sublink T' of T also laces O. □

In what follows, we will in fact need only a weaker version of Corollary 5.7, in which we omit the assumption of triviality of the lacing pair.

5.5 The inclusion $\mathcal{F}^{as}_{3n} \subset \mathcal{F}^{Y}_{2n} \otimes \mathbb{Z}[1/2]$

We will need a modification of Corollary 4.2 and Lemma 4.4. Returning to their proofs, we notice that both statements can be stated for Y–links lacing a fixed link in M:

[2] In fact, we will not need this type of leaf, but prefer to formulate the definition in its full generality

Corollary 5.8 *Fix a link L in a manifold M. Let G be a Y–link of degree d lacing L and let G' be obtained from G by sliding an edge of G along a knot $K \subset M \setminus L \setminus G$. Suppose that for every Y–link Γ of degree $(d+1)$ lacing L one has $[M, L, \Gamma] \underset{k}{=} 0$. Then $[M, G'] \underset{k}{=} [M, G]$.*

Lemma 5.9 *Fix a link L in a manifold M. Let G be a Y–link of degree d lacing L and let G' be obtained from G by twisting the framing of some edge by a half twist. Suppose that for every Y–link Γ of degree $(d+1)$ lacing L one has $[M, L, \Gamma] \underset{k}{=} 0$. Then $[M, G'] \underset{k}{=} -[M, G]$*

Moreover, the following version of Lemma 4.8 holds:

Lemma 5.10 *Let (O, G) be a lacing pair in a manifold M with G of degree d. Suppose that some disc D_i bounding a component of O intersects G in just one point, which belongs to a leaf of G. Suppose also that for any lacing pair (O, Γ) with Γ of degree greater than d we have $[M, O, \Gamma] \underset{k}{=} 0$. Then $2[M, G] \underset{k}{=} 0$.*

Proof Proceeding similarly to the proof of Lemma 4.8, we rotate O together with the trivial leaf linked with it. This adds a half twist to the framing of the adjacent edge of G, while preserving the isotopy class of O and G. Thus $[S^3, O, G] \underset{k}{=} -[S^3, O, G]$ by Lemma 5.9. □

We are in a position to prove the second inclusion theorem.

Theorem 5.11 *For each integer n we have $\mathcal{F}^{as}_{3n} \subset \mathcal{F}^{Y}_{2n} \otimes \mathbb{Z}[1/2]$.*

Proof Let O be a lacing pair in S^3 with O having at least n components. In view of Corollary 5.7, it suffices to prove that $[S^3, O, G] \in \mathcal{F}^{Y}_{2n} \otimes \mathbb{Z}[1/2]$. We proceed by downward induction on the degree d of G. If $d \geq 2n$, then obviously $[S^3, O, G] \in \mathcal{F}^{Y}_{2n}$ and the theorem follows. Suppose now that the inclusion holds for all Y–links of degree higher than d and let us prove it for a Y–link G of degree d.

By a repeated use of Corollary 5.8 we can reduce the problem to the case when none of the edges of G pass through the discs D_i bounding the components of O.

If for some i the disc D_i does not intersect G, then $[S^3, O, G] = 0 \in \mathcal{F}^{Y}_{2n}$ by Lemma 2.2, and we are done.

If some disc D_i intersects G in exactly one point belonging to a leaf l of G, the statement follows from Lemma 5.10 (applicable by the induction assumption).

We are left with the case when each component of O is linked with at least two leaves of G. But each leaf of G can be linked with at most one component of O. Therefore, G should have at least $6n$ leaves, ie, at least $2n$ components. Hence $[S^3, O, G] \in \mathcal{F}^Y_{2n}$, and the theorem follows. □

Remark 5.12 Over the integers, a simplified version of the above counting argument leads to an inclusion $[S^3, O, G] \in \mathcal{F}^Y_n$ that does not require Lemmas 5.9 and 5.10. Indeed, we proceed as in the proof of Theorem 5.11 above, first reducing the problem to the case when no edges of G pass through the discs D_i, and then noticing that if some D_i does not intersect G, then $[S^3, O, G] = 0$ by Lemma 2.2. Otherwise, each of the $3n$ discs D_i intersects at least one leaf of G. Each leaf of G can intersect at most one disc, so G should have at least $3n$ leaves, ie, at least n components, hence $[S^3, O, G] \in \mathcal{F}^Y_n$.

6 The equivalence of the $\mathcal{F}^b$ and $\mathcal{F}^Y$ filtrations

6.1 Plan of the proof

The section is arranged in the following way.

