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Flag Structures on Seifert Manifolds

Thierry Barbot

C.N.R.S, ENS Lyon, UMPA, UMR 5669
46, allée d'Italie, 69364 Lyon, France

Email: barbot@umpa.ens-lyon.fr

Abstract

We consider faithful projective actions of a cocompact lattice of $SL(2,\mathbf{R})$ on the projective plane, with the following property: there is a common fixed point, which is a saddle fixed point for every element of infinite order of the the group. Typical examples of such an action are linear actions, ie, when the action arises from a morphism of the group into $GL(2,\mathbf{R})$, viewed as the group of linear transformations of a copy of the affine plane in $\mathbf{R}P^2$. We prove that in the general situation, such an action is always topologically linearisable, and that the linearisation is Lipschitz if and only if it is projective. This result is obtained through the study of a certain family of flag structures on Seifert manifolds. As a corollary, we deduce some dynamical properties of the transversely affine flows obtained by deformations of horocyclic flows. In particular, these flows are not minimal.

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1 Introduction

Let $\bar{\Gamma}$ be the fundamental group of a closed surface with negative Euler characteristic. It admits many interesting actions on the sphere S^2:

- conformal actions through morphisms $\bar{\Gamma} \to PSL(2, \mathbf{C})$,

- projective actions on the sphere of half-directions in $\mathbf{R}^3$ through morphisms $\bar{\Gamma} \to GL(3, \mathbf{R})$.

We have one natural family of morphisms from $\bar{\Gamma}$ into $PSL(2, \mathbf{C})$, and two natural families of morphisms from $\bar{\Gamma}$ into $GL(3, \mathbf{R})$:

(1) *Fuchsian morphisms:* fuchsian morphisms are faithful morphisms from $\bar{\Gamma}$ into $PSL(2, \mathbf{R}) \subset PSL(2, \mathbf{C})$, with image a cocompact discrete subgroup of $PSL(2, \mathbf{R})$. In this case, the domain of discontinuity of the corresponding action of $\bar{\Gamma}$ is the union of two discs, and these two discs have the same boundary, which is nothing but the natural embedding of the boundary of the Poincaré disc $\mathbf{H}^2$ into the boundary of the hyperbolic 3–space $\mathbf{H}^3$. Moreover, on every component of the domain of discontinuity, the action of $\bar{\Gamma}$ is topologically conjugate to the action by isometries through $PSL(2, \mathbf{R})$ on the Poincaré disc (the topological conjugacy is actually quasi-conformal) and the action on the common boundary of these discs is conjugate to the natural action of $\bar{\Gamma}$ through $PSL(2, \mathbf{R})$ on the projective line $\mathbf{R}P^1$. Finally, all these actions on the whole sphere through $PSL(2, \mathbf{R})$ are quasi-conformally conjugate one to the other.

(2) *Lorentzian morphisms:* a lorentzian morphism is a faithful morphism $\bar{\Gamma} \to SO_0(2,1) \subset GL(3, \mathbf{R})$ whose image is a cocompact lattice of $SO_0(2,1)$, the group of linear transformations of determinant 1 preserving the Lorentzian cone of $\mathbf{R}^3$. Observe that such a morphism corresponds to a fuchsian morphism via the isomorphism $SO_0(2,1) \approx PSL(2, \mathbf{R})$. The action on the projective plane associated to a lorentzian morphism has the following properties:
- it preserves an ellipse, on which the restricted action is conjugate to the projective action on $\mathbf{R}P^1$ through the associated fuchsian morphism,
- it preserves a disc, whose boundary is the $\bar{\Gamma}$–invariant ellipse. This disc is actually the projective Klein model of the Poincaré disc, the action of $\bar{\Gamma}$ on it is conjugate to the associated fuchsian action of $\bar{\Gamma}$ on the Poincaré disc,
- it preserves a Möbius band (the complement of the closure of the invariant disc). The action on it is topologically transitive (ie, there is a dense $\bar{\Gamma}$–orbit). We have no need here to describe further this nice action.

Moreover, all the lorentzian actions are topologically conjugate one to the other, and the conjugacy is Hölder continuous (we won't give any justification here of this assertion, since it requires developments which are far away from the real topic of this paper).

(3) *Special linear morphisms:* they are the faithful morphisms $\bar{\Gamma} \to SL(2, \mathbf{R})$, where $SL(2, \mathbf{R})$ is considered here as the group $SL \subset SL(3, \mathbf{R})$ of matrices of positive determinant and of the form:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Moreover we require that the image of the morphism is a lattice in SL. Then, the action of $\bar{\Gamma}$ on the projective plane has a common fixed point, an invariant projective line, and an invariant punctured affine plane. The action on the invariant line is the usual projective action on $\mathbf{R}P^1$ through the natural projection $SL \to PSL(2, \mathbf{R})$, and the action on the punctured affine plane is the usual linear action. This action is minimal (every orbit is dense) and uniquely ergodic (there is an unique invariant measure up to constant factors). Contrary to the preceding cases, the action highly depends on the morphism into SL: two morphisms induce topologically conjugate actions if and only if they are conjugate by an inner automorphism in the target SL.

We are interested in the small deformations of these actions arising by perturbations of the morphisms into $PSL(2, \mathbf{C})$ or $GL(3, \mathbf{R})$. We list below the main properties of these deformed actions; we will see later how to justify all these claims.

(1) *Quasi-fuchsian actions:* morphisms from $\bar{\Gamma}$ into $PSL(2, \mathbf{C})$ which are small deformations of fuchsian morphisms are *quasi-fuchsian*: this essentially means that their associated actions on the sphere are quasi-conformally conjugate to fuchsian actions. They all preserve a Jordan curve, this Jordan curve is rectifiable if and only if it is a great circle, in which case the action is actually fuchsian (see for example [30], chapter 7).

(2) *Convex projective actions:* we mean by this the actions arising from morphisms from $\bar{\Gamma}$ into $GL(3, \mathbf{R})$ near lorentzian projective morphisms. Such an action still preserves a strictly convex subset of $\mathbf{R}P^2$ whose boundary is a Jordan curve of class C^1 (it is of class C^2 if and only if the action

is conjugate in $PGL(3, \mathbf{R})$ to a lorentzian action, see [5]). Moreover, all these actions are still topologically conjugate one to the other[1].

(3) *Hyperbolic actions:* these are the real topic of this paper, thus we discuss them below in more detail.

Hyperbolic actions arise from morphisms from $\bar{\Gamma}$ into $PGL(3, \mathbf{R})$ which are deformations of special linear morphisms. Actually, we will not consider all these deformations; we will restrict ourselves to the deformations for which the deformed action has still an invariant point: they correspond to morphisms into the group Af_0^* of matrices of the form:

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ x & y & 1 \end{pmatrix}$$

where A is a 2×2–matrix of positive determinant (we will say that the matrix A is the linear part, and that (x, y) is the translation part). This group is in a natural way dual to the group Af_0 of orientation preserving affine transformations of the plane: the space of projective lines in $\mathbf{R}P^2$ is a projective plane too, and the dual action of Af_0^* on this dual projective plane preserves a projective copy of the affine plane.

Small deformations $\bar{\Gamma} \to Af_0^*$ of special linear morphisms all satisfy the following properties (cf Lemma 2.1):

- the morphism $\bar{\Gamma} \to Af_0^*$ is injective,

- the common fixed point is a fixed point of saddle type for every non-trivial element of $\bar{\Gamma}$. Equivalently, the image of every non-trivial element of $\bar{\Gamma}$ in the dual group Af_0 is a hyperbolic affine transformation.

Morphisms $\rho\colon \bar{\Gamma} \to Af_0^*$ satisfying the properties above are called *hyperbolic.* In the special case where the translation part (x, y) is zero for every element, we say that the hyperbolic action is *horocyclic* (we will soon justify this terminology). Observe that the conjugacy by homotheties of the form

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[1]In this case, we have the additional remarkable fact: in the variety of morphisms $\bar{\Gamma} \to PGL(3, \mathbf{R})$, the morphisms belonging to the whole connected component of the lorentzian morphisms (the so-called Hitchin component) induce the same action on the projective plane up to topogical conjugacy [15].

does not modify the linear parts, but multiply the translation part (x, y) by e^t. It follows that hyperbolic morphisms can all be considered as small deformations of horocyclic morphisms (cf Proposition 3.5).

Hyperbolic morphisms can be defined in another way: we call the *unimodular linear part of* ρ the projection in $SL(2, \mathbf{R})$ of the linear part of the morphism; we denote it by ρ_0. For every element γ of $\bar{\Gamma}$, let $\bar{u}(\gamma)$ be the logarithm of the determinant of the linear part of $\rho(\gamma)$ (as an linear transformation of the plane). It induces an element of $H^1(\bar{\Gamma}, \mathbf{R})$. On the other hand, $H^1(\bar{\Gamma}, \mathbf{R})$ is isomorphic to $H^1(\Sigma, \mathbf{R})$, where Σ is the quotient of the Poincaré disc by the projection of $\rho_0(\bar{\Gamma})$ in $PSL(2, \mathbf{R})$. The surface Σ is naturally equipped with a hyperbolic metric, and thus, we can consider the *stable norm on* $H^1(\Sigma, \mathbf{R})$ (this stable norm depends on ρ_0) Then (Remark 2.2), the morphism ρ is hyperbolic if and only if the morphism ρ_0 is fuchsian (ie, has a dicrete cocompact image), and if the stable norm of $\bar{u}$ is less than $\frac{1}{2}$. We call hyperbolic every projective action of $\bar{\Gamma}$ induced by a hyperbolic morphism. The main result of this paper is (Corollaries 4.14, 4.18):

Theorem A *Every hyperbolic action of* $\bar{\Gamma}$ *is topologically conjugate to the projective horocyclic action of its linear part. The conjugacy is Lipschitz if and only if it is a projective transformation.*

As a corollary, any hyperbolic action preserves an annulus on which it is uniquely ergodic, and the two boundary components of this annulus are respectively the common fixed point and an invariant Jordan curve (Corollary 4.15). We give below a computed picture of such a Jordan curve:

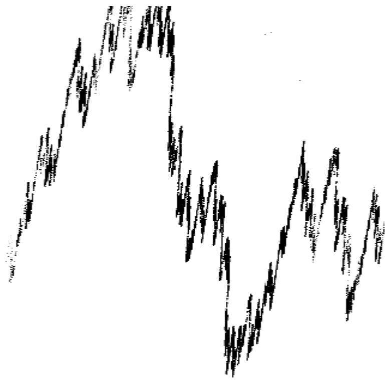

Figure 1: A zoom on the invariant Jordan curve

The studies of all these deformations have a common feature: we have to transpose the problem to a 3–dimensional object.

(1) *The case of fuchsian actions:* in this case, the key idea is to consider the quotient of hyperbolic 3–space $\mathbf{H}^3$ by $\bar{\Gamma}$ (viewed as a subgroup of $PSL(2,\mathbf{C}) \approx Isom(\mathbf{H}^3)$). It is a hyperbolic 3–manifold, homeomorphic to the product of a surface Σ by $]0,1[$. The action in $\mathbf{H}^3$ has a finite fundamental polyhedron (see [24], chapter 4). The fuchsian morphism can be considered as the holonomy morphism of this hyperbolic manifold. It is well-known that any deformation of the holonomy corresponds to a deformation of the hyperbolic structure (this is a general fact about (G,X)–structures, see for example [18], [9]). According to [24], Theorem 10.1, the deformed action still has a finite sided polyhedron. It follows then that the domain of discontinuity of the deformed action contains two invariant discs, and then, that the action is quasi-fuchsian, ie, that it is quasi-conformally conjugate to a fuchsian action ([24], section 3.2). (Quasi-conformal stability of quasi-fuchsian groups is also proved by L Bers in [6], using different tools).

(2) *The case of convex projective actions:* the deformations of lorentzian cones can be understood by the following method: the invariant disc is the projection in $\mathbf{R}P^2$ of the lorentzian cone. Add to the cocompact lattice in $SO_0(2,1)$ any homothety of $\mathbf{R}^3$ of non-constant factor. We obtain a new group which acts freely, properly and cocompactly on the lorentzian cone. The quotient of this action is a closed 3–manifold, equipped with a *radiant affine structure,* ie, a $(GL(3,\mathbf{R}),\mathbf{R}^3)$–structure. It follows from a Theorem of J L Koszul [22] that for any deformation of the holonomy morphism, the corresponding deformed radiant affine manifold is still the quotient of some convex open cone in $\mathbf{R}^3$. It provides the invariant strictly convex subset in $\mathbf{R}P^2$. We won't discuss here why the $\bar{\Gamma}$–action is still conjugate to the lorentzian action.

(3) *The case of hyperbolic actions:* we will deal with this case by considering *flag manifolds.*

A flag manifold is a closed 3–manifold equipped with a (G,X)–structure where the model space X is the flag variety, ie, the set of pairs (x,d), where x is a point of the projective plane, and d is an oriented projective line through x. The group G to be considered is the group $PGL(3,\mathbf{R})$ of projective transformations. A typical example of such a structure is given by the projectivisation of the tangent bundle of a 2–dimensional real projective orbifold. This family is fairly well-understood, thanks to the classification of compact real projective surfaces (see [11, 12, 13, 14]). Anyway, the flag manifolds we will consider here are of different nature.

The prototypes of the flag manifolds we will consider here are obtained in the following way: consider the GL_0–invariant copy of the affine plane $\mathbf{R}^2$ in $\mathbf{R}P^2$, and let 0 be the fixed point of GL_0 in $\mathbf{R}^2$. Let X_0 be the open subset of X formed by the pairs (x,d), where x belongs to $\mathbf{R}^2\setminus\{0\}$, and d is a projective line containing x but not 0. Then, the subgroup $SL(2,\mathbf{R})\subset Af_0^*\subset PGL(3,\mathbf{R})$ acts simply transitively on X_0. Therefore, if $\rho_0\colon\bar{\Gamma}\to SL(2,\mathbf{R})$ is a faithful morphism with discrete and cocompact image, the $\bar{\Gamma}$–action on X_0 through ρ_0 is free and properly discontinuous. The quotient of this action is a flag manifold, homeomorphic to the unitary tangent bundle of a surface. Actually, it follows from a Theorem of F Salein that horocyclic actions on X_0 are free and properly discontinuous too (Corollary 3.4). We call *canonical Goldman flag manifolds* all the quotient manifolds of actions obtained in this way. In this case, the morphism ρ_0 is not strictly speaking the holonomy morphism of the flag structure, because $\bar{\Gamma}$ is not the fundamental group Γ of the flag manifold, but the quotient of it by its center. We will actually consider the morphism $\Gamma\to GL_0$ induced by ρ_0; and we will still denote it by ρ_0. Then, the definition of hyperbolic morphism has to be generalised for morphisms $\Gamma\to Af_0^*$ (cf section 2.1).

By deforming the morphism ρ_0, we obtain new flag manifolds. Small deformations still satisfy:

- the ambient flag manifold is homeomorphic to the unitary tangent bundle of a surface,

- the holonomy morphism is hyperbolic,

- the image of the developing map is contained in X_∞, the open subspace of X formed by the pairs (x,d) where x belongs to $\mathbf{R}P^2\setminus\{0\}$ and where d does not contain 0 (see section 3).

We call flag manifolds satisfying these 3 properties *Goldman flag manifolds* . The main step for the proof of Theorem A is the following theorem (section 4):

Theorem B *Let M be a Goldman flag manifold with holonomy morphism ρ. Then, M is the quotient of an open subset $X(\rho)$ of $X_\infty\subset X$ which has the following description: there is a Jordan curve $\Lambda(\rho)$ in $\mathbf{R}P^2$ which does not contain the common fixed point 0, and $X(\rho)$ is the set of pairs (x,d) where x belongs to $\mathbf{R}P^2\setminus(\Lambda(\rho)\cup\{0\})$ and d does not contain 0.*

Any flag manifold inherits two 1–dimensional foliations, that we call the *tautological foliations.* They arise from the $PGL(3,\mathbf{R})$–invariant tautological foliations on X whose leaves are the (x,d) where x and d respectively remain fixed. The tautological foliations are naturally transversely real projective. We observe only in this introduction that projectivisations of tangent bundles of real projective orbifolds can be characterized as the flag manifolds such that one of their tautological foliations has only compact leaves (this observation has no incidence in the present work).

In the case of canonical Goldman flag manifolds, the tautological foliations are transversely affine. Actually, they are the horocyclic foliations associated to the exotic Anosov flows defined in [17]. This justifies our terminology "horocyclic actions", the fact that horocyclic actions are uniquely ergodic (since horocyclic foliations of exotic Anosov flows are uniquely ergodic [7]), and the non-conjugacy between different horocyclic actions (since horocyclic foliations are rigid (cf [1])).

When the Goldman flag manifold is *pure,* ie, when it is not isomorphic to a canonical flag Goldman manifold, one of these foliations is no longer transversely affine; in fact we understand this foliation quite well, since it is topologically conjugate to an exotic horocyclic foliation (Theorem 5.1).

The situation is different for the other tautological foliation: they have been first introduced by W Goldman, which defined them as the flows obtained by deformation of horocyclic foliations amongst transversely affine foliations on a given Seifert manifold M (the two definitions coincide, see Proposition 4.1 and the following discussion). For this reason, we call these foliations *Goldman foliations,* and we extend this terminology to the ambient flag manifold. As observed by S Matsumoto [25], nothing is known about the dynamical properties of pure Goldman foliations, even when they preserve a transverse parallel volume form. As a consequence of this work, we can prove (section 5.2):

Theorem C *Goldman foliations are not minimal.*

Hence, the dynamical properties of Goldman foliations are drastically different from the dynamical properties of horocyclic foliations.

Finally, many questions on the subject are still open. The presentation of these problems is the conent of the last section (Conclusion) of this paper.

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2 Preliminaries

2.1 Notation

M is an oriented closed 3–manifold. We denote by $p: \widetilde{M} \to M$ a universal covering and Γ the Galois group of this covering, ie, the fundamental group of M.

We denote by $\mathbf{R}P^2$ the usual projective plane, and $\mathbf{R}P^2_*$ its dual: $\mathbf{R}P^2_*$ is the set of projective lines in $\mathbf{R}P^2$. Let $\kappa: \mathbf{R}P^2 \to \mathbf{R}P^2_*$ be the duality map induced by the identification of $\mathbf{R}^3$ with its own dual, mapping the canonical basis of $\mathbf{R}^3$ to its canonical dual base. Since $\mathbf{R}^3$ is also the dual space of its own dual, we obtain by the same way an isomorphism $\kappa^*: \mathbf{R}P^2_* \to \mathbf{R}P^2$, which is the inverse of κ.

We denote by X the flag variety: this is the subset of $\mathbf{R}P^2 \times \mathbf{R}P^2_*$ formed by the pairs (x,d) where d is an projective line containing x. Let p_1 and p_2 be the projections of X over $\mathbf{R}P^2$ and $\mathbf{R}P^2_*$. The flag variety X is naturally identified with the projectivisation of the tangent bundle of $\mathbf{R}P^2$. Let Θ be the orientation preserving involution of X defined by $\Theta(x,d) = (\kappa^*(d), \kappa(x))$.

Let $PGL(3,\mathbf{R})$ be the group of projective automorphisms of $\mathbf{R}P^2$. The differential of the action of $PGL(3,\mathbf{R})$ on $\mathbf{R}P^2$ induces an orientation preserving action on X. Consider the Cartan involution on $GL(3,\mathbf{R})$ mapping a matrix to the inverse of its transposed matrix. It induces an involution θ of $PGL(3,\mathbf{R})$. We have the equivariance relation $\Theta \circ A = \theta(A) \circ \Theta$ for any element A of $PGL(3,\mathbf{R})$.

A flag structure on M is a $(PGL(3,\mathbf{R}), X)$–structure on M in the sense of [28]. We denote by $\mathcal{D}: \widetilde{M} \to X$ its developing map, and by $\rho: \Gamma \to PGL(3,\mathbf{R})$ its holonomy morphism. The compositions of $\mathcal{D}$ and ρ by Θ and θ define another flag structure on M: the *dual flag structure.* In general, a flag structure is not isomorphic to its dual.

On X, we have two natural one dimensional foliations by circles: the foliations whose leaves are the fibers of p_1 and p_2. We call them respectively the *first* and the *second tautological foliation.* They are both preserved by the action of $PGL(3,\mathbf{R})$. Therefore, they induce on each manifold equipped with a flag structure two foliations that we still call the first and second tautological foliations. The first (respectively second) tautological foliation is the second (respectively first) tautological foliation of the dual flag structure. Observe that these foliations are transversely real projective. Observe also that they are nowhere collinear, and that the plane field that contains both is a contact plane field.

Consider the usual embedding of the affine plane $\mathbf{R}^2$ in $P^2\mathbf{R}$. We denote by 0 the origin of $\mathbf{R}^2$. The boundary of $\mathbf{R}^2$ in $\mathbf{R}P^2$ is the projective line $\kappa(0)$, the line at infinity. We denote it by d_∞. It is naturally identified with the set $\mathbf{R}P^1$ of lines in $\mathbf{R}^2$ through 0. We identify thus the group of transformations of the plane with the group of projective transformations preserving the line d_∞. Let Af_0 be the group of orientation preserving affine transformations. The elements of Af_0 are the projections in $PGL(3,\mathbf{R})$ of matrices of the form:

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & u \\ & & v \\ 0 & 0 & 1 \end{pmatrix}$$

where A belongs GL_0, the group of 2×2 matrix with positive determinant. The group GL_0 is the stabilizer in Af_0 of the point 0. We denote by SL the subgroup formed by the elements of GL_0 of determinant 1 (as a group of linear transformation of the plane; equivalently, SL is the derived subgroup of GL_0), by $p_0\colon \widetilde{SL}\to SL$ the universal covering map, and by PSL the quotient of SL by its center $\{\pm Id\}$.

Let Γ be a cocompact lattice of $\widetilde{SL}$. Let H be the center of Γ. We select a generator h of $H\approx\mathbf{Z}$. Let $\bar\Gamma$ be the quotient of Γ by H. We denote by $\rho_0\colon \Gamma\to\bar\Gamma\subset SL\subset Af_0$ the quotient map: this is the restriction of p_0 to Γ.

Let $\mathcal{R}(\Gamma)$ be the space of representations of Γ into Af_0. It has a natural structure of an algebraic variety.

Let $Rep(\Gamma,PSL)$ be the space of morphisms of Γ into PSL. The elements of $Rep(\Gamma,PSL)$ vanishing on H form a subspace that we denote by $Rep(\bar\Gamma,PSL)$. As suggested by the notation, $Rep(\bar\Gamma,PSL)$ can be identified with the space of representations of $\bar\Gamma$ into PSL.

By taking the linear part of $\rho(\gamma)$, and then projecting in $PGL_0\approx PSL$, we define an open map $\lambda\colon \mathcal{R}(\Gamma)\to Rep(\Gamma,PSL)$. We call $\lambda(\rho)$ the *projectivised linear part of* ρ.

An element ρ of $\mathcal{R}(\Gamma)$ is *hyperbolic* if it satisfies the following conditions:

- the kernel of $\lambda(\rho)$ is H,

- for every element γ of Γ which has no non-trivial power belonging to H, $\rho(\gamma)$ has two real eigenvalues, one of absolute value strictly greater than 1, and the other of absolute value strictly less than 1. In other words, $\rho(\gamma)$ has a fixed point of saddle type.

Observe that this definition is dual to the definition given in the introduction. A typical example of hyperbolic representations is ρ_0. We denote by $\mathcal{R}_h(\Gamma)$ the set of elements of $\mathcal{R}(\Gamma)$ which are hyperbolic.

Let $\mathcal{T}(\bar{\Gamma})$ be the space of *cocompact fuchsian representations of* $\bar{\Gamma}$ *into* PSL, ie, injective representations with a discrete and cocompact image in PSL. It is well-known that it is a connected component of the space $Rep(\bar{\Gamma}, PSL)$ of all representations $\bar{\Gamma} \to PSL$.

Lemma 2.1 *$\mathcal{R}_h(\Gamma)$ is an open subset of $\mathcal{R}(\Gamma)$. Its image by λ is $\mathcal{T}(\bar{\Gamma})$.*

Proof Let $Rep_0(\Gamma, PSL)$ be the subspace of $Rep(\Gamma, PSL)$ formed by the morphisms ρ with non-abelian image. This is an open subspace. For any element ρ of $Rep(\Gamma, PSL)$, the image of ρ is contained in the centralizer of $\rho(h)$. But the centralizers of non-trivial elements of PSL are all abelian, thus $Rep_0(\Gamma, PSL)$ is an open subset of $Rep(\bar{\Gamma}, PSL)$. Moreover, $Rep_0(\Gamma, PSL)$ obviously contains $\mathcal{T}(\bar{\Gamma})$.

Take any element ρ of $\mathcal{R}_h(\Gamma)$. Since $\bar{\Gamma}$ is not abelian, and since the kernel of ρ is contained in H, $\lambda(\rho)$ belongs to $Rep(\bar{\Gamma}, PSL)$. Moreover, $\lambda(\rho)\colon \bar{\Gamma} \to PSL$ is injective. Let N_0 be the identity component of the closure of $\lambda(\rho)(\bar{\Gamma})$ in PSL. Then, $\lambda(\rho)^{-1}(N_0 \cap \lambda(\rho)(\bar{\Gamma}))$ is a normal subgroup of $\bar{\Gamma}$. Hence, either it is contained in the center H, or it is not solvable. In the second case, N_0 is not solvable too: it must contain elliptic elements with arbitrarly small rotation angle. But $\rho(\Gamma)$ contains then many elliptic elements with rotation angles arbitrarly small: this is a contradiction since ρ is hyperbolic.