In the first part we shall prove an inclusion $\mathcal{F}^Y_{2n} \subset \mathcal{F}^b_n$. This is done in two steps. By Corollary 5.4 the space $\mathcal{F}^Y_d$ is generated by all $[S^3, O, T]$, where T is a trivial d–component Y–link in S^3, and the link O laces T. In the first step, we take $d = 2n^2$, construct a "good" sublink $B \subset O$ with at least n components, and show that this implies $\mathcal{F}^Y_{2n^2} \subset \mathcal{F}^b_n$. Building on this result, in the second step we prove a stronger inclusion $\mathcal{F}^Y_{2n} \subset \mathcal{F}^b_n$ using the IHX relation.

In the second part we shall prove an opposite inclusion $\mathcal{F}^b_n \subset \mathcal{F}^Y_{2n} \otimes \mathbb{Z}[1/2]$. We present any n–component boundary link as an image of a trivial unimodular link O under surgery on a Y–link G, linking O in a special way, which we will call "1–lacing". Using this presentation together with an appropriately modified version of Lemma 5.10, we prove that $\mathcal{F}^b_n \subset \mathcal{F}^Y_{2n} \otimes \mathbb{Z}[1/2]$ by an easy counting argument.

6.2 Constructing a boundary sublink

Let T be a trivial $2n^2$ component Y–link in M and let O be a link which laces T. Below we construct a sublink $B \subset O$ with at least n components, which will be a boundary link in the manifold, obtained from $M \smallsetminus (O \smallsetminus B)$ by surgery on T. This construction is based on a notion of a good sublink.

A link B in M is *good*, if B laces T and the following conditions hold:

- No Y–graph of T is linked with B by all three leaves.
- If some Y–graph of T is linked with B by two leaves, then it is linked with just one component of B.

The possible structure of B in a neighborhood of a component of T is shown in Figure 18.

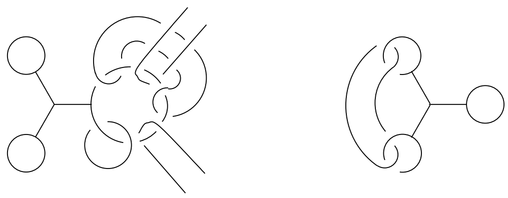


Figure 18: The structure of a good link near a Y–graph

Lemma 6.1 *Let T be a trivial Y–link in M and B be a good sublink. Then the image B' of B in M_T is a boundary link in a neighborhood of $T \cup D$, where D is a union of discs bounding the components of B.*

Proof Cut out a tubular neighborhood N of T; this is a disjoint union of genus 3 handlebodies. Components of B bound non-intersecting discs in M, which intersect the boundary ∂N by either one or two circles, which are meridians of N. Perform surgery on T by cutting, twisting by an appropriate element of the mapping class group, and gluing back the handlebodies N. This transforms the meridians into some other curves on ∂N. It suffices to show that for each handlebody N_i of N these curves on ∂N_i are bounding inside N_i.

Let m be a meridian on ∂N_i. The surgery on this Y–graph corresponds to the action of an element of the Torelli subgroup (which acts trivially on $H_1(N_i)$), so the image of m still bounds a surface Σ_m. Now, recall that B is a good link, so there are only two possible configurations of the meridians. Firstly, ∂N_i may contain just two meridians m and m', corresponding to the same component b of B. Then, smoothing the intersections of Σ_m with $\Sigma_{m'}$, we obtain a surface bounding $m \cup m'$. Secondly, these meridians may appear only on one of the three handles of ∂N_i. Pick one of the meridians m; all other meridians on ∂N_i may be considered as small push-offs of m in the normal direction. Thus their images under the Y–surgery, together with the corresponding surfaces, can be obtained by a similar push-offs. This finishes the proof of the lemma. □

Corollary 6.2 *Let T be a trivail Y–link in M and B be an n–component good link with the union D of discs bounding B. Then $[M, L \cup B, T] \in \mathcal{F}_n^b$ for any link L in $M \smallsetminus (T \cup D)$.*

Proof By Lemma 6.1, the image of B under surgery on $T' \subset T$ bounds in the complement of L. Thus it will bound also in $(M_{T'})_{L'}$, for any $L' \subset L$. □

Lemma 6.3 *Let T be a trivial $2n^2$–component Y–link in M and let O be a link, which laces T and links all leaves of T. Then there exists a good sublink B of O with at least n components.*

We will say that two components of O (respectively, two leaves of T) are *neighboring*, if there is a Y–graph (respectively, a component of O) which is linked with both of them.

Proof Suppose that there is a leaf of T, which has at least n neighbors, apart from the other leaves of the same Y–graph. Then the corresponding components of O comprise B. It remains to consider the case when each leaf of T has less than n neighbors belonging to other Y–graphs. Pick an arbitrary component B_1 of O and remove from O all neighboring components of B_1. Repeat this step, each time picking a new component B_i of the remaining link, until there are no more components left. Finally, take $B = \cup_i B_i$.