Therefore, $\lambda(\rho)^{-1}(N_0 \cap \rho(\bar{\Gamma}))$ is trivial, ie, $\rho(\Gamma)$ is discrete. Since $\lambda(\rho)(\bar{\Gamma})$ is isomorphic to $\bar{\Gamma}$, its cohomological dimension is two. Hence, it is a cocompact subgroup of PSL, and $\lambda(\mathcal{R}_h(\bar{\Gamma}))$ is contained in $\mathcal{T}(\bar{\Gamma})$. The lemma follows. □

Remark 2.2 Lemma 2.1 enables us to give a method for defining all hyperbolic morphisms: take any cocompact fuchsian group $\bar{\Gamma}$ in PSL, and let $\tilde{\Gamma}$ be the preimage by p_0 of $\bar{\Gamma}$. Take any finite index subgroup Γ of $\tilde{\Gamma}$. Denote by ρ_0 the restriction of p_0 to Γ. Take now any morphism u from Γ into the multiplicative group $\mathbf{R} \setminus 0$. We can now define a new morphism $\rho_u\colon \Gamma \to GL_0$ just by requiring $\rho_u(\gamma) = u(\gamma)\rho_0(\gamma)$. Actually, all the ρ_u are nothing but the elements of the fiber of λ containing ρ_0. The absolute value of the morphism u is a morphism $|u|\colon \Gamma \to \mathbf{R}^+$. Since h admits a non-trivial power belonging to the commutator subgroup $[\Gamma, \Gamma]$, $|u|$ is trivial on H; therefore, it induces a morphism $\bar{u}\colon \bar{\Gamma} \to \mathbf{R}^+$.

Now, the following claim is easy to check: *the morphism ρ_u is hyperbolic if and only if for any non-elliptic element γ of $\bar{\Gamma}$, the absolute value $\bar{u}(\gamma)$ belongs to $]r(\gamma)^{-1}, r(\gamma)[$, where $r(\gamma)$ is the spectral radius of γ.*

This condition can be expressed in a more elegant way: the logarithm of $\bar{u}$ is a morphism $L_u\colon \bar{\Gamma} \to \mathbf{R}$, ie, an element of $H^1(\bar{\Sigma}, \mathbf{R})$. On this cohomology space, we have the *stable norm* (cf [2]) which is defined as follows: for any hyperbolic element γ of $\bar{\Gamma}$, let $t(\gamma)$ be the double of the logarithm of $r(\gamma)$ (this is the length of the closed geodesic associated to $\bar{\Gamma}$ in the quotient of the Poincaré disc by $\bar{\Gamma}$). For any element $\hat{\gamma}$ of $H_1(\bar{\Sigma}, \mathbf{Z})$, and for any positive integer n, let $t_n(\hat{\gamma})$ the infimum of the $\frac{t(\gamma)}{n}$ where γ describes all the elements of Γ representing $n\hat{\gamma}$. The limit of $t_n(\hat{\gamma})$ exists, it is the *stable norm of $\hat{\gamma}$ in $H_1(\bar{\Sigma}, \mathbf{Z})$*. This norm is extended in an unique way on all $H_1(\bar{\Gamma}, \mathbf{R})$; the dual of it is the *stable norm of $H^1(\bar{\Sigma}, \mathbf{R})$*. The proof of the following claim is left to the reader: *the representation ρ_u is hyperbolic if and only if the stable norm of L_u is strictly less than $\frac{1}{2}$.*

Remark 2.3 According to Selberg's Theorem, asserting that any finitely generated linear group admits a finite index subgroup without torsion, for any hyperbolic representation $\rho\colon \Gamma \to Af_0$, there exists a finite index subgroup Γ' of Γ on which ρ restricts as a hyperbolic representation. This hyperbolic representation has the following properties:

- its kernel is precisely the center of Γ',
- every non-trivial element of $\rho(\Gamma')$ is hyperbolic.

Let Af_0^* be the dual $\theta(Af_0)$ of Af_0. Since Af_0 preserves the line at infinity d_∞, the group Af_0^* fixes the point 0 in $\mathbf{R}P^2$. It preserves also the open set X_∞ whose elements are the pairs (x, d), where x is a point of $\mathbf{R}P^2 \setminus 0$, and d a line containing x but not 0. Observe that the fundamental group of X_∞ is infinite cyclic. The group GL_0 (which is equal to its dual $\theta(GL_0)$) preserves the subset $X_0 \subset X_\infty$ where (x, d) belongs to X_0 if and only if x belongs to $\mathbf{R}^2 \setminus \{0\}$, and d does not contain 0. Actually, the action of SL on X_0 is simply transitive. A representation $\rho\colon \Gamma \to Af_0^*$ is said to be hyperbolic if it is the dual representation of an element of $\mathcal{R}_h(\Gamma)$. Equivalently, it means that the point 0 is a fixed point of saddle type of every $\rho(\gamma)$, when γ is of of infinite order. Such a representation is given by a morphism $\rho_1\colon \Gamma \to GL_0$ and two cocycles u and v such that $\rho(\gamma)$ is the projection in $PGL(3, \mathbf{R})$ of:

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\rho_1(\gamma)$}} & 0 \\ & & 0 \\ u(\gamma) & v(\gamma) & 1 \end{pmatrix}$$

The morphism ρ_1 is the linear part of ρ. It is a *horocyclic morphism.*

An Af_0–foliation is a foliation admitting a transverse $(Af_0, \mathbf{R}^2)$–structure.

2.2 Convex and non-convex sets

Here, we collect some elementary facts on affine manifolds.

Definition 2.4 Let X be a flat affine manifold. An open subset U of X is convex if any pair (x, y) of points of U are extremities of some linear path contained in U. The exponential $\mathcal{E}_x$ of a point x of X is the open subset of X formed by the points which are extremities of linear paths starting from x.

The following lemmas are well-known. A good reference is [10].

Lemma 2.5 *The developing map of a flat convex simply connected affine manifold is a homeomorphism onto its image.* $\square$

Lemma 2.6 *Let X be a connected flat affine manifold. If the exponential of every point of X is convex, then X is convex.* $\square$

Lemma 2.7 *Let X be a flat affine simply connected manifold. Let U and V be two convex subsets of X. If $U \cap V$ is not empty, the restriction of the developing map $\mathcal{D}$ to $U \cup V$ is a homeomorphism over $\mathcal{D}(U) \cup \mathcal{D}(V)$.* $\square$

Lemma 2.8 *Let U be an open star-shaped neighborhood of a point x in the plane. If U is not convex, then it contains two points y and z such that:*

- *x, y and z are not collinear,*
- *the closed triangle with vertices x, y and z is not contained in U,*
- *the open triangle with vertices x, y and z, and the sides $[x, y]$, $[x, z]$, are contained in U.*

Proof Let y' and z be two points of U such that the segment $[y', z]$ is not contained in U. Observe that x, y' and z are not collinear. Let T_0 be the closed triangle of vertices x, y' and z. For any real t in the interval $[0, 1]$, let y_t be the point $ty' + (1-t)x$. Let I be the set of parameters t for which the segment $[y_t, z]$ is contained in U. It is open, non-empty since 0 belongs to it, and does not contain 1. Let t be a boundary point of I: the points y_t and z have the properties required by the lemma. $\square$

3 Existence of flag structures

Let M be a *principal Seifert manifold,* ie, the left quotient of $\widetilde{SL}$ by a cocompact lattice Γ. Let $\bar{\Gamma}$ be the projection of $p_0(\Gamma)$ in PSL. Topologically, M is a Seifert bundle over the hyperbolic orbifold $\bar{\Sigma}$, quotient of the Poincaré disc by $\bar{\Gamma}$.

Choose any element v of X_0. Consider the map $\widetilde{SL} \to X_0 \subset X$ that maps g to $p_0(g)(v)$, and the morphism $\rho_0 \colon \Gamma \to PGL(3, \mathbf{R})$, which is the composition of p_0 with the inclusion $SL \subset Af_0^* \subset PGL(3, \mathbf{R})$. They are the developing map and holonomy morphism of some flag structure on M. Observe that this structure does not depend on the choice of v. We call the flag structures obtained in this way the *unimodular canonical flag structures.*

We are concerned here with the deformations of unimodular canonical flag structures. Let $t \mapsto \rho_t$ be a deformation of ρ_0 inside $PGL(3, \mathbf{R})$, where the parameter t belongs to $[0,1]$. As we recalled in the introduction, for small t, the morphisms ρ_t is the holonomy morphism of some new flag structure. Moreover, these deformed flag structures near the canonical one are well-defined up to isotopy by their holonomy morphisms. We are interested by the deformations of ρ_0 inside Af_0^*, ie, where all the ρ_t are morphisms from Γ into Af_0^*. Then, according to Lemma 2.1, for small t, ρ_t is a hyperbolic representation.

Denote by $\mathcal{D}_t$ the developing maps of the flag structures realizing the holonomy morphisms ρ_t. They vary continuously in the compact open topology of maps $\widetilde{SL} \to X$. Let K be a compact fundamental domain of the action of Γ on $\widetilde{SL}$. For small t, $\mathcal{D}_t$ is near $\mathcal{D}_0$ in the compact–open topology, and since $\mathcal{D}_0(K)$ is a compact subset of X_0, $\mathcal{D}_t(K)$ is still a compact subset of X_0. But the whole image of $\mathcal{D}_t$ is the $\rho_t(\Gamma)$–saturated of $\mathcal{D}_t(K)$, therefore, it is contained in X_∞.

All the discussion above shows that the deformed flag structures we considered are Goldman flag structures in the following meaning:

Definition 3.1 A *Goldman flag structure* is a flag structure on a principal Seifert manifold such that:

- its holonomy morphism is a hyperbolic representation into Af_0^*,
- the image of its developing map is contained in the open subset X_∞.

A Goldman flag structure is *pure* if its holonomy group does not fix a projective line.

The arguments above show that Goldman flag structures form an open subset of the space of flag structures on M with holonomy group contained in Af_0^*.

We are now concerned with the problem of the existence of Goldman flag structure on the manifold M which are not unimodular canonical flag structures. The case of non pure Goldman flag manifolds follows from a result of F Salein in the following way: consider any morphism u from the cocompact fuchsian group $\bar{\Gamma}$ into $\mathbf{R}^+$, and consider the new subgroup $\bar{\Gamma}_u$ of GL_0 obtained by replacing γ by the multiplication of γ by the homothety of factor $\bar{u}(\gamma)$. The logarithm of the absolute value of $\bar{u}$ induces a morphism $L_u\colon \bar{\Gamma} \to \mathbf{R}$. Then:

Theorem 3.2 [29] *The action of $\bar{\Gamma}_u$ on X_0 is free and proper if and only the stable norm of L_u of u is less than $\frac{1}{2}$.* □

Remark 3.3 Actually, this Theorem is not stated in this form in [29]: F Salein considered the following action of $\bar{\Gamma}$ on PSL: every element γ maps an element g of PSL on $\gamma g \Delta(\gamma)$, where $\Delta(\gamma)$ is the diagonal matrix with diagonal coefficients $e^{L_u(\gamma)}$, $e^{-L_u(\gamma)}$. Then he proved that this action is free and proper if and only if the stable norm of $2L_u$ is less than 1 (Théorème 3.4 of [29]). But the action that we consider here is a double covering of the action considered by F Salein: indeed, using the fact that SL acts freely and transitively on X_0, we identify X_0 with SL, and then project on PSL. This double covering is an equivariant map.

Corollary 3.4 *For any hyperbolic representation $\rho\colon \Gamma \to GL_0$, the action of $\rho(\Gamma)$ on X_0 is free and proper.*

Proof This is a corollary of Theorem 3.2 and of Remark 2.2. Proposition 4.19 will give another proof of this fact. □

The quotients of X_0 by hyperbolic subgroups of GL_0 are called *canonical Goldman flag manifolds.*

Proposition 3.5 *Every hyperbolic representation is the holonomy representation of some Goldman flag structure, which is a small deformation of a canonical flag structure.*

Proof Let $\rho\colon \Gamma \to Af_0^*$ be a hyperbolic morphism. If $\rho(\Gamma)$ is contained in GL_0, the proposition follows from the Corollary 3.4: the quotient of X_0 by $\rho(\Gamma)$ is a non-pure Goldman flag manifold.

Consider now the case where $\rho(\Gamma)$ is not contained in GL_0. Conjugating ρ by a homothety of factor s amounts to multiplying the translational part of ρ^* by s. Therefore, if s is small enough, the conjugate of ρ is close to its linear part (the conjugacy does not affect this linear part). Therefore, this conjugate is the holonomy of some deformation of a canonical flag structure, ie, a Goldman flag structure. Now, conjugating back by the homothety of factor s^{-1} corresponds to multiplying the developing map of this flag structure by s^{-1}. □

Remark 3.6 A corollary of Theorem B will be that the holonomy morphism characterizes the Goldman flag structures, ie, two Goldman flag structures whose holonomy morphisms are conjugate in Af_0^* are isomorphic. As a corollary, using Proposition 3.5, Goldman flag structures are all deformations of canonical flag structures.

Definition 3.7 A Goldman foliation is the second tautological foliation of a Goldman flag structure.

Proposition 3.8 *Goldman foliations are Af_0–foliation.*

Proof As we observed previously, the second tautological foliation of a flag manifold is transversely projective. The holonomy morphism of this projective structure is the holonomy morphism of the flag structure, and its developing map is the composition of the developing map of the flag structure with the projection p_2 of X onto $\mathbf{R}P^2$. For flag manifolds, the dual holonomy group is by definition in Af_0, and the image of the developing map is contained in X_∞. The proposition follows since $p_2(X_\infty)$ is the affine plane $\mathbf{R}^2$. □

Remark 3.9 Obvious examples of non-pure Goldman flag manifolds are the canonical ones. They are actually the only ones. When the holonomy group is contained in SL, this follows from the proposition 3.8 and from the classification of SL–foliations by S Matsumoto [25]. Theorem B provides the proof in all the cases.