Let us establish a lower bound for the number of these steps. Each B_i is linked with at most two Y–graphs, leaves of which have altogether at most $6(n-1) - 2 = 6n - 8$ other neighboring leaves. Thus the removal of B_i and of the link components neighboring B_i may unlink at most $6n-2$ leaves of T. In the beginning, O was linked with all $6n^2$ leaves of G. Therefore, the number of steps is at least $\frac{6n^2}{6n-2} > n$, thus B has more than n components. □

Remark 6.4 In fact, one can assume that the degree of T is just $6n$, but the construction of B in this case is significantly more complicated.

6.3 The inclusion $\mathcal{F}_{2n}^Y \subset \mathcal{F}_n^b$

Let us start with a weaker inclusion:

Proposition 6.5 *For each integer n, we have $\mathcal{F}^Y_{2n^2} \subset \mathcal{F}^b_n$.*

Proof Let T be a trivial $2n^2$–component Y–link in S^3 and O be an arbitrary link lacing T. In view of Corollary 5.4, it suffices to prove that $[S^3, O, T] \in \mathcal{F}^b_n$. If some leaf of T is not linked with O, then $[S^3, O, T] = 0 \in \mathcal{F}^b_n$. Otherwise, all leaves of T are linked with O and we can use Lemma 6.3 to find a good sublink B of O with at least n components. It remains to apply Lemma 6.1 for $M = S^3$ and $L = O \smallsetminus B$. □

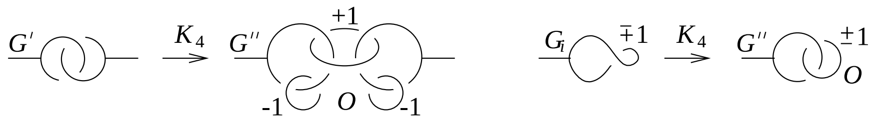
Now we are in a position to prove a stronger result:

Theorem 6.6 *For each integer n, we have $\mathcal{F}^Y_{2n} \subset \mathcal{F}^b_n$.*

Proof Let G be a clover of degree $d \geq 2n$ in S^3. We proceed by the downward induction on d. If $d \geq 2n^2$, then $[S^3, G] \in \mathcal{F}^b_n$ by Proposition 6.5, and the theorem follows. Suppose that the statement holds for any clover of degree higher than d, and let us prove it for a clover G of degree d. Note that by the induction assumption we can use Theorem 4.13. Thus it suffices to prove the statement for a clover G which is a disjoint union of a degree $2k \leq d$ clover G' with no leaves and $d - 2k$ copies G_i of a Y–graph with three special leaves.

We call a path in a connected graph *maximal*, if it is connected, passes along each edge at most once, and contains the maximal number of edges. A path in G' is maximal, if it is maximal in each of its connected components. The number v of vertices of G' which do not belong to a maximal path is called the *length-defect* of G'. If v is positive, pick a vertex of G' which does not belong to a maximal path, but is connected to it by an edge. Applying to this edge the IHX relation of Theorem 4.11, we obtain two clovers, each of which has the length defect $v - 1$. Hence it suffices to prove the theorem for $v = 0$. Notice that if $v = 0$, there are at least k edges of G' without common ends. Thus, including G_i's, there are at least $d - k$ edges of G without common ends.

Cut all edges of G' by Theorem 2.4. Now, unlink all pairs of newly created leaves and change the framing of all leaves of G_i's to 0 by K_4, as shown below:



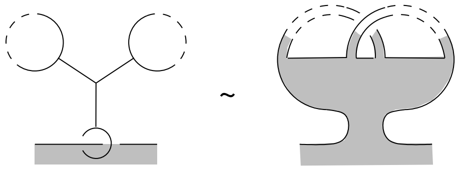
Denote by O the resulting link and by T the d–component Y–link obtained from T. Clearly the link O laces T and

$$[S^3, G] = \pm[S^3_O, T] = \pm \sum_{O' \subset O} (-1)^{|O'|}[S^3, O', T].$$

We should show that $[S^3, O', T] \in \mathcal{F}_n^b$ for each $O' \subset O$. If some leaf of T is unlinked with O', then this leaf is trivial and $[S^3, O', T] = 0$. Otherwise, by the construction of O, there are at least $d - k \geq d/2 \geq n$ non-neighboring components of O' (since there were at least $d - k$ edges of G without common ends). These components comprise a good sublink of O'. The theorem now follows from Corollary 6.2. □