Remark 3.10 In the case of unimodular canonical flag structures, the Goldman foliation is induced by the right action on M, the left quotient of $X_0 \approx SL$ by Γ, by the unipotent subgroup:

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

In other words, it is the horocyclic foliation of the Anosov flow induced by diagonal matrices.

Similarly, Goldman foliations associated to non-unimodular canonical Goldman flag structures are horocyclic foliations associated to some Anosov flows: the *exotic Anosov flows* introduced in [17]. Exotic Anosov flows are characterized by the following property: they are, with the suspensions of linear hyperbolic automorphisms of the torus, the only Anosov flows on closed 3–manifolds admitting a smooth splitting. For this reason, we call these GL_0–foliations *exotic horocyclic foliations*.

We discuss now the problem of deformation of canonical flag structures: what are the canonical flag structure which can be deformed to pure Goldman flag structures? According to the remark 3.6, this question amounts to identifying Seifert manifolds admitting pure Goldman structures.

The generator h of the center of Γ is mapped by ρ_0 on the identity matrix Id, or its opposite $-Id$. In the first case, Γ is said *adapted,* in the second one, Γ is *forbidden.* For example, the fundamental group of the unit tangent bundle M_0 of $\bar{\Sigma}$ is of the forbidden type. Γ is adapted if and only if the finite covering $M \to M_0$ is of even index.

Then, Γ admits a presentation, with $2g+r+1$ generators a_i, b_i $(i=1...g)$, q_j $(j=1...r)$ and h, satisfying the relations

$$[a_1,b_1]...[a_g,b_g]q_1...q_r = h^e, q_j^{\alpha_j} = h^{\beta_j}, [h,a_i] = [h,b_i] = [h,q_j] = 1$$

Proposition 3.11 *The canonical flag structure associated to Γ can be deformed to a pure Goldman flag structure if and only if Γ is adapted.*

As a corollary, the canonical flag structure on the unit tangent bundle of a hyperbolic orbifold cannot be deformed to a pure Goldman flag structure. But its double covering along the fibers can be deformed non-trivially.

Proof of 3.11 We need to understand when the morphism ρ_0 can be deformed in Af_0^* to morphisms which do not preserve a projective line. Dually, this is equivalent to seeing when there are morphisms $\rho\colon \Gamma \to Af_0$ without common fixed point.

We first deal with the forbidden case: in this case, the center of the holonomy group $\rho_0(\Gamma)$ is not trivial: it contains $-Id$. For any perturbation ρ, $\rho(h)$ remains an order two element of Af_0, ie, conjugate to $-Id$. Since $\rho(h)$ commutes with every element of $\rho(\Gamma)$, its unique fixed point is preserved by all $\rho(\Gamma)$. Hence, the flag structure is not pure.

Consider now the adapted case: then, $\rho_0(h) = Id$. We have to find $2g$ values in Af_0 for the $\rho(a_i)$'s and the $\rho(b_i)$'s, r values for the $\rho(q_j)$'s such that $\rho(q_j)^{\alpha_j} = Id$, and satisfying the relation $(*)$ below:

$$[\rho(a_1), \rho(b_1)]...[\rho(a_g), \rho(b_g)]\rho(q_1)...\rho(q_r) = Id \qquad (*)$$

We realize this by adding small translation parts to the $\rho_0(a_i)$, $\rho(b_i)$ and $\rho(q_j)$, ie, we try to find ρ with the same linear part than ρ_0. Adding a translationnal part to $\rho(q_j)$ does not affect the property of being of order α_j (here α_j is bigger than 2!) and equation $(*)$ depends linearly on the added translational parts (the linear part ρ_0 being fixed). The number of indeterminates is $2(2g+r)$, therefore, the space of solutions is of dimension at least $4g+2r-2$. Amongst them, the radiants ones—ie, fixing a point of the plane—are the conjugates of ρ_0 by affine conjugacies whose linear parts commute with ρ_0, ie, by compositions of homotheties and translations. The space of radiant solutions is thus of dimension 3. Therefore, the dimension of the space of Goldman deformations is at least $4g+2r-5$. But the inequality $4g+2r > 5$ is always true for hyperbolic orbifolds. □

4 Description of Goldman flag manifolds

Proposition 4.1 *Let Φ be an Af_0–foliation on a closed 3–manifold M. Assume that Φ is transverse to a transversely projective foliation $\mathcal{F}$ on M of codimension one. Assume moreover that the dual of the transverse holonomy of $\mathcal{F}$ coincides with the projectivised linear part of the transverse holonomy of Φ. Then, Φ is the second tautological foliation of some flag structure on M.*

Proof Let $\xi\colon \widetilde{M} \to \mathbf{R}^2$ be the developing map of the transverse structure of Φ, and $\tau\colon \widetilde{M} \to \mathbf{R}P^1$ be the developing map for the transverse structure of $\mathcal{F}$. Let $\rho\colon \Gamma \to Af_0$ be the holonomy morphism of the transverse structure of Φ. By hypothesis, the holonomy morphism associated to $\mathcal{F}$ is the dual ρ_0^*, where ρ_0 is the linear part of ρ. Define $\mathcal{D}\colon \widetilde{M} \to X_\infty$ as follows: for any element m of $\widetilde{M}$, $\mathcal{D}(m)$ is the pair (x, d), where d is equal to $\xi(m) \in \mathbf{R}^2 \subset \mathbf{R}P^2_*$, and where x is the point of $\mathbf{R}P^2$ corresponding to the line in $\mathbf{R}P^2$ containing $\xi(m)$ and parallel to the direction $\tau(m)$. Since Φ and $\mathcal{F}$ are transverse, $\mathcal{D}$ is a local homeomorphism. It is clearly equivariant with respect to the actions on $\widetilde{M}$ and X of Γ. It is the developing map of the required flag structure. □

As we will see below (section 4.1), Goldman manifolds are typical illustrations of this proposition: the Af_0–foliations associated to a Goldman manifold are transverse to a foliation satisfying the hypothesis of Proposition 4.1. Any small Af_0–deformation of the Af_0–foliation (amongst the category of Af_0–foliations), the linear part of the holonomy being preserved, still remains transverse to the foliation. Therefore, Proposition 4.1 applied in this context proves the equivalence of the definition of Goldman foliations we have given here with the definition introduced by Goldman, defining them as the affine perturbations of horocyclic foliations. Actually, the existence of the transverse foliation is the key ingredient which allows us to study Goldman manifolds.

Let M be a Goldman flag manifold. As usual, let Γ be the fundamental group of M, let $\mathcal{D}$ be the developing map of the flag structure, and let $\rho\colon \Gamma \to Af_0^*$ be the holonomy morphism, which is assumed to be hyperbolic. In order to prove Theorem B, we can replace M by any finite covering of itself, ie, replace Γ by any finite index subgroup of itself. In particular, thanks to Remark 2.3, we can assume that the kernel of ρ is H, and that $\rho(\Gamma)$ has no element of finite order.

Let ρ_0 be the projectivised linear part of ρ. The morphisms ρ and ρ_0 induce morphisms on the surface group $\bar{\Gamma}$, the quotient of Γ by H. We will sometimes denote these induced morphisms abusively by ρ and ρ_0. Let $\Omega \subset X_\infty$ be the image of $\mathcal{D}$.

Let Φ be the Goldman foliation: it is an Af_0–foliation, its holonomy morphism being ρ, and its developing map being $\mathcal{D}_2 = p_2 \circ \mathcal{D}$. Let $\widetilde{\Phi}$ be the lifting of Φ to the universal covering $\widetilde{M}$ of M.

4.1 The affine foliation

On X_∞, we can define the following codimension one foliation $\mathcal{F}_0$: two points (x,d) and (x',d') of X_∞ are on the same leaf if and only if there is a line containing 0, x and x'. The space of leaves of $\mathcal{F}_0$ is $\mathbf{R}P^1$. Moreover, every leaf of $\mathcal{F}_0$ is naturally equipped with an affine structure and for this structure, the leaf is isomorphic to the plane through the projection p_2. The foliation $\mathcal{F}_0$ is Af_0^*–invariant; therefore it induces a regular foliation $\mathcal{F}$ on M. Up to finite coverings, $\mathcal{F}$ is orientable and transversely orientable. It is a transversely projective foliation: there is a developing map $\tau\colon \widetilde{M} \to \mathbf{R}P^1$ and a holonomy morphism $\rho_0\colon \Gamma \to PSL$. Observe that, as our notation suggests, ρ_0 is the projectivised linear part of ρ. The developing map τ is the map associating to x the leaf of $\mathcal{F}_0$ containing $\mathcal{D}(x)$.

Let $\widetilde{\mathcal{F}}$ denote the lifting of $\mathcal{F}$ to $\widetilde{M}$. Let $\mathcal{Q}$ be the leaf-space of $\widetilde{\mathcal{F}}$: the fundamental group Γ acts on it.

Observe that every leaf F of the foliation $\mathcal{F}$ has a natural affine structure, whose developing map is the restriction of $\mathcal{D}_2$ to any leaf of $\widetilde{\mathcal{F}}$ above F.

Lemma 4.2 *The foliation $\mathcal{F}$ is taut, ie, $\mathcal{F}$ admits no Reeb component.*

Proof Assume that $\mathcal{F}$ admits a Reeb component. Let F be the boundary torus of this Reeb component: the inclusion of $\pi_1(F)$ in Γ is non-injective. Thus, the natural affine structure of F has a non-injective holonomy morphism, and every element of infinite order of the holonomy group is hyperbolic. This is in contradiction with the classification of affine structures on the torus [26]. □

It follows from a Theorem of W Thurston [31] that $\mathcal{F}$ is a suspension. In particular, the leaf space $\mathcal{Q}$ is homeomorphic to the real line, and the developing map τ induces a cyclic covering $\mathcal{Q} \to \mathbf{R}P^1$. The natural action of Γ on the leaf space $\mathcal{Q}$ is conjugate to a lifting of the action of the cocompact fuchsian group $\rho_0(\Gamma) \subset PSL$ on $\mathbf{R}P^1$. It follows that the Γ–stabilizer of a point in $\mathcal{Q}$ is trivial or cyclic. Moreover, the Γ–orbits in $\mathcal{Q}$ are dense. In terms of $\mathcal{F}$: every leaf of $\mathcal{F}$ is a plane or a cylinder, and is dense in M.

Let K be a compact fundamental domain for the action of Γ on $\widetilde{M}$. Let g be any Γ–invariant metric on $\widetilde{M}$. We fix a flat euclidian metric dy^2 on $\mathbf{R}^2$. This is equivalent to selecting an ellipse field $y \mapsto \bar{E}(y)$ on the plane preserved by translations.

If the ellipses are chosen sufficiently small, the following fact is true: for any element x of K, there is an unique open subset $E(x)$ of the leaf through x such that:

- it contains x,
- the restriction of $\mathcal{D}_2 = p_2 \circ \mathcal{D}$ to $E(x)$ is injective,
- the image of $E(x)$ by $\mathcal{D}_2$ is $\bar{E}(\mathcal{D}_2(x))$,
- the g–diameter of $E(x)$ is less than 1.

Since the dual morphism ρ^* is hyperbolic, there exist a real positive ϵ such that the following fact is true: *for any element γ of Γ and for any element y of $\mathbf{R}^2$, the iterate $\rho^*(\gamma)y$ is the middle point of an affine segment $\bar{\sigma}(\gamma y)$ of length 2ϵ which is contained in the ellipse $\rho^*(\gamma)\bar{E}(y)$* (all these metrics properties are relative to the fixed euclidean metric dy^2).

Lemma 4.3 *Every leaf of $\mathcal{F}$, equipped with its affine structure, is convex.*

Proof Let $\widetilde{F}$ be a leaf of $\widetilde{\mathcal{F}}$. According to Lemma 2.6, if $\widetilde{F}$ is not convex, there is an element x of $\widetilde{F}$ for which the exponential $\mathcal{E}_x$ is not convex. Let U be the image of $\mathcal{E}_x$ by $\mathcal{D}_2$: the restriction of $\mathcal{D}_2$ to $\mathcal{E}_x$ is an affine homeomorphism over U. Hence, U is not convex. According to 2.8, there are two points y and z in $\mathcal{E}_x$, and a closed subset k of the segment $]\mathcal{D}_2(y), \mathcal{D}_2(z)[$ such that the closed triangle T with vertices $\mathcal{D}_2(x)$, $\mathcal{D}_2(y)$ and $\mathcal{D}_2(z)$ is contained in U, except at k. Modifying the choice of x and restricting to a smaller triangle if necessary, we can assume that the dy^2–diameter of T is as small as we want. In particular, we can assume that for every point y' sufficiently near to k, any segment centered at y' and of length 2ϵ must intersect $[\mathcal{D}_2(x), \mathcal{D}_2(y)] \cup [\mathcal{D}_2(x), \mathcal{D}_2(z)]]$.

Let V be the subset of $\mathcal{E}_x$ that is mapped by $\mathcal{D}_2$ to $T \setminus k$, and let v be the compact subset of V that is mapped onto $[\mathcal{D}_2(x), \mathcal{D}_2(y)] \cup [\mathcal{D}_2(x), \mathcal{D}_2(z)]]Lebesgue$. Let τ be a segment in V such that $\mathcal{D}_2(\tau)$ is a segment $[\mathcal{D}_2(x), t[$, where t belongs to k. Let t_n be a sequence of points in τ such that $\mathcal{D}_2(t_n)$ converge to t. For every index n, there exists an element γ_n of Γ and an element x_n of K such that $t_n = \gamma_n x_n$.

We claim that the sequence t_n escapes from any compact subset of $\widetilde{F}$. Indeed, if this is not true, extracting a subsequence if necessary, we can assume that t_n converges to some point $\bar{t}$ of $\widetilde{F}$. Clearly, $\mathcal{D}_2(\bar{t})$ is equal to t. Let W be a convex neighborhood of $\bar{t}$ in $\widetilde{F}$ such that the restriction of $\mathcal{D}_2$ to it is injective. According to Lemma 2.7, the restriction of $\mathcal{D}_2$ to $V \cup W$ is a homeomorphism to $T \cup \mathcal{D}_2(W)$. It follows that the path τ can be completed as a closed path joining x to $\bar{t}$. Hence, $\bar{t}$ belongs to $\mathcal{E}_x$, ie, t belongs to U. Contradiction.

Therefore the t_n go to infinity. Their g–distances in $\widetilde{F}$ to the compact set v tend to infinity. When n is sufficiently big, this distance is bigger than 1. Therefore, none of the ellipses $E_n = \gamma_n E(x_n)$ intersects v, since their g–diameter are less than 1. On the other hand, since $\mathcal{D}_2(E_n) = \rho^*(\gamma_n)\bar{E}(\mathcal{D}_2(x_n))$ contains the segment $\bar{\sigma}(t_n)$ of length 2ϵ, the ellipse $\mathcal{D}_2(E_n)$ intersects $[\mathcal{D}_2(x), \mathcal{D}_2(y)] \cup [\mathcal{D}_2(x), \mathcal{D}_2(z)]]Lebesgue$. According to the lemma 2.7, it follows that E_n intersects v. Contradiction. □

In the following lemma, we call any open subset of the affine plane bounded by two parallel lines a *strip*.

Lemma 4.4 *The leaves of $\widetilde{\mathcal{F}}$ are affinely isomorphic to the affine plane, or to an affine half plane, or to a strip.*

Proof Let $\widetilde{F}$ be the universal covering of a leaf of $\mathcal{L}$. According to Lemmas 4.3 and 2.5, the restriction of $\mathcal{D}_2$ to $\widetilde{F}$ is a homeomorphism onto a convex subset U of the plane. In order to prove the proposition, we just have to see that the boundary components of the convex U are lines. Assume that this is not the case. Then there is a closed half plane P such that the intersection of P with the closure of U is a compact convex set K whose boundary is the union of a segment $]x,y[$ contained in U and a convex curve c contained in ∂U. We obtain a contradiction as in the proof of the Lemma 4.3 by considering ellipses centered at points t_n of $\widetilde{F}$ such that $\mathcal{D}_2(t_n)$ converges to some point t of c: for sufficiently big n, these ellipses, containing segments whose length is bounded by below, must intersect c. This leads to a contradiction with the Lemma 2.7. □

The developing map τ induces a finite covering of the quotient of $\mathcal{Q}$ by the center of Γ over the circle $\mathbf{R}P^1$. Let n be the degree of this covering. Consider $p_n\colon X^n_\infty \to X_\infty$, the finite covering of X_∞ of degree n. Let $\widehat{M}$ be the quotient of $\widetilde{M}$ by the center of Γ. The map $\mathcal{D}$ induces a map $\widehat{\mathcal{D}}$ from $\widehat{M}$ into X^n_∞. The action of $\rho(\Gamma)$ on X_∞ lifts to an action of $\bar{\Gamma}$ on X^n_∞ for which $\widehat{\mathcal{D}}$ is equivariant. Obviously, $p_n(\Omega^n) = \Omega$.

Proposition 4.5 *The map $\widehat{\mathcal{D}}$ is a homeomorphism onto some open subset Ω^n of X^n_∞.*

Proof This follows from the injectivity of $\mathcal{D}_2$ on every leaf of $\widetilde{\mathcal{F}}$ and from the fact that τ is a cyclic covering over $\mathbf{R}P^1$. □

The content of the following sections is to identify the form of Ω^n. It is not yet clear for example that Ω^n is a cyclic covering over Ω.

4.2 Affine description of the cylindrical leaves

Let F_0 be the lifting of a cylindrical leaf of $\mathcal{F}$. The set of elements of Γ preserving F_0 is a subgroup generated by an element γ_0 of infinite order. Since $\rho(\gamma_0)$ is a hyperbolic element of $PGL(3,\mathbf{R})$, F_0 is an attracting or repelling fixed point of γ_0 in $\mathcal{Q}$. We choose γ_0 such that F_0 is a attracting fixed point of γ_0. Observe that the fixed points of γ_0 in $\mathcal{Q}$ are discrete, infinite in number, and alternatively attracting and repelling. We denote by F_1 the lowest fixed point of γ_0 greater than F_0.

Lemma 4.6 *$\mathcal{D}_2(F_0)$ is a half-plane. Its boundary $d(F_0)$ is a line preserved by $\rho^*(\gamma_0)$.*

Proof This follows directly from Lemma 4.4 and the fact that $\rho^*(\gamma_0)$ is a hyperbolic affine transformation acting freely on $\mathcal{D}_2(F_0)$. $\square$

Definition 4.7 For every leaf F of $\widetilde{\mathcal{F}}$ we define $]F_0, F[$ as the set of leaves in $\mathcal{Q}$ which separates F from G. The interval $[F_0, F]$ is the union of $]F_0, F[$ with $\{F_0, F\}$. We define Ω_F as the subset of elements of F_0, whose $\widetilde{\Phi}$–leaf intersects F.

Another equivalent definition of $[F_0, F]$ is to consider it as the set of leaves of $\widetilde{\mathcal{F}}$ meeting every path joining F_0 to F.

Lemma 4.8 *The sets Ω_F are convex open subsets of F_0.*

Proof $\Omega_F \approx \mathcal{D}_2(\Omega_F)$ is the intersection of the $\mathcal{D}_2(L)$, where L is in $[F_0, F]$. It is therefore an intersection of half-planes (maybe empty) (observe that a strip is the intersection of two half-planes, and we can omit the leaves L whose $\mathcal{D}_2$–image are the whole plane since they make no new contribution to the intersection). $\square$

Lemma 4.9 *Let F be an element of $]F_0, F_1[$. Consider the sequence of convex subsets of F_0, indexed by positive integers n, formed by the $\gamma_0^n \Omega_F$. This is an increasing sequence under inclusion. Moreover, the union of these convex subsets is the whole of F_0 and the interior of their intersection is not empty.*

Proof Let F' be a leaf such that $\Omega_{F'}$ is not empty (for example, this is true if F' is near F_0). Since $F_0 \prec \gamma_0 F' \prec F'$, then $\gamma_0 \Omega_{F'} = \Omega_{\gamma_0 F'}$ contains $\Omega_{F'}$. Since the $\gamma_0^n F'$ converge to F_0 when n tend to $+\infty$, the union of the $\gamma_0^n \Omega_{F'}$ is the whole F_0. Observe that for any F in $]F_0, F_1[$, there exists some integer n such that $\gamma_0^n F'$ is greater than F. Therefore, Ω_F is not empty since it contains $\gamma_0^n \Omega_{F'}$.

Since the action of Γ on $\mathcal{Q}$ is a lifting of the action of a cocompact fuchsian group on $\mathbf{R}P^1$, there is an element γ_1 in Γ, fixing two leaves F_0' and F_1', such that $]F_0', F_1'[$ contains no other fixed point of γ_1, but containing F_0 and F_1. What we did above for the pair (γ_0, F_0) can be applied to the pair (γ_1, F_0'): the set of $\widetilde{\Phi}$–leaves meeting both F_0' and F_1 is not empty. Since all these $\widetilde{\Phi}$–leaves meet F_0 and F_1, Ω_{F_1} is not empty. The intersection between the Ω_F contains Ω_{F_1}, Therefore, its interior is not empty. $\square$

Remember that we assumed that F_0 is an *attracting* fixed point of γ_0.

Corollary 4.10 *The boundary line of F_0 is the unstable line of $\rho^*(\gamma_0)$, ie, it is parallel to the eigenspace associated to the eigenvalue of $\rho^*(\gamma_0)$ of absolute value greater than 1.*

Proof of 4.10 Assume that the Lemma is false. Take some leaf F in $]F_0, F_1[$. According to Lemma 4.9, the $\gamma_0^n\Omega_F$ for positive n form an increasing sequence of convex sets whose union is the whole of F_0. This is possible only if the convex set Ω_F is a strip containing $d(F_0)$ in its boundary. But then the intersection of the $\gamma_0^n\Omega_F$ would be empty: this contradicts 4.9. □

Corollary 4.11 *No leaf of $\widetilde{\mathcal{F}}$ is a strip.*

Proof Assume that some leaf F is a strip. Then it admits at least two iterates γF and $\gamma' F$ in $]F_0, F_1[$. One of them, let's say γF, disconnects F_0 from the other ($\gamma' F$). We can choose these iterates such that the strips $\mathcal{D}_2(\gamma F)$ and $\mathcal{D}_2(\gamma' F)$ are not parallel. Then, the intersection of these two strips is a parallelogram. But, since γF disconnects F_0 from $\gamma' F$, this parallelogram contains $\mathcal{D}_2(\Omega_{\gamma' F})$, According to Lemma 4.9, the intersection between the positive $\rho^*(\gamma_0)$–iterates of this parallelogram must have a non-empty interior. But this is clearly impossible: for any parallelogram P of the plane, the intersection of the positive $\rho^*(\gamma_0)$–iterates of P is either empty, either a subinterval of the unstable line of $\rho^*(\gamma_0)$. □

4.3 Description of the image and the limit set

Let $U(\rho)$ be the image of $\mathcal{D}_1 = p_1 \circ \mathcal{D}$. This is a subset of $\mathbf{R}P^2 \setminus 0$. Let $\mathcal{G}_0$ be the projection in $\mathbf{R}P^2$ of $\mathcal{F}_0$: this is the foliation whose regular leaves are the $d \setminus 0$, where d is any projective line in $\mathbf{R}P^2$ containing 0.

Lemma 4.12 *$U(\rho)$ is not the whole of $\mathbf{R}P^2 \setminus 0$.*

Proof Let γ_0 be an element of Γ of infinite order admitting fixed points in $\mathcal{Q}$. Let d_0 be the attracting fixed point of $\rho(\gamma_0)$ in $\mathbf{R}P^2$, and g_0 the leaf of $\mathcal{G}_0$ containing d_0: in $\mathbf{R}P^1$, the space of leaves of $\mathcal{G}_0$, g_0 is an attracting fixed point of $\rho_0(\gamma_0)$. Since the action of γ_0 on $\mathcal{Q}$ is a lifting of the action of $\rho_0(\gamma_0)$ on $\mathbf{R}P^1$ and since this action admits a fixed point, every leaf of $\widetilde{\mathcal{F}}$ whose image is contained in g_0 is an attracting fixed point of γ_0. According to Corollary 4.10, the image by $\mathcal{D}_2$ of such a leaf is an half-plane whose boundary is the unstable line of $\rho^*(\gamma_0)$, ie d_0. It follows that d_0 is not in $U(\rho)$. □

There is a morphism $\rho_1 \colon \Gamma \to GL_0$ and two maps u and v from Γ into $\mathbf{R}$ such that the morphism $\rho \colon \Gamma \to PGL(3, \mathbf{R})$ is induced by a morphism of the form:

$$\rho(\gamma) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\rho_1(\gamma)$}} & 0 \\ & & 0 \\ u(\gamma) & v(\gamma) & 1 \end{pmatrix}$$

The open set $U(\rho)$ is a set of lines in $\mathbf{R}^3$. Their union minus the origin is $\rho(\Gamma)$–invariant open cone in $\mathbf{R}^3$. We denote this by $\mathcal{U}(\rho)$.