6.4 Undoing a boundary link

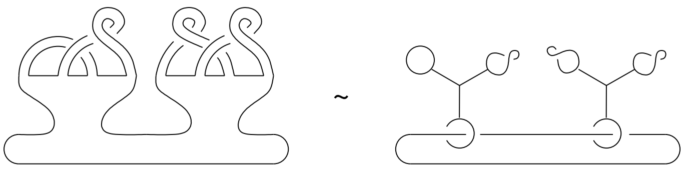
Let B be a boundary link in M, and fix a surface $\Sigma = \cup_i \Sigma_i$ bounded by B. We would like to present B by a trivial unimodular link bounding a collection of discs, together with a Y–link, lacing it in a rather special way. Namely, we will say that a Y–link *b–laces the link B*, and each of its components intersects Σ in at most one point, which belongs to a trivial leaf. By the classification of surfaces, we can assume that a surface Σ_i bounding each component is an embedding of a connected sum of discs with two attached bands. Lemma 5.1 shows how such a disc with two (possibly linked and knotted) bands can be obtained from a standard disc by surgery on a Y–graph b–lacing it:



A pair (O, G) consisting of a trivial link O and a Y–link G in M is called a *b–lacing pair*, if G b–laces O for $\Sigma_i = D_i$ being discs bounding the components of O. Exchanging pairs of bands to b–lacing Y–graphs (or vice versa) as above, we obtain

Corollary 6.7 *For any n–component boundary link B in a manifold M there exists a b–lacing pair (O, G) such that surgery on G transforms O into B. Conversly, for any b–lacing pair (O, G), surgery on G transforms O into a boundary link.*

We illustrate this construction on an example of a genus 2 surface bounding a trefoil:



Remark 6.8 A presentation of a boundary link by a b–lacing pair (O,G) encodes an information about the Seifert surface Σ of B and the Seifert form. Indeed, each Y–graph G_i of G corresponds to a pair of bands, ie, to a handle h_i of Σ. The cores b_{2i-1}, b_{2i} of these bands, being appropriately oriented and closed in the disc in a standard way, comprise a preferred basis of $H_1(\Sigma)$. This observation allows one to deduce that the Seifert matrix is given by $\mathrm{lk}(b_i,b_j)+\delta_{i+1,j}$, where $\mathrm{lk}(b_i,b_j)$ is the linking matrix of $2n$ non-trivial leaves of G. We will investigate some applications of this construction in a future paper.

Corollary 6.9 *The space $\mathcal{F}^b_n$ is generated by all $[S^3,O,G]$, where (O,G) is a b–lacing pair in S^3 and O has at least n components.*

Proof By Corollary 6.7, $\mathcal{F}^b_n$ is generated by all $[S^3_G,O]$, with G and O as above. It remains to notice that $S^3_G=\sum_{G'\subset G}(-1)^{|G'|}[S^3,G']$, and that any Y–sublink G' of G also b–laces O. □

6.5 The inclusion $\mathcal{F}^b_n\subset\mathcal{F}^Y_{2n}$

Returning to the proof of Lemma 5.10 (and Lemmas 5.9, 4.4 used in it), we notice that it may be restated for b–lacing pairs in M. Thus the following modified version of Lemma 5.10 holds:

Lemma 6.10 *Let (O,G) be a b–lacing pair in a manifold M with G of degree d. Suppose that some disc D_i bounding a component of O intersects G in just one point. Suppose also that for any b–lacing pair (O,Γ) with Γ of degree greater than d we have $[M,O,\Gamma]\underset{k}{=}0$. Then $2[M,G]\underset{k}{=}0$.*

We are ready to prove the last inclusion theorem.

Theorem 6.11 *For each integer n we have $\mathcal{F}^b_n\subset\mathcal{F}^Y_{2n}\otimes\mathbb{Z}[1/2]$.*

Proof Let (O,G) be a b–lacing pair in S^3 such that O has at least n components. We will show by downward induction on the degree d of G that $[S^3,O,G]\in\mathcal{F}^Y_{2n}\otimes\mathbb{Z}[1/2]$. If $d\geq 2n$, then obviously $[S^3,O,G]\in\mathcal{F}^Y_{2n}$. Inductively suppose that the statement holds for any b–lacing pair (O,Γ) in S^3 with Γ of degree greater than d.

If for some i the disc D_i bounding a component of O does not intersect G, then obviously $[S^3,O,G]=0\in\mathcal{F}^Y_{2n}$ and we are done. If for some i the disc

D_i intersects G in exactly one point, the statement follows from Lemma 6.10 (applicable by the induction assumption). We are left with the case when each disc D_i intersects G in at least two points. Since G b–laces O, these points should belong to the leaves of different Y–graphs, which do not intersect any other disc D_j, $j \neq i$. Therefore, G should have at least $2n$ components, and the theorem follows. □

Remark 6.12 The inclusion of Theorem 6.11 and its proof are valid not just for $\mathbb{Z}$HS, but for arbitrary 3–manifolds.

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