For any point $w = (x, y)$ in $\mathbf{R}^2 \setminus 0$, consider the set of real numbers z such that (x, y, z) belongs to $\mathcal{U}(\rho)$. We denote it by $I(w)$; it is an open subset of $\mathbf{R}$. According to Lemma 4.4 and Corollary 4.11, for every leaf F of $\widetilde{\mathcal{F}}$, the image $\mathcal{D}_2(F)$ is the whole plane or a half-plane. In both cases, d_∞ meets the boundary of $\mathcal{D}_2(F)$. It means that the point 0 is an extremity of $\mathcal{D}_1(F)$. It follows that that for every w in $\mathbf{R}^2 \setminus 0$, $I(w)$ contains an interval of the form $]-\infty, t[$, and another of the form $]t, +\infty[$. We denote by $\delta^-(w)$ (respectively $\delta^+(w)$) the supremun (respectively the infimum) of the real numbers t for which $]-\infty, t[$ (respectively $]t, +\infty[$) is contained in $I(w)$. These maps have the following properties:

- $\delta^-(w) \leq \delta^+(w)$ or $\delta^-(w) = +\infty = -\delta^+(w)$,
- $\delta^+(w) = -\delta^-(-w)$ (because $-\mathcal{U}(\rho) = \mathcal{U}(\rho)$),
- δ^- is lower semi-continuous (l.s.c.), and δ^+ is upper semi-continuous (u.s.c.) (because $\mathcal{U}(\rho)$ is open),
- δ^+ and δ^- are homogeneous of degree 1 (because $\mathcal{U}(\rho)$ is a cone).
- since $\mathcal{U}(\rho)$ is $\rho(\Gamma)$–invariant, for every element γ of Γ, they both satisfy:

$$\delta^{\pm}(\rho_1(\gamma)(x, y)) = \delta^{\pm}(x, y) + u(\gamma)x + v(\gamma)y \quad (*)$$

Proposition 4.13 *The maps δ^+ and δ^- are equal and take only finite values.*

Proof For any w in $\mathbf{R}^2 \setminus 0$, let $\Delta(w)$ be the difference $\delta^+(w) - \delta^-(w)$. The map $\Delta \colon \mathbf{R}^2 \setminus 0 \to \mathbf{R} \cup \{-\infty\}$ is u.s.c. and, according to equation $(*)$ above, Δ *is* $\rho_1(\Gamma)$*–invariant.* On the other hand, the quotient of X_0 by $\rho_1(\Gamma)$ is a canonical flag manifold $M(\rho_1)$, whose second tautological foliation is an exotic horocyclic foliations (see Remark 3.10). Therefore, Δ induces an u.s.c. function $\bar{\Delta} \colon M(\rho_1) \to \mathbf{R} \cup \{-\infty\}$ which is invariant along the leaves of the exotic horocyclic foliation. Since $M(\rho_1)$ is compact, and since $\bar{\Delta}$ is u.s.c. $\bar{\Delta}$ attains its maximal value and the locus where it attains this maximal value is closed. Since horocyclic foliations of Anosov flows are notoriously minimal (when the flow is not a suspension, and this is the case here; see Theorem 1.8 of [27]), it follows that Δ is constant. Since it is homogeneous of degree 1, the constant value of Δ is either $-\infty$ or 0. If the constant value is $-\infty$, then δ^+ and δ^- have infinite value everywhere. Then, all the (x,y,z), for (x,y) describing $\mathbf{R}^2 \setminus 0$, belong to $\mathcal{U}(\rho)$. This contradicts Lemma 4.12 and proves the proposition. □

Corollary 4.14 *The actions of $\rho(\Gamma)$ and $\rho_1(\Gamma)$ on the projective plane are topologically conjugate, ie, there is a homeomorphism $f\colon \mathbf{R}P^2 \to \mathbf{R}P^2$ satisfying the following equivariance property:*

$$\forall \gamma \in \Gamma \quad f \circ \rho_1(\gamma) = \rho(\gamma) \circ f$$

Moreover, the conjugacy f is unique up to composition on the left by homotheties.

Proof According to the Proposition 4.13, the map $\delta^+ = \delta^-$ is continuous, since it is u.s.c. and l.s.c. at the same time. Consider the following map of $\mathbf{R}^3$ minus the z–axis into itself:

$$(x,y,z) \mapsto (x,y,z+\delta^+(x,y))$$

Since δ^+ is homogeneous of degree one, and since $\delta^+(-w) = -\delta^-(w) = -\delta^+(w)$, this map induces a homeomorphism of $\mathbf{R}P^2 \setminus \{0\}$ onto itself. This homeomorphism extends as a homeomorphism f of $\mathbf{R}P^2$ onto itself by setting $f(0) = 0$. Equation $(*)$ above implies the required Γ–equivariance of f. If f' is another topological conjugacy, then $f'^{-1} \circ f$ is a transformation of $\mathbf{R}^2 \setminus 0$ commuting with the linear action of $\rho_1(\Gamma)$. Then, according to the rigidity of horocyclic flows, $f'^{-1} \circ f$ must be a homothety (see [1] for the case of geodesic flows, the case of exotic Anosov flows is similar). □

Let $\Lambda(\rho)$ be the image by f of the GL_0–invariant projective line. This is a Jordan curve.

Corollary 4.15 *The curve $\Lambda(\rho)$ is the closure of the union of the repelling fixed points of elements of $\rho(\Gamma)$. It is the complement in $\mathbf{R}P^2$ of the disc $U(\rho) \cup \{0\}$. The action of $\rho(\Gamma)$ on $\Lambda(\rho)$ is topologically conjugate to the projective action of the fuchsian group $\lambda(\rho)(\Gamma)$ on the projective plane $\mathbf{R}P^1$. The action of $\rho(\Gamma)$ on $U(\rho)$ is uniquely ergodic.*

Proof Using the equivariant map f, it is enough to check all these statements in the case of canonical flag manifolds, in which they are easily established. □

Lemma 4.16 *Two hyperbolic actions of Γ are topologically conjugate if and only if their linear parts are conjugate in GL_0. The curve $\Lambda(\rho)$ is a projective line if and only if the conjugacy f with the linear part is projective.*

Proof The first part is a corollary of the rigidity of exotic horocyclic flows. For the second part, when $\Lambda(\rho)$ is a projective line, there is a projective transformation g mapping $U(\rho)$ to $\mathbf{R}^2 \setminus 0$, and thus mapping the ρ–action of Γ to some linear action. Then, $g \circ f$ is a topological conjugacy between two linear actions. By the first part, by modifying g, we can assume that $g \circ f$ commutes with the linear action of $\rho_1(\Gamma)$ on $\mathbf{R}^2 \setminus 0$. By Lemma 4.14 it is a homothety; therefore, f is projective. □

Lemma 4.17 *If the map $\delta^+ = \delta^-$ is differentiable on a set of non zero Lebesgue measure, then the conjugacy f is a projective transformation.*

Proof The idea of the proof is due to A Zeghib. In the hypothesis of the lemma, since the action of $\rho_1(\Gamma)$ on $\mathbf{R}R^2 \setminus 0$ is uniquely ergodic, δ^+ is differentiable almost everywhere. We can then define an equivariant measurable map τ: $\mathbf{R}P^1 \to \mathbf{R}^2 \subset \mathbf{R}P^2_*$ defined almost everywhere, by associating to every $[x;y]$ the projective line tangent to $\Lambda(\rho)$ at the ray $\bar{f}([x;y]) = [x;y;\delta^+(x,y)]$: observe that $\tau([x;y])$ never contains 0. Let $\mathcal{P}$ be the product $\mathbf{R}P^1 \times \mathbf{R}P^1$ minus the diagonal. Observe that the diagonal action of $\rho(\Gamma)$ on $\mathcal{P}$ admits an ergodic invariant measure equivalent to the Lebesgue measure. We say that a subset of $\mathcal{P}$ is conull if the measure of its complement in $\mathcal{P}$ is 0. The crucial and classical observation is that this ergodicity property implies that there is no measurable equivariant map from $\mathcal{P}$ into a topological space where Γ acts freely and properly discontinuously.

Assume that the set of pairs (θ,θ') for which θ' does not belong to $\tau(\theta)$ is conull. Then, its intersection with its image by the flip map $(\theta,\theta') \mapsto (\theta',\theta)$ is conull, and its intersection with all its $\bar{\Gamma}$–iterates also. Thus, there is a conull $\rho(\Gamma)$–invariant subset $\mathcal{E}$ of $\mathcal{P}$ of pairs (θ,θ') for which the projective lines $\tau(\theta)$ and $\tau(\theta')$ intersect at some point $x(\theta,\theta')$ different from $\bar{f}(\theta)$ and $\bar{f}(\theta')$. We have then two cases: either almost every $x(\theta,\theta')$ belongs to $\Lambda(\rho)$ or almost all of them belongs to $U(\rho)$. In the first case, we obtain a $\rho(\Gamma)$–equivariant map from $\mathcal{E}$ into the set of distinct triples of points of $\mathbf{R}P^1$. Since the action of $\bar{\Gamma}$ on this set of triples is free and properly discontinuous, we obtain a contradiction with the ergodic argument discussed above. In the second case, the map associating to a pair (θ,θ') the flag $(x(\theta,\theta'),\tau(\theta))$ is an equivariant map from $\mathcal{E}$ into $X(\rho)$. According to Proposition 4.19 below, we obtain once more a contradiction with the ergodic argument.

Therefore, the measure of the set of pairs (θ, θ') for which the line $\tau(\theta)$ contains θ' is conull. Then, by Fubini's Theorem, there is an element θ of $\mathbf{R}P^1$ such that for almost all θ' in $\mathbf{R}P^1$, $\bar{f}(\theta')$ belongs to $\tau(\theta)$. But the intersection of $\Lambda(\rho)$ with $\tau(\theta)$ is closed, and $\bar{f}$ is continuous: it follows that $\Lambda(\rho)$ must be equal to $\tau(\theta)$. We conclude by applying Lemma 4.16. □

Corollary 4.18 *The Jordan curve $\Lambda(\rho)$ is Lipschitz if and only if it is a projective line, ie, if and only if the conjugacy f is projective.*

Proof this follows from Lemma 4.17 since Lipschitz maps are differentiable almost everywhere. □

4.4 Properness of the action

We define $X(\rho)$ as the intersection of X_∞ with the preimage by p_1 of $U(\rho)$.

Proposition 4.19 *The action of $\rho(\Gamma)$ on $X(\rho)$ is free and properly discontinuous.*

Proof The action of $\rho(\Gamma)$ on $U(\rho)$ is conjugate to the action of $\rho_1(\Gamma)$ on the punctured affine plane. Therefore, it is free, and the action of $\rho(\Gamma)$ on $X(\rho)$ is free. Remember also that by replacing Γ by a finite index subgroup, we can assume that all the non-trivial elements of $\rho(\Gamma)$ are hyperbolics.

Since $\rho(\Gamma)$ is discrete in Af_0^*, we just have to establish the properness of its action on $X(\rho)$. Assume *a contrario* that it is not the case: there are elements (x_n, d_n), (x'_n, d'_n) of $X(\rho)$, and elements g_n of $\rho(\Gamma)$ such that:

- $(x'_n, d'_n) = g_n(x_n, d_n)$,
- the (x_n, d_n) converge to some (x, d) in $X(\rho)$,
- the (x'_n, d'_n) converge to some (x', d') in $X(\rho)$,
- the g_n escape from any compact subset of Af_0^*.

Define $g_n^* = \theta(g_n)$: they escape from any compact subset of Af_0 too. As elements of $Af_0 \subset PGL(3, \mathbf{R})$, the g_n^* are representated by 3×3–matrices of the form:

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B_n}} & u_n \\ & & v_n \\ 0 & 0 & 1 \end{pmatrix}$$

For any vector subspace E of $\mathbf{R}^3$ (or its dual), we denote by $S(E)$ its projection in $\mathbf{R}P^2$ (or $\mathbf{R}P^2_*$). We see $GL(3,\mathbf{R})$ as a subset of $M(3,\mathbf{R})$, the algebra of 3×3–matrices. Denote by $\|\|_0$ the operator norm on $M(3,\mathbf{R})$; let $\mathcal{B}$ be the unit ball of this norm.

Extracting a subsequence if necessary, we can assume that the sequences $h^n = \frac{g_n}{\|g_n\|_0}$ and $h^*_n = \frac{g^*_n}{\|g^*_n\|_0}$ converge respectively to $\bar{g}$ and $\bar{g}^*$ in $\mathcal{B}$.

A fundamental fact is the following claim: *the norm* $\|g^*_n\|_0$ *tends to* $+\infty$. Indeed: remember the discussion in Remark 2.2. The linear part B_n is of the form $\bar{u}(\gamma_n)\rho^*_0(\gamma_n)$, where $\rho^*_0\colon \Gamma \to SL$ is the composition of the linear part of $\rho^*\colon \Gamma \to Af^*_0$ with the projection of GL_0 over SL, and $\bar{u}\colon \Gamma \to \mathbf{R}^+$ is a morphism. The projection of $\rho^*_0(\gamma)$ in PSL is the projectivised linear part $\lambda(\rho^*)(\gamma)$. Let $t(\gamma)$ be the logarithm of the spectral radius of $\rho^*_0(\gamma)$: it is also the logarithm of the spectral radius of $\lambda(\rho^*)(\gamma)$. Since $\lambda(\rho^*)(\Gamma)$ is a cocompact fuchsian group, $t(\gamma_n)$ tends to $+\infty$ when n goes to infinity. Let now $L_u(\gamma)$ be the logarithm of the absolute value of $\bar{u}(\gamma)$. Since ρ^* is hyperbolic, the stable norm of the morphism induced on $\bar{\Gamma}$ by L_u is less than $\frac{1}{2}$; let $0 < C < 1$ be the double of this norm: by definition of the stable norm, the absolute value of $L_u(\gamma_n)$ is less than $Ct(\gamma_n)$. It follows that $t(\gamma_n) \pm L_u(\gamma_n)$ is bigger than $(1-C)t(\gamma_n)$, and thus, that $t(\gamma_n) \pm L_u(\gamma_n)$ tends to $+\infty$ with n. But the absolute value of the eigenvalues of B_n are the exponentials of $t(\gamma_n) \pm L_u(\gamma_n)$ and of $-t(\gamma_n) \pm L_u(\gamma_n)$. It follows that one of these eigenvalues tends to $+\infty$, and therefore, that the norm of B_n tends to $+\infty$.

Hence, $\bar{g}^*$ is of the form:

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B}} & u \\ & & v \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, the image I^* and the kernel K^* of $\bar{g}^*$ are proper subspaces of $\mathbf{R}^3$, and $S(I^*)$ is contained in the line at infinity.

Similar considerations show that the norm of g_n tends to $+\infty$, that the image I and the kernel K of $\bar{g}$ are proper subspaces, and that $S(K)$ contains the point 0.

For every index n, the products $h^t_n h^*_n$ and $h^*_n h^t_n$ (where h^t_n is the transposed matrix of h_n) are both equals to $\frac{id}{\|g_n\|_0\|g^*_n\|_0}$. Hence, when n goes to infinity, we obtain:

$$\bar{g}^t \circ \bar{g}^* = 0 = \bar{g}^* \circ \bar{g}^t$$

The transposed matrix $\bar{g}^t$ of $\bar{g}$ has to be considered as a linear endomorphism of the dual of $\mathbf{R}^3$: $\bar{g}^t$ maps a linear form φ on $\varphi \circ \bar{g}$. The elements of its kernel are the linear forms whose kernels contain I, and the elements of its image are the linear forms whose kernels contain K. From the equalities above, we obtain that *any element d of $S(I^*)$, viewed as a projective line in $\mathbf{R}P^2$, contains $S(I)$, and that any element d of $\mathbf{R}P^2_*$ containing $S(K)$ necessarily belongs to $S(K^*)$.*

Observe that $\bar{g}$ (respectively $\bar{g}^*$) defines a map from $\mathbf{R}P^2 \setminus S(K)$ (respectively $\mathbf{R}P^2_* \setminus S(K^*)$) into $S(I) \subset \mathbf{R}P^2$ (respectively $S(I^*) \subset \mathbf{R}P^2_*$). We claim: *$g_n$ converges on $\mathbf{R}P^2 \setminus S(K)$ to the map $\bar{g}$. This convergence is uniform on the compact subsets of $\mathbf{R}P^2 \setminus S(K)$.* Indeed: consider any compact subset $\mathcal{K}$ in $\mathbf{R}P^2 \setminus S(K)$. We denote by $\|$ the euclidean norm of $\mathbf{R}^3$. Observe that g_n and $h_n = \frac{g_n}{\|g_n\|_0}$ have the same actions on $\mathbf{R}P^2$. For every element x of $\mathbf{R}^3$ representing an element of $\mathcal{K}$ we have:

$$\begin{aligned}\left\|\frac{h_n(x)}{\|h_n(x)\|} - \frac{\bar{g}(x)}{\|\bar{g}(x)\|}\right\| &\leq \left\|\frac{h_n(x)}{\|h_n(x)\|} - \frac{h_n(x)}{\|\bar{g}(x)\|}\right\| + \left\|\frac{h_n(x)}{\|\bar{g}(x)\|} - \frac{\bar{g}(x)}{\|\bar{g}(x)\|}\right\| \\ &\leq \frac{\big|\|h_n(x)\| - \|\bar{g}(x)\|\big|}{\|\bar{g}(x)\|} + \frac{\|h_n(x) - \bar{g}(x)\|}{\|\bar{g}(x)\|} \\ &\leq \frac{2}{\|\bar{g}(x)\|}\|h_n(x) - \bar{g}(x)\|\end{aligned}$$

The claim now follows from the fact that $\|\bar{g}(x)\|$ is bounded from below by a positive constant valid for all the points of the unit ball of $\mathbf{R}^3$ representing elements of $\mathcal{K}$.

The similar property for g_n^* is also true.

It follows that if d does not belong to $S(K^*)$, then d' belongs to $S(I^*)$. But this is impossible since $S(I^*)$ is contained in the line at infinity. Hence, d belongs to $S(K^*)$. Assume that $S(K^*)$ is reduced to $\{d\}$. Then, according to the property of uniform convergence, we see that for any small disc D in $\mathbf{R}P^2_*$ containing d, the closure of D is contained in g_n^*D when n is sufficiently big. It follows that g_n^* admits a repelling fixed point in D: contradiction.

Thus, $S(K^*)$ is a projective line, and $S(I^*)$ a single point. Consider now $S(K)$: it contains 0. We have seen that any projective line containing $S(K)$ belongs to $S(K^*)$: therefore, if $S(K)$ was reduced to $\{0\}$, $S(K^*)$ would be the line d_∞: this is impossible since d belongs to $S(K^*)$ and not to d_∞. Therefore, $S(K)$ is a projective line, and $S(I)$ a single point. The Jordan curve $\Lambda(\rho)$ does not contain 0, therefore, it is not $S(K)$. It follows that there is a point x_0 of $\Lambda(\rho)$ which does not belong to $S(K)$. The iterates $g_n x_0$ all belong to $\Lambda(\rho)$ and converge to $S(I)$: it follows that $S(I)$ belongs to $\Lambda(\rho)$. Hence, x' is not $S(I)$, this implies that x belongs to $S(K)$. We use once more the fact that any projective line containing $S(K)$ belongs to $S(K^*)$: this shows that $S(K)$, as a point in $\mathbf{R}P^2_*$, belongs to $S(K^*)$. Dually, $S(K^*)$, as a point in $\mathbf{R}P^2$, belongs to $S(K)$. Then, d and $S(K)$ have two distinct points in common: x and the point $S(K^*)$. They must be equal. But $S(K)$ contains 0, and, by hypothesis, d does not contain 0. This is a contradiction. $\square$

Since $U(\rho)$ is topologically an annulus, $X(\rho)$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^2$. It follows that the quotient $M(\rho)$ of $X(\rho)$ by $\rho(\Gamma)$ is a $K(\Pi, 1)$. We deduce from homological considerations that $M(\rho)$ is a *compact* 3–manifold. Now, $\mathcal{D}$ induces a local homeomorphism of M in $M(\rho)$. Since both are compact 3–manifolds, this induced map is a finite covering. We have proved Theorem B.

5 The tautological foliations

In this section, we study the tautological foliations associated to Goldman flag structures. We are only interested in dynamical properties which are not perturbed by finite coverings. Therefore, we can, and we do, assume that M is the quotient of $X(\rho)$ by $\rho(\Gamma)$, where ρ is the holonomy morphism. Therefore, the holonomy group is isomorphic to $\bar{\Gamma}$, the quotient of the fundamental group Γ by its center H. From now on, we denote $\rho(\Gamma)$ by $\bar{\Gamma}$. We can assume that $\bar{\Gamma}$ has no torsion.

We call Ψ the first tautological foliation, and Φ the second one. We will see that their dynamical behaviors are quite different. We call their liftings in the covering $X(\rho)$ of M, $\hat{\Psi}$ and $\hat{\Phi}$. Observe that these foliations are orientable since Af_0 preserves any orientation of $\mathbf{R}^2$.

5.1 Study of the first tautological foliation

We need to consider another foliation on M: as a topological manifold, M is homeomorphic to the left quotient of SL by the linear part $\bar{\Gamma}_0 \subset GL_0$ of $\bar{\Gamma}$. On this quotient, which we denote by M_l, we have the horocyclic flow Ψ_0^t, induced by the right action of unipotent matrices. Observe that the operation of "taking the linear part" defines an isomorphism $\bar{\Gamma} \to \bar{\Gamma}_0$.

Theorem 5.1 *Ψ is topologically conjugate to the horocyclic foliation Ψ_0.*

Proof According to Corollary 4.15, there is a topological conjugacy f between the action of $\bar{\Gamma}_0$ on $\mathbf{R}^2 \setminus \{0\}$ and the action of $\bar{\Gamma}$ on $U(\rho)$. But the pairs $(\mathbf{R}^2 \setminus \{0\}, \bar{\Gamma}_0)$ and $(U(\rho), \bar{\Gamma})$ can be interpreted as the leaf spaces of Ψ_0 and Ψ respectively. The holonomy covering of the leaves of Ψ and Ψ_0 are contractible.

Hence, (M_l, Ψ_0) and (M, Ψ) are representatives of the classifying spaces of their transverse holonomy groupoids. $(\mathbf{R}^2 \setminus \{0\}, \bar{\Gamma})$ and $(U(\rho), \bar{\Gamma})$ are also representatives of these classifying spaces. Since they are conjugate, and by uniqueness of classifying spaces modulo equivalence, *there exists a homotopy equivalence $F: M_l \to M$ mapping every leaf of Ψ_0 into a leaf of Ψ.* Moreover, F lifts to some mapping $\hat{F}$ between the coverings X_0 and $X(\rho)$, which induces f at the level of the leaf spaces. In particular, F maps two different leaves of Ψ_0 into two different leaves of Ψ (for the notion of classifying spaces, and for all the arguments used here, we refer to [20]). The problem is that this map has no reason to be injective along the leaves of Ψ_0.

We will modify F along the leaves of Ψ_0 in order to correct this imperfection. This idea of diffusion process along the leaves seems due to M Gromov. It has been used in [4], [25], and previously in [16].

First, we choose arbitrary parametrisations Ψ_0^t and Ψ^s of the foliations. Since Ψ_0^t has no periodic orbit, we have a continuous map $u: M_l \times \mathbf{R} \to \mathbf{R}$ satisfying:

$$\forall t \in \mathbf{R} \quad \forall x \in M \qquad F(\Psi_0^t x) = \Psi^{u(t,x)}(F(x))$$

u is a cocycle, ie, for every element x of M_l:

$$\begin{array}{rrcl} \forall s, t \in \mathbf{R} & u(t+s, x) & = & u(t, \Psi_0^s x) + u(s, x) \\ \forall t \in \mathbf{R} & u(0, x) & = & 0 \end{array}$$

The main lemma is:

Lemma 5.2 *There is a real $T > 0$ such that, for any element x of M_l, the quantity $u(T, x)$ is not zero.*

Assume that Lemma 5.2 is true. Let T be the real given by the lemma. We define:

$$\forall x \in M_l \quad u_T(x) = \frac{1}{T} \int_0^T u(s, x)\, ds$$

Then, we define $F_T: M_l \to M$:

$$F_T(x) = \Psi^{u_T(x)}(F(x))$$

This map has the same properties than F. Moreover

$$F_T(h^t(x)) = \Psi^{v_T(t,x)}(F_T(x))$$

where:

$$v_T(t, x) = \frac{1}{T} \int_t^{T+t} u(s, x)\, ds$$

The derivation of v_T with respect to t is:

$$\begin{aligned}\frac{\partial}{\partial t}v_T(t,x) &= \frac{1}{T}[u(T+t,x)-u(t,x)]\\ &= \frac{1}{T}u(T,\Psi_0^t(x))\end{aligned}$$

According to our choice of T, this is never zero. It follows that F_T is injective along the leaves of Ψ_0, and therefore injective. Since it is a homotopy equivalence, it is a topological conjugacy between Ψ_0 and Ψ. Therefore, in order to finish the proof of Theorem 5.1, we just have to prove 5.2:

Proof of 5.2 Assume that Lemma 5.2 is not true. Then, there is a sequence of increasing real numbers t_n, converging to $+\infty$, and a sequence of points x_n in M_l such that the $u(t_n,x_n)$ are zero. We can assume that x_n converges to some point of M_l. Remember that SL is naturally identified with X_0. The x_n lift in X_0 to pairs (y_n,d_n) where the y_n are points in $\mathbf{R}^2\setminus\{0\}$ and the d_n are projective lines through y_n (but not 0) converging to some element $(\bar{y},\bar{d})$ of X_0. The $\Psi_0^{t_n}(x_n)$ lift to pairs (y_n,d_n'). Since t_n go towards infinity, and since the y_n converge to $\bar{y}$, the d_n' converge to the projective line containing both 0 and $\bar{y}$. Now, the nullity of $u(x_n,t_n)$ means that the $\hat{F}(y_n,d_n)$ and $\hat{F}(y_n,d_n')$ are equal for every integer n. We denote by (y_n',d_n'') this common value.

Since M_l is compact, there are elements γ_n of $\bar{\Gamma}$ such that (γ_0^n being the linear part of $\rho(\gamma_n)$) the $\gamma_0^n(y_n,d_n')$ converge to an element $(\bar{y}_\infty,\bar{d}_\infty)$ of X_0. Denote by $(\bar{y}',\bar{d}')$ and $(\bar{y}'_\infty,\bar{d}'_\infty)$ the images by $\hat{F}$ of $(\bar{y},\bar{d})$ and $(\bar{y}_\infty,\bar{d}_\infty)$. Then, we have:

- the (y_n',d_n'') converge to the element $(\bar{y}',\bar{d}')$ of $X(\rho)$,
- the $\gamma_n(y_n',d_n'')$ converge to the element $(\bar{y}'_\infty,\bar{d}'_\infty)$ of $X(\rho)$.

According to Proposition 4.19, the γ_n are finite in number. But this is impossible, since the d_n' converge to the projective line $(\bar{y},0)$ and the $\gamma_0^n d_n'$ converge to the projective line $\bar{d}$. The lemma and the theorem are proven. □

Remark 5.3 According to Lemma 4.17, we have found a new family of different differentiable structures on M_l for which the horocyclic foliation remains analytic. The non-triviality of the moduli of differentiable structures for a given foliation is never an easy task; this problem has to be compared with the fact that on a given closed surface there is one and only one differentiable structure. Up to our knowledge, the only examples of foliations with many differentiable structures previously known were the structurally stable ones, and horocyclic foliations are very far from being structurally stable!

5.2 Study of the first tautological foliation: the Goldman foliation

We prove here the Theorem C. It is an immediate consequence of Lemmas 5.6, 5.7 and 5.8 below. Let Φ be a Goldman foliation on a *pure* Goldman manifold M. Fix any parametrisation Φ^t of Φ, and any auxiliary Riemannian metric on M.

Definition 5.4 The flow Φ^t is called *non-expansive* at a point x of M if, for every $\epsilon > 0$, there is an element y of M and an increasing homeomorphism $v: \mathbf{R} \to \mathbf{R}$ such that:

- y is not on the Φ^t–orbit of x,
- for any time t, the distance between $\Phi^t(x)$ and $\Phi^{v(t)}(y)$ is less than ϵ.

The set of points where Φ^t is non-expansive is called the *non-expansiveness locus*, and denoted by $\mathcal{N}$. Its complement in M is called the *expansiveness locus* of Φ^t, and denoted by $\mathcal{E}$. The sets $\mathcal{E}$ and $\mathcal{N}$ are both Φ^t–invariant.

Definition 5.5 For any element (x,d) of $X(\rho)$, the connected component of $d \cap U(\rho)$ containing d is denoted by $]\alpha(x,d), \beta(x,d)[$.

Observe that $\alpha(x,d)$ and $\beta(x,d)$ are elements of $\Lambda(\rho)$. We choose the notation so that $\beta(x,d)$ (respectively $\alpha(x,d)$) is the limit when t goes to $+\infty$ (respectively $-\infty$) of $p_1(\hat{\Phi}^t(x,d))$.

Lemma 5.6 *If $\alpha(x,d) \neq \beta(x,d)$, then the projection m of (x,d) in M belongs to the expansiveness locus.*

Proof A rigorous and detailed exposition would be long and tedious. We prefer to indicate the main argument.

Let K be a compact fundamental domain for the action of $\bar{\Gamma}$ on $X(\rho)$. We consider an ellipse field $\hat{E}$ on K similar to the ellipse field introduced in the section 4.1: we fix a euclidean metric on $\mathbf{R}^2 \subset \mathbf{R}P^2_*$, ie, an ellipse E_0 and $\hat{E}(x,d)$ is an open neighborhood of (x,d), on which p_2 is a homeomorphism with image the translated of E_0 centered at d. We extend $\hat{E}$ on the whole $X(\rho)$: for any element (x,d) of $X(\rho)$, the ellipse $\hat{E}(x,d)$ is $\gamma^{-1}\hat{E}(y,d')$, where $\gamma(x,d) = (y,d')$ (it can be multidefined for some (x,d), but this has no incidence for our reasoning). Define $E^t = p_2(\hat{E}(\hat{\Phi}^t(x,d)))$: they are ellipses in $\mathbf{R}^2$, centered at d. Moreover, they all contain a subsegment σ_t centered at d and of length at least 2ϵ (for the auxiliary euclidean metric on $\mathbf{R}^2$).

For every t, let F_t be the p_2–projection of the leaf of $\widetilde{\mathcal{F}}$ containing $\hat{\Phi}^t(x,d)$. It is a half-plane containing E^t, bounded by some line $x(t)$. When t goes towards $+\infty$, the lines $x(t)$ converge to the line $\beta(x,d)$. Since x belongs to this limit line, we see that the ellipses E^t are more and more flattened, and converge to the "degenerated ellipse" $x^+ = \beta(x,d)$.

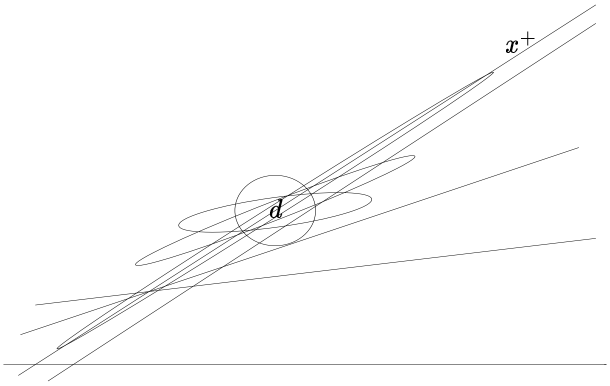

Figure 2: Ellipses are flattened

By the same argument, we see that when t goes to $-\infty$, the ellipses converge to $x^- = \alpha(x,d)$. When $\alpha(x,d) \neq \beta(x,d)$, as is assumed in this lemma, the lines x^+ and x^- are transverse one to the other. Therefore, the intersection of all the ellipses E^t as t varies over all the real numbers, the positive and the negative, reduces to d. It follows that (x,d) belongs to the expansiveness locus: indeed, if the Φ^t–orbit of a point (x',d') remains near the Φ^t–orbit of (x,d), the projection $p_2(x',d') = d'$ must belong to all the ellipses E^t, and, therefore, d' is equal to d. □

Let $\hat{W}$ be the interior of the set of elements (x,d) of $X(\rho)$ for which $\alpha(x,d) \neq \beta(x,d)$. It projects to an open subset W of M. Let $\mathcal{M}$ be the complement of W in M.

Lemma 5.7 *$\hat{W}$ is not empty.*

Proof Let γ_0 be any element of $\bar{\Gamma}$. In $\mathbf{R}P^2$, it admits 3 fixed points: 0, which is of saddle type and two others which are contained in some projective line d in $\mathbf{R}P^2$ which is γ_0–invariant. Observe that d meets $U(\rho)$: if not, $\Lambda(\rho)$ would be contained in d, and we excluded this case while restricting ourselves to pure Goldman structures.

Let x be any element of $d \cap U(\rho)$. Then, (x,d) belongs to $X(\rho)$. If $\alpha(x,d) = \beta(x,d)$, then $d \cap U(\rho)$ is a complete affine line. Therefore, it must contain at least one fixed point of γ_0, but this is impossible since the fixed points of γ_0 are outside $U(\rho)$.

Let c be the subarc of $\Lambda(\rho)$ bounded by $\alpha(x,d)$ and $\beta(x,d)$ such that the image of c by the projection of $\mathbf{R}P^2 \setminus 0$ along the leaves of $\mathcal{G}_0$ coincides with the image of $[\alpha(x,d), \beta(x,d)]$. The union of c with $[\alpha(x,d), \beta(x,d)]$ is a Jordan curve bounding some open subset V of $U(\rho)$ (see figure 3). Obviously, the points (x',d') where x' belongs to V and d' is a projective line which doesn't meet 0 or $[\alpha(x,d), \beta(x,d)]$ all satisfy $\alpha(x',d') \neq \beta(x',d')$, and their union is an open subset of $X(\rho)$. The lemma follows. □

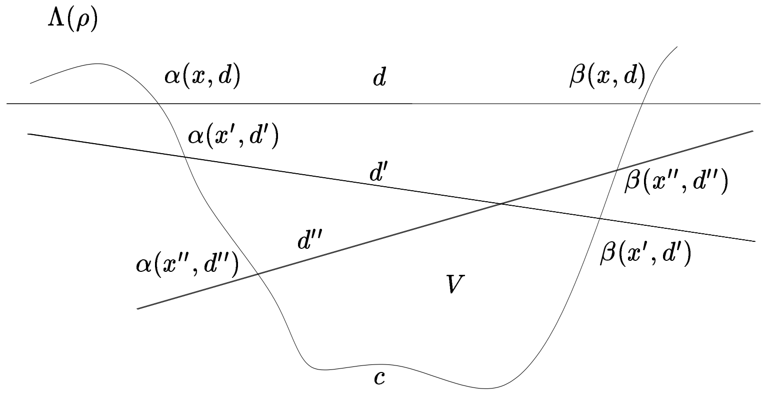

Figure 3: Exhibiting elements of W

Lemma 5.8 *$\hat{W}$ is not the whole of $X(\rho)$.*

Proof If not, by Lemma 5.6, the flow Φ^t is expansive. According to [8], it is topologically equivalent to an Anosov flow. By [16], up to finite coverings, Φ^t is topologically equivalent to the geodesic flow on the unitary tangent bundle of a hyperbolic riemmanian surface S. There are many ways to see the impossibility of that. For example: up to finite coverings, the flow $\hat{\Psi}^t$ on $X(\rho)$ must be topologically equivalent to the geodesic flow of the Poincaré disc, and the orbit space of this geodesic flow is the complement of the diagonal in $\mathbf{R}P^1 \times \mathbf{R}P^1$. Therefore, the leaf space $Q_{\hat{\Phi}}$ of $\hat{\Psi}$ is homeomorphic to the annulus, in particular, it satisfies the Hausdorff separation property.

We observe now that Ψ^t is topologically equivalent to its inverse $\hat{\Psi}^{-t}$ (this property is valid for any $\mathbf{R}$–covered Anosov flow whithout cross-section, see

Theorem C of [4]). In particular, for every perodic orbit, there is another periodic orbit which is freely homotopic to the initial periodic orbit, but with the inverse orientation. Let γ_0 be any element of $\bar{\Gamma}$ preserving an orbit θ_1 of $\hat{\Psi}$ (it exists since the geodesic flow has many periodic orbits). According to the discussion above, there is another orbit θ_2 which is preserved by γ_0. Let x_1 and x_2 be respectively the repelling and attracting fixed point of γ_0 in $\mathbf{R}P^2$, and let $d_0 = (x_1, x_2)$ be the projective line containing them. It is the unique γ_0–invariant projective line which does not contain 0, therefore, it must contain $p_1(\theta_1)$ and $p_1(\theta_2)$. By γ_0–invariance, it follows that $p_1(\theta_1)$ and $p_1(\theta_2)$ are the two connected components of $d_0 \setminus \{x_1, x_2\}$. Consider now the intersections of regular leaves of $\mathcal{G}_0$ with $U(\rho)$: we see them as oriented rays starting from 0 and reaching $\Lambda(\rho)$. There are four γ_0–invariant rays, ending at x_1 and x_2. The others rays falls into the two following exclusive possibilities:

- they meet $p_1(\theta_1)$ or $p_1(\theta_2)$,
- they meet $\Lambda(\rho)$ before meeting $p_1(\theta_1)$ or $p_1(\theta_2)$.

It is easy to see that the union of the rays of the first category contains the union of two triangles with vertices 0, x_1 and x_2. In other words, there is an affine half-plane T contained in $U(\rho)$ bounded by d_0 and another projective line containing 0 and x_i (where $i = 1$ or 2). Let d be any projective line containing x_i and intersecting T: the affine line $d \setminus \{x_i\}$ is the projection by p_1 of some leaf of $\hat{\Psi}$. This line d can be arbitrarly close to d_0, and thus, arbitrarly close to $p_1(\theta_1)$ and $p_1(\theta_2)$. This is in contradiction with the Hausdorff separation property in $Q_{\hat{\Psi}}$. □

Remark 5.9 The proof of 5.8 we propose here uses very deep results. We don't know a more elementary one.

6 Conclusion

Pure Goldman foliations are good examples of analytic foliations, satisfying strong properties, but which remain quite mysterious. Many questions remain open. It would be worthwhile to know a little more about them.

Question 1 Does a Goldman foliation admit periodic orbits? Observe that such a periodic orbit is necessarily hyperbolic. Observe also that if a pure Goldman foliation has no periodic orbits and every element of the affine holonomy group is of determinant 1, it would be a non-minimal analytical volume preserving foliation without periodic orbit on a 3–manifold. Such foliations are not so easy to construct, the first known example being the Kuperberg foliation [23].

Question 2 We proved that a pure Goldman foliation is not minimal by exhibiting a non-trivial closed invariant subset $\mathcal{M}$. Is $\mathcal{M}$ itself minimal? How can we describe the dynamic of the Goldman foliation on $\mathcal{M}$?

Question 3 Being conjugate to horocyclic foliations, the first tautological foliations of pure Goldman flag manifolds are uniquely ergodic: there is a unique invariant measure. When is this measure absolutely continuous with respect to the Lebesgue measure?

Question 4 What are the ergodic properties of Goldman foliations?

Question 5 We can suspect, from the expansiveness of a pure Goldman flow outside $\mathcal{M}$, that its measure entropy is positive. Is this true? This question is related to the question 1, since, according to a theorem by A Katok, the entropy of a regular flow on a closed 3–manifold without periodic orbit is zero [21]. The positivity of the entropy would follow if we could show that the Lyapounov exponents are not all zero almost everywhere. The paper [19] of Y Guivarc'h establishes some results in this direction. Unfortunately, they apply to groups of projective transformations which do not preserve any projective subspaces, which is certainly not the case for the groups we have considered here.

Question 6 We know that the Jordan curve $\Lambda(\rho)$ is not Lipschitz. But we can wonder what is its regularity. Is it Hölder? Is it rectifiable?

Question 7 In Theorem B, can we withdraw the assumption forcing the image of the developing image to be contained in X_∞? In other words, is it true that any flag structure on a Seifert manifold, for which the holonomy morphism is hyperbolic, is a finite covering of M_H? The answer is expected to be yes.

Question 8 We only considered deformations of holonomy groups inside Af_0. What happens for general deformations inside the whole $SL(3,\mathbf{R})$? Do they still act freely and properly discontinuously on some open subset of X with compact quotient?

